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Application of the circle method on multidimensional limit-periodic functions

by

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1. Introduction and notation. *Limit-periodic functions* $f : \mathbb{N} \to \mathbb{C}$ are limits of periodic functions under the Besicovitch seminorm

$$\|f\|_2 := \limsup_{N\to\infty} \left(\frac{1}{N}\sum_{n\le N} |f(n)|^2\right)^{1/2}.$$

These functions appear naturally in number-theoretical problems and a famous example is the indicator function of square-free numbers. For some general properties of limit-periodic functions, the interested reader is referred to the book [S].

In forthcoming work (see also [B]) Brüdern shows among other things that binary additive problems with limit-periodic functions are within the grasp of the circle method. Thereby he gives alternative characterisations for limit-periodicity.

We will extend some of his results to higher dimensions using elementary functional analysis and the circle method.

Before we can state the results, we need notation and some definitions.

Vectors $\mathbf{x} = (x_1, \dots, x_d)$ in $\mathbb{N}^d$ or $\mathbb{R}^d$ will be written in bold face, and in particular $\mathbf{1} = (1, \dots, 1)$. The relations $\le$ and $\equiv$ (congruence modulo q) are to be understood componentwise, and $|\mathbf{x}| := \max |x_i|$.

For $p \ge 1$ and $f : \mathbb{N}^d \to \mathbb{C}$, the Besicovitch seminorms are given by

$$\|f\|_p := \limsup_{N\to\infty} \left(\frac{1}{N^d}\sum_{|\mathbf{n}|\le N} |f(\mathbf{n})|^p\right)^{1/p}, \tag{1.1}$$

and the function spaces

$$\mathcal{L}_p^d := \{f : \mathbb{N}^d \to \mathbb{C} : \|f\|_p < \infty\}$$

are defined as usual.

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A function $f : \mathbb{N}^d \to \mathbb{C}$ is called *periodic* if there is a $q \in \mathbb{N}$ such that $f(\mathbf{n}) = f(\mathbf{m})$ whenever $\mathbf{n} \equiv \mathbf{m} \bmod q$. A function $f : \mathbb{N}^d \to \mathbb{C}$ is called *limit-periodic* (in $\mathcal{L}_p^d$) if there is a sequence of periodic functions f_m with $\lim_{m\to\infty} \|f - f_m\|_p = 0$. The space of limit-periodic functions is denoted by $\mathcal{D}_p^d$. We will give an interesting example of a multidimensional limit-periodic function in Section 5.

Surprisingly, we have to deal with a directed scalar product to understand the Fourier analysis of limit-periodic functions. For $N \in \mathbb{N}$ and a direction $\mathbf{w}$ in

$$K^d := \{\mathbf{x} \in \mathbb{R}^d : |\mathbf{x}| = 1,\, x_i > 0\}, \tag{1.2}$$

let $\Theta(\mathbf{w}, N)$ be the number of elements in the rectangle $\{\mathbf{n} \in \mathbb{N}^d : \mathbf{n} \le \mathbf{w}N\}$. For f and g in suitable function spaces, the limit

$$\langle f, g\rangle_{\mathbf{w}} := \lim_{N\to\infty} \frac{1}{\Theta(\mathbf{w}, N)} \sum_{\mathbf{n}\le\mathbf{w}N} f(\mathbf{n})\overline{g(\mathbf{n})} \tag{1.3}$$

will exist, and we refer to it as "scalar product", although it is not definite. In the special case $\mathbf{w} = \mathbf{1}$ we have $\Theta(\mathbf{1}, N) = N^d$ and get the standard scalar product $\langle f, g\rangle$. If g is the constant function $g(\mathbf{n}) = 1$, we obtain the mean value $M(f)$ of f.

The auxiliary functions

$$e_{\boldsymbol{\alpha}}(\mathbf{n}) := e(\boldsymbol{\alpha}\cdot\mathbf{n}) := \exp(2\pi i\, \boldsymbol{\alpha}\cdot\mathbf{n}) \qquad \text{for } \boldsymbol{\alpha} \in \mathbb{R}^d,$$
$$\psi_{q,\mathbf{a}}(\mathbf{n}) := \begin{cases} 1 & \text{if } \mathbf{n} \equiv \mathbf{a} \bmod q, \\ 0 & \text{else} \end{cases} \qquad \text{for } q \in \mathbb{N} \text{ and } \mathbf{a} \in \mathbb{N}^d$$

will be useful because the set $\{e_{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \mathbb{R}^d/\mathbb{Z}^d\}$ is an orthonormal basis with regard to our scalar product in the space of periodic functions. The functions $\psi_{q,\mathbf{a}}$ on the other hand provide a basis with the additional property of decreasing norm for $q \to \infty$.

We can only hope to be able to apply the cirle method to a function f if the scalar products $\langle f, e_{\mathbf{a}/q}\rangle$ exist. For this reason, we define

$$\mathcal{V}_p^d := \{f \in \mathcal{L}_p^d : \langle f, e_{\mathbf{a}/q}\rangle \text{ exists for all } q \in \mathbb{N},\, \mathbf{a} \in \mathbb{N}^d\}.$$

But it turns out to be an insufficient condition for $d \ge 2$ as we will see in Section 6. The function f has to be even in

$$\mathcal{W}_p^d := \{f \in \mathcal{V}_p^d : \langle f, e_{\mathbf{a}/q}\rangle_{\mathbf{w}} = \langle f, e_{\mathbf{a}/q}\rangle \text{ for all } \mathbf{w} \in K^d,\, q \in \mathbb{N},\, \mathbf{a} \in \mathbb{N}^d\}.$$

The generalised Fourier coefficients $\langle f, e_{\mathbf{a}/q}\rangle_{\mathbf{w}}$ have to be independent of $\mathbf{w}$. But fortunately this is true for limit-periodic functions, as we will see in Section 2.

In Sections 3 and 4 we will look at the case $p = 2$. One tool in the analysis of functions in $\mathcal{L}_2^d$ is the exponential sum

$$S_f(\boldsymbol{\alpha}) := \sum_{|\mathbf{n}|\le N} f(\mathbf{n})\overline{e_{\boldsymbol{\alpha}}(\mathbf{n})}, \tag{1.4}$$

which satisfies the orthogonality relation

$$\sum_{|\mathbf{n}|\le N} |f(\mathbf{n})|^2 = \int_{[0,1]^d} |S_f(\boldsymbol{\alpha})|^2\, d\boldsymbol{\alpha}. \tag{1.5}$$

The integral on the right-hand side can be evaluated by the circle method. In this context, the major arcs $\mathfrak{M} = \mathfrak{M}(Q,N)$ are defined by

$$\mathfrak{M} = \bigcup_{q\le Q} \bigcup_{\substack{|\mathbf{a}|\le q\\ (\mathbf{a};q)=1}} \{\boldsymbol{\alpha}\in\mathbb{R}^d : |\boldsymbol{\alpha}-\mathbf{a}/q| \le Q/N\}, \tag{1.6}$$

where $Q = Q(N) \le N^{1/4}$ is a monotone and unbounded function in N. The minor arcs $\mathfrak{m}$ are the complement of $\mathfrak{M}$ in $(Q/N, 1+Q/N]^d$.

