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Lehmer-type congruences for lacunary harmonic sums modulo p^2

by

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1. Introduction. Wolstenholme's well-known harmonic series congruence asserts that

$$\sum_{k=1}^{p-1} \frac{1}{k} \equiv 0 \pmod{p^2} \tag{1.1}$$

for each prime $p \geq 5$. With the help of (1.1), Wolstenholme [9] proved that

$$\binom{mp}{np} \equiv \binom{m}{n} \pmod{p^3}$$

for any $m, n \geq 1$ and prime $p \geq 5$. In 1938, Lehmer [2] discovered the following interesting congruence:

$$\sum_{j=1}^{(p-1)/2} \frac{1}{j} \equiv -\frac{2^p - 2}{p} + \frac{(2^{p-1} - 1)^2}{p} \pmod{p^2}$$

for each prime $p \geq 3$.

Define

$$\mathcal{H}_{r,m}(n) = \sum_{\substack{1 \leq k \leq n \\ k \equiv r \ (\mathrm{mod}\ m)}} \frac{1}{k}.$$

Clearly, with the help of (1.1), Lehmer's congruence can be rewritten as

$$\mathcal{H}_{p,2}(p-1) \equiv \frac{2^{p-1} - 1}{p} - \frac{(2^{p-1} - 1)^2}{2p} \pmod{p}. \tag{1.2}$$

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In fact, Lehmer also proved three other congruences of the same nature:

$$\mathcal{H}_{p,3}(p-1) \equiv \frac{3^{p-1}-1}{2p} - \frac{(3^{p-1}-1)^2}{4p} \pmod{p^2}, \tag{1.3}$$

$$\mathcal{H}_{p,4}(p-1) \equiv \frac{3(2^{p-1}-1)}{4p} - \frac{3(2^{p-1}-1)^2}{8p} \pmod{p^2}, \tag{1.4}$$

$$\begin{aligned}\mathcal{H}_{p,6}(p-1) \equiv{}& \frac{2^{p-1}-1}{3p} + \frac{3^{p-1}-1}{4p} - \frac{(2^{p-1}-1)^2}{6p} \\ &- \frac{(3^{p-1}-1)^2}{8p} \pmod{p^2},\end{aligned} \tag{1.5}$$

where $p \ge 5$ is a prime. The proofs of (1.2)–(1.5) are based on the values of the Bernoulli polynomial $B_{p(p-1)}(x)$ at $x = 1/2, 1/3, 1/4, 1/6$.

However, no other congruence for $\mathcal{H}_{p,m}(p-1)$ modulo p^2 is known, mainly because there are no known closed forms for $B_{p(p-1)}(n/m)$ when $m \ne 1, 2, 3, 4, 6$. Some Lehmer-type congruences modulo p (not modulo p^2!) have been proved in [3]–[8]. In this paper, we shall investigate Lehmer-type congruences modulo p^2.

Define

$$\mathcal{T}_{r,m}(n) = \sum_{\substack{0\le k\le n \\ k\equiv r \pmod m}} \binom{n}{k} \quad \text{and} \quad \mathcal{T}^*_{r,m}(n) = \sum_{\substack{0\le k\le n \\ k\equiv r \pmod m}} (-1)^k \binom{n}{k}.$$

Clearly $\mathcal{T}^*_{r,m}(n) = (-1)^n \mathcal{T}^*_{n-r,m}(n)$ and

$$\mathcal{T}^*_{r,m}(n) = \begin{cases} (-1)^r \mathcal{T}_{r,m}(n) & \text{if } m \text{ is even,} \\ (-1)^r(\mathcal{T}_{r,2m}(n) - \mathcal{T}_{m+r,2m}(n)) & \text{if } m \text{ is odd.} \end{cases}$$

As we shall see soon, if $p \ne m$, it is not difficult to show that

$$\mathcal{H}_{r,m}(p-1) \equiv \frac{\delta_{r,m}(p) - \mathcal{T}^*_{r,m}(p)}{p} \pmod{p},$$

where

$$\delta_{r,m}(p) = \begin{cases} 1 & \text{if } r \equiv 0 \pmod{m}, \\ -1 & \text{if } r \equiv p \pmod{m}, \\ 0 & \text{otherwise.} \end{cases} \tag{1.6}$$

THEOREM 1.1. *Let $m \ge 2$ be an integer and let $p \ge 5$ be a prime with $p \ne m$. Then*

$$\mathcal{H}_{p,m}(p-1) \equiv -\frac{2\mathcal{T}^*_{p,m}(p) + 2}{p} + \frac{\mathcal{T}^*_{p,m}(2p) + 2}{4p} \pmod{p^2}. \tag{1.7}$$

