ACTA ARITHMETICA
149.4 (2011)

Note on a product formula for the Bayad function and a law of quadratic reciprocity

by

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1. Introduction. This study is basically motivated by a study of Bayad [1] who constructed some elliptic functions and proved product formulas for them. Moreover, he used these product formulas to prove an explicit quadratic reciprocity law in an imaginary quadratic number field. However, Bayad's argument contains some flawed parts, and his quadratic reciprocity law also requires a modification.

Our first aim is to prove a generalized product formula for the elliptic function f_Ω treated in [1], correcting the flaws in Bayad's argument (see Theorem 3.1). Here we investigate certain quantities ξ_Ω and ε_Ω defined in relation to our product formula for f_Ω. The fundamental properties of the Klein function $\mathcal{K}_\Omega$ and the Jacobi form D_Ω listed in Section 2 enable us to give explicit expressions of ξ_Ω and ε_Ω using the values of $\mathcal{K}_\Omega$. In this way, we prove that ε_Ω has a cocycle property (see Theorem 4.3). In a certain case ε_Ω defines a character of order 2 or 4 in a ring of integers of an imaginary quadratic number field.

Next, we use our product formula to prove an explicit quadratic reciprocity law (see Theorem 6.1), which also corrects and refines Bayad's reciprocity law of [1]. Our method uses the classical results of Eisenstein ([4], [5]). It is remarkable that our reciprocity law has a quite similar form to a formula established by Hajir–Villegas [6], and the comparison of these formulas raises some interesting problems (see Theorems 6.3 and 6.4).

2. Terminology and reformulation of the Bayad function. By a $\mathbb{C}$-*lattice* we mean a free $\mathbb{Z}$-module of rank 2 which spans $\mathbb{C}$ over $\mathbb{R}$. For a

2010 *Mathematics Subject Classification*: 11A15, 11F03, 11F20, 11G05.
Key words and phrases: complex multiplication, product formula, reciprocity law.

DOI: 10.4064/aa149-4-2

$\mathbb{C}$-lattice Ω with $\mathbb{Z}$-basis $\{\omega_1, \omega_2\}$ such that $\mathrm{Im}(\omega_1/\omega_2) > 0$,

$$a(\Omega) := \frac{1}{2i}\begin{vmatrix}\omega_1 & \omega_2 \\ \overline{\omega}_1 & \overline{\omega}_2\end{vmatrix} = |\omega_2|^2 \mathrm{Im}\left(\frac{\omega_1}{\omega_2}\right)$$

is a positive real number representing the area of a fundamental parallelogram of Ω which depends only on Ω. Let E_Ω be the $\mathbb{R}$-bilinear form defined by

$$E_\Omega(u,v) := \frac{1}{2ia(\Omega)}(\overline{u}v - u\overline{v}) \quad \text{for } (u,v) \in \mathbb{C} \times \mathbb{C}.$$

Then E_Ω is integral-valued on $\Omega \times \Omega$ and in particular $E_\Omega(\omega_1, \omega_2) = -1$ for any basis $\{\omega_1, \omega_2\}$ of Ω such that $\mathrm{Im}(\omega_1/\omega_2) > 0$.

Here we review briefly the Klein function $\mathcal{K}_\Omega$, the Jacobi form D_Ω and their fundamental properties, quoting mainly from Bayad–Ayala [2]. For details, one should also refer to Kubert [7], Kubert–Lang [8] and Lang [10]. The *Klein function* $\mathcal{K}_\Omega$ attached to a $\mathbb{C}$-lattice Ω is defined by the infinite product

$$\mathcal{K}_\Omega(z) = ze^{-\frac{1}{2}z\eta(z,\Omega)} \prod_{\omega \in \Omega \setminus \{0\}} \left(1 - \frac{z}{\omega}\right) e^{\frac{z}{\omega} + \frac{1}{2}(\frac{z}{\omega})^2}$$

for any $z \in \mathbb{C}$, where $\eta(z,\Omega)$ denotes the Weierstrass–Legendre eta function attached to Ω. The function $\mathcal{K}_\Omega$ has the following fundamental properties:

(K1) For $\rho \in \Omega$, $\mathcal{K}_\Omega(z+\rho) = \chi_\Omega(\rho) e(E_\Omega(\rho, z)/2)\mathcal{K}_\Omega(z)$, where

$$\chi_\Omega(\rho) = \begin{cases} 1 & \text{if } \rho \in 2\Omega, \\ -1 & \text{if } \rho \in \Omega \setminus 2\Omega, \end{cases}$$

and $e(x) = e^{2\pi i x}$ for $x \in \mathbb{R}$.

(K2) $\mathcal{K}_\Omega(z)$ is homogeneous of degree 1, that is,

$$\mathcal{K}_{\lambda\Omega}(\lambda z) = \lambda \mathcal{K}_\Omega(z) \quad \text{for } \lambda \in \mathbb{C}^\times = \mathbb{C} \setminus \{0\}.$$

In particular, $\mathcal{K}_\Omega(-z) = -\mathcal{K}_\Omega(z)$.

(K3) $\mathcal{K}_\Omega(z)$ has principal part z as z tends to 0, that is,

$$\lim_{z \to 0} \frac{\mathcal{K}_\Omega(z)}{z} = 1.$$

Let Ω and Λ be two $\mathbb{C}$-lattices such that $\Omega \subset \Lambda$, and $\mathcal{R}$ be any complete system of representatives of Λ/Ω. Then the following product formula holds:

$$\text{(K4)} \qquad \mathcal{K}_\Lambda(z) = e\Big(E_\Omega\Big(z, \sum_{\substack{x \in \mathcal{R} \\ x \notin \Omega}} x\Big)/2\Big) \mathcal{K}_\Omega(z) \prod_{\substack{x \in \mathcal{R} \\ x \notin \Omega}} \frac{\mathcal{K}_\Omega(z+x)}{\mathcal{K}_\Omega(x)}.$$

The *Jacobi form* D_Ω associated with $\mathcal{K}_\Omega$ is defined by

$$D_\Omega(z;\varphi) = e(E_\Omega(z,\varphi)/2)\frac{\mathcal{K}_\Omega(z+\varphi)}{\mathcal{K}_\Omega(z)\mathcal{K}_\Omega(\varphi)} \quad \text{for } z, \varphi \in \mathbb{C} \setminus \Omega.$$

$D_\Omega(z;\varphi)$ has the following fundamental properties:

(D1) $D_\Omega(z;\varphi+\rho) = D_\Omega(z;\varphi)$ for any $\rho \in \Omega$.

(D2) $D_\Omega(z+\rho;\varphi) = e(E_\Omega(\rho,\varphi))D_\Omega(z;\varphi)$ for any $\rho \in \Omega$.

(D3) $D_\Omega(z;\varphi) = e(E_\Omega(z,\varphi))D_\Omega(\varphi;z)$.

(D4) $D_\Omega(z;\varphi)$ is homogeneous of degree -1, that is,

$$D_{\lambda\Omega}(\lambda z;\lambda\varphi) = \lambda^{-1}D_\Omega(z;\varphi) \quad \text{for } \lambda \in \mathbb{C}^\times.$$

(D5) $D_\Omega(z;\varphi)$ has principal part $1/z$ as z tends to 0, that is,

$$\lim_{z\to 0} zD_\Omega(z;\varphi) = 1.$$

Let Ω, Λ and $\mathcal{R}$ be as before. Then the main theorem in [2] gives the following product formulas:

(D6) For any z and $\varphi \in \mathbb{C}\setminus\Lambda$,

$$D_\Lambda(z;\varphi) = \frac{\mathcal{K}_\Omega(\varphi)^{[\Lambda:\Omega]}}{\mathcal{K}_\Lambda(\varphi)}\prod_{x\in\mathcal{R}} D_\Omega(z+x;\varphi)e(-E_\Omega(x,\varphi)).$$

