ACTA ARITHMETICA
121.4 (2006)

On values of a modular form on $\Gamma_0(N)$

by

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1. Introduction. The values of a modular function at certain points play a role in modular form theory. Especially, those of $j(z)$ are related to other number theoretical objects, where $j(z)$ is the usual elliptic modular function on $\mathrm{SL}_2(\mathbb{Z})$. For example, if a complex number τ is a Heegner point, i.e. $\tau=(-b+\sqrt{b^2-4ac})/2a$ with $a,b,c\in\mathbb{Z}$, $\gcd(a,b,c)=1$ and $b^2-4ac<0$, then the Hurwitz–Kronecker class number is related to the trace of $j(\tau)$ that is called a singular modulus in [9].

After Borcherds' work [2] on the infinite product expansion of modular forms with no divisor except cusps and Heegner points, some results give connections between the values of a modular function at divisor points and the exponents of the infinite product expansion of modular forms. Bruinier, Kohnen and Ono provided a relation between the infinite product expansion of a modular form f and the values of a certain meromorphic modular form at points in the divisor of f (see [3]). Ahlgren gave analogues of these results for modular forms on $\Gamma_0(p)$ for $p\in\{2,3,5,7,13\}$ (see [1]). Their results are restricted to the genus zero group.

In this paper, we give analogues of their results for $\Gamma_0(N)$ for square free N, and also describe the values of a modular function at certain points. To do this we consider a certain sequence of modular functions. In Section 2, we state the connection between the exponents of an infinite product expansion of a modular form f and the values of a modular function at divisor points of f, where $\theta f=\frac{1}{2\pi i}f'(z)$. In Section 3, we define (l,N)-type sequences of modular functions and give some related identities. In Section 4, we obtain congruence properties of values of a modular function.

2. The values of a modular function at divisor points. Suppose that N is a square free positive integer. The group $\Gamma_0(N)$ is the congruence

2000 *Mathematics Subject Classification*: Primary 11F11; Secondary 11F33.
Key words and phrases: modular forms, one variable, congruences for modular forms.
This work is partially supported by KOSEF R01-2003-00011596-0.

subgroup of $\mathrm{SL}_2(\mathbb{Z})$ defined as

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \,\middle|\, c \equiv 0 \pmod N \right\}.$$

Let Γ denote $\mathrm{SL}_2(\mathbb{Z})$ and $\mathcal{F}_N$ be a fundamental domain for the action of $\Gamma_0(N)$ on $\mathbb{H}$. We denote the set of distinct cusps by S_N,

$$S_N = \{1/w \mid w \neq N \text{ and } w \mid N\} \cup \{0, \infty\}.$$

From now on, we suppose that if t is a cusp point, then t is in S_N. The period of q-expansion at t is denoted by N_t, where N_t is given by the following way:

$$N_t = \begin{cases} Nt & \text{if } t \in S_N \setminus \{0, \infty\}, \\ 1 & \text{if } t = \infty, \\ N & \text{if } t = 0. \end{cases}$$

Adjoining the cusps to $\Gamma_0(N) \setminus \mathbb{H}$, we obtain a compact Riemann surface $X_0(N)$. For $\tau \in \mathbb{H} \cup S_N$, let Q_τ be the image of τ under the canonical map from $\mathbb{H} \cup S_N$ to $X_0(N)$.

Suppose G is a meromorphic modular form of weight 2 on $\Gamma_0(N)$. The residue of G at Q_τ on $X_0(N)$, denoted by $\mathrm{Res}_{Q_\tau} Gdz$, is well defined since we have the canonical correspondence between meromorphic modular forms of weight 2 on $\Gamma_0(N)$ and meromorphic 1-forms of $X_0(N)$. If $\mathrm{Res}_\tau G$ denotes the residue of G at τ on $\mathbb{H}$, then for $\tau \in \mathbb{H}$ we obtain

$$\mathrm{Res}_{Q_\tau} Gdz = \frac{1}{l_\tau} \mathrm{Res}_\tau G.$$

Here, l_τ is the order of the isotropy group at τ. In particular, if f is a meromorphic modular form of weight k on $\Gamma_0(N)$ and $G = \frac{\theta f}{f}$, then the residue of G at Q_τ on $\tau \in \mathbb{H}$ is computed from the order of zero or pole of f at $\tau \in \mathbb{H}$. The latter is denoted by $\nu_\tau^{(N)}(f)$ and has the form

$$\nu_\tau^{(N)}(f) = \frac{1}{l_\tau} \mathrm{ord}_\tau(f),$$

where $\mathrm{ord}_\tau(f)$ denotes the order of zero or pole of f at τ as a complex function on $\mathbb{H}$. Then we have

$$2\pi i \cdot \mathrm{Res}_{Q_\tau} \frac{\theta f}{f} = \nu_\tau^{(N)}(f). \tag{2.1}$$