Let us see how (1.2) follows from Theorem 1.1. Clearly, $\mathcal{T}^*_{0,2}(n) = 2^{n-1}$ and $\mathcal{T}^*_{1,2}(n) = -2^{n-1}$. Hence in view of (1.7), for any prime $p \ge 5$,

$$\begin{aligned}\mathcal{H}_{p,2}(p-1) &\equiv -\frac{2\mathcal{T}^*_{p,2}(p)+2}{p} + \frac{\mathcal{T}^*_{p,2}(2p)+2}{4p} \\ &= \frac{2^p-2}{p} - \frac{2^{2p-1}-2}{4p} \pmod{p^2}.\end{aligned}$$

In [7], Sun showed that $\mathcal{T}_{r,m}(n)$ can be expressed in terms of some linearly recurrent sequences with orders not exceeding $\phi(m)/2$, where ϕ is the Euler totient function. Thus in view of Theorem 1.1, for each m, we always have a Lehmer-type congruence for $\mathcal{H}_{p,m}(p-1)$ modulo p^2, involving some linearly recurrent sequences.

However, as we shall see later, (1.7) is not suitable to derive (1.3), (1.4) and (1.5). So we need the following theorem.

Theorem 1.2. *Let $m \geq 2$ be an integer and let $p \geq 3$ be a prime with $p \neq m$. Then*

$$\begin{aligned}(1.8)\qquad &\mathcal{H}_{p,m}(p-1) \\ &\equiv -\frac{\mathcal{T}^*_{p,m}(2p)+2}{4p} - \frac{p}{2}\sum_{\substack{1\leq r\leq m\\ 2r\not\equiv p \,(\mathrm{mod}\, m)}} \mathcal{H}_{r,m}(p-1)^2 \pmod{p^2}.\end{aligned}$$

When $m=3$, we have $\mathcal{T}^*_{p,3}(2p) = -2\cdot 3^{p-1}$ (cf. [3, Theorem 1.9] and [7, Theorem 3.2]). Thus by (1.8), we get

$$\begin{aligned}\mathcal{H}_{p,3}(p-1) &\equiv -\frac{\mathcal{T}^*_{p,3}(2p)+2}{4p} - p\left(\frac{\mathcal{T}^*_{p,3}(2p)+2}{4p}\right)^2 \\ &= \frac{3^{p-1}-1}{2p} - \frac{(3^{p-1}-1)^2}{4p} \pmod{p^2},\end{aligned}$$

since

$$\mathcal{H}_{0,3}(p-1) \equiv -\mathcal{H}_{p,3}(p-1) \equiv \frac{\mathcal{T}^*_{p,3}(2p)+2}{4p} \pmod{p}.$$

Let us apply Theorem 1.2 to obtain more congruences of Lehmer's type. The Fibonacci numbers $F_0, F_1, F_2, \ldots$ are given by $F_0 = 0$, $F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for every $n \geq 2$. It is well-known that $F_p \equiv \left(\frac{5}{p}\right) \pmod{p}$ and $F_{p-(\frac{5}{p})} \equiv 0 \pmod{p}$ for any prime $p \neq 2, 5$, where $\left(\frac{\cdot}{p}\right)$ is the Legendre symbol. Williams [8] proved that

$$\frac{2}{5}\sum_{1\leq k\leq 4p/5-1} \frac{(-1)^k}{k} \equiv \frac{F_{p-(\frac{5}{p})}}{p} \pmod{p}$$

for prime $p \neq 2, 5$. Subsequently Sun and Sun [6, Corollary 3] proved that

$$\mathcal{H}_{2p,5}(p-1) \equiv -\mathcal{H}_{-p,5}(p-1) \equiv -\frac{F_{p-(\frac{5}{p})}}{2p} \pmod{p}. \tag{1.9}$$

We have the following mod p^2 congruence involving Fibonacci numbers.

THEOREM 1.3. *Suppose that $p > 5$ is a prime. Then*

$$\mathcal{H}_{p,5}(p-1) \equiv \frac{5^{(p-1)/2}F_p - 1}{p} - \frac{5^{p-1}F_{2p-(\frac{5}{p})} - 1}{4p} \pmod{p^2}. \tag{1.10}$$

The Pell numbers $P_0, P_1, P_2, \ldots$ are given by $P_0 = 0$, $P_1 = 1$ and $P_n = 2P_{n-1} + P_{n-2}$ for $n \geq 2$. It is known that $P_p \equiv \left(\frac{2}{p}\right) \pmod{p}$ and $P_{p-(\frac{2}{p})} \equiv 0 \pmod{p}$ for every odd prime p. In [4], Sun proved that

$$(-1)^{(p-1)/2} \sum_{1\leq k\leq (p+1)/4} \frac{(-1)^k}{2k-1} \equiv -\frac{1}{4}\sum_{k=1}^{(p-1)/2} \frac{2^k}{k} \equiv \frac{P_{p-(\frac{2}{p})}}{p} \pmod{p}$$

for any odd prime p. Similarly, we have a Lehmer-type congruence involving Pell numbers.