(D7) For any $z \in \mathbb{C}\setminus\Lambda$,

$$\prod_{x\in\mathcal{R},\, x\notin\Omega} D_\Omega(z;x)^{-1} = \frac{\mathcal{K}_\Omega(z)^{[\Lambda:\Omega]}}{\mathcal{K}_\Lambda(z)}.$$

In relation to the Weierstrass $\wp$-function, the following formulas hold:

(D8) For any $z,\varphi \in \mathbb{C}\setminus\Omega$, $\wp_\Omega(z)-\wp_\Omega(\varphi) = D_\Omega(z;\varphi)D_\Omega(z;-\varphi)$.

(D9) For any $z \in \mathbb{C}\setminus\Omega$, $\wp'_\Omega(z) = -2\prod_\varphi D_\Omega(z;\varphi)$, where φ runs over the set of representatives of $\frac{1}{2}\Omega/\Omega$ such that $\varphi\notin\Omega$.

The *Bayad function* f_Ω attached to a $\mathbb{C}$-lattice Ω with basis $\{\omega_1,\omega_2\}$ is originally defined by

$$f_\Omega(z) = C\frac{\wp_\Omega(z)-\wp_\Omega((\omega_1+\omega_2)/2)}{\wp'_\Omega(z)}$$

with a constant C such that

$$C^2 = \frac{2\wp''_\Omega(\omega_2/2)}{\wp_\Omega(\omega_2/2)-\wp_\Omega((\omega_1+\omega_2)/2)}$$

(see [1]). Of course the definition of f_Ω depends on the choice of a basis $\{\omega_1,\omega_2\}$ of Ω. Using (D7) and (D8), $f_\Omega(z)$ can be rewritten as follows:

$$\begin{aligned}
f_\Omega(z) &= -\frac{C}{2}\,\frac{D_\Omega(z;(\omega_1+\omega_2)/2)}{D_\Omega(z;\omega_1/2)D_\Omega(z;\omega_2/2)}\\
&= -\frac{C}{2}\,\frac{\mathcal{K}_\Omega(\omega_1/2)\mathcal{K}_\Omega(\omega_2/2)}{\mathcal{K}_\Omega((\omega_1+\omega_2)/2)}\cdot\frac{\mathcal{K}_\Omega(z+(\omega_1+\omega_2)/2)\mathcal{K}_\Omega(z)}{\mathcal{K}_\Omega(z+\omega_1/2)\mathcal{K}_\Omega(z+\omega_2/2)}\\
&:= C_1\frac{\mathcal{K}_\Omega(z+(\omega_1+\omega_2)/2)\mathcal{K}_\Omega(z)}{\mathcal{K}_\Omega(z+\omega_1/2)\mathcal{K}_\Omega(z+\omega_2/2)}.
\end{aligned}$$

A short calculation (using (D8) and (D9)) gives $C_1 = \pm e\big(\frac{1}{8}E_\Omega(\omega_2,\omega_1)\big)$. Hence we may adopt

$$f_\Omega(z) = e\left(\frac{1}{8}E_\Omega(\omega_2,\omega_1)\right)\frac{\mathcal{K}_\Omega(z+(\omega_1+\omega_2)/2)\mathcal{K}_\Omega(z)}{\mathcal{K}_\Omega(z+\omega_1/2)\mathcal{K}_\Omega(z+\omega_2/2)}. \tag{2.1}$$

as the definition of the Bayad function. As is easily seen, f_Ω is an Ω-elliptic function and its divisor on $\mathbb{C}/\Omega$ is

$$(f_\Omega) = \left(\frac{\omega_1+\omega_2}{2}\right) + (0) - \left(\frac{\omega_1}{2}\right) - \left(\frac{\omega_2}{2}\right). \tag{2.2}$$

The following lemma is immediate from the definition (2.1) and the formula (K1) (see also Theorem 1.8 in [1]).

Lemma 2.1. *Under the above notation, we have*

(1) $f_\Omega(z)\cdot f_\Omega(z+\omega_1/2) = 1$, (2) $f_\Omega(z)\cdot f_\Omega(z+\omega_2/2) = -1$.