We introduce some notations to describe $\mathrm{Res}_{Q_t} Gdz$ at every cusp t. First, recall the usual slash operator $f(z)|_k\gamma$ given as

$$f(z)|_k\gamma = \det(\gamma)^{k/2}(cz+d)^{-k} f(\gamma z),$$

where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2^+(\mathbb{Q})$ and γz denotes $\frac{az+b}{cz+d}$. From now on, q denotes

$e^{2\pi iz}$. We define a matrix $\gamma_t^{(N)}$ in the following way:

$$\gamma_t^{(N)} := \begin{cases} \begin{pmatrix} 1 & 0 \\ 1/t & 1 \end{pmatrix} \begin{pmatrix} N_t & 0 \\ 0 & 1 \end{pmatrix} & \text{if } 1/t \mid N, \\ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} N & 0 \\ 0 & 1 \end{pmatrix} & \text{if } t = 0, \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{if } t = \infty. \end{cases}$$

If at each cusp, G has the Fourier expansion of the form

$$G(z)|_2\gamma_t^{(N)} = \sum_{n=m_t}^{\infty} a_t(n)q^n \quad \text{at } \infty,$$

then we have

$$\mathrm{Res}_{Q_t} G dz = \frac{a_t(0)}{2\pi i} \quad \text{for } t \in S_N. \tag{2.2}$$

From now on, we assume that all of the meromorphic modular functions in this paper may have poles only at cusps. Let g be a meromorphic modular function on $\Gamma_0(N)$ which at each cusp has the q-expansion of the form

$$g(z)|_0\gamma_t^{(N)} = \sum_{n=\mu_t}^{\infty} r_t(n)q^n \quad \text{at } \infty.$$

Using these results, we can obtain a connection between the Fourier coefficients of $\frac{\theta f}{f}$ and the values of a meromorphic modular function at the divisor points of f, where f is a meromorphic modular form of weight k on $\Gamma_0(N)$.

Theorem 2.1. *Suppose that $f(z)$ is a meromorphic modular form of weight k on $\Gamma_0(N)$ with square free N. For each cusp t let $\{c_t(n)\}_{n=1}^{\infty}$ be the complex numbers for which*

$$f|_k\gamma_t^{(N)} = \alpha_t q^{h_t} \prod_{n=1}^{\infty} (1-q^n)^{c_t(n)} \quad \textit{for a complex number } \alpha_t. \tag{2.3}$$

If $g(z)$ is a meromorphic modular function on $\Gamma_0(N)$ with possible poles only at cusps, then

$$\sum_{t\in S_N} \sum_{\mu_t \le n < 0} \Big(\sum_{d|-n} c_t(d)d\Big) r_t(n) = \sum_{\tau \in \mathcal{F}_N} \nu_\tau^{(N)}(f) g(\tau) + \sum_{t\in S_N} \left(h_t - N_t \frac{k}{12}\right) r_t(0) + P_g,$$

where

$$P_g = \sum_{t\in S_N} 2kN_t \sum_{\mu_t\le n<0} \sigma(-n/N_t) r_t(n)$$

and $\sigma(n) = \sum_{0<d|n} d$.

Proof. We begin by stating a lemma which was proved by Eholzer and Skoruppa in [4].

LEMMA 2.2. *Suppose that* $f = \sum_{n=h}^{\infty} a(n)q^n$ *is a meromorphic modular function in a neighborhood of* $q = 0$ *and that* $a(h) = 1$. *Then there are uniquely determined complex numbers* $c(n)$ *such that*

$$f = q^h \prod_{n=1}^{\infty} (1-q^n)^{c(n)},$$

where the product converges in a small neighborhood of $q = 0$. *Moreover, the following identity is true*

$$\frac{\theta f}{f} = h - \sum_{n\ge 1} \sum_{d|n} c(d) d q^n.$$

Let

$$F(z) = \frac{\theta f(z)}{f(z)} - \frac{k}{12} E_2(z).$$

Here, $E_2(z)$ is the usual normalized Eisenstein series of weight 2 defined by

$$E_2(z) = 1 - 24 \sum_{n\ge 1} \sigma(n) q^n.$$

The function $F(z)$ is a meromorphic modular form of weight 2 on $\Gamma_0(N)$. Its Fourier expansion at $t \in S_N$ is given by

