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The number of powers of 2 in a representation of large odd integers

by

LIQUN HU and LI YANG (Nanchang)

1. Introduction. In 1951 and 1953, Linnik established the "almost Goldbach" result that each large even integer N is a sum of two primes p_1, p_2 and a bounded number of powers of 2,

$$N = p_1 + p_2 + 2^{\nu_1} + \cdots + 2^{\nu_k}; \tag{1.1}$$

here (and throughout) p and ν, with or without subscripts, denote prime numbers and positive integers respectively. Later Gallagher [1] established a stronger result by a different method. An explicit value for the number k of powers of 2 was first establish by Liu, Liu and Wang [11], who found that $k = 54000$ is acceptable. The value of k was subsequently improved by Li [5], Wang [18], and Li [6]. Recently Heath-Brown and Puchta [4] applied a rather different approach to this problem and showed that $k = 13$ is acceptable.

In 1923, Hardy and Littlewood [3] conjectured that each integer N can be written as

$$N = p + n_1^2 + n_2^2,$$

and Linnik [7, 8] proved this conjecture. In view of this result, it seems reasonable to conjecture that each large $N \equiv 0$ or $1 \pmod 3$ is a sum of a prime and two squares of primes,

$$N = p_1 + p_2^2 + p_3^2.$$

But current technologies lack the power to solve it. As an analogous result, Liu, Liu and Zhan [12] studied the number of solutions of the equation

$$N = p_1 + p_2^2 + p_3^2 + 2^{\nu_1} + \cdots + 2^{\nu_k}. \tag{1.2}$$

They showed, in particular, that there is a positive constant k_0 such that for $k \geq k_0$, every large odd integer is a sum of a prime, two squares of primes and k powers of 2. In [9] it is shown that $k_0 = 22000$ is acceptable in (1.2).

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In this paper we sharpen this result considerably by establishing the following theorem.

THEOREM 1.1. *Every large odd integer is the sum of a prime, two squares of primes and* 12000 *powers of* 2.

This theorem implies that there is a set $\Im$ of integers $n \le x$ of cardinality only $O(\log^{12000} x)$ such that every large even integer $N \le x$ can be written as $N = p_1 + p_2^2 + p_3^2 + n$, with p_1, p_2, p_3 being primes and $n \in \Im$. Thus our result can be compared with another approximation to the conjecture $N = p_1 + p_2^2 + p_3^2$. In [17], Wang proved that with at most $O(N^{5/12+\epsilon})$ exceptions, all positive odd integers $n \equiv 0$ or $1 \pmod 3$ not exceeding N can be written as $n = p_1 + p_2^2 + p_3^2$.

Notation. As usual, $\varphi(n)$ and $\mu(n)$ stand for the Euler and Möbius functions respectively. N is a large integer and $L = \log_2 N$. The letter ϵ denotes a positive constant which is arbitrarily small.

2. Outline of the method. In this section we will give the proof of Theorem 1.1. Our proof depends essentially on Theorem 1.2 below, which will be established by the circle method. In order to apply the circle method, we set

$$P = N^{1/6-\epsilon}, \qquad Q = N/(PL^{14}), \qquad M = NL^{-14}. \tag{2.1}$$

By Dirichlet's lemma on rational approximation, each $\alpha \in [1/Q, 1+1/Q]$ may be written in the form

$$\alpha = a/q + \lambda, \qquad |\lambda| \le 1/(qQ), \tag{2.2}$$

for some integers a, q with $1 \le a \le q \le Q$ and $(a, q) = 1$. We denote by $\mathcal{M}(a, q)$ the set of α's satisfying (2.2), and define the major arcs $\mathcal{M}$ and minor arcs $C(\mathcal{M})$ as follows:

$$\mathcal{M} = \bigcup_{q \le P} \bigcup_{\substack{a=1 \\ (a,q)=1}}^{q} \mathcal{M}(a, q), \qquad C(\mathcal{M}) = [1/Q, 1 + 1/Q] \setminus \mathcal{M}. \tag{2.3}$$

It follows from $2P \le Q$ that the major arcs $\mathcal{M}(a, q)$ are mutually disjoint. Let

$$T_1(\alpha) = \sum_{p \le N} \log p \cdot e(p\alpha), \qquad S_1(\alpha) = \sum_{p^2 \le N} \log p \cdot e(p^2\alpha), \tag{2.4}$$

$$T(\alpha) = \sum_{M \le p \le N} \log p \cdot e(p\alpha), \qquad S(\alpha) = \sum_{M \le p^2 \le N} \log p \cdot e(p^2\alpha), \tag{2.5}$$

$$G(\alpha) = \sum_{2^\nu \le N} e(2^\nu \alpha),$$

and

$$r_k(N) = \sum_{N=p_1+p_2^2+p_3^2+2^{\nu_1}+\cdots+2^{\nu_k}} (\log p_1)(\log p_2)(\log p_3). \tag{2.6}$$

Then $r_k(N)$ can be written as

$$\begin{aligned} r_k(N) &= \int_0^1 S_1^2(\alpha)T_1(\alpha)G^k(\alpha)e(-N\alpha)\,d\alpha \\ &= \Big\{\int_{\mathcal{M}} + \int_{C(\mathcal{M})}\Big\} S_1^2(\alpha)T_1(\alpha)G^k(\alpha)e(-N\alpha)\,d\alpha. \end{aligned} \tag{2.7}$$

In the course of proof of Theorem 1.1 we will use the following result.

Lemma 2.1. *Let $\mathcal{M}$ be as in (2.3) with P determined by (2.1). Then for $2 \le n \le N$, we have*

$$\int_{\mathcal{M}} S^2(\alpha)T(\alpha)e(-n\alpha)\,d\alpha = \frac{\pi}{4}\sigma_1(n)n + O\Big(\frac{N}{\log N}\Big). \tag{2.8}$$

Here $\sigma_1(n)$ is defined in §4, and satisfies $\sigma_1(n) \gg 1$.