For a function $f \in \mathcal{V}_2^d$ let

$$\mathfrak{S}_f := \sum_{q=1}^{\infty} \sum_{\substack{|\mathbf{a}|\le q\\ (\mathbf{a};q)=1}} |\langle f, e_{\mathbf{a}/q}\rangle|^2$$

be the singular series of f, where $(\mathbf{a};q) := \gcd(a_1,\dots,a_d,q)$. This is just the sum of all rational Fourier coefficients of f. Therefore, by Bessel's inequality, this series is bounded by $\|f\|_2^2$ and convergent as we shall show in Section 3 below (see Lemma 3.1). We give an asymptotic formula for the contribution of $\mathfrak{M}$ to (1.5).

THEOREM 1.1. *For all $f \in \mathcal{W}_2^d$ there is a function $Q(N)\to\infty$ with*

$$\int_{\mathfrak{M}} |S_f(\boldsymbol{\alpha})|^2\, d\boldsymbol{\alpha} = \mathfrak{S}_f N^d + o(N^d).$$

In Landau's O-, o-notation the constants may depend on the functions f and $Q(N)$ of course, but we will suppress this dependence here.

If the function f is regular, we can save a small power of N in the term $o(N^d)$ (see [BGPVW] for a one-dimensional example). But even for general limit-periodic functions, $o(N^d)$ is the best we can hope for.

This allows us to show that limit-periodic functions can be characterised by their contribution on the minor arcs.

THEOREM 1.2. *A function $f \in \mathcal{L}_2^d$ is limit-periodic if and only if $f \in \mathcal{W}_2^d$, and for all $Q(N)\to\infty$ we have $\int_{\mathfrak{m}} |S_f(\boldsymbol{\alpha})|^2\, d\boldsymbol{\alpha} = o(N^d)$.*

We will deduce Theorem 1.2 from Theorem 1.1 by a functional-analytic argument in Section 3. It can be used to deal with binary problems involving limit-periodic functions, as we will see in Section 5.

2. Basic lemmata. In this section we will deduce some important properties of limit-periodic functions from periodic ones by the continuity of the scalar product. The proofs are all quite standard but it is not easy to find an adequate reference, so we will give a sketch of every proof.

LEMMA 2.1. *Let* $\mathbf{w} \in K^d$ *and for* $m \in \mathbb{N}$ *let* $f, f_m \in \mathcal{L}_p^d$. *If* $p > 1$ *let* $g, g_m \in \mathcal{L}_{p'}^d$ *with the dual index* p' *given by* $1/p + 1/p' = 1$.

(i) *If* $\langle f, g\rangle_{\mathbf{w}}$ *exists, then* $|\langle f, g\rangle_{\mathbf{w}}| \le \frac{1}{w_1\cdots w_d}\|f\|_p \cdot \|g\|_{p'}$.
(ii) *If* $\lim_{m\to\infty}\|f - f_m\|_p = 0$, $\lim_{m\to\infty}\|g - g_m\|_{p'} = 0$, *and* $\langle f_m, g_m\rangle_{\mathbf{w}}$ *exist for all* $m \in \mathbb{N}$, *then* $\langle f, g\rangle_{\mathbf{w}}$ *exists and satisfies* $\langle f, g\rangle_{\mathbf{w}} = \lim_{m\to\infty}\langle f_m, g_m\rangle_{\mathbf{w}}$.
(iii) *If* $M(f_m)$ *exists for all* $m \in \mathbb{N}$ *and* $\lim_{m\to\infty}\|f - f_m\|_p = 0$, *then* $M(f)$ *exists and* $M(f) = \lim_{m\to\infty} M(f_m)$. *The case* $p = 1$ *is allowed.*

Proof. First, we see that for $N \ge \max\{1/w_i : 1 \le i \le d\}$,

$$\Theta(\mathbf{w}, N) = \prod_{i=1}^{d}[w_i N] \ge \prod_{i=1}^{d} w_i\Big(N - \frac{1}{w_i}\Big),$$

where $[x] := \max\{n \in \mathbb{N} : n \le x\}$. Since $\mathbf{n} \le \mathbf{w}N$ implies $|\mathbf{n}| \le N$, we can use Hölder's inequality and the estimate above to bound $|\langle f, g\rangle_{\mathbf{w}}|$ by

$$\frac{1}{w_1\cdots w_d}\limsup_{N\to\infty}\prod_{i=1}^{d}(N - 1/w_i)^{-1}\Big(\sum_{|\mathbf{n}|\le N}|f(\mathbf{n})|^p\Big)^{1/p}\Big(\sum_{|\mathbf{n}|\le N}|g(\mathbf{n})|^{p'}\Big)^{1/p'}$$

and thus gain statement (i). Applying this inequality to the situation in (ii), we see by standard arguments that $\langle f_m, g_m\rangle_{\mathbf{w}}$ is a Cauchy sequence in $\mathbb{C}$ and that the limit $G = \lim_{m\to\infty}\langle f_m, g_m\rangle_{\mathbf{w}}$ exists. To compare the finite approximation on $\langle f, g\rangle_{\mathbf{w}}$ with G, we split the expression as follows:

$$\begin{aligned}&\Big|G - \frac{1}{\Theta(\mathbf{w}, N)}\sum_{\mathbf{n}\le\mathbf{w}N} f(\mathbf{n})\overline{g(\mathbf{n})}\Big| \\ &\quad\le |G - \langle f_m, g_m\rangle_{\mathbf{w}}| + \Big|\langle f_m, g_m\rangle_{\mathbf{w}} - \frac{1}{\Theta(\mathbf{w}, N)}\sum_{\mathbf{n}\le\mathbf{w}N} f_m(\mathbf{n})\overline{g_m(\mathbf{n})}\Big| \\ &\qquad + \Big|\frac{1}{\Theta(\mathbf{w}, N)}\sum_{\mathbf{n}\le\mathbf{w}N} f_m(\mathbf{n})\overline{g_m(\mathbf{n})} - \frac{1}{\Theta(\mathbf{w}, N)}\sum_{\mathbf{n}\le\mathbf{w}N} f(\mathbf{n})\overline{g(\mathbf{n})}\Big|.\end{aligned}$$

The first term is small due to the definition of G when m is large enough, while the second one is small because the limits $\langle f_m, g_m\rangle_{\mathbf{w}}$ exist. The last one can be treated by $\lim_{m\to\infty}\|f - f_m\|_p = 0$, $\lim_{m\to\infty}\|g - g_m\|_{p'} = 0$ and Hölder's inequality. Choose $m = m(\epsilon)$ large enough; then for $N \ge N_0(m, \epsilon)$ the difference is bounded by ϵ.

Since $|M(f)| \le \|f\|_1 \le \|f\|_p$, a similar calculation yields (iii). ∎

The next three lemmata give some basic properties, which can be transferred from periodic functions to limit-periodic functions by Lemma 2.1.