$$\begin{aligned} F(z)|_2\gamma_t^{(N)} &= \left(\frac{\theta f(z)}{f(z)}\right)\Big|_2 \gamma_t^{(N)} - \frac{k}{12} E_2(z)|_2\gamma_t^{(N)} \\ &= \frac{\theta(f|_k\gamma_t^{(N)})}{f|_k\gamma_t^{(N)}} - \frac{k}{12} N_t E_2(N_t z) \\ &= h_t - \sum_{n\ge 1}\sum_{d|n} c_t(d) d q^n - \frac{k}{12} N_t + 2kN_t \sum_{n\ge 1} \sigma(n/N_t) q^n. \end{aligned}$$

Since $F(z)g(z)dz$ is a meromorphic 1-form on $X_0(N)$, we deduce that for $t \in S_N$,

$$\begin{aligned} 2\pi i \operatorname{Res}_{Q_t} F(z)g(z)dz = \left(h_t - N_t \frac{k}{12}\right) r_t(0) + 2kN_t \sum_{\mu_t\le n<0} \sigma(-n/N_t) r_t(n) \\ - \sum_{\mu_t\le n<0} \Big(\sum_{d|-n} c_t(d) d\Big) r_t(n) \end{aligned}$$

from (2.2).

Next we compute $\mathrm{Res}_{Q_\tau} F(z)g(z)dz$ for $\tau \in \mathbb{H}$. For each $\tau \in \mathbb{H}$, from (2.1) we find that

$$2\pi i \,\mathrm{Res}_{Q_\tau} F(z)g(z)dz = 2\pi i \,\frac{1}{l_\tau}\,\mathrm{Res}_\tau \frac{\theta f(z)}{f(z)}\, g(z) = \nu_\tau^{(N)}(f)g(\tau)$$

since $E_2(z)$ and $g(z)$ are holomorphic on $\mathbb{H}$.

The residue theorem implies that

$$2\pi i \sum_{Q_\tau \in X_0(N)} \mathrm{Res}_{Q_\tau} F(z)g(z)dz = 0$$

since $X_0(N)$ is a compact Riemann surface. This completes the proof of Theorem 2.1. ∎

REMARK 2.3. Suppose that $j(z)$ is the usual j-function

$$j(z) = \frac{1}{q} + 744 + 196884q + \cdots$$

and $j_m(z) = (j(z) - 744) \,|\, T_0(m)$, where $T_0(m)$ is the normalized mth weight zero Hecke operator. When $g(z) = j_n(z)$, Kohnen and Ahlgren point out that Theorem 2.1 can be extended to every finite index subgroup G of Γ in the following way. Let S be the set of equivalence classes of cusps of G and h_s denote the period of $f(z)|_k\gamma_s$ where $s = \gamma_s\infty$. Let $\{c_s(n)\}_{n=1}^{\infty}$ be the complex numbers for which

$$f|_k\gamma_s = q_s^{\mu_s} \prod_{n=1}^{\infty} (1 - q_s^n)^{c_s(n)},$$

where $q_s = q^{1/h_s}$. Using the residue theorem with

$$j_m(z)\biggl(\frac{\theta(\prod_{\gamma\in G\backslash\Gamma} f(z)|_k\gamma)}{\prod_{\gamma\in G\backslash\Gamma} f(z)|_k\gamma} - [\Gamma : G]\,\frac{k}{12}\,E_2(z)\biggr)$$
$$= j_m(z)\biggl(\sum_{\gamma\in G\backslash\Gamma} \frac{\theta(f(z)|_k\gamma)}{f(z)|_k\gamma} - [\Gamma : G]\,\frac{k}{12}\,E_2(z)\biggr),$$

we obtain

$$\sum_{\tau\in G\backslash\mathbb{H}} \nu_\tau(f) j_m(\tau) + 2k\sigma(m)[\Gamma : G] = \sum_{s\in S}\sum_{d|mh_s} dc_s(d),$$

where $\nu_\tau(f) = (1/l_\tau)\,\mathrm{ord}_\tau(f)$ and l_τ is the order of the isotropy subgroup of G at τ.