This is Theorem 2 in Wang [17].

On the minor arcs, we also need estimates for the measure of the set

$$\mathcal{E}_\lambda = \{\alpha \in (0,1] : |G(\alpha)| \ge \lambda L\}.$$

The following lemma is due to Heath-Brown and Puchta [4].

Lemma 2.2. *Let*

$$G_h(\alpha) = \sum_{0\le n\le h-1} e(2^n\alpha)$$

and

$$F(\xi,h) = \frac{1}{2^h}\sum_{r=0}^{2^h-1} \exp\Big\{\xi \mathrm{Re}\Big(G_h\Big(\frac{r}{2^h}\Big)\Big)\Big\}.$$

Then

$$\mathrm{meas}(\mathcal{E}_\lambda) \le N^{-E(\lambda)},$$

where

$$E(\lambda) = \frac{\xi\lambda}{\log 2} - \frac{\log F(\xi,h)}{h\log 2} - \frac{\epsilon}{\log 2}$$

holds for any $h \in \mathbb{N}$, any $\xi > 0$ and $\epsilon > 0$.

On the minor arcs, the new result of Ren [15] (see Lemma 5.2 below) on exponential sums over primes will also be applied.

In Section 3 we shall give some lemmas which will be used in this paper. The relevant singular series will be discussed in Section 4. Finally we will complete the proof of Theorem 1.1 in the last section.

3. Some lemmas. In order to deal with the minor arcs, we need to estimate the number of solutions of the equations

$$p_1^2 + p_2^2 - 2^{\nu_1} - 2^{\nu_2} = p_3^2 + p_4^2 - 2^{\nu_3} - 2^{\nu_4}$$

and

$$p_1 + 2^{\nu_1} = p_2 + 2^{\nu_2}.$$

We have the following lemmas:

LEMMA 3.1 (see Lemma 4.1 in [13]).

$$\int_0^1 |S_1(\alpha)G(\alpha)|^4\,d\alpha \le c_5\frac{\pi^2}{16}NL^4,$$

where

$$c_5 \le \left(\frac{44^4\cdot 101\cdot 43}{25\cdot 3} + \frac{2^3}{\pi^2}\log^2 2\right)(1+\epsilon)^9.$$

LEMMA 3.2 (see Lemma 10 in [14]).

$$\int_0^1 |T_1(\alpha)G(\alpha)|^2\,d\alpha \le 2c_3NL^2, \quad \textit{where} \quad c_3 \le 5.3636.$$

Now we can get a certain lower estimate on the integral on the major arcs by the circle method.

By Lemma 2.1 we have

$$\int_{\mathcal{M}} S^2(\alpha)T(\alpha)e(-n\alpha)\,d\alpha = \frac{\pi}{4}\sigma_1(n)n + O(NL^{-1}), \tag{3.1}$$

where

$$\sigma_1(n) = \sum_{q\le P}\frac{\mu(q)}{\varphi^3(q)}\sum_{\substack{a=1\\(a,q)=1}}^{q} C^2(a,q)e(-an/q)$$

and $C(a,q)$ is defined in §4.

Now,

$$\int_{\mathcal{M}} (S^2(\alpha)T(\alpha) - S_1^2(\alpha)T_1(\alpha))e(-n\alpha)\,d\alpha$$
$$\ll \int_{\mathcal{M}} |S_1^2(\alpha)|\,|T(\alpha)-T_1(\alpha)|\,d\alpha + \int_0^1 |S^2(\alpha)-S_1^2(\alpha)|\,|T(\alpha)|\,d\alpha$$
$$=: H_1 + H_2.$$

By Cauchy's inequality we have

$$H_1 \le \Big(\int_0^1\Big|\sum_{p\le M}\log p\cdot e(p\alpha)\Big|^2\,d\alpha\Big)^{1/2}\Big(\int_0^1 |S_1(\alpha)|^4\,d\alpha\Big)^{1/2} =: H_{11}^{1/2}H_{12}^{1/2},$$

where

$$H_{11} = \int_0^1 \Big| \sum_{p \le M} \log p \cdot e(p\alpha) \Big|^2 \, d\alpha \ll L^2 M,$$

$$H_{12} = \int_0^1 |S_1(\alpha)|^4 \, d\alpha \ll L^4 Z(N).$$

Here $Z(N)$ is the number of solutions of the equation

$$p_1^2 + p_2^2 = p_3^2 + p_4^2, \tag{3.2}$$

and p_j $(1 \le j \le 4)$ are primes. By [16], the number of solutions of (3.2) satisfying $p_1p_2 \ne p_3p_4$ is $O(NL^3)$. By the prime number theorem, there are $O(NL^{-2})$ obvious solutions satisfying $p_1p_2 = p_3p_4$. Thus

$$H_{12} \le NL^2. \tag{3.3}$$

Then

$$H_1 \le (ML^2)^{1/2}(NL^2)^{1/2} \ll NL^{-5}.$$

We have

$$\begin{aligned}
H_2 &= \int_0^1 |S^2(\alpha) - S_1^2(\alpha)| \, |T(\alpha)| \, d\alpha \\
&\le \Big(\int_0^1 |S^2(\alpha) - S_1^2(\alpha)|^2 \, d\alpha\Big)^{1/2} \Big(\int_0^1 |T(\alpha)^2| \, d\alpha\Big)^{1/2} \\
&= \Big(\int_0^1 |S(\alpha) - S_1(\alpha)|^2 |S(\alpha) + S_1(\alpha)|^2 \, d\alpha\Big)^{1/2} \Big(\int_0^1 |T(\alpha)|^2 \, d\alpha\Big)^{1/2} \\
&\le \Big(\int_0^1 |S(\alpha) - S_1(\alpha)|^4 \, d\alpha\Big)^{1/4} \Big(\int_0^1 |S(\alpha) + S_1(\alpha)|^4 \, d\alpha\Big)^{1/4} \Big(\int_0^1 |T(\alpha)|^2 \, d\alpha\Big)^{1/2} \\
&=: H_{21}^{1/2} H_{22}^{1/4} H_{23}^{1/4},
\end{aligned}$$

where

$$H_{21} = \int_0^1 |T(\alpha)|^2 \, d\alpha \ll L^2 N, \qquad H_{22} = \int_0^1 |S(\alpha) - S_1(\alpha)|^4 \, d\alpha \ll L^4 Z(M).$$