LEMMA 2.2. *For $f \in \mathcal{D}_p^d$ the mean value $M(f)$ exists. When $p > 1$ and $g \in \mathcal{V}_{p'}^d$ $(1/p + 1/p' = 1)$ the scalar product $\langle f, g\rangle$ exists and is given by*

$$\langle f, g\rangle = \sum_{q=1}^{\infty} \sum_{\substack{|\mathbf{a}|\le q \\ (\mathbf{a};q)=1}} \langle f, e_{\mathbf{a}/q}\rangle\langle e_{\mathbf{a}/q}, g\rangle.$$

In particular, $\mathcal{D}_p^d \subseteq \mathcal{V}_p^d$.

Proof. Use Lemma 2.1, linearity and observe that this lemma is clear for $f = e_{\mathbf{a}/q}$. ■

It will be important that $\langle f, e_{\mathbf{a}/q}\rangle_{\mathbf{w}}$ is even independent of $\mathbf{w}$ for $f \in \mathcal{D}_p^d$.

LEMMA 2.3. *We have $\mathcal{D}_p^d \subset \mathcal{W}_p^d$.*

Proof. Due to Lemma 2.1, it suffices to show that $\langle f, e_{\mathbf{a}/q}\rangle_{\mathbf{w}}$ is independent of $\mathbf{w}$ for periodic functions f. Using the tensor product property $e(\boldsymbol{\alpha}\cdot\mathbf{n}) = \prod_{i=1}^{d} e(\alpha_i n_i)$ we can verify that $\langle e_{\boldsymbol{\alpha}}, e_{\boldsymbol{\beta}}\rangle_{\mathbf{w}} = \langle e_{\boldsymbol{\alpha}}, e_{\boldsymbol{\beta}}\rangle$ for $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^d$. The result follows by linearity and the fact that periodic functions are linear combinations of functions $e_{\boldsymbol{\alpha}}$ with $\boldsymbol{\alpha} \in \mathbb{Q}^d$. ■

As the periodic functions form a $\mathbb{C}$-algebra, we also get a multiplicative structure for limit-periodic functions.

LEMMA 2.4. *Let $f \in \mathcal{D}_p^d$ and $g \in \mathcal{D}_{\hat{p}}^d$ be limit-periodic functions and $r \ge 1$ with $1/p + 1/\hat{p} = 1/r$. Then the product fg is also limit-periodic with $fg \in \mathcal{D}_r^d$.*

Proof. A variant of Hölder's inequality states that $\|fg\|_r \le \|f\|_p\|g\|_{\hat{p}}$ for functions $f \in \mathcal{L}_p^d$ and $g \in \mathcal{D}_{\hat{p}}^d$. Let (f_m) and (g_m) be sequences of periodic functions which approximate f and g. Apply this estimate to the right side of $\|fg - f_m g_m\|_r \le \|f(g - g_m)\|_r + \|(f - f_m)g_m\|_r$ and use the fact that convergent series have bounded norms. ■

3. Parseval's formula. In this section we will turn to the case $p = 2$ and give a characterisation of limit-periodic functions in $\mathcal{L}_2^d$ by functional analysis. The following proposition may be viewed as a generalised Parseval's formula for functions in $\mathcal{D}_2^d \subseteq \mathcal{L}_2^d$.

PROPOSITION. *A function $f \in \mathcal{L}_2^d$ is limit-periodic if and only if $f \in \mathcal{V}_2^d$ and $\|f\|_2^2 = \mathfrak{S}_f$.*

It can be used to deduce Theorem 1.2 from Theorem 1.1, which we shall do first.

Proof of Theorem 1.2. Lemma 2.3 gives $\mathcal{D}_2^d \subseteq \mathcal{W}_2^d$ and therefore it suffices to show the part about the minor arc integral. We may choose $Q(N)$ to be a slowly growing function in N. (The theorem is true if it is true for these functions $Q(N)$.) By inserting formula (1.5) in Theorem 1.1 we get

$$\int_{\mathfrak{m}} |S_f(\boldsymbol{\alpha})|^2\,d\boldsymbol{\alpha} = \int_{[0,1]^d} |S_f(\boldsymbol{\alpha})|^2\,d\boldsymbol{\alpha} - \int_{\mathfrak{M}} |S_f(\boldsymbol{\alpha})|^2\,d\boldsymbol{\alpha}$$
$$= \Big(\frac{1}{N^d}\sum_{|\mathbf{n}|\le N} |f(\mathbf{n})|^2 - \mathfrak{S}_f\Big)N^d + o(N^d).$$

If f is limit-periodic, we have the relation $\mathfrak{S}_f = \|f\|_2^2$, and thus the integral over the minor arcs is of order $o(N^d)$. Otherwise, we obtain $\mathfrak{S}_f < \|f\|_2^2$, and the first term is no longer of order $o(N^d)$. ∎

The proof of the Proposition itself is quite standard but has some small subtleties because the scalar product is not defined on the whole space $\mathcal{L}_2^d$. We will give a sketch of the proof for the sake of completeness by decomposition into the next two lemmata.

For a function $f \in \mathcal{V}_2^d$, we look at the periodic approximation

$$F_m := \sum_{q\le m}\sum_{\substack{|\mathbf{a}|\le q\\ (\mathbf{a};q)=1}} \langle f, e_{\mathbf{a}/q}\rangle e_{\mathbf{a}/q}$$

and the truncated singular series

$$\mathfrak{S}_f(m) := \sum_{q\le m}\sum_{\substack{|\mathbf{a}|\le q\\ (\mathbf{a};q)=1}} |\langle f, e_{\mathbf{a}/q}\rangle|^2.$$

Lemma 3.1. *For $f \in \mathcal{V}_2^d$ and $m \in \mathbb{N}$ we have $\|f-F_m\|_2^2 = \|f\|_2^2 - \mathfrak{S}_f(m)$. In particular, $\mathfrak{S}_f$ exists, and $\mathfrak{S}_f \le \|f\|_2^2$.*

Proof. Noting that $f \in \mathcal{V}_2^d$, we get by direct calculation

$$\|f - F_m\|_2^2 = \|f\|_2^2 - \langle f, F_m\rangle - \langle F_m, f\rangle + \langle F_m, F_m\rangle.$$

The orthogonality of $\{e_{\boldsymbol{\alpha}} : \boldsymbol{\alpha} \in \mathbb{R}^d/\mathbb{Z}^d\}$ provides

$$\langle f, F_m\rangle = \langle F_m, f\rangle = \langle F_m, F_m\rangle = \mathfrak{S}_f(m).$$

The convergence of $\mathfrak{S}_f$ and the estimate $\mathfrak{S}_f \le \|f\|_2^2$ are due to the positivity of $\|f - F_m\|_2^2$. ∎

For periodic functions f of period q, we have the identities $f = F_m$ and $\mathfrak{S}_f(m) = \mathfrak{S}_f$ for $m \ge q$. Thus, we get the equation

$$\|f\|_2^2 = \mathfrak{S}_f, \tag{3.1}$$

which can be transferred to limit-periodic functions by the following lemma.