We give an example of application of Theorem 2.1 for a meromorphic modular function $j_m^{(p)} = q^{-m} + \sum_{n=0}^{\infty} a_n q^n$ whose order of pole at every cusp except ∞ is bounded by a fixed constant for every positive integer m.

EXAMPLE 2.4. Let

$$\phi_p(z) := \left(\frac{\eta(z)}{\eta(pz)}\right)^{24u/(p-1)} \qquad \text{for a prime number } p,$$

where $u = (p-1)/\gcd(p-1, 24)$ and $\eta(z)$ is defined by the product

$$\eta(z) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n).$$

For each $m \geq 1$, let $j_m^{(p)}$ be a polynomial in $\phi_p(z)$ and $j(z)$ defined by

$$(2.4) \qquad j_m^{(p)}(z) := \phi_p(z)^{[m/u]} j(z)^{m-[m/u]u} + \sum_{n=0}^{m-1} b_m^{(p)}(n)\phi_p(z)^{[n/u]} j(z)^{n-[n/u]u},$$

where the constants $b_m^{(p)}(n)$ are so chosen that $j_m^{(p)}$ has the q-expansion of the form

$$j_m^{(p)}(z) = q^{-m} + \alpha_m^{(p)}(0) + \sum_{n=1}^{\infty}\alpha_m^{(p)}(n)q^n \qquad \text{at } \infty,$$

and its q-expansion at 0 has no constant term. Let $(L_{ij})_{(m+1)\times(m+1)}$ be the $(m+1)\times(m+1)$ matrix whose ij-entry is equal to the $(-j)$th coefficient of $\phi_p(z)^{[(i-1)/u]} j(z)^{(i-1)-[(i-1)/u]u}$. Since $(L_{ij})_{(m+1)\times(m+1)}$ is invertible, $b_m^{(p)}(i)$ is uniquely determined for $0 \leq i \leq m-1$. So, $j_m^{(p)}(z)$ is well defined. Let the q-expansion of $j_m^{(p)}(z)|_0\left(\begin{smallmatrix} 0 & -1 \\ p & 0 \end{smallmatrix}\right)$ at ∞ be

$$j_m^{(p)}(z)|_0\left(\begin{smallmatrix} 0 & -1 \\ p & 0 \end{smallmatrix}\right) = \sum_{n=m_0}^{\infty}\beta_m^{(p)}(n)q^n \qquad (\beta_m^{(p)}(0) = 0).$$

First, note that $-p(u-1) \leq m_0$. Secondly, $j_m^{(p)}$ is holomorphic on $\mathbb{H}$ since $\phi_p(z)$ is holomorphic on $\mathbb{H}$ and $j_m^{(p)}$ is a polynomial in $j(z)$ and $\phi_p(z)$. So, if $f(z)$ is a meromorphic modular form of weight k on $\Gamma_0(p)$, then for each $m \geq 1$ we have

$$\begin{aligned} \sum_{d|m} c_\infty(d)d + \sum_{m_0 \leq n < 0}\Big(\sum_{d|-n} c_0(d)d\Big)\beta_m^{(p)}(n) \\ = \sum_{\tau \in \mathcal{F}_p}\nu_\tau^{(p)}(f) j_m^{(p)}(\tau) + \left(h - \frac{k}{12}\right)\alpha_m^{(p)}(0) + 2k\sigma(m) \\ + 2kp\sum_{m_0 \leq n < 0}\sigma(-n/p)\beta_m^{(p)}(n). \end{aligned}$$

The sequence $\{j_m^{(p)}\}$ is an example of an (l, p)-type sequence defined in the next section.

3. Identities related to an (l,p)-type sequences. We begin by defining a class of sequences of modular functions.

DEFINITION 3.1. Let $\{g_m^{(N)}\}$ be a sequence of modular functions on $\Gamma_0(N)$. We call $\{g_m^{(N)}\}$ an (l,N)-*type sequence* if:

(1) for every $m \geq 1$, the function $g_m^{(N)}$ is holomorphic on $\mathbb{H}$,
(2) $g_m^{(N)} = q^{-m} + \sum_{n=0}^{\infty} a_m(n)q^n$ at ∞,
(3) $-l = \inf\{m_t \mid t \in S_N \setminus \{\infty\} \text{ and } m \geq 1\} \gg -\infty$ if $N \neq 1$, $l = -1$ if $N = 1$,

where $g_m^{(N)}|_0\gamma_t^{(N)} = \sum_{n=m_t}^{\infty} a_m^t(n)q^n$ at ∞ and $a_m^t(m_t) \neq 0$.