By (3.3), we have

$$H_{22} \ll ML^2 = NL^{-12},$$

$$H_{23} = \int_0^1 |S(\alpha) + S_1(\alpha)|^4 \, d\alpha \ll \int_0^1 |S(\alpha)|^4 \, d\alpha + \int_0^1 |S_1(\alpha)|^4 \, d\alpha.$$

We know

$$\int_0^1 |S_1(\alpha)|^4\,d\alpha = \sum_{\substack{p_1^2+p_2^2=p_3^2+p_4^2\\ M\le p_i^2\le N}} \log p_1\cdots\log p_4 \le \sum_{\substack{p_1^2+p_2^2=p_3^2+p_4^2\\ p_i^2\le N}} \log p_1\cdots\log p_4$$
$$= \int_0^1 |S(\alpha)|^4\,d\alpha,$$

so that

$$H_{23} \ll \int_0^1 |S(\alpha)|^4\,d\alpha = \sum_{\substack{p_1^2+p_2^2=p_3^2+p_4^2\\ p_i^2\le N}} \log p_1\cdots\log p_4 \ll L^4 Z(N) \ll NL^2.$$

Then

$$H_2 \le H_{21}^{1/2} H_{22}^{1/4} H_{23}^{1/4} \le (NL^2)^{1/2}(NL^{-12})^{1/4}(NL^2)^{1/4} \ll NL^{-1},$$

and consequently

$$\int_{\mathcal{M}} (S^2(\alpha)T(\alpha) - S_1^2(\alpha)T_1(\alpha))e(-n\alpha)\,d\alpha = O(NL^{-1}).$$

Thus

$$\int_{\mathcal{M}} S_1^2(\alpha)T_1(\alpha)e(-n\alpha)\,d\alpha = \int_{\mathcal{M}} S^2(\alpha)T(\alpha)e(-n\alpha)\,d\alpha + O(NL^{-1}).$$

Define

$$\varXi(N,k) = \{n : n = N - 2^{\nu_1} - \cdots - 2^{\nu_k}\}.$$

We have

$$\sum_{n\in\varXi(N,k)} \int_{\mathcal{M}} T_1(\alpha)S_1(\alpha)e(-n\alpha)\,d\alpha = \frac{\pi}{4}\sum_{n\in\varXi(N,k)} \sigma_1(n)n + O(NL^{k-1}).$$

For simplicity, we set

$$\mathcal{A} = \left\{n = N - 2^{\nu_1} - \cdots - 2^{\nu_k} : \nu_i \le \log_2\left(\frac{N}{kL}\right),\ 1\le i\le k\right\}.$$

Thus we have

$$\int_{\mathcal{M}} S_1^2(\alpha)T_1(\alpha)e(-n\alpha)\,d\alpha = \frac{\pi}{4}\sum_{n\in\varXi(N,k)} \sigma_1(n)n + O(NL^{k-1}) \tag{3.4}$$

$$\begin{aligned}
&\geq \frac{\pi}{4}\sum_{n\in\mathcal{A}}\sigma_1(n)n + O(NL^{k-1})\\
&= \frac{\pi}{4}\sum_{n\in\mathcal{A}}\sigma_1(n)(N - 2^{\nu_1} - \cdots - 2^{\nu_k}) + O(NL^{k-1})\\
&\geq \frac{\pi}{4}N(1-L^{-1})\sum_{n\in\mathcal{A}}\sigma_1(n) + O(NL^{k-1})\\
&\geq \frac{\pi}{4}N(1-\delta)\sum_{n\in\mathcal{A}}\sigma_1(n) + O(NL^{k-1}),
\end{aligned}$$

where $\delta \geq 0$ is a sufficiently small positive constant. Now, we discuss the singular series $\sigma_1(n)$.

4. Singular series. We need some lemmas:

LEMMA 4.1 (see [10, Lemma 4]). *If α is a rational number of odd denominator q and $1 < \xi(q) < L$, then*

$$|G(\alpha)| \leq \left(1 - \frac{1}{\xi(q)\csc^2(\pi/8)} + \frac{2}{L}\right)L.$$

Here $\xi(q)$ is the least positive integer which satisfies

$$2^{\xi} \equiv 1 \pmod q$$

for the given odd q.

LEMMA 4.2. *Let $A(q) = \prod_{p|q} A(p)$, where*

$$A(p) = \begin{cases} \sqrt{p}+1 & \text{if } p \equiv 1 \pmod 4,\\ \sqrt{p+1} & \text{if } p \equiv -1 \pmod 4.\end{cases} \tag{4.1}$$

Then

$$\sum_{\xi(q)\leq x}\frac{\mu^2(q)}{\varphi^3(q)}A^2(q)q \leq c_1\log^2 x, \qquad \sum_{\xi(q)\leq x}\frac{\mu^2(q)}{\varphi^3(q)}A^2(q) \leq c_2\log^{1.5} x,$$

with

$$c_1 = 5.287076611, \qquad c_2 = 3.803.$$

Proof. Let

$$X = \prod_{\xi\leq x}(2^{\xi}-1).$$

Then $q \mid X$, if $\xi(q) \leq x$. And obviously $2 \nmid X$ and $X \leq 2^{x^2}$. We have

$$\sum_{\xi(q)\leq x}\frac{\mu^2(q)}{\varphi^3(q)}A^2(q)q \leq \sum_{q|X}\frac{\mu^2(q)}{\varphi^3(q)}A^2(q)q \leq \prod_{p|X}\left(1 + \frac{A^2(p)}{\varphi^3(p)}p\right).$$