3. Product formula for f_Ω. Hereafter we consider the case where Ω admits complex multiplication. Let O be an order in an imaginary quadratic number field Σ and let Ω be a fixed proper O-ideal. For any fixed element α in O, we define

$$\mathrm{Ker}(\alpha) = \mathrm{Ker}_\Omega(\alpha) := \{x \in \mathbb{C}/\Omega \mid \alpha x = 0\}.$$

We call an element x in $\mathrm{Ker}(\alpha)$ an *α-division point* of Ω. In particular, $x \in \mathrm{Ker}(\alpha)$ is called a *primitive* α-division point of Ω if $\alpha_1 x \neq 0$ for any $\alpha_1 \in O$ such that $\alpha_1 \notin \alpha O$. Plainly $\mathrm{Ker}(\alpha) = \alpha^{-1}\Omega/\Omega$ and this is a finite group of order $N\alpha$, where $N\alpha$ is the absolute norm of α. Moreover, if x_α is a fixed primitive α-division point of Ω, then $\mathrm{Ker}(\alpha) = \{rx_\alpha \mid r \bmod \alpha O,\ r \in O\}$. Sometimes, we use the notation $\mathrm{Ker}(\alpha)$ identifying it with a complete set of representatives of $\alpha^{-1}\Omega/\Omega$.

Let $J^*(2)$ be the set $\{\alpha \in O \mid \alpha \text{ coprime to } 2O\}$. Bayad stated a product formula for f_Ω ([1, Theorem 1.8]). However, his formula is valid only under the additional assumption $\alpha \equiv 1 \pmod{2O}$, not stated in [1]. Here we give a more general product formula for f_Ω, which corrects and refines the formula of Bayad.

Theorem 3.1. *For any α in $J^*(2)$,*

$$f_\Omega(\alpha z)\frac{D_\Omega^2(\alpha z;\alpha(\omega_1+\omega_2)/2)}{D_\Omega^2(\alpha z;(\omega_1+\omega_2)/2)} = \xi_\Omega(\alpha)\prod_{x\in\mathrm{Ker}(\alpha)} f_\Omega(z+x),$$

where $\xi_\Omega(\alpha)$ is given by

$$\xi_\Omega(\alpha) = \alpha\prod_{\substack{x\in\mathrm{Ker}(\alpha)\\ x\neq 0}}(f_\Omega(x))^{-1} = \alpha\prod_{\substack{x\in\mathrm{Ker}(\alpha)\\ x\neq 0}} f_\Omega(x+\omega_1/2).$$

In particular, if $\alpha \equiv 1 \pmod{2O}$, then $\xi_\Omega(\alpha) = \pm 1$.

Proof. By the formulas (D1) and (D8),

$$\frac{D_\Omega^2(\alpha z;\alpha(\omega_1+\omega_2)/2)}{D_\Omega^2(\alpha z;(\omega_1+\omega_2)/2)}=\frac{\wp_\Omega(\alpha z)-\wp_\Omega(\alpha(\omega_1+\omega_2)/2)}{\wp_\Omega(\alpha z)-\wp_\Omega((\omega_1+\omega_2)/2)}$$

for any $\alpha\in J^*(2)$, and this is an Ω-elliptic function. By a tedious check using (2.2), we see that the two Ω-elliptic functions

$$z\mapsto f_\Omega(\alpha z)\frac{D_\Omega^2(\alpha z;\alpha(\omega_1+\omega_2)/2)}{D_\Omega^2(\alpha z;(\omega_1+\omega_2)/2)}\quad\text{and}\quad z\mapsto\prod_{x\in\mathrm{Ker}(\alpha)}f_\Omega(z+x)$$

have the common divisor

$$\sum_{x\in\mathrm{Ker}(\alpha)}\left[\left(\frac{\omega_1+\omega_2}{2}+x\right)+(x)-\left(\frac{\omega_1}{2}+x\right)-\left(\frac{\omega_2}{2}+x\right)\right]$$