The Riemann–Roch theorem implies that for each positive integer N there exists an integer I such that for any $l \geq I$ we can find an (l,N)-type sequence.

We define a function related to a sequence $\{g_m^{(N)}\}$ of modular functions on $\Gamma_0(N)$. Let

$$[g_m^{(N)}]_\tau(z) = \sum_{n=1}^{\infty} g_n^{(N)}(\tau)q^n \quad \text{for } \tau \in \mathbb{H}$$

and

$$[g_m^{(N)}]_t(z) = \sum_{n=1}^{\infty} a_n^t(0)q^n \quad \text{for } t \in S_N \setminus \{\infty\},$$

where $g_m^{(N)}(z)|_0\gamma_t^{(N)} = \sum_{n=m_t}^{\infty} a_m^t(n)q^n$. For notational convenience we define

$$[g_m^{(N)}]_\infty(z) = -1 + \sum_{n=1}^{\infty} a_n(0)q^n.$$

Though in general $[g_m^{(N)}]_\tau(z)$ is not a meromorphic modular form, we give some examples where $[g_m^{(N)}]_\tau(z)$ is a meromorphic modular form.

EXAMPLE 3.2 (see [3]). Let $j_m(z) = (j(z) - 744) \mid T_0(m)$. A sequence $\{j_m(z)\}$ is a $(-1,1)$-type sequence, and we have

$$[j_m]_\tau(z) = \frac{E_4^2(z)E_6(z)}{\Delta(z)} \cdot \frac{1}{j(z) - j(\tau)} \quad \text{for } \tau \in \mathbb{H},$$

where $E_k(z)$ is the usual normalized Eisenstein series of weight k.

EXAMPLE 3.3. The sequence $\{j_m^{(2)}\}$ defined in Example 2.4 is a $(-1,2)$-type sequence. Let $\omega = (1+\sqrt{-3})/2$ and $f(z) = E_4(2z)$. It is well known that $E_4(2z)$ has only a zero at $\omega/2$ in $\mathcal{F}_2$. A complex number $c_t(n)$ is defined from (2.3). Then we have

$$\sum_{d|n} c_\infty(d)d = j_n^{(2)}(\omega/2) - 16\sigma(n/2) + 16\sigma(n),$$

since $\{j_m^{(2)}\}$ is a $(-1,2)$-type sequence. Using this result and Ramanujan's identity (see [6]), we obtain

$$[j_m^{(2)}]_\omega(z) = \frac{2}{3}\left(\frac{E_6(2z)}{E_4(2z)} - 2E_2(2z) + E_2(z)\right).$$

In fact, when $p \in \{2,3,5,7,13\}$, $[j_m^{(p)}]_\tau(z)$ is a meromorphic modular form for $\tau \in \mathbb{H}$. This can be deduced from Theorem 3.4 (or see [1]).

It can be shown that the θ-operator plays an important role in the theory of modular forms and p-adic modular forms (see [7], [8]). Theorem 3.4 gives an expression of $\frac{\theta f}{f}$ through $[g_m^{(N)}]_\tau$ and $[g_m^{(N)}]_\infty$.

THEOREM 3.4. *Suppose that $f(z)$ is a modular form of weight k on $\Gamma_0(N)$, and that $\{g_m^{(N)}\}$ is an (l,N)-type sequence. For each cusp t let $\{c_t(n)\}_{n=1}^\infty$ be the complex numbers for which*

$$f|_k\gamma_t^{(N)} = \alpha_t q^{h_t}\prod_{n=1}^\infty (1-q^n)^{c_t(n)} \quad \text{for a complex number } \alpha_t.$$

Then

$$\begin{aligned}
&-\frac{\theta f(z)}{f(z)} + \frac{k}{12}E_2(z)\\
&\quad = \sum_{\tau\in\mathcal{F}_N} \nu_\tau^{(p)}(f)[g_m^{(N)}]_\tau(z) + \sum_{t\in S_N}\left(h_t - N_t\frac{k}{12}\right)[g_m^{(N)}]_t(z)\\
&\qquad - \sum_{m=1}^\infty\Bigg(\sum_{t\in S_N\setminus\{\infty\}}\sum_{m_t\le n<0}\sum_{d|-n} c_t(d)a_m^t(n)d - (P_{g_m^{(N)}} - 2k\sigma(m))\Bigg)q^m,
\end{aligned}$$

where $g_m^{(N)}(z)|_0\gamma_t^{(N)} = \sum_{n=m_t}^\infty a_m^t(n)q^n$.