Lemma 3.2. *The functional $\mathfrak{S} : \mathcal{V}_2^d \to \mathbb{R}; f \mapsto \mathfrak{S}_f$ is continuous.*

Proof. For $f, g \in \mathcal{V}_2^d$, by the triangle inequality and Cauchy's inequality,

$$|\mathfrak{S}_f - \mathfrak{S}_g| \le \sum_{q=1}^{\infty} \sum_{\substack{|\mathbf{a}|\le q \\ (\mathbf{a};q)=1}} |\langle f, e_{\mathbf{a}/q}\rangle\langle e_{\mathbf{a}/q}, f-g\rangle + \langle f-g, e_{\mathbf{a}/q}\rangle\langle e_{\mathbf{a}/q}, g\rangle|$$
$$\le (\mathfrak{S}_f)^{1/2}(\mathfrak{S}_{f-g})^{1/2} + (\mathfrak{S}_{f-g})^{1/2}(\mathfrak{S}_g)^{1/2}.$$

Recalling the estimate $\mathfrak{S}_f \le \|f\|_2^2$ from Lemma 3.1, we obtain the result. ■

Finally, we put all together and get the proof of the Proposition.

Proof of Proposition. Lemma 2.2 gives $\mathcal{D}_2^d \subseteq \mathcal{V}_2^d$. Let f_k be a sequence of periodic functions converging to $f \in \mathcal{D}_2^d$. Then (3.1) provides $\|f_k\|_2^2 = \mathfrak{S}_{f_k}$. Since both sides are continuous, taking the limit $k \to \infty$ gives the result.

If now $\|f\|_2^2 = \mathfrak{S}_f$ is assumed, then we get the convergence $\|f-F_m\|_2^2 \to 0$ for $m \to \infty$ from Lemma 3.1. As the functions F_m are periodic, f is limit-periodic. ■

4. Application of the circle method. Now, we focus on the proof of Theorem 1.1. The evaluation of the major arc integral

$$\int_{\mathfrak{M}} |S_f(\boldsymbol{\alpha})|^2\, d\boldsymbol{\alpha} = \sum_{q\le Q} \sum_{\substack{|\mathbf{a}|\le q \\ (\mathbf{a};q)=1}} \int_{|\boldsymbol{\alpha}-\mathbf{a}/q|\le Q/N} |S_f(\boldsymbol{\alpha})|^2\, d\boldsymbol{\alpha} \tag{4.1}$$

can be reduced to the approximation of $S_f(\boldsymbol{\alpha})$ in the neighbourhood of rational points with small denominator. The main ingredient is the following lemma.

LEMMA 4.1. *For functions $f \in \mathcal{W}_2^d$, we have the asymptotic formula*

$$\sum_{\mathbf{n}\le\mathbf{x}} f(\mathbf{n})\psi_{q,\mathbf{a}}(\mathbf{n}) = \langle f, \psi_{q,\mathbf{a}}\rangle \sum_{\mathbf{n}\le\mathbf{x}} 1 + o(N^d)$$

uniformly in $\mathbf{a} \in \mathbb{N}^d$, $q \in \mathbb{N}$, and $|\mathbf{x}| \le N$.

Proof. We define

$$T := \frac{1}{N^d}\Big|\sum_{\mathbf{n}\le\mathbf{x}} f(\mathbf{n})\psi_{q,\mathbf{a}}(\mathbf{n}) - \langle f, \psi_{q,\mathbf{a}}\rangle \sum_{\mathbf{n}\le\mathbf{x}} 1\Big|.$$

The dependence of T on $N, \mathbf{x}, q$, and $\mathbf{a}$ is suppressed. We have to show that for every $\epsilon > 0$ there is an $N_0 \in \mathbb{N}$ so that for all $N \ge N_0$, $q \in \mathbb{N}$, $\mathbf{a} \in \mathbb{N}^d$, and $|\mathbf{x}| \le N$, we have $T < \epsilon$.

Due to the fact that $\|f\|_2 < \infty$, there is a constant $c_f > 0$ such that

$$\frac{1}{N^d}\sum_{\mathbf{n}\le\mathbf{x}} |f(\mathbf{n})|^2 \le \frac{1}{N^d}\sum_{|\mathbf{n}|\le N} |f(\mathbf{n})|^2 \le c_f$$

for all $N \in \mathbb{N}$ and $|\mathbf{x}| \le N$.

Let us first look at the case where $q > q_0$ is large. Using the triangle inequality and Cauchy's inequality as well as Lemma 2.1(i), we get

$$T \leq \bigg(\frac{1}{N^d}\sum_{\mathbf{n}\leq\mathbf{x}}|f(\mathbf{n})|^2\bigg)^{1/2}\bigg(\frac{1}{N^d}\sum_{\mathbf{n}\leq\mathbf{x}}\psi_{q,\mathbf{a}}(\mathbf{n})\bigg)^{1/2} + \|f\|_2\|\psi_{q,\mathbf{a}}\|_2\frac{1}{N^d}\sum_{\mathbf{n}\leq\mathbf{x}}1.$$

For $N > q$ we can estimate $N^{-d}\sum_{\mathbf{n}\leq\mathbf{x}}\psi_{q,\mathbf{a}}(\mathbf{n})$ by $(2/q)^d$ while for $N \leq q$ this expression can be estimated by $1/N^d$. We end up with

$$T \leq \sqrt{c_f}\,\sqrt{\max\{(2/q)^d, 1/N^d\}} + \sqrt{c_f}\,\sqrt{1/q^d} < \epsilon$$

for all $q \geq q_0$ and $N \geq N_1$ if we choose q_0 and N_1 large enough. To obtain the uniformity in q and $\mathbf{a}$, it now suffices to show the estimate for any fixed $q < q_0$ and $|\mathbf{a}| \leq q$.

The special case of small $\mathbf{x}$ has to be treated separately before we embark on the general proof. Let $\kappa > 0$; then for $|\mathbf{x}| \leq \kappa N$ we get $N^{-d}\sum_{\mathbf{n}\leq\mathbf{x}}\psi_{q,\mathbf{a}}(\mathbf{n}) \leq \kappa^d$ and $N^{-d}\sum_{\mathbf{n}\leq\mathbf{x}}1 \leq \kappa^d$. This provides the estimate

$$T \leq \sqrt{c_f}\,\kappa^{d/2} + \sqrt{c_f}\,\kappa^d < \epsilon$$

if we choose κ so small that each term is less than $\epsilon/2$.

For $\mathbf{x}$ with $|\mathbf{x}| > \kappa N$, choose eventually a parameter $\eta > 0$ with $d\sqrt{c_f}\,\eta < \epsilon/3$ and $dc_f\eta < \epsilon^2/9$. Now, we can approximate our directions in K^d (see (1.2)) by a finite set of points $\mathbf{w}_j \in K^d$ such that for every $\mathbf{w} \in K^d$ there is a $\mathbf{w}_j$ with $|\mathbf{w} - \mathbf{w}_j| < \eta$ (K^d is relatively compact). As $f \in \mathcal{W}_2^d$, it is possible to choose $N_0 \geq N_1$ such that

$$\bigg|\frac{1}{\Theta(\mathbf{w}_j, M)}\sum_{\mathbf{n}\leq\mathbf{w}_j M} f(\mathbf{n})\psi_{q,\mathbf{a}}(\mathbf{n}) - \langle f, \psi_{q,\mathbf{a}}\rangle\bigg| < \epsilon/3 \tag{4.2}$$

for all $\mathbf{w}_j$, $N \geq N_0$, and $M > \kappa N$.