If $p \geq 16$,

$$1+\frac{A^2(p)}{\varphi^3(p)}p<1+\frac{2}{p-1}.$$

By (4.1), we have

$$\begin{aligned}\prod_{3\leq p|X}\left(1+\frac{A^2(p)p}{(p-1)^3}\right) &\leq \frac{5}{4}\prod_{3\leq p|X}\left(1+\frac{2}{p-1}\right)=\prod_{p|2X}\left(1+\frac{2}{p-1}\right)\cdot\frac{5}{12}\\ &\leq \prod_{p|2X}\left(1+\frac{1}{p-1}\right)^2\cdot\frac{5}{12}\\ &=\frac{(2X)^2}{\varphi^2(2X)}\frac{5}{12}\leq\frac{5}{12}\cdot 4e^{2\gamma}\log^2 x=: c_1\log^2 x.\end{aligned}$$

Using Lemma 5 in [10], we obtain

$$\frac{2X}{\varphi(2X)}<2e^{\gamma}\log x. \tag{4.2}$$

Noting $1.7810 < e^{\gamma} < 1.78108$, we have

$$\sum_{\xi(q)\leq x}\frac{\mu^2(q)A^2(q)q}{\varphi^3(q)}<c_1\log^2 x,\quad \text{where}\quad c_1=5.287076611.$$

Now we prove the second inequality. We have

$$\sum_{\xi(q)\leq x}\frac{\mu^2(q)}{\varphi^2(q)}A^2(q)\leq\sum_{q|X}\frac{\mu^2(q)}{\varphi^2(q)}A^2(q)=\prod_{p|X}\left(1+\frac{A^2(p)}{\varphi^2(p)}\right).$$

It is easy to see that for $p \geq 25$,

$$1+\frac{A^2(p)}{(p-1)^2}<1+\frac{1.5}{p-1}\leq\left(1+\frac{1}{p-1}\right)^{1.5}.$$

Therefore

$$\begin{aligned}\prod_{3\leq p|X}\left(1+\frac{A^2(p)}{\varphi^2(p)}\right) &\leq \prod_{3\leq p|X}\left(1+\frac{1.5}{p-1}\right)\cdot(1.413867968)\\ &=\prod_{p|2X}\left(1+\frac{1.5}{p-1}\right)\cdot\left(\frac{1}{2.5}\cdot 1.413867968\right)\\ &\leq\prod_{p|2X}\left(1+\frac{1}{p-1}\right)^{1.5}\cdot\left(\frac{1}{2.5}\cdot 1.413867968\right)\\ &<(2e^{\gamma})^{1.5}\log^{1.5}x\cdot\left(\frac{1}{2.5}\cdot 1.413867968\right)<3.803\log^{1.5}x.\end{aligned}$$

Here we have used (3.1).

Lemma 4.3. *For odd q and $k \geq 2$, we have*

$$r_{kk}(0) \leq 2L^{2k-2} \quad \textit{and} \quad r_{kk}(n) \leq L^{2k-1}\left(1 + \frac{L}{\xi(q)}\right),$$

where r_{kk} denotes the number of n's which can be represented as

$$n = 2^{\nu_1} + \cdots + 2^{\nu_k} - 2^{\mu_1} - \cdots - 2^{\mu_k} \qquad (1 \leq \nu_i, \mu_i \leq L).$$

In order to get an estimate of $\sum_{n\in\mathcal{A}} \sigma_1(n)$, we need to estimate the following sum first:

$$\sum_{\substack{3\leq q\leq R\\ 2\nmid q}} \frac{\mu^2(q)}{\varphi^3(q)} \sum_{\substack{a=1\\ (a,q)=1}}^{q} |C^2(a,q)|\,|G^k(a/q)|.$$

To estimate this sum, we divide it according to the length of the period $\xi(q)$ into two parts as follows:

$$(4.3) \qquad \sum_{\substack{3\leq q\leq R\\ 2\nmid q}} \frac{\mu^2(q)}{\varphi^3(q)} \sum_{\substack{a=1\\ (a,q)=1}}^{q} |C^2(a,q)|\,|G^k(a/q)|$$
$$= \Big(\sum_{\substack{3\leq q\leq R\\ \xi(q)\leq E}} + \sum_{\substack{3\leq q\leq R\\ \xi(q)> E}}\Big) \frac{\mu^2(q)}{\varphi^3(q)} \sum_{\substack{a=1\\ (a,q)=1}}^{q} |C^2(a,q)|\,|G^k(a/q)|,$$

where $E \leq L$ is a constant.

Use (see [17, Lemma 6.1])

$$C(a,q) = \chi(a)S(q,1) - 1,$$

where $\chi(a)$ is the Legendre symbol and the Gauss sum $S(q,1)$ satisfies

$$(4.4) \qquad S(q,1) = \begin{cases} \sqrt{q}, & q \equiv 1 \pmod 4,\\ i\sqrt{q}, & q \equiv -1 \pmod 4. \end{cases}$$

As $C(a,q)$ is multiplicative, we know that if $p_1, \ldots, p_s$ are primes with $q = p_1 \cdots p_s$, then

$$C(a,q) = C(a_1,p_1)\cdots C(a_s,p_s),$$

where $a_i = aq/p_i$. Thus for $1 \leq i \leq s$, if $p_i \equiv 1 \pmod 4$,

$$|C(a_i,p_i)| = |\pm\sqrt{p_i} - 1| \leq \sqrt{p_i} + 1.$$

If $p_i \equiv -1 \pmod 4$,

$$|C(a_i,p_i)| = |\pm i\sqrt{p_i} - 1| \leq \sqrt{p_i} + 1.$$

So we have

$$|C(a_i,p_i)| \le A(p_i), \qquad |C(a,q)| \le \prod_{p|q} A(p) = A(q).$$

Then by Lemmas 4.1 and 4.2, the first sum on the right of (4.3) can be estimated as