Proof. From Theorem 2.1 we have

$$\begin{aligned}
\frac{\theta f(z)}{f(z)} &= h_\infty - \sum_{m=1}^\infty\sum_{d|m} c_\infty(d)dq^m\\
&= h_\infty - \sum_{m=1}^\infty\left(\sum_{\tau\in\mathcal{F}_N}\nu_\tau^{(p)}(f)g_m^{(N)}(\tau) + \sum_{t\in S_N}\left(h_t - N_t\frac{k}{12}\right)a_m^t(0)\right)q^m
\end{aligned}$$

$$
\begin{aligned}
&-\sum_{m=1}^{\infty}\Big(2k\sigma(m)+\sum_{t\in S_N\setminus\{\infty\}}\sum_{m_t\le n\le 0}\Big(2kN_t\sigma(-n/N_t)-\sum_{d\mid -n}c_t(d)d\Big)a_m^t(n)\Big)q^m\\
&= -\sum_{\tau\in\mathcal{F}_N}\nu_\tau^p(f)[g_m^{(N)}]_\tau(z)-\sum_{t\in S_N}\left(h_t-N_t\,\frac{k}{12}\right)[g_m^{(N)}]_t(z)+\frac{k}{12}\,E_2(z)\\
&\quad+\sum_{m=1}^{\infty}\Big(\sum_{t\in S_N\setminus\{\infty\}}\sum_{m_t\le n<0}\sum_{d\mid n}c_t(d)a_m^t(n)d-(P_{g_m^{(N)}}-2k\sigma(m))\Big)q^m.
\end{aligned}
$$

So, the result follows. ∎

When $\{g_m^{(p)}\}$ is an (l,p)-type sequence for a prime p, we can obtain an explicit formula for the sum of $[g_m^{(p)}]_\tau(z)$ and $[g_m^{(p)}]_\infty(z)$.

THEOREM 3.5. *Suppose that $\{g_m^{(p)}\}$ is an (l,p)-type sequence for a prime p and an integer $l\ge -1$. Let*

$$
u=\frac{p-1}{\gcd(p-1,24)},\qquad v=\max\left\{1,\left[\frac{l}{u}\right]+1\right\}.
$$

Then for $\tau_0\in\mathbb{H}$ we have

$$
\sum_{\tau\in W_p}[g_m^{(p)}]_\tau(z)-uv[g_m^{(p)}]_\infty(z)=\frac{uv}{p-1}\cdot\frac{\eta(z)^{24uv/(p-1)}(pE_2(pz)-E_2(z))}{\eta(z)^{24uv/(p-1)}-\tau_0\eta(pz)^{24uv/(p-1)}},
$$

where $W_p=\{\tau\in\mathcal{F}_p\mid \phi_p(z)^v=\tau_0,\ \phi_p(z):=(\eta(z)/\eta(pz))^{24u/(p-1)}\}$.

Proof. To apply Theorem 3.4 we take $f(z)=\phi_p(\tau)^v-\tau_0$ for $\tau_0\in\mathbb{H}$. Suppose that $g_m^{(p)}(z)$ has the q-expansion

$$
\begin{aligned}
g_m^{(p)}&=q^{-m}+\sum_{n=0}^{\infty}a_m^{\infty}(n)q^n\quad\text{ at }\infty,\\
g_m^{(p)}|_0\gamma_0^{(p)}&=\sum_{n=m_0}^{\infty}a_m^{0}(n)q^n\quad\text{ at }\infty.
\end{aligned}
$$

For each cusp t let $\{c_t(n)\}_{n=1}^{\infty}$ be the complex numbers for which

$$
f|_k\gamma_t^{(N)}=\alpha_t q^{h_t}\prod_{n=1}^{\infty}(1-q^n)^{c_t(n)}\qquad\text{for a complex number }\alpha_t.
$$

Here, $h_\infty=-uv$ and $h_0=0$. Note that the nth Fourier coefficient of $\phi_p(\tau)^v|_0\gamma_0^{(p)}$ is zero for every $n\le l$. This implies that $\theta(f|_0\gamma_0)$ has a zero of order at least $l+1$ at ∞. So, the function $\frac{\theta f}{f}\big|_2\gamma_0$ has the q-expansion

$$
\frac{\theta f}{f}\bigg|_2\gamma_0=\frac{\theta(f|_0\gamma_0)}{f|_0\gamma_0}=\sum_{d\mid(l+1)}c_0(d)dq^{l+1}+O(q^{l+2}).
$$