THEOREM 1.4. *Suppose that $p > 3$ is a prime. Then*

$$\begin{aligned}\mathcal{H}_{p,8}(p-1) \equiv\ & \frac{2^{2p-4} + 2^{p-3} + 2^{(p-3)/2}P_p - 1}{p} \\ & - \frac{2^{4p-6} + 2^{2p-4} + 2^{p-2}P_{2p-(\frac{2}{p})} - 1}{4p} \pmod{p^2}.\end{aligned} \tag{1.11}$$

We shall prove Theorems 1.1 and 1.2 in Section 2, and the proofs of Theorems 1.3 and 1.4 will be given in Section 3.

2. Proof of Theorems 1.1 and 1.2

LEMMA 2.1. *Suppose that p is a prime. Then*

$$\begin{aligned}&\frac{1}{p} \sum_{\substack{1\leq k\leq p-1 \\ k\equiv r \pmod{m}}} (-a)^k \binom{p}{k} \\ &\qquad \equiv -\sum_{\substack{1\leq k\leq p-1 \\ k\equiv r \pmod{m}}} \frac{a^k}{k} + p \sum_{\substack{1\leq j<k\leq p-1 \\ k\equiv r \pmod{m}}} \frac{a^k}{jk} \pmod{p^2}\end{aligned} \tag{2.1}$$

and

$$\frac{1}{2p}\sum_{\substack{1\le k\le 2p-1,\,k\ne p\\ k\equiv r\,(\mathrm{mod}\,m)}}(-a)^k\binom{2p}{k}$$
$$\equiv -\sum_{\substack{1\le k\le p-1\\ k\equiv r\,(\mathrm{mod}\,m)}}\frac{a^k}{k}-\sum_{\substack{1\le k\le p-1\\ k\equiv 2p-r\,(\mathrm{mod}\,m)}}\frac{a^{2p-k}}{k}$$
$$+2p\sum_{\substack{1\le j<k\le p-1\\ k\equiv r\,(\mathrm{mod}\,m)}}\frac{a^k}{jk}+2p\sum_{\substack{1\le j<k\le p-1\\ k\equiv 2p-r\,(\mathrm{mod}\,m)}}\frac{a^{2p-k}}{jk}\pmod{p^2}. \tag{2.2}$$

Proof. We have

$$\frac{1}{p}\sum_{\substack{1\le k\le p-1\\ k\equiv r\,(\mathrm{mod}\,m)}}(-a)^k\binom{p}{k}=\sum_{\substack{1\le k\le p-1\\ k\equiv r\,(\mathrm{mod}\,m)}}\frac{(-a)^k}{k}\prod_{j=1}^{k-1}\left(\frac{p}{j}-1\right)$$
$$\equiv -\sum_{\substack{1\le k\le p-1\\ k\equiv r\,(\mathrm{mod}\,m)}}\frac{a^k}{k}+\sum_{\substack{2\le k\le p-1\\ k\equiv r\,(\mathrm{mod}\,m)}}\frac{a^k}{k}\sum_{j=1}^{k-1}\frac{p}{j}\pmod{p^2}.$$

Similarly,

$$\frac{1}{2p}\sum_{\substack{1\le k\le 2p-1,\,k\ne p\\ k\equiv r\,(\mathrm{mod}\,m)}}(-a)^k\binom{2p}{k}$$
$$=\sum_{\substack{1\le k\le p-1\\ k\equiv r\,(\mathrm{mod}\,m)}}\frac{(-a)^k}{k}\binom{2p-1}{k-1}+\sum_{\substack{1\le k\le p-1\\ k\equiv 2p-r\,(\mathrm{mod}\,m)}}\frac{(-a)^{2p-k}}{2p-k}\binom{2p-1}{k}.$$

We have

$$\sum_{\substack{1\le k\le p-1\\ k\equiv r\,(\mathrm{mod}\,m)}}\frac{(-a)^k}{k}\binom{2p-1}{k-1}\equiv -\sum_{\substack{1\le k\le p-1\\ k\equiv r\,(\mathrm{mod}\,m)}}\frac{a^k}{k}+2p\sum_{\substack{1\le j<k\le p-1\\ k\equiv r\,(\mathrm{mod}\,m)}}\frac{a^k}{jk}\pmod{p^2}.$$