$$\sum_{\substack{3\le q\le R\\ \xi(q)\le E}} \le L^k\Big(1-\frac{1}{E\csc^2(\pi/8)}\Big)^k \sum_{\substack{3\le q\le R\\ \xi(q)\le E}} \frac{\mu^2(q)}{\varphi^3(q)}A^2(q) \tag{4.5}$$

$$=: c_2\log^{1.5} E\Big(1-\frac{1}{E\csc^2(\pi/8)}\Big)^k L^k.$$

We use Lemma 4.3 to estimate the other sum of (4.3). If $k=2m$,

$$\sum_{\substack{a=1\\ (a,q)=1}}^{q} |G(a/q)|^k = \sum_{\substack{a=1\\ (a,q)=1}}^{q} |G(a/q)|^{2m} \le \sum_{a=1}^{q}|G(a/q)|^{2m} = q\sum_{q|n} r_{m,m}(n)$$

$$\le qL^{2m-1}(1+L/\xi(q)) = qL^{k-1}+qL^k/\xi(q).$$

For $k=2m+1$, we also have

$$\sum_{\substack{a=1\\ (a,q)=1}}^{q} |G(a/q)|^k = \sum_{\substack{a=1\\ (a,q)=1}}^{q} |G(a/q)|^{2m+1} \le L \sum_{\substack{a=1\\ (a,q)=1}}^{q} |G(a/q)|^{2m}$$

$$\le L(qL^{2m-1}(1+L/\xi(q))) = qL^{k-1}+qL^k/\xi(q).$$

Therefore for any $k\in\mathbb{Z}^+$, we have

$$\sum_{\substack{a=1\\ (a,q)=1}}^{q} |G(a/q)|^k \le qL^{k-1}+qL^k/\xi(q).$$

From the above estimates we obtain

$$\sum_{\substack{3\le q\le R\\ \xi(q)>E}} \le \sum_{\substack{3\le q\le R\\ \xi(q)>E}} \frac{\mu^2(q)}{\varphi^3(q)} \sum_{\substack{a=1\\ (a,q)=1}}^{q} |C^2(a,q)|\,|G^k(a/q)| \tag{4.6}$$

$$\le \sum_{\substack{3\le q\le R\\ \xi(q)>E}} \frac{\mu^2(q)}{\varphi^3(q)}A^2(q) \sum_{\substack{a=1\\ (a,q)=1}}^{q} |G^k(a/q)|$$

$$\le L^{k-1}\sum_{\substack{3\le q\le R\\ \xi(q)>E}} \frac{\mu^2(q)}{\varphi^3(q)}A^2(q)q + L^k \sum_{\substack{3\le q\le R\\ \xi(q)>E}} \frac{\mu^2(q)}{\varphi^3(q)}A^2(q)\frac{q}{\xi(q)}.$$

The first sum on the RHS of (4.6) is $\ll \log^2 R$. We use integration by parts and Lemma 4.1 to show that the second sum is

$$\leq L^k \sum_{m>E} \frac{1}{m} \sum_{\xi(q)=m} \frac{\mu^2(q)}{\varphi^3(q)} A^2(q)q = L^k \int_E^\infty \frac{1}{t^2} \Big(\sum_{\xi(q)\leq t} \frac{\mu^2(q)}{\varphi^3(q)} A^2(q)q \Big)\, dt$$
$$\leq L^k c_1 \int_E^\infty \frac{\log^2 t}{t^2}\, dt = c_1 \Big(\frac{\log^2 E}{E} + \frac{2\log E}{E} + \frac{2}{E} \Big) L^k.$$

Thus

$$\sum_{\substack{3\leq q\leq R\\ \xi(q)>E}} \leq c_1 \Big(\frac{\log^2 E}{E} + \frac{2\log E}{E} + \frac{2}{E} \Big) L^k + O(L^{k-1}\log^2 R).$$

Combining (4.5) and (4.6), we get

$$\begin{aligned}(4.7)\qquad &\sum_{\substack{3\leq q\leq R\\ 2\nmid q}} \frac{\mu^2(q)}{\varphi^3(q)} \sum_{\substack{a=1\\ (a,q)=1}}^{q} |C^2(a,q)|\,|G^k(a/q)| \\ &\leq c_1 \Big(\frac{\log^2 E}{E} + \frac{2\log E}{E} + \frac{2}{E} \Big) L^k \\ &\quad + c_2 \log^{1.5} E \Big(1 - \frac{1}{E\csc^2(\pi/8)} \Big)^k L^k + O(L^{k-1}\log^2 R).\end{aligned}$$

We take

$$\sigma_1(n) = \sum_{q\leq P} \frac{\mu(q)}{\varphi^3(q)} \sum_{\substack{a=1\\ (a,q)=1}}^{q} C^2(a,q)e(-an/q),$$
$$\sigma_0(n) = \sum_{q=1}^{+\infty} \frac{\mu(q)}{\varphi^3(q)} \sum_{\substack{a=1\\ (a,q)=1}}^{q} C^2(a,q)e(-an/q).$$

Then $\sigma_0(n) = \sigma_1(n) + \sum_{q>P}$. By (4.4) we have

$$\begin{aligned}(4.8)\qquad &\sum_{a=1}^{p-1} C^2(a,p)e(-an/p) = \sum_{a=1}^{p-1} (\chi(a)S(p,1)-1)^2 e(-an/p) \\ &= (S^2(p,1)+1) \sum_{a=1}^{p-1} e(-an/p) - 2S(p,1) \sum_{a=1}^{p-1} \chi(a)e(-an/p) \\ &:= \sum{}^{1} + \sum{}^{2}.\end{aligned}$$

For $\sum^1$, we need the following result:

$$\sum_{a=1}^{p-1} e(-an/p) = \begin{cases} p-1, & p \mid n, \\ -1, & p \nmid n. \end{cases}.$$

Using this result and (4.4), we obtain

$$\sum^1 = \begin{cases} p^2-1, & p \equiv 1 \pmod 4, \quad p \mid n, \\ -(p-1)^2, & p \equiv -1 \pmod 4, \; p \mid n, \\ -(p+1), & p \equiv 1 \pmod 4, \quad p \nmid n, \\ (p-1), & p \equiv -1 \pmod 4, \; p \nmid n. \end{cases} \tag{4.9}$$

For $\sum^2$, when $p \mid n$,

$$\sum_{a=1}^{p-1} \chi(a) = 0,$$

thus $\sum^2 = 0$ and $\left(\frac{n}{p}\right) = 0$.