Since $\{g_m^{(p)}(z)\}$ is an (l,p)-type sequence, for every $m \geq 1$ we obtain

$$\sum_{m_0 \leq n < 0} \Big(\sum_{d|-n} c_0(d)d\Big) a_m^0(n) = 0.$$

Moreover, for every $m \geq 1$, we have $P_{g_m^{(p)}} = 0$ by noting that the weight of f is zero. Therefore, from Theorem 3.4 we obtain

$$\begin{aligned}\sum_{\tau \in W_p} [g_m^{(p)}]_\tau(z) - uv[g_m^{(p)}]_\infty(z) &= -\frac{\theta f}{f} \\ &= \frac{24uv}{p-1} \cdot \frac{\phi_p(z)^v \big(\frac{p}{2\pi i}\frac{\eta'(pz)}{\eta(pz)} - \frac{1}{2\pi i}\frac{\eta'(z)}{\eta(z)}\big)}{\phi_p(z)^v - \tau_0} \\ &= \frac{uv}{p-1} \cdot \frac{\eta(z)^{24uv/(p-1)}(pE_2(pz) - E_2(z))}{\eta(z)^{24uv/(p-1)} - \tau_0 \eta(pz)^{24uv/(p-1)}}\end{aligned}$$

since

$$\frac{\eta'(z)}{\eta(z)} = \frac{2\pi i}{24} E_2(z). \quad \blacksquare$$

Remark 3.6. Suppose that $p \in \{2,3,5,7,13\}$. Take $g_m^{(p)} = \{j_m^{(p)}\}$ defined in Example 2.4. The sequence $\{j_m^{(p)}\}$ is a $(-1,p)$-type sequence, and

$$[j_m^{(p)}]_\infty(z) = \frac{-1}{p-1}\,(E_2(z) - pE_2(pz)).$$

Take $\tau_0 = \phi_p(\tau)$ for a fixed $\tau \in \mathbb{H}$. Since ϕ_p is bijective from $\mathcal{F}_p$ to $\mathbb{C} \setminus \{0\}$, we have $W_p = \{\tau\}$. Then we obtain

$$[j_m^{(p)}]_\tau(z) = \frac{pE_2(pz) - E_2(z)}{p-1}\bigg(\frac{\eta(z)^{24/(p-1)}}{\eta(z)^{24/(p-1)} - \phi_p(\tau)\eta(pz)^{24/(p-1)}} - 1\bigg)$$

from Theorem 3.5. So, in this case we recover the result given in [1]. Furthermore, Theorem 1 in [1] turns out to be a special case of Theorem 3.5.

4. Values of modular functions at certain points. In this section, we introduce some congruence properties of $j_m^{(p)}(\tau)$ at the zero divisor of E_k. The von Staudt–Clausen Theorem (see p. 153 in [5]) gives the congruence properties of $E_k(z)$: for each $k \geq 4$,

$$E_k \equiv 1 \Big(\bmod 4 \prod_{\substack{(e-1)|k \\ 5 \leq e \text{ prime}}} e\Big).$$

From this identity, when $p \in \{2,3,5,7,13\}$, we obtain congruence properties of the values of $j_m^{(p)}$ defined in Example 2.4 at a certain point.

COROLLARY 4.1. *Suppose that $p \in \{2,3,5,7,13\}$. Let f be a normalized modular form of weight δ such that*

$$f = \sum_{i=1}^{h} c_i \prod_{j=1}^{t_i} E_{k_{ij}}(L_{ij}z) \quad \text{for } c_i \in \mathbb{Z} \text{ and } L_{ij} \mid p,$$

where $k_{ij} \geq 4$ is a positive even integer. Let k be the greatest common divisor of the k_{ij}. If $\gcd(e, p-1) = 1$ for every e such that $e \geq 5$ and $(e-1) \mid k$, then

$$\sum_{\tau \in \mathcal{F}_p} \nu_\tau^{(p)}(f) j_m^{(p)}(\tau) \equiv \frac{2\delta p}{p-1}\,\sigma(m(p-1)/p) - \frac{2\delta p}{p-1}\,\sigma(m(p-1)) \Big(\mathrm{mod} \prod_{\substack{(e-1)|k \\ 5 \leq e\, prime}} e\Big).$$