The vector $\mathbf{x}/|\mathbf{x}|$ is in K^d. Therefore, $|\mathbf{x}/|\mathbf{x}| - \mathbf{w}_j| < \eta$ for some $\mathbf{w}_j$. With this approximation, we obtain the estimate

$$\begin{aligned}\Big|\sum_{\mathbf{n}\leq\mathbf{x}} f(\mathbf{n})\psi_{q,\mathbf{a}}(\mathbf{n}) - \langle f, \psi_{q,\mathbf{a}}\rangle\sum_{\mathbf{n}\leq\mathbf{x}}1\Big| &\leq \Big|\sum_{\mathbf{n}\leq\mathbf{x}} f(\mathbf{n})\psi_{q,\mathbf{a}}(\mathbf{n}) - \sum_{\mathbf{n}\leq\mathbf{w}_j|\mathbf{x}|} f(\mathbf{n})\psi_{q,\mathbf{a}}(\mathbf{n})\Big| \\ + \Big|\sum_{\mathbf{n}\leq\mathbf{w}_j|\mathbf{x}|} f(\mathbf{n})\psi_{q,\mathbf{a}}(\mathbf{n}) - \langle f, \psi_{q,\mathbf{a}}\rangle\sum_{\mathbf{n}\leq\mathbf{w}_j|\mathbf{x}|}1\Big| &+ |\langle f, \psi_{q,\mathbf{a}}\rangle|\Big|\sum_{\mathbf{n}\leq\mathbf{w}_j|\mathbf{x}|}1 - \sum_{\mathbf{n}\leq\mathbf{x}}1\Big|.\end{aligned}$$

By (4.2) and the restriction $|\mathbf{x}| > \kappa N$, the middle term does not exceed $N^d\epsilon/3$. We apply Cauchy's inequality to the first term and see that it is bounded by

$$\Big|\sum_{|\mathbf{n}|\leq N} f(\mathbf{n})\psi_{q,\mathbf{a}}(\mathbf{n})(1_{\mathbf{n}\leq\mathbf{x}} - 1_{\mathbf{n}\leq\mathbf{w}_j|\mathbf{x}|})\Big| \leq \sqrt{c_f}\,N^{d/2}\Big(\sum_{|\mathbf{n}|\leq N}|1_{\mathbf{n}\leq\mathbf{x}} - 1_{\mathbf{n}\leq\mathbf{w}_j|\mathbf{x}|}|\Big)^{1/2},$$

where $1_{\mathbf{n}\le\mathbf{x}} = 1$ if the condition $\mathbf{n} \le \mathbf{x}$ is satisfied and $1_{\mathbf{n}\le\mathbf{x}} = 0$ otherwise. Write $\mathbf{y} = \mathbf{w}_j|\mathbf{x}|$. Then the sum on the right can be estimated by

$$\sum_{|\mathbf{n}|\le N}\sum_{i=1}^{d}|1_{n_i\le x_i} - 1_{n_i\le y_i}| \le \sum_{i=1}^{d}|x_i - y_i|N^{d-1} \le d|\mathbf{x}-\mathbf{y}|N^{d-1}.$$

Because of $\big|\mathbf{x} - \mathbf{w}_j|\mathbf{x}|\big| < \eta N$ and the choice of η, we obtain the desired estimate. The inequality $|\langle f, \psi_{q,\mathbf{a}}\rangle| \le \sqrt{c_f}$ allows us to apply this calculation also to the third term. This concludes the proof. ∎

Lemma 4.1 used with the formula $e_{\mathbf{a}/q} = \sum_{|\mathbf{b}|\le q} e_{\mathbf{a}/q}(\mathbf{b})\psi_{q,\mathbf{b}}$ provides

$$\sum_{\mathbf{n}\le\mathbf{x}} f(\mathbf{n})\overline{e_{\mathbf{a}/q}(\mathbf{n})} = \langle f, e_{\mathbf{a}/q}\rangle\sum_{\mathbf{n}\le\mathbf{x}} 1 + q^d\cdot o(N^d). \tag{4.3}$$

For $\boldsymbol{\alpha}$ in the major arcs, we get an asymptotic expression by using summation by parts. In order to describe this in the multidimensional setting, we need some more notation.

Multiindices $\tau \in I_d := \{0,1\}^d$ are used to make some choice operation. We write $|\tau| := \tau_1 + \cdots + \tau_d$ and $\tau = (\sigma, \tau_d)$ with $\sigma \in I_{d-1}$. We need the same decomposition for $\mathbf{x} = (\mathbf{y}, x_d)$, a d-dimensional vector of variables, to define the differential operator $\partial_{\mathbf{x}}^{\tau}$ inductively by

$$\partial_{\mathbf{x}}^{\tau} := \begin{cases} \partial_{\mathbf{y}}^{\sigma} & \text{if } \tau_d = 0,\\ \partial_{x_d}\partial_{\mathbf{y}}^{\sigma} & \text{if } \tau_d = 1,\end{cases}$$

where $\partial_x := \partial/\partial x$. For $d = 1$ the symbol $\partial_{\mathbf{y}}^{\sigma}$ with empty indices $\mathbf{y}$ and σ is to be understood as the identity. Using additionally the vector $\mathbf{N} = (\mathbf{M}, N_d) \in \mathbb{R}^d$ and the notation $[\mathbf{1},\mathbf{N}] = \{\mathbf{x}\in\mathbb{R}^d : 1 \le x_i \le N_i\}$, we can define an integral operator in a similar way by

$$\int_{[\mathbf{1},\mathbf{N}]}^{\tau} f(\mathbf{x})\,d\mathbf{x} := \begin{cases} \int_{[\mathbf{1},\mathbf{M}]}^{\sigma} f(\mathbf{y}, N_d)\,d\mathbf{y} & \text{if } \tau_d = 0,\\ \int_1^{N_d}\int_{[\mathbf{1},\mathbf{M}]}^{\sigma} f(\mathbf{y}, x_d)\,d\mathbf{y}\,dx_d & \text{if } \tau_d = 1.\end{cases}$$

In the case $d = 1$, the integral over $\mathbf{y}$ has to be ignored.

As this is only a new notation for well-known operators, the linearity is preserved and can be shown by induction.