Also,

$$\sum_{\substack{1\le k\le p-1\\ k\equiv 2p-r\,(\mathrm{mod}\,m)}}\frac{(-a)^{2p-k}}{2p-k}\binom{2p-1}{k}$$
$$\equiv\sum_{\substack{1\le k\le p-1\\ k\equiv 2p-r\,(\mathrm{mod}\,m)}}\frac{a^{2p-k}}{2p-k}-2p\sum_{\substack{1\le k\le p-1\\ k\equiv 2p-r\,(\mathrm{mod}\,m)}}\frac{a^{2p-k}}{2p-k}\sum_{j=1}^{k}\frac{1}{j}$$

$$\equiv -\sum_{\substack{1\le k\le p-1\\ k\equiv 2p-r\ (\mathrm{mod}\ m)}}\left(\frac{a^{2p-k}}{k}+2p\cdot\frac{a^{2-k}}{k^2}\right)+2p\sum_{\substack{1\le j\le k\le p-1\\ k\equiv 2p-r\ (\mathrm{mod}\ m)}}\frac{a^{2p-k}}{jk}\ (\mathrm{mod}\ p^2).$$

We are done. ∎

Define

$$\mathcal{S}_{r,m}(n)=\sum_{\substack{2\le k\le n\\ k\equiv r\ (\mathrm{mod}\ m)}}\frac{1}{k}\sum_{j=1}^{k-1}\frac{1}{j}.$$

Substituting $a=1$ in (2.1), we get

COROLLARY 2.1. *Suppose that $m\ge 2$ and p is an odd prime with $p\ne m$. Then*

$$\mathcal{H}_{r,m}(p-1)\equiv -\frac{\mathcal{T}^*_{r,m}(p)-\delta_{r,m}(p)}{p}+p\mathcal{S}_{r,m}(p-1)\ (\mathrm{mod}\ p^2), \tag{2.3}$$

where $\delta_{r,m}(p)$ is as defined in (1.6). *In particular,*

$$\mathcal{H}_{p,m}(p-1)\equiv -\frac{\mathcal{T}^*_{p,m}(p)+1}{p}+p\mathcal{S}_{p,m}(p-1)\ (\mathrm{mod}\ p^2). \tag{2.4}$$

Substituting $r=p,p+m/2$ and $a=1$ in (2.2) and noting that $\binom{2p}{p}\equiv 2\ (\mathrm{mod}\ p^3)$, we have

COROLLARY 2.2. *Suppose that $m\ge 2$ and $p\ge 5$ is a prime with $p\ne m$. Then*

$$\mathcal{H}_{p,m}(p-1)\equiv -\frac{\mathcal{T}^*_{p,m}(2p)+2}{4p}+2p\mathcal{S}_{p,m}(p-1)\ (\mathrm{mod}\ p^2). \tag{2.5}$$

Furthermore, if m is even, then

$$\mathcal{H}_{p+m/2,m}(p-1)\equiv -\frac{\mathcal{T}^*_{p+m/2,m}(2p)}{4p}+2p\mathcal{S}_{p+m/2,m}(p-1)\ (\mathrm{mod}\ p^2). \tag{2.6}$$

Combining (2.4) and (2.5), we get

$$p\mathcal{S}_{p,m}(p-1)\equiv -\frac{\mathcal{T}^*_{p,m}(p)+1}{p}+\frac{\mathcal{T}^*_{p,m}(2p)+2}{4p}\ (\mathrm{mod}\ p^2),$$

and Theorem 1.1 easily follows.

LEMMA 2.2.

$$\sum_{r=1}^{m}\mathcal{T}^*_{r,m}(n)\mathcal{T}^*_{r+s,m}(n)=(-1)^n\mathcal{T}^*_{n+s,m}(2n).$$

Proof. Let ζ be a primitive mth root of unity. Clearly,

$$\mathcal{T}^*_{r,m}(n)=\frac{1}{m}\sum_{k=0}^{n}(-1)^k\binom{n}{k}\sum_{t=1}^{m}\zeta^{(k-r)t}=\frac{1}{m}\sum_{t=1}^{m}\zeta^{-rt}(1-\zeta^t)^n.$$