For $p \nmid n$, we introduce

$$F(n) = \sum_{a=1}^{p} \left(\frac{a}{p}\right) e(-an/p).$$

It is easy to see that $F(n) = \left(\frac{n}{p}\right) F(1)$. On the other hand,

$$\begin{aligned} S(p,1) &= \sum_{m=1}^{p} e(m^2/p) = \sum_{m=1}^{p-1} e(m^2/p) + 1 \\ &= \sum_{a=1}^{p-1} \left(1 + \left(\frac{a}{p}\right)\right) e(a/p) + 1 \\ &= \sum_{a=1}^{p-1} \left(\frac{a}{p}\right) e(a/p) = F(-1). \end{aligned}$$

Hence

$$\sum^2 = -2|S^2(p,1)| \left(\frac{n}{p}\right) = \begin{cases} -2p, & \left(\frac{n}{p}\right) = 1, \; p \equiv 1 \pmod 4, \\ -2p, & \left(\frac{n}{p}\right) = 1, \; p \equiv -1 \pmod 4, \\ 2p, & \left(\frac{n}{p}\right) = -1, \; p \equiv 1 \pmod 4, \\ 2p, & \left(\frac{n}{p}\right) = -1, \; p \equiv -1 \pmod 4. \end{cases} \tag{4.10}$$

By (4.4) and (4.7)–(4.9), we have

$$\sum{}^1+\sum{}^2 \tag{4.11}$$
$$=\begin{cases} p^2-1, & p\equiv 1 \pmod 4,\ p\,|\,n,\\ -(p-1)^2, & p\equiv -1 \pmod 4,\ p\,|\,n,\\ -(p+1)-2p=-3p-1, & \left(\frac{n}{p}\right)=1,\ p\equiv 1 \pmod 4,\\ (p-1)-2p=-(p+1), & \left(\frac{n}{p}\right)=1,\ p\equiv -1 \pmod 4,\\ -(p+1)+2p=p-1, & \left(\frac{n}{p}\right)=-1,\ p\equiv 1 \pmod 4,\\ (p-1)+2p=3p-1, & \left(\frac{n}{p}\right)=-1,\ p\equiv -1 \pmod 4. \end{cases}$$

Since

$$\frac{\mu(q)}{\varphi^3(q)}\sum_{\substack{a=1\\(a,q)=1}}^{q} C^2(a,q)e(-an/q)$$

is multiplicative, we define

$$A(n,q)=\frac{\mu(q)}{\varphi^3(q)}\sum_{\substack{a=1\\(a,q)=1}}^{q} C^2(a,q)e(-an/q).$$

We have

$$\sigma_0(n)=\sum_{q=1}^{+\infty}\frac{\mu(q)}{\varphi^3(q)}\sum_{\substack{a=1\\(a,q)=1}}^{q} C^2(a,q)e(-an/q)=\prod_p(1+A(n,p)).$$

We can easily see that $1+A(n,2)=0$ if n is even, and $1+A(n,2)>0$ if n is odd. By (4.10), we have $1+A(n,3)=0$ if $n\equiv 2 \pmod 3$, and $1+A(n,3)>0$ otherwise. By (4.8) and (4.10), for $p>3$,

$$|A(n,p)|\le\begin{cases}\dfrac{3p+1}{(p-1)^3}, & p\nmid n,\\[2mm] \dfrac{p^2-1}{(p-1)^3}, & p\,|\,n.\end{cases}$$

Then

$$|A(n,q)|\le 2\prod_{\substack{p|q\\p\nmid n}}\frac{25}{p^2}\prod_{\substack{p|q\\p|n}}\frac{25}{p}=2\prod_{p|q}\frac{25}{p^2}\prod_{\substack{p|q\\p|n}}p\ll q^{-2+\epsilon}(q,n).$$

Thus

$$\sum_{q>x}|A(n,q)|\ll\sum_{s|n}s\sum_{st>x}(st)^{-2+\epsilon}\ll x^{-1+\epsilon}d(n).$$

Therefore

$$\sigma_1(n) = \sigma_0(n) + O(P^{-1+\epsilon}).$$

It can be easily verified that if we exclude the case of $n \equiv 2 \pmod 3$ then for odd n we have

$$\sigma_0(n) = \prod_p (1 + A(n,p)) \geq c > 0.$$

So when N is sufficiently large, we have $\sigma_1(n) > 0$. Let

$$\sigma(n) = \sum_{q \leq R} \frac{\mu(q)}{\varphi^3(q)} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} C^2(a,q) e(-an/q),$$

$$\sigma_1(n) = \sum_{q \leq R} + \sum_{R<q\leq P} = \sigma(n) + \sum_{R<q\leq P}.$$

Similarly, we have

$$\sum_{R<q\leq P} \leq \sum_{R<q\leq\infty} = O(R^{-1+2\epsilon}),$$

where we take $\epsilon = \frac{\log\log\log R}{\log N}$, $R = o(N)$. Thus

$$\sum_{n\in\mathcal{A}} \sigma_1(n) = \sum_{n\in\mathcal{A}} \sigma(n) + O(L^k R^{-1+2\epsilon}).$$

Define

$$\begin{aligned}\sum_{n\in\mathcal{A}} \sigma(n) &= \sum_{q \leq R} \frac{\mu(q)}{\varphi^3(q)} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} C^2(a,q) e(-aN/q) \Bigg(\sum_{\nu=1}^{\log_2(\frac{N}{kL})} e\Big(\frac{a}{q} 2^\nu\Big)\Bigg)^k \\ &= \sum_{q \leq R} \frac{\mu(q)}{\varphi^3(q)} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} C^2(a,q) e(-aN/q) G^k(a/q) \\ &:= \sum_{q\leq R} B(q,N).\end{aligned}$$