on $\mathbb{C}/\Omega$, and therefore they differ at most by a non-zero constant multiple. Put

$$f_\Omega(\alpha z)\frac{D_\Omega^2(\alpha z;\alpha(\omega_1+\omega_2)/2)}{D_\Omega^2(\alpha z;(\omega_1+\omega_2)/2)}=c\prod_{x\in\mathrm{Ker}(\alpha)}f_\Omega(z+x),$$

for some c in $\mathbb{C}^\times$. Then (D5) and the fact that $\lim_{z\to0}f(\alpha z)/f(z)=\alpha$ show that $c=\xi_\Omega(\alpha)$. In particular, if $\alpha\equiv1\pmod{2O}$, then the quotient factor $D_\Omega^2(\alpha z;\alpha(\omega_1+\omega_2)/2)/D_\Omega^2(\alpha z;(\omega_1+\omega_2)/2)$ can be removed and we have $\xi_\Omega^2(\alpha)=1$ as in [1].

In the rest of this section, we consider the value $\xi_\Omega(\alpha)$ more precisely. Let $-d=-d_\Sigma f^2$ be the discriminant of O, where $-d_\Sigma$ is the discriminant of the maximal order O_Σ of Σ and $f=[O_\Sigma:O]$. Now there are three possibilities for the multiplicative group $(O/2O)^\times$:

(a) $(O/2O)^\times\cong\{1\}$, when $d\equiv7\pmod8$.
(b) $(O/2O)^\times$ is a cyclic group of order 2, when $d\equiv0\pmod4$.
(c) $(O/2O)^\times$ is a cyclic group of order 3, when $d\equiv3\pmod8$.

In case (a), since $\alpha\in J^*(2)\Leftrightarrow\alpha\equiv1\pmod{2O}$, we have $\xi_\Omega(\alpha)=\pm1$.

In case (b), we may choose a basis $\{\omega_1,\omega_2\}$ of Ω so that $\omega_1/2$ and $\omega_2/2$ represent two distinct primitive 2-division points of Ω, and $(\omega_1+\omega_2)/2$ another non-zero 2-division point. Then for any $\alpha\in J^*(2)$ such that $\alpha\not\equiv1\pmod{2O}$, as elements of $\mathbb{C}/\Omega$,

$$\alpha\frac{\omega_1}{2}=\frac{\omega_2}{2},\qquad\alpha\frac{\omega_2}{2}=\frac{\omega_1}{2},\qquad\alpha\frac{\omega_1+\omega_2}{2}=\frac{\omega_1+\omega_2}{2},$$

and also in this case the D_Ω-factor in Theorem 3.1 can be deleted. However, note that $\xi_\Omega(\alpha) = \pm\sqrt{-1}$, because by Lemma 2.1 and Theorem 3.1,

$$\begin{aligned}-1 &= f_\Omega(\alpha z) f_\Omega(\alpha z + \omega_2/2) = f_\Omega(\alpha z) f_\Omega(\alpha(z + \omega_1/2)) \\ &= \xi_\Omega^2(\alpha) \prod_{v \in \mathrm{Ker}(\alpha)} f_\Omega(z+v) f_\Omega(z + v + \omega_1/2) = \xi_\Omega^2(\alpha).\end{aligned}$$

In case (c), the set $\{\omega_1/2, \omega_2/2, (\omega_1+\omega_2)/2\}$ represents all primitive 2-division points of Ω. Then for any $\alpha \in J^*(2)$ such that $\alpha \not\equiv 1 \pmod{2O}$, two subcases are possible:

$$\text{(I)} \quad \alpha\frac{\omega_1+\omega_2}{2} = \frac{\omega_1}{2}, \quad \alpha\frac{\omega_1}{2} = \frac{\omega_2}{2}, \quad \alpha\frac{\omega_2}{2} = \frac{\omega_1+\omega_2}{2},$$

$$\text{(II)} \quad \alpha\frac{\omega_1+\omega_2}{2} = \frac{\omega_2}{2}, \quad \alpha\frac{\omega_2}{2} = \frac{\omega_1}{2}, \quad \alpha\frac{\omega_1}{2} = \frac{\omega_1+\omega_2}{2}.$$