Hence,

$$\sum_{r=1}^{m} \mathcal{T}^*_{r,m}(n)\mathcal{T}^*_{r+s,m}(n) = \sum_{r=1}^{m} \frac{1}{m^2} \sum_{1\le t_1,t_2\le m} \zeta^{-r(t_1+t_2)-st_2}(1-\zeta^{t_1})^n(1-\zeta^{t_2})^n$$
$$= \frac{1}{m} \sum_{\substack{1\le t_1,t_2\le m\\ t_1+t_2=m}} \zeta^{-st_2}(1-\zeta^{t_1})^n(1-\zeta^{t_2})^n$$
$$= \frac{(-1)^n}{m} \sum_{t=1}^{m} \zeta^{-(n+s)t}(1-\zeta^t)^{2n} = (-1)^n \mathcal{T}^*_{n+s,m}(2n). \blacksquare$$

Substituting $s = 0$ and $n = p$ in Lemma 2.2, we have

$$\mathcal{T}^*_{p,m}(2p) = -\sum_{r=1}^{m} \mathcal{T}^*_{r,m}(p)^2.$$

Thus by noting that $\mathcal{T}^*_{0,m}(p) = -\mathcal{T}^*_{r,m}(p)$, we get

$$\mathcal{S}_{p,m}(p-1) \equiv -\frac{\mathcal{T}^*_{p,m}(p)+1}{p^2} + \frac{\mathcal{T}^*_{p,m}(2p)+2}{4p^2} \tag{2.7}$$
$$= \frac{\mathcal{T}^*_{0,m}(p)-1}{2p^2} - \frac{\mathcal{T}^*_{p,m}(p)+1}{2p^2} - \frac{\sum_{r=1}^{m} \mathcal{T}^*_{r,m}(p)^2 - 2}{4p^2}$$
$$= -\sum_{\substack{1\le r\le m\\ r\not\equiv 0,p \,(\mathrm{mod}\, m)}} \frac{\mathcal{T}^*_{r,m}(p)^2}{4p^2} - \frac{(\mathcal{T}^*_{0,m}(p)-1)^2}{4p^2} - \frac{(\mathcal{T}^*_{p,m}(p)+1)^2}{4p^2}$$
$$\equiv -\frac{1}{4}\sum_{r=1}^{m} \mathcal{H}_{r,m}(p-1)^2 \pmod{p}.$$

Since $\mathcal{H}_{p-r,m}(p-1) \equiv -\mathcal{H}_{r,m}(p-1) \pmod{p}$, $\mathcal{H}_{r,m}(p-1) \equiv 0 \pmod{p}$ provided that $2r \equiv p \pmod{m}$. So we also have

$$\mathcal{S}_{p,m}(p-1) \equiv -\frac{1}{4} \sum_{\substack{1\le r\le m\\ 2r\not\equiv p \,(\mathrm{mod}\, m)}} \mathcal{H}_{r,m}(p-1)^2 \pmod{p}. \tag{2.8}$$

Thus by (2.5), Theorem 1.2 follows.

3. Fermat's quotient and Pell's quotient. Let L_n be the Lucas numbers given by $L_0 = 2$, $L_1 = 1$ and $L_n = L_{n-1} + L_{n-2}$ for $n \ge 2$. We require the following result of Sun and Sun on $\mathcal{T}_{r,10}(n)$.

LEMMA 3.1 ([6, Theorem 1]). *Let n be a positive odd integer. If $n \equiv 1 \pmod{4}$, then*

$$\begin{aligned}
10\mathcal{T}_{(n-1)/2,10}(n) &= 2^n + L_{n+1} + 5^{(n+3)/4}F_{(n+1)/2},\\
10\mathcal{T}_{(n+3)/2,10}(n) &= 2^n - L_{n-1} + 5^{(n+3)/4}F_{(n-1)/2},\\
10\mathcal{T}_{(n+7)/2,10}(n) &= 2^n - L_{n-1} - 5^{(n+3)/4}F_{(n-1)/2},\\
10\mathcal{T}_{(n+11)/2,10}(n) &= 2^n + L_{n+1} - 5^{(n+3)/4}F_{(n+1)/2}.
\end{aligned}$$

If $n \equiv 3 \pmod 4$, *then*

$$\begin{aligned}
10\mathcal{T}_{(n-1)/2,10}(n) &= 2^n + L_{n+1} + 5^{(n+1)/4}L_{(n+1)/2},\\
10\mathcal{T}_{(n+3)/2,10}(n) &= 2^n - L_{n-1} + 5^{(n+1)/4}L_{(n-1)/2},\\
10\mathcal{T}_{(n+7)/2,10}(n) &= 2^n - L_{n-1} - 5^{(n+1)/4}L_{(n-1)/2},\\
10\mathcal{T}_{(n+11)/2,10}(n) &= 2^n + L_{n+1} - 5^{(n+1)/4}L_{(n+1)/2}.
\end{aligned}$$