Thus

$$\begin{aligned}\sum_{n\in\mathcal{A}} \sigma(n) &= 2L^k + \sum_{3\leq q\leq R} = 2L^k + \sum_{\substack{3\leq q\leq R \\ 2\nmid q}} B(q,N) + \sum_{\substack{3\leq q\leq R/2 \\ 2\nmid q}} B(q,N) \\ &:= 2L^k + \sum{}^3 + \sum{}^4.\end{aligned} \tag{4.12}$$

By (4.6), we have

$$\Big|\sum^3+\sum^4\Big| \le 2c_1L^k\bigg(\frac{\log^2 E}{E}+\frac{2\log E}{E}+\frac{2}{E}\bigg)$$
$$+\,2c_2\log^{1.5}E\bigg(1-\frac{1}{E\csc^2(\pi/8)}\bigg)^k L^k + O(L^{k-1}\log^2 R),$$

with

$$c_1 = 5.287076611, \qquad c_2 = 3.803.$$

Take

$$R = \exp\bigg(\frac{\sqrt{\log N}}{\log\log N}\bigg);$$

then the quantity in the above O symbol is $O(L^k(\log\log N)^{-2})$.

Combining (3.4) and (4.11), we have

$$\int_{\mathcal{M}} \ge \frac{\pi}{2}NL^k + \Big(\sum^5+\sum^6\Big), \tag{4.13}$$

where

$$\Big|\sum^5+\sum^6\Big| = \frac{\pi}{4}N(1-\delta)\Big|\sum^3+\sum^4\Big|$$
$$\le 2c_1'NL^k\bigg(\frac{\log^2 E}{E}+\frac{2\log E}{E}+\frac{2}{E}\bigg)$$
$$+\,2c_2\log^{1.5}E\bigg(1-\frac{1}{E\csc^2(\pi/8)}\bigg)^k NL^k + O(L^{k-1}\log^2 R),$$
$$c_1' = 4.152460187, \qquad c_2 = 3.803.$$

5. Proof of Theorem 1.1. In order to apply Lemma 2.2, we need to find an optimal λ such that $E(\lambda) > 19/24$. Thus we have to compute

$$F(\xi,h) = \frac{1}{2^h}\sum_{r=0}^{2^h-1}\exp\bigg\{\xi\cdot\sum_{i=1}^{h}\cos\bigg(\frac{2\pi r}{2^i}\bigg)\bigg\}$$

to optimize ξ. Use Mathematica 4.1 on a PC and run the following procedure:

a=N[Sum[Cos[$2\pi r/2^i$], $\{i,1.22\}$]];
b=Apply[Plus, Table[Exp[ξ*a],$\{r,0,2^{22}-1\}$]];
(Log[b/2^{22}]/22/Log[2]+19/24)*Log[2]/ξ.

We can take $\xi = 1.21$, $h = 22$ in Lemma 2.2 to get

LEMMA 5.1. *Let $E(\lambda)$ be as in Lemma 2.2. Then*

$$E(0.910707) > 19/24 + 10^{-10}.$$

LEMMA 5.2. *Let $S_1(\alpha)$ be as in (2.4) and let $\alpha = a/q+\lambda$ satisfy $(a,q)=1$ and $\lambda\in\mathbb{R}$. Then*

$$S_1(\alpha) \ll N^{1/4+\epsilon}\sqrt{q(1+|\lambda|N)} + N^{2/5+\epsilon} + \frac{N^{1/2+\epsilon}}{\sqrt{q(1+|\lambda|N)}}.$$

Proof. This is a special case of Theorem 1.1 in [9].

Now we prove the main result of this paper.

Proof of Theorem 1.1. Let $\mathcal{E}_\lambda$ be as in Lemma 2.2 and $\mathcal{M}$ as in (2.3) with P, Q determined by (2.1). Then (2.7) becomes

$$r_k(N) = \int_0^1 S_1^2(\alpha)T_1(\alpha)G^k(\alpha)e(-N\alpha)\,d\alpha = \Big\{\int_{\mathcal{M}} + \int_{C(\mathcal{M})\cap\mathcal{E}_\lambda} + \int_{C(\mathcal{M})\cap C(\mathcal{E}_\lambda)}\Big\}. \tag{5.1}$$

We can see in (4.13) the estimation of the first integral on the right-hand side.

By Dirichlet's Lemma on rational approximations each $\alpha \in [0,1]$ can be written as $\alpha = a/q + \lambda$, $(a,q) = 1$, with

$$1 \le q \le Q_0 = N^{3/4}, \qquad |\lambda| \le 1/(qQ_0).$$

Let $\mathcal{N}$ be the set of $\alpha \in C(\mathcal{M})$ satisfying $\alpha = a/q + \lambda$, $(a,q) = 1$, where

$$P_0 = N^{1/4} < q \le Q_0, \qquad |\lambda| \le 1/(qQ_0).$$

On $\mathcal{N}$, we apply Ghosh's result in [2], which states that

$$\max_{\alpha\in\mathcal{N}} |S_1(\alpha)| \ll N^{1/2+\epsilon}P_0^{-1/4} + N^{7/16+\epsilon} + N^{1/4+\epsilon}Q_0^{1/4} \ll N^{1/2-1/16+\epsilon}. \tag{5.2}$$

Let $\mathcal{J}$ be the complement of $\mathcal{N}$ in $C(\mathcal{M})$, so that $C(\mathcal{M}) = \mathcal{J} \cup \mathcal{N}$. For $\alpha \in \mathcal{J}$, we have either

$$P < q \le P_0, \qquad |\lambda| \le 1/(qQ_0),$$

or

$$q \le P, \qquad 1/(qQ) < |\lambda| \le 1/(qQ_0).$$

In either case, we have

$$N^{1/12-\epsilon} \ll \sqrt{q(1+|\lambda|N)} \ll N^{1/8}.$$

Therefore, Lemma 5.2 gives

$$\max_{\alpha\in\mathcal{J}} |S_1(\alpha)| \ll N^{5/12+\epsilon}. \tag{5.3}$$

Combining (5.2) and (5.3), we have

$$\max_{\alpha\in C(\mathcal{M})} |S_1(\alpha)| \ll N^{1/2-1/16+\epsilon}.$$

Thus the second integral in (5.1) satisfies

$$\Big|\int_{C(\mathcal{M})\cap\mathcal{E}_\lambda}\Big| \ll N^{-E(\lambda)}N^{2-5/24+2\epsilon}L^k \ll NL^{k-1}, \tag{5.4}$$

where we used Lemma 5.1.