Lemma 4.2 (Summation by parts). *Let $f : \mathbb{R}^d \to \mathbb{C}$ be d times continuously differentiable and $g : \mathbb{N}^d \to \mathbb{C}$. Then*

$$\sum_{\mathbf{n}\le\mathbf{N}} f(\mathbf{n})\cdot g(\mathbf{n}) = \sum_{\tau\in I_d}(-1)^{|\tau|}\int_{[\mathbf{1},\mathbf{N}]}^{\tau}\partial_{\mathbf{x}}^{\tau} f(\mathbf{x})\cdot\sum_{\mathbf{n}\le\mathbf{x}} g(\mathbf{n})\,d\mathbf{x}.$$

Proof. We will give a proof by induction. The case $d = 1$ is well-known and can be found, for example, in [S, p. 2]. If $d \ge 2$, we use the notation

$\mathbf{n} = (\mathbf{k}, n_d)$ with $\mathbf{k} \in \mathbb{N}^{d-1}$ and, as above, $\mathbf{N} = (\mathbf{M}, N_d)$, $\mathbf{x} = (\mathbf{y}, x_d)$ and $\tau = (\sigma, \tau_d)$. Separating one summation, we obtain

$$\begin{aligned}\sum_{\mathbf{n}\le\mathbf{N}} f(\mathbf{n})\cdot g(\mathbf{n}) &= \sum_{n_d\le N_d}\sum_{\mathbf{k}\le\mathbf{M}} f(\mathbf{k},n_d)\cdot g(\mathbf{k},n_d)\\ &= \sum_{n_d\le N_d}\sum_{\sigma\in I_{d-1}} (-1)^{|\sigma|} \int\limits_{[\mathbf{1},\mathbf{M}]}^{\sigma} \partial_{\mathbf{y}}^{\sigma} f(\mathbf{y},n_d)\cdot\sum_{\mathbf{k}\le\mathbf{y}} g(\mathbf{k},n_d)\,d\mathbf{y}\end{aligned}$$

where we have applied our induction hypothesis to the inner sum over $d-1$ variables. Now, we can invoke the one-dimensional formula for the summation over n_d inside the integral to find that the above equals

$$\begin{aligned}&\sum_{\sigma\in I_{d-1}} (-1)^{|\sigma|} \int\limits_{[\mathbf{1},\mathbf{M}]}^{\sigma} \sum_{\mathbf{k}\le\mathbf{y}} \Big[\sum_{n_d\le N_d} \partial_{\mathbf{y}}^{\sigma} f(\mathbf{y},n_d)\cdot g(\mathbf{k},n_d)\Big]\,d\mathbf{y}\\ &\quad= \sum_{\sigma\in I_{d-1}} (-1)^{|\sigma|}\\ &\qquad\times \int\limits_{[\mathbf{1},\mathbf{M}]}^{\sigma} \sum_{\mathbf{k}\le\mathbf{y}} \Big[\sum_{\tau_d\in I_1} (-1)^{\tau_d} \int\limits_{[1,N_d]}^{\tau_d} \partial_{x_d}^{\tau_d}\partial_{\mathbf{y}}^{\sigma} f(\mathbf{y},x_d)\cdot \sum_{n_d\le x_d} g(\mathbf{k},n_d)\,dx_d\Big]\,d\mathbf{y}.\end{aligned}$$

Rearranging and combining the terms, we end up with the expression

$$\begin{aligned}&\sum_{\sigma\in I_{d-1}}\sum_{\tau_d\in I_1} (-1)^{\tau_d}(-1)^{|\sigma|}\\ &\qquad\times \int\limits_{[\mathbf{1},\mathbf{M}]}^{\sigma}\int\limits_{[1,N_d]}^{\tau_d} \partial_{x_d}^{\tau_d}\partial_{\mathbf{y}}^{\sigma} f(\mathbf{y},x_d)\cdot\sum_{\mathbf{k}\le\mathbf{y}}\sum_{n_d\le x_d} g(\mathbf{k},n_d)\,dx_d\,d\mathbf{y}\\ &\quad= \sum_{\tau\in I_d} (-1)^{|\tau|} \int\limits_{[\mathbf{1},\mathbf{N}]}^{\tau} \partial_{\mathbf{x}}^{\tau} f(\mathbf{x})\cdot\sum_{\mathbf{n}\le\mathbf{x}} g(\mathbf{n})\,d\mathbf{x}. \quad\blacksquare\end{aligned}$$

At this point, we are in a position to give the approximation for the exponential sum on the major arcs.

LEMMA 4.3. *Let $S_f(\boldsymbol{\alpha})$ be the exponential sum for $f \in \mathcal{W}_p^d$. Then*

$$S_f(\mathbf{a}/q+\boldsymbol{\beta}) = \langle f, e_{\mathbf{a}/q}\rangle \sum_{|\mathbf{n}|\le N} \overline{e(\boldsymbol{\beta}\cdot\mathbf{n})} + Q^{2d}\cdot o(N^d)$$

for $\boldsymbol{\alpha} = \mathbf{a}/q+\boldsymbol{\beta} \in \mathfrak{M}$ with $|\boldsymbol{\beta}| \le Q/N$.

Proof. We write

$$S_f(\mathbf{a}/q+\boldsymbol{\beta}) = \sum_{|\mathbf{n}|\le N} f(\mathbf{n})\overline{e_{\mathbf{a}/q+\boldsymbol{\beta}}(\mathbf{n})} = \sum_{|\mathbf{n}|\le N} \overline{e_{\boldsymbol{\beta}}(\mathbf{n})} f(\mathbf{n})\overline{e_{\mathbf{a}/q}(\mathbf{n})}$$

and apply Lemma 4.2 to get

$$= \sum_{\tau\in I_d} (-1)^{|\tau|} \int\limits_{[\mathbf{1},\mathbf{N}]}^{\tau} \partial_{\mathbf{x}}^{\tau} \overline{e_{\boldsymbol{\beta}}(\mathbf{x})} \cdot \sum_{\mathbf{n}\le\mathbf{x}} f(\mathbf{n}) \overline{e_{\mathbf{a}/q}(\mathbf{n})}\, d\mathbf{x}.$$

Next, we insert formula (4.3) to obtain

$$\begin{aligned} S_f(\mathbf{a}/q+\boldsymbol{\beta}) = & \sum_{\tau\in I_d} (-1)^{|\tau|} \int\limits_{[\mathbf{1},\mathbf{N}]}^{\tau} \partial_{\mathbf{x}}^{\tau} \overline{e_{\boldsymbol{\beta}}(\mathbf{x})} \langle f, e_{\mathbf{a}/q}\rangle \sum_{\mathbf{n}\le\mathbf{x}} 1\, d\mathbf{x} \\ & + \sum_{\tau\in I_d} (-1)^{|\tau|} \int\limits_{[\mathbf{1},\mathbf{N}]}^{\tau} (-2\pi i)^{|\tau|} \boldsymbol{\beta}^{\tau} \overline{e_{\boldsymbol{\beta}}(\mathbf{x})} \cdot q^d \cdot o(N^d)\, d\mathbf{x}, \end{aligned}$$

where $\boldsymbol{\beta}^{\tau} := \prod_{\tau_i=1} \beta_i$ for a vector $\boldsymbol{\beta}\in\mathbb{R}^d$ and $\tau\in I_d$.

The first integral can be treated with Lemma 4.2 by doing the calculation backwards. The second one is estimated with $|\boldsymbol{\beta}| < Q/N$ and $|e_{\boldsymbol{\beta}}(\mathbf{x})| \le 1$, yielding

$$\begin{aligned} S_f(\mathbf{a}/q+\boldsymbol{\beta}) &= \langle f, e_{\mathbf{a}/q}\rangle \sum_{|\mathbf{n}|\le N} \overline{e_{\boldsymbol{\beta}}(\mathbf{n})} + q^d \cdot o\Big(\sum_{\tau\in I_d} (Q/N)^{|\tau|} N^{d+|\tau|}\Big) \\ &= \langle f, e_{\mathbf{a}/q}\rangle \sum_{|\mathbf{n}|\le N} \overline{e_{\boldsymbol{\beta}}(\mathbf{n})} + Q^{2d} \cdot o(N^d). \ \blacksquare \end{aligned}$$

Now we can prove Theorem 1.1.