Furthermore, for every odd n,

$$10\mathcal{T}_{(n+13)/2,10}(n) = 2^n - 2L_n.$$

For each odd $n \geq 1$, since

$$\mathcal{T}^*_{n,m}(2n) = \mathcal{T}^*_{n,m}(2n-1) - \mathcal{T}^*_{n-1,m}(2n-1) = -2\mathcal{T}^*_{n-1,m}(2n-1),$$
$$\mathcal{T}^*_{n+m,2m}(2n) = \mathcal{T}^*_{n+m,2m}(2n-1) - \mathcal{T}^*_{n+m-1,m}(2n-1) = -2\mathcal{T}^*_{n+m-1,m}(2n-1),$$

by Lemma 3.1 we get

$$\mathcal{T}^*_{n,5}(2n) = -2\cdot 5^{(n-1)/2}F_n. \tag{3.1}$$

Let $p > 5$ be a prime. By (1.8),

$$\mathcal{H}_{p,5}(p-1) \equiv -\frac{\mathcal{T}^*_{p,5}(2p)+2}{4p} - p(\mathcal{H}_{p,5}(p-1)^2 + \mathcal{H}_{2p,5}(p-1)^2) \pmod{p^2}.$$

By (1.9), we have

$$\begin{aligned}
\mathcal{H}_{p,5}(p-1) &\equiv \frac{5^{(p-1)/2}F_p - 1}{2p} - p\biggl(\Bigl(\frac{F_{p-(\frac{5}{p})}}{2p}\Bigr)^2 + \Bigl(\frac{5^{(p-1)/2}F_p - 1}{2p}\Bigr)^2\biggr)\\
&\equiv \frac{5^{(p-1)/2}F_p - 1}{2p} - p\biggl(5^{p-1}\Bigl(\frac{F_{p-(\frac{5}{p})}}{2p}\Bigr)^2 + \Bigl(\frac{5^{(p-1)/2}F_p - 1}{2p}\Bigr)^2\biggr)\\
&= \frac{5^{(p-1)/2}F_p - 1}{p} - \frac{5^{p-1}(F^2_{p-(\frac{5}{p})} + F_p^2) - 1}{4p}\\
&= \frac{5^{(p-1)/2}F_p - 1}{p} - \frac{5^{p-1}F_{2p-(\frac{5}{p})} - 1}{4p} \pmod{p^2},
\end{aligned}$$

where in the last step we use the fact that $F_{2n-1} = F_n^2 + F_{n-1}^2$. Thus the proof of Theorem 1.3 is complete.

REMARK. Similarly, we can get

$$\sum_{\substack{1\le k\le p-1\\ k\equiv p\,(\mathrm{mod}\,5)}}\frac{(-1)^k}{k} \tag{3.2}$$
$$\equiv \frac{5(2^{4p-1}-2^{2p+3})+12L_{4p}+L_{4p-4(\frac{5}{p})}-112L_{2p}-4L_{2p-2(\frac{5}{p})}+378}{400p} \pmod{p^2}.$$

Let Q_n be the Pell–Lucas numbers given by $Q_0=2$, $Q_1=2$ and $Q_n=2Q_{n-1}+Q_{n-2}$ for each $n\ge 2$. For $\mathcal{T}_{r,8}(n)$, Sun proved

LEMMA 3.2 ([4, Theorem 2.2]). *Let n be a positive odd integer. If $n\equiv 1 \pmod 4$, then*

$$\begin{aligned}
8\mathcal{T}_{(n-1)/2,8}(n) &= 2^n+2^{(n+1)/2}+2^{(n+7)/4}P_{(n+1)/2},\\
8\mathcal{T}_{(n+3)/2,8}(n) &= 2^n-2^{(n+1)/2}+2^{(n+7)/4}P_{(n-1)/2},\\
8\mathcal{T}_{(n+7)/2,8}(n) &= 2^n-2^{(n+1)/2}-2^{(n+7)/4}P_{(n-1)/2},\\
8\mathcal{T}_{(n+11)/2,8}(n) &= 2^n+2^{(n+1)/2}-2^{(n+7)/4}P_{(n+1)/2}.
\end{aligned}$$

If $n\equiv 3 \pmod 4$, then

$$\begin{aligned}
8\mathcal{T}_{(n-1)/2,8}(n) &= 2^n+2^{(n+1)/2}+2^{(n+1)/4}Q_{(n+1)/2},\\
8\mathcal{T}_{(n+3)/2,8}(n) &= 2^n-2^{(n+1)/2}+2^{(n+1)/4}Q_{(n-1)/2},\\
8\mathcal{T}_{(n+7)/2,8}(n) &= 2^n-2^{(n+1)/2}-2^{(n+1)/4}Q_{(n-1)/2},\\
8\mathcal{T}_{(n+11)/2,8}(n) &= 2^n+2^{(n+1)/2}-2^{(n+1)/4}Q_{(n+1)/2}.
\end{aligned}$$