Proof. Using the von Staudt–Clausen Theorem, for even $r \geq 4$ we have

$$E_r \equiv 1 \Big(\mathrm{mod}\, 4 \prod_{\substack{(e-1)|r \\ 5 \leq e \text{ prime}}} e\Big).$$

This identity implies

$$\frac{\theta f}{f} \equiv 0 \Big(\mathrm{mod}\, 4 \prod_{\substack{(e-1)|k \\ 5 \leq e \text{ prime}}} e\Big).$$

From Theorem 2.1 we obtain

$$\sum_{\tau \in \mathcal{F}_p} \nu_\tau^{(p)}(f) j_m^{(p)}(\tau) \equiv \frac{2\delta p}{p-1}\,\sigma(m(p-1)/p) - \frac{2\delta p}{p-1}\,\sigma(m(p-1)) \Big(\mathrm{mod} \prod_{\substack{(e-1)|k \\ 5 \leq e \text{ prime}}} e\Big). \blacksquare$$

COROLLARY 4.2. *Suppose that $p \in \{2,3,5,7,13\}$. Let $j_m^{(p)}$ be a modular function defined in Example 2.4 and let ω denote $(1+\sqrt{-3})/2$.*

(I) *Let $T := \{\gamma\omega \in \mathcal{F}_p \mid \gamma \in \mathrm{SL}_2(\mathbb{Z})\}$ and M be a positive integer not divisible by a prime $\mathcal{P} \equiv 1 \ (\mathrm{mod}\, 3)$. If $\gcd(p-1, M) = 1$, then for some positive real number $\xi(M)$ we have*

$$\sharp\Big\{1 \leq m \leq X \,\Big|\, 3\Big(\sum_{\tau \in T} \frac{1}{l_\tau}\, j_m^{(p)}(\tau)\Big) \equiv \frac{24}{p-1}\,(p\sigma(m/p) - \sigma(m))\ (\mathrm{mod}\, M)\Big\} = \mathcal{O}\Big(\frac{X}{(\log X)^{\xi(M)}}\Big).$$

(II) *Let* $T := \{\gamma i \in \mathcal{F}_p \mid \gamma \in \mathrm{SL}_2(\mathbb{Z})\}$ *and* M *be a positive integer not divisible by a prime* $\mathcal{P} \equiv 1 \pmod 4$. *If* $\gcd(p-1, M) = 1$, *then for some positive real number* $\xi(M)$ *we obtain*

$$\sharp\left\{1 \le m \le X \,\middle|\, 2\Big(\sum_{\tau \in T} \frac{1}{l_\tau} j_m^{(p)}(\tau)\Big) \equiv \frac{24}{p-1}\,(p\sigma(m/p) - \sigma(m)) \pmod M\right\} = \mathcal{O}\left(\frac{X}{(\log X)^{\xi(M)}}\right).$$

Proof. Let $j_m(z) = (j(z) - 744) \,|\, T_0(m)$. Applying Theorem 2.1 to $j_m^{(p)}$, we obtain

$$3\Big(\sum_{\tau \in \mathcal{F}_N} \nu_\tau^{(p)}(E_4) j_m^{(p)}(\tau)\Big) - \frac{24}{p-1}\,(p\sigma(m/p) - \sigma(m)) = j_m(\omega),$$

$$2\Big(\sum_{\tau \in \mathcal{F}_p} \nu_\tau^{(p)}(E_6) j_m^{(p)}(\tau)\Big) - \frac{24}{p-1}\,(p\sigma(m/p) - \sigma(m)) = j_m(i).$$

Theorem 6 in [3] implies that for each $\tau \in \{i, \omega\}$ there is a positive number $\xi(M)$ for which

$$\sharp\{1 \le m \le X \mid j_m(\tau) \equiv 0 \pmod M\} = \mathcal{O}\left(\frac{X}{(\log X)^{\xi(M)}}\right),$$

where M is as in (I) if $\tau = \omega$, and as in (II) if $\tau = i$. Therefore, the corollary can be proved by the method of Theorem 6 in [3]. So, we omit the details. ∎

Acknowledgements. I would like to thank Professors W. Kohnen and S. Ahlgren for helpful comments and allowing me to mention their results. I am grateful to Professor Y. Choie for the various instructive discussions and introducing me to this object. I also appreciate the referee for a careful reading of the manuscript and for helpful comments.

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Received on 18.6.2004
and in revised form on 9.9.2005 (4792)