Proof of Theorem 1.1. Squaring the formula of Lemma 4.3 and using the estimates $|\langle f, e_{\mathbf{a}/q}\rangle| \le \|f\|_2$ and $|\sum_{|\mathbf{n}|\le N} e(\boldsymbol{\beta}\cdot\mathbf{n})| \le N^d$, we get

$$|S_f(\mathbf{a}/q+\boldsymbol{\beta})|^2 = |\langle f, e_{\mathbf{a}/q}\rangle|^2 \Big| \sum_{|\mathbf{n}|\le N} e(\boldsymbol{\beta}\cdot\mathbf{n}) \Big|^2 + Q^{4d} \cdot o(N^{2d}). \tag{4.4}$$

The sum splits up into one-dimensional parts. We have

$$\Big| \sum_{|\mathbf{n}|\le N} e(\boldsymbol{\beta}\cdot\mathbf{n}) \Big| = \prod_{i=1}^{d} \Big| \sum_{n_i\le N} e(\beta_i n_i) \Big|,$$

which can be estimated by making use of the geometric sum formula and the inequalities $|\beta| \le |1-e(\beta)|$ as well as $|e(\beta)| \le 1$, valid for $\beta\in[-1/2,1/2]$. Thus, we have

$$\Big| \sum_{n_i\le N} e(\beta_i n_i) \Big| = \left| \frac{1-e(\beta_i(N+1))}{1-e(\beta_i)} \right| \le \frac{2}{|\beta_i|}.$$

This allows us to extend the integration over the major arc ($|\boldsymbol{\beta}| \le Q/N$) to an integration over $[-1/2,1/2]^d$ and to estimate the error when $|\boldsymbol{\beta}| > Q/N$.

This case occurs if there exists a β_j with $|\beta_j| > Q/N$ and we can bound the contribution of the large $\boldsymbol{\beta}$ by

$$\begin{aligned}\int\limits_{|\boldsymbol{\beta}|>Q/N} \Big|\sum_{|\mathbf{n}|\le N} e(\boldsymbol{\beta}\cdot\mathbf{n})\Big|^2\,d\boldsymbol{\beta} &\le \sum_{j=1}^{d}\int\limits_{|\beta_j|>Q/N}\Big|\sum_{|\mathbf{n}|\le N} e(\boldsymbol{\beta}\cdot\mathbf{n})\Big|^2\,d\boldsymbol{\beta}\\ &\le \sum_{j=1}^{d}\prod_{i\ne j}\int\limits_{\beta_i\in[-1/2,1/2]}\Big|\sum_{n_i\le N} e(\beta_i n_i)\Big|^2\,d\beta_i\cdot\int\limits_{|\beta_j|>Q/N}\frac{4}{|\beta_j|^2}\,d\beta_j\\ &= O(N^{d-1}\cdot N/Q) = O(N^d/Q),\end{aligned}$$

where a complete integral over $[-1/2,1/2]$ yields N by the orthogonality relation (1.5).

If we put all together and insert equation (4.4) into (4.1), we get

$$\begin{aligned}&\sum_{q\le Q}\sum_{\substack{|\mathbf{a}|\le q\\(\mathbf{a};q)=1}}\int\limits_{|\boldsymbol{\alpha}-\mathbf{a}/q|\le Q/N}|S_f(\boldsymbol{\alpha})|^2\,d\boldsymbol{\alpha}\\ &=\sum_{q\le Q}\sum_{\substack{|\mathbf{a}|\le q\\(\mathbf{a};q)=1}}\Big(|\langle f,e_{\mathbf{a}/q}\rangle|^2\int\limits_{|\boldsymbol{\beta}|\le Q/N}\Big|\sum_{|\mathbf{n}|\le N}e(\boldsymbol{\beta}\cdot\mathbf{n})\Big|^2\,d\boldsymbol{\beta}+Q^{5d}\cdot o(N^d)\Big)\\ &=\mathfrak{S}_f(Q)\Big(\int\limits_{[-1/2,1/2]^d}\Big|\sum_{|\mathbf{n}|\le N}e(\boldsymbol{\beta}\cdot\mathbf{n})\Big|^2\,d\boldsymbol{\beta}+O(N^d/Q)\Big)+Q^{7d}\cdot o(N^d).\end{aligned}$$

The remaining integral gives N^d by the orthogonality relation (1.5). Eventually, we obtain

$$\int\limits_{\mathfrak{M}}|S_f(\boldsymbol{\alpha})|^2\,d\boldsymbol{\alpha}=\mathfrak{S}_f(Q)(N^d+O(N^d/Q))+Q^{7d}\cdot o(N^d).$$

Now, choose for Q a function in N which goes to infinity, but still satisfies $Q^{7d}\cdot o(N^d)=o(N^d)$. Noting that the left-hand side above is bounded by $(\|f\|_2^2+\epsilon)N^d$ for $N\ge N_0(\epsilon)$, we get once more the estimate $\mathfrak{S}_f\le\|f\|_2^2$ and the desired asymptotic formula. ∎

5. Examples and applications. First we give a natural example for a limit-periodic function in dimension $d\ge 2$. Let $\tau\in I_d=\{0,1\}^d$ be a multiindex with $|\tau|\ge 2$. Define

$$\gamma_\tau(\mathbf{n})=\begin{cases}1 & \text{if } \gcd(\mathbf{n}_\tau)=1,\\ 0 & \text{else,}\end{cases}$$

where we use the notation $\mathbf{n}_\tau=(n_1\tau_1,\dots,n_d\tau_d)$.

To see that the function is indeed limit-periodic, we note that it can be written as

$$\gamma_\tau(\mathbf{n}) = \sum_{e|\gcd(\mathbf{n}_\tau)} \mu(e) = \sum_{e=1}^{\infty} \mu(e) \prod_{\tau_i=1} \psi_{e,0}(n_i) = \prod_{\pi} (1 - \psi_{\pi,\mathbf{0}}(\mathbf{n}_\tau)),$$

where μ is the Möbius function and the product goes over all primes π. The condition $|\tau| \geq 2$ and the multiplicative structure ensures convergence in $\mathcal{L}_p^d$ for all $p \geq 1$.

More generally let G be a finite set of such multiindices. Then the function

$$\gamma_G = \prod_{\tau \in G} \gamma_\tau$$

is also limit-periodic in $\mathcal{L}_p^d$ for all $p \geq 1$ due to Lemma 2.4.

If we choose $G = \{\tau \in I_d : |\tau| = 2\}$ for example, we get the indicator function of the set of integers with pairwise coprime components.

Taking products is one possibility to get new examples of limit-periodic functions. Another is to take linear transformations $f_{A,\mathbf{m}}(\mathbf{n}) := f(A\mathbf{n} + \mathbf{m})$ with $A \in \mathbb{N}^{c \times d}, \mathbf{m} \in \mathbb{N}^c$ and $f \in \mathcal{D}_2^c$.