Thus for every odd $n\ge 1$ we have

$$\mathcal{T}^*_{n,8}(2n) = -2^{2n-3}-2^{n-2}-2^{(n-1)/2}P_n, \tag{3.3}$$
$$\mathcal{T}^*_{n+4,8}(2n) = -2^{2n-3}-2^{n-2}+2^{(n-1)/2}P_n. \tag{3.4}$$

Applying (1.8),

$$\mathcal{H}_{p,8}(p-1)\equiv -\frac{\mathcal{T}^*_{p,8}(2p)+2}{4p}-p\sum_{0\le j\le 3}\mathcal{H}_{p+2j,8}(p)^2 \pmod{p^2}.$$

By (2.5) and (2.6), we have

$$\mathcal{H}_{p,8}(p-1)\equiv -\frac{\mathcal{T}^*_{p,8}(2p)+2}{4p} \pmod{p},$$
$$\mathcal{H}_{p+4,8}(p-1)\equiv -\frac{\mathcal{T}^*_{p+4,8}(2p)}{4p} \pmod{p}.$$

In view of (2.3) and Lemma 3.2,

$$\mathcal{H}_{p+2,8}(p-1)^2+\mathcal{H}_{p+6,8}(p-1)^2 \tag{3.5}$$
$$\equiv \begin{cases} \displaystyle\sum_{i=0}^{1}\frac{1}{p^2}\bigg(2^{p-3}-\Big(\frac{2}{p}\Big)2^{(p-5)/2}+(-1)^i2^{(p-5)/4}P_{(p-(\frac{2}{p}))/2}\bigg)^2 \pmod p & \\ \hfill \text{if } p\equiv 1 \pmod 4, & \\ \displaystyle\sum_{i=0}^{1}\frac{1}{p^2}\bigg(2^{p-3}-\Big(\frac{2}{p}\Big)2^{(p-5)/2}+(-1)^i2^{(p-11)/4}Q_{(p-(\frac{2}{p}))/2}\bigg)^2 \pmod p & \\ \hfill \text{if } p\equiv 3 \pmod 4. & \end{cases}$$

LEMMA 3.3. *Let p be an odd prime. Then*

$$\begin{cases} P_{(p-1)/2}\equiv 0 \pmod p, & \\ P_{(p+1)/2}\equiv (-1)^{(p-1)/8}2^{(p-1)/4} \pmod p, & \textit{if } p\equiv 1 \pmod 8, \\ P_{(p-1)/2}\equiv (-1)^{(p-3)/8}2^{(p-3)/4} \pmod p, & \\ P_{(p+1)/2}\equiv (-1)^{(p+5)/8}2^{(p-3)/4} \pmod p, & \textit{if } p\equiv 3 \pmod 8, \\ P_{(p-1)/2}\equiv (-1)^{(p-5)/8}2^{(p-1)/4} \pmod p, & \\ P_{(p+1)/2}\equiv 0 \pmod p, & \textit{if } p\equiv 5 \pmod 8, \\ P_{(p-1)/2}\equiv (-1)^{(p+1)/8}2^{(p-3)/4} \pmod p, & \\ P_{(p+1)/2}\equiv (-1)^{(p+1)/8}2^{(p-3)/4} \pmod p, & \textit{if } p\equiv 7 \pmod 8, \end{cases} \tag{3.6}$$

and

$$\begin{cases} Q_{(p-1)/2}\equiv (-1)^{(p-1)/8}2^{(p+3)/4} \pmod p, & \\ Q_{(p+1)/2}\equiv (-1)^{(p-1)/8}2^{(p+3)/4} \pmod p, & \textit{if } p\equiv 1 \pmod 8, \\ Q_{(p-1)/2}\equiv (-1)^{(p+5)/8}2^{(p+5)/4} \pmod p, & \\ Q_{(p+1)/2}\equiv 0 \pmod p, & \textit{if } p\equiv 3 \pmod 8, \\ Q_{(p-1)/2}\equiv (-1)^{(p+3)/8}2^{(p+3)/4} \pmod p, & \\ Q_{(p+1)/2}\equiv (-1)^{(p-5)/8}2^{(p+3)/4} \pmod p, & \textit{if } p\equiv 5 \pmod 8, \\ Q_{(p-1)/2}\equiv 0 \pmod p, & \\ Q_{(p+1)/2}\equiv (-1)^{(p+1)/8}2^{(p+1)/4} \pmod p, & \textit{if } p\equiv 7 \pmod 8. \end{cases} \tag{3.7}$$