Using the definition of $\mathcal{E}_\lambda$ and Lemmas 3.1 and 3.2, the last integral in (5.1) can be estimated as

$$\begin{aligned}(5.5)\qquad \Big|\int\limits_{C(\mathcal{M})\cap C(\mathcal{E}_\lambda)}\Big| &\le (\lambda L)^{k-3}\Big(\int\limits_0^1 |T_1(\alpha)G(\alpha)|^2\,d\alpha\Big)^{1/2}\Big(\int\limits_0^1 |S_1(\alpha)G(\alpha)|^4\,d\alpha\Big)^{1/2}\\ &\le 54638\lambda^{k-3}NL^k.\end{aligned}$$

Combining (5.4) and (5.5), we have

$$(5.6)\qquad \Big|\int\limits_{C(\mathcal{M})}\Big| \le 54638\lambda^{k-3}NL^k + O(NL^{k-1}).$$

Setting $E = 300$, and inserting (4.13), (5.4), (5.5) into (5.1), we find that if $k \ge 12000$, then

$$\begin{aligned} r_k(N) &= \int\limits_0^1 S_1^2(\alpha)T_1(\alpha)G^k(\alpha)e(-N\alpha)\,d\alpha\\ &= \Big\{\int\limits_{\mathcal{M}} + \int\limits_{C(\mathcal{M})\cap\mathcal{E}_\lambda} + \int\limits_{C(\mathcal{M})\cap C(\mathcal{E}_\lambda)}\Big\}\\ &\ge \frac{\pi}{2}NL^k - 2c_1'NL^k\Big(\frac{\log^2 E}{E} + \frac{2\log E}{E} + \frac{2}{E}\Big)\\ &\quad - 2c_2\log^{1.5}E\Big(1 - \frac{1}{E\csc^2(\pi/8)}\Big)^k NL^k\\ &\quad - 54638\lambda^{k-3}NL^k + O(NL^{k-1})\\ &> 0.\end{aligned}$$

Here

$$c_1' = 4.152460187,\qquad c_2 = 3.803,\qquad \lambda = 0.910707.$$

This completes the proof of Theorem 1.1.

References

[1] P. X. Gallagher, *Primes and powers of 2*, Invent. Math. 29 (1975), 125–142.

[2] A. Ghosh, *The distribution of* αp^2 *modulo 1*, Proc. London Math. Soc. (3) 42 (1981), 252–269.

[3] G. H. Hardy and J. E. Littlewood, *Some problems of partitio numerorum III: On the expression of a number as a sum of primes*, Acta Math. 44 (1923), 1–70.

[4] D. R. Heath-Brown and J. C. Puchta, *Integers represented as a sum of primes and powers of two*, Asian J. Math. 6 (2002), 535–565.

[5] H. Z. Li, *The number of powers of 2 in a representation of large even integers by sums of such powers and of two primes*, Acta Arith. 92 (2000), 229–237.
[6] —, *The number of powers of 2 in a representation of large even integers by sums of such powers and of two primes (II)*, ibid. 96 (2001), 369–379.
[7] Yu. V. Linnik, *Hardy–Littlewood problem on the reprensentation as the sum of a prime and two squares*, Dokl. Akad. Nauk SSSR 124 (1959), 29–30 (in Russian).
[8] —, *An asymptotic formula in an addition problem of Hardy and Littlewood*, Izv. Akad. Nauk SSSR Ser. Mat. 24 (1960), 629–706 (in Russian).
[9] T. Liu, *A sum of a prime, two squares of primes and 22000 power of 2*, PhD Thesis, 2002.
[10] Y. Liu, M. C. Liu and T. Z. Wang, *The number of powers of 2 in a representation of large even integers (1)*, Sci. China Ser. A 41 (1998), 386–398.
[11] —, —, —, *The number of powers of 2 in a representation of large even integers (2)*, ibid. 41 (1998), 1255–1271.
[12] J. Y. Liu, M. C. Liu and T. Zhan, *Square of primes and powers of 2*, Monatsh. Math. 128 (1999), 283–313.
[13] J. Y. Liu and G. S. Lü, *Four squares of primes and 165 powers of 2*, Acta Arith. 114 (2004), 55–70.
[14] J. Pintz and I. Z. Ruzsa, *On Linnik's approximation to Goldbach's problem, I*, ibid. 109 (2003), 169–194.
[15] X. M. Ren, *On exponential sums over primes and application in the Waring–Goldbach problem*, to appear.
[16] G. J. Rieger, *Über die Summe aus einem Quadrat und einem Primzahlquadrat*, J. Reine Angew. Math. 231 (1968), 89–100.
[17] M. Q. Wang, *On the exceptional set in additive prime number theory and caps between consecutive primes*, PhD Thesis, 2004.
[18] T. Z. Wang, *On Linnik's almost Goldbach theorem*, Sci. China Ser. A 42 (1999), 1155–1172.

Liqun Hu, Li Yang
Department of Mathematics
Nanchang University
Nanchang, Jiangxi 330031, P.R. China
E-mail: huliqun8888@163.com
kongzixu@163.com

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