Now we want to see how Theorem 1.2 (and the proof of Theorem 1.1) may be used to solve some binary additive problems. Lemma 2.2 allows us to compute scalar products $\langle f, g \rangle$ of functions $f \in \mathcal{D}_p^d$ and $g \in \mathcal{V}_{p'}^d$. But when it comes to more elaborate binary problems, such as the evaluation of the sum

$$\sum_{\mathbf{n}+\mathbf{m}=\mathbf{k}} f(\mathbf{n}) g(\mathbf{m}),$$

functional analysis gives no simple answer. In the special case of $f \in \mathcal{D}_2^d$ and $g \in \mathcal{W}_2^d$ however, we can immediately write down an asymptotic formula for the sum above:

$$\sum_{\mathbf{n}+\mathbf{m}=\mathbf{k}} f(\mathbf{n}) g(\mathbf{m}) \\ = k_1 \cdots k_d \sum_{q=1}^{\infty} \sum_{\substack{|\mathbf{a}| \leq q \\ (\mathbf{a};q)=1}} \langle f, e_{\mathbf{a}/q} \rangle \langle g, e_{\mathbf{a}/q} \rangle e(\mathbf{k} \cdot \mathbf{a}/q) + o(|\mathbf{k}|^d).$$

This can be achieved by expressing the sum as

$$\int_{[0,1]^d} S(\boldsymbol{\alpha}) T(\boldsymbol{\alpha}) e(\mathbf{k} \cdot \boldsymbol{\alpha}) \, d\boldsymbol{\alpha},$$

where S and T are the exponential sums of f and g respectively with summation constraint $\mathbf{n} \leq \mathbf{k}$. Then we split the integral into major and minor arcs. The minor arcs give a contribution of $o(|\mathbf{k}|^d)$ due to $f \in \mathcal{D}_2^d$ when we apply the Cauchy–Schwarz inequality and Theorem 1.2. The major arcs can be evaluated similarly to the procedure in the proof of Theorem 1.1 and give the main term in the asymptotics.

Now a few words about the space $\mathcal{W}_2^d$. First of all we have $\mathcal{D}_2^d \subseteq \mathcal{W}_2^d$ from Lemma 2.3. Therefore, we get many interesting examples of functions $g \in \mathcal{W}_2^d$ by looking at the space $\mathcal{D}_2^d$.

Many other examples can be found if we use the tensor product. We define $(g_1 \otimes g_2)(\mathbf{n}) = g_1(\mathbf{n}_1)g_2(\mathbf{n}_2)$, where $\mathbf{n} = (\mathbf{n}_1, \mathbf{n}_2)$ and $g_i \in \mathcal{W}_2^{d_i}$ ($i \in \{1,2\}$, $d_1 + d_2 = d$). Then we obtain $g_1 \otimes g_2 \in \mathcal{W}_2^d$.

Let $\mathcal{V}_2^{\otimes d}$ be the closure of the space of linear combinations of functions g with $g(\mathbf{n}) = \prod_{i=1}^d g_i(n_i)$ and $g_i \in \mathcal{V}_2^1$. Then $\mathcal{V}_2^{\otimes d} \subseteq \mathcal{W}_2^d$ by the identities $\mathcal{V}_2^1 = \mathcal{W}_2^1$, $\|g\|_p = \prod_{i=1}^d \|g_i\|_p$ and Lemma 2.1.

6. A counterexample. Theorem 1.1 is not always true for functions $f \in \mathcal{V}_2^d$. To see this, we look at the counterexample $f : \mathbb{N}_0^2 \to \mathbb{R}$ given by

$$f(n_1, n_2) := \begin{cases} 1 & \text{if } n_1 > n_2, \\ -1 & \text{if } n_1 \le n_2. \end{cases}$$

The symmetry $f(n_1, n_2) = -f(n_2, n_1)$ $(n_1 \neq n_2)$ simplifies the calculation of

$$\langle f, \psi_{q,\mathbf{a}} \rangle = \lim_{N\to\infty} \frac{1}{N^2} \sum_{\substack{|\mathbf{n}| \le N-1 \\ \mathbf{n} \equiv \mathbf{a} \bmod q}} f(\mathbf{n}) = 0.$$

A direct consequence is that $\langle f, e_{\mathbf{a}/q} \rangle = 0$ for every $q \in \mathbb{N}$, $|\mathbf{a}| \le q$, and thus we have $\mathfrak{S}_f = 0$. On the other hand, we are also able to calculate the exponential sum for the parameter $N-1$ explicitly, using the geometric sum formula.

For $\alpha_2 \notin \mathbb{Z}$, $\alpha_1 \notin \mathbb{Z}$, and $\alpha_1 + \alpha_2 \notin \mathbb{Z}$ we get

$$\overline{S_f(\boldsymbol{\alpha})} = \frac{1}{e(\alpha_2) - 1} \left(2 \frac{e(\alpha_1 N + \alpha_2 N) - 1}{e(\alpha_1 + \alpha_2) - 1} - (e(\alpha_2 N) + 1) \frac{e(\alpha_1 N) - 1}{e(\alpha_1) - 1} \right).$$

Let $\alpha_1 = \beta_1$, $\alpha_2 = 1/N + \beta_2$, and $|\beta_i| < \delta/N$, $\beta_1 \neq 0$ for some $1/100 > \delta > 0$. Substituting and using the 1-periodicity of e gives

$$\overline{S_f(\boldsymbol{\alpha})} = \frac{1}{e(1/N + \beta_2) - 1} \times \left(2 \frac{e(\beta_1 N + \beta_2 N) - 1}{e(1/N + \beta_1 + \beta_2) - 1} - (e(\beta_2 N) + 1) \frac{e(\beta_1 N) - 1}{e(\beta_1) - 1} \right).$$

The (rough) estimates

$$
\begin{gathered}
|e(1/N+\beta_1)-1| \le 20/N, \\
|e(1/N+\beta_1+\beta_2)-1| \ge 1/N, \\
|e(\beta_2 N)+1| \ge 1, \\
|e(\beta_1 N+\beta_2 N)-1| \le 20\delta, \\
\left|\frac{1-e(\beta_1 N)}{1-e(\beta_1)}\right| \ge N/2
\end{gathered}
$$

are valid when $N \ge 10$ and imply the lower bound

$$|S_f(\beta_1, 1/N+\beta_2)| \ge (N/20)(-40\delta N + N/2) \ge N^2/200.$$

Since the δ/N-neighbourhood of $(0,1/N)$ is part of the major arcs, we get

$$\int_{\mathfrak{M}} |S_f(\boldsymbol{\alpha})|^2\, d\boldsymbol{\alpha} \ge (N^2/200)^2 \cdot (\delta/N)^2 = \eta N^2$$

with some $\eta > 0$.

7. Further directions and generalisations. An obvious generalisation of limit-periodicity are almost-periodic functions. A function $f \in \mathcal{L}_2^d$ is *almost-periodic* if it is the limit of linear combinations of functions $e_{\boldsymbol{\alpha}}$ in the Besicovitch seminorm. Theorem 1.2 and the relevant lemmata generalise to this function space with the appropriate adaptations (see also [P]).

Sometimes, when the function f under consideration is regular enough, it is possible to improve significantly on the $o(N^d)$-bounds in this paper. This is possible for the indicator function of k-free numbers and should be possible for our functions γ_G, too. Possible applications are binary problems involving prime numbers and multidimensional sets with similar structural properties.

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