Proof. The congruences in (3.6) were obtained by Sun [4, Theorem 2.3]. Those in (3.7) follow from (3.6), by noting that $Q_n = 2P_{n+1}-2P_n$ and $Q_{n+1}=2P_{n+1}+2P_n$. ∎

Since $P_{(p-(\frac{2}{p}))/2}Q_{(p-(\frac{2}{p}))/2} = P_{p-(\frac{2}{p})}$, by (3.5) we have

$$\mathcal{H}_{p+2,8}(p-1)^2 + \mathcal{H}_{p+6,8}(p-1)^2 \equiv \frac{2^{p-1}\big(2^{(p-1)/2} - (\frac{2}{p})\big)^2 + P^2_{p-(\frac{2}{p})}}{8p} \pmod p.$$

Note that

$$\frac{2^{p-1}-1}{p} = \frac{\big(2^{(p-1)/2}+(\frac{2}{p})\big)\big(2^{(p-1)/2}-(\frac{2}{p})\big)}{p} \equiv 2\left(\frac{2}{p}\right)\frac{2^{(p-1)/2}-(\frac{2}{p})}{p} \pmod p.$$

Hence,

$$\begin{aligned}\mathcal{H}_{p,8}(p-1) &\equiv \frac{2^{2p-4}+2^{p-3}+2^{(p-3)/2}P_p-1}{p} - \frac{(2^{2p-3}+2^{p-2}+2^{(p-1)/2}P_p)^2-4}{16p}\\ &\quad - \frac{(2^{2p-3}+2^{p-2}-2^{(p-1)/2}P_p)^2}{16p} - \frac{2^{p-1}\big(2^{(p-1)/2}-(\frac{2}{p})\big)^2+2^{p-1}P^2_{p-(\frac{2}{p})}}{8p}\\ &\equiv \frac{2^{2p-4}+2^{p-3}+2^{(p-3)/2}P_p-1}{p} - \frac{2^{4p-6}+2^{2p-4}+2^{p-2}P_{2p-(\frac{2}{p})}-1}{4p} \pmod{p^2},\end{aligned}$$

by noting that

$$P^2_{p-(\frac{2}{p})} + P_p^2 = P_{2p-(\frac{2}{p})}.$$

This concludes the proof of Theorem 1.4. ∎

Remark. The Bernoulli polynomials $B_n(x)$ $(n = 0,1,2,\ldots)$ are given by

$$\frac{te^{xt}}{e^t-1} = \sum_{n=0}^{\infty}\frac{B_n(x)}{n!}t^n.$$

In particular, the Bernoulli numbers B_n are $B_n(0)$. Granville and Sun [1] proved that

$$\begin{aligned}B_{p-1}\left(\frac{\{p\}_5}{5}\right) - B_{p-1} &\equiv \frac{5}{4p}F_{p-(\frac{5}{p})} + \frac{5^p-5}{4p} \pmod p,\\ B_{p-1}\left(\frac{\{p\}_8}{8}\right) - B_{p-1} &\equiv \frac{2}{p}P_{p-(\frac{2}{p})} + \frac{2^{p+1}-4}{p} \pmod p,\end{aligned}$$

for any prime $p \neq 2,5$, where $\{p\}_m$ denotes the least non-negative residue of p modulo m. In [5, Theorem 3.3], Sun also proved that

$$m\mathcal{H}_{p,m}(p-1) \equiv \frac{B_{2p-2}(\{p\}_m/m) - B_{2p-2}}{2p-2} - 2\frac{B_{p-1}(\{p\}_m/m) - B_{p-1}}{p-1} \pmod{p^2}.$$

Using Theorems 1.3 and 1.4 we deduce that

$$\frac{B_{p(p-1)}(\{p\}_5/5) - B_{p(p-1)}}{5p(p-1)} \equiv -\frac{5^{(p-1)/2}F_p - 1}{p} + \frac{5^{p-1}F_{2p-(\frac{5}{p})} - 1}{4p} \pmod{p^2} \tag{3.8}$$

for any prime $p > 5$, and

$$\frac{B_{p(p-1)}(\{p\}_8/8) - B_{p(p-1)}}{8p(p-1)} \equiv -\frac{2^{2p-4} + 2^{p-3} + 2^{(p-3)/2}P_p - 1}{p} + \frac{2^{4p-6} + 2^{2p-4} + 2^{p-2}P_{2p-(\frac{2}{p})} - 1}{4p} \pmod{p^2} \tag{3.9}$$

for any prime $p \geq 5$.

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