In subcase (I), by Lemma 2.1 and Theorem 3.1, we have

$$\begin{aligned}-1 &= f_\Omega(\alpha z) f_\Omega(\alpha z + \omega_2/2) = f_\Omega(\alpha z) f_\Omega(\alpha(z + \omega_1/2)) \\ &= \xi_\Omega^2(\alpha) \frac{D_\Omega^2(\alpha z; (\omega_1+\omega_2)/2)}{D_\Omega^2(\alpha z; \alpha(\omega_1+\omega_2)/2)} \cdot \frac{D_\Omega^2(\alpha(z+\omega_1/2); (\omega_1+\omega_2)/2)}{D_\Omega^2(\alpha(z+\omega_1/2); \alpha(\omega_1+\omega_2)/2)} \\ &\quad \times \prod_{v \in \mathrm{Ker}(\alpha)} f_\Omega(z+v) f(z+v+\omega_1/2) \\ &= \xi_\Omega^2(\alpha) \frac{D_\Omega^2(\alpha z; (\omega_1+\omega_2)/2)}{D_\Omega^2(\alpha z; \alpha(\omega_1+\omega_2)/2)} \cdot \frac{D_\Omega^2(\alpha(z+\omega_1/2); (\omega_1+\omega_2)/2)}{D_\Omega^2(\alpha(z+\omega_1/2); \alpha(\omega_1+\omega_2)/2)}.\end{aligned}$$

Moreover, taking the limit $z \to 0$, we have

$$\xi_\Omega^2(\alpha) = -\frac{D_\Omega^2(\alpha\omega_1/2; \alpha(\omega_1+\omega_2)/2)}{D_\Omega^2(\alpha\omega_1/2; (\omega_1+\omega_2)/2)}.$$

Similarly, in subcase (II), we have

$$\xi_\Omega^2(\alpha) = -\frac{D_\Omega^2(\alpha\omega_2/2; \alpha(\omega_1+\omega_2)/2)}{D_\Omega^2(\alpha\omega_2/2; (\omega_1+\omega_2)/2)}.$$

Hence by a simple check (using (D1) and (D2)), we can summarize that in case (c),

$$\xi_\Omega^2(\alpha) = -\frac{D_\Omega^2(\alpha^2(\omega_1+\omega_2)/2; \alpha(\omega_1+\omega_2)/2)}{D_\Omega^2(\alpha^2(\omega_1+\omega_2)/2; (\omega_1+\omega_2)/2)}. \tag{3.1}$$

4. The characters ε_Ω and $\tilde{\varepsilon}_\Omega$. Let O and Ω be as in Section 3, and fix a basis $\{\omega_1, \omega_2\}$ of Ω. Then, using (2.1), (K2) and (K4), we have

$$\begin{aligned}
&\prod_{\substack{x\in\mathrm{Ker}(\alpha)\\ x\neq 0}} f_\Omega(x)\\
&= e\Big(\frac{1}{8}(N\alpha-1)E_\Omega(\omega_2,\omega_1)\Big)\prod_{\substack{x\in\mathrm{Ker}(\alpha)\\ x\neq 0}}\frac{\mathcal{K}_\Omega(x)\mathcal{K}_\Omega(x+(\omega_1+\omega_2)/2)}{\mathcal{K}_\Omega(x+\omega_1/2)\mathcal{K}_\Omega(x+\omega_2/2)}\\
&= e\Big(\frac{1}{8}(N\alpha-1)E_\Omega(\omega_2,\omega_1)\Big)\\
&\quad\times\prod_{\substack{x\in\mathrm{Ker}(\alpha)\\ x\neq 0}}\frac{\mathcal{K}_\Omega(x+(\omega_1+\omega_2)/2)}{\mathcal{K}_\Omega(x)}\,\frac{\mathcal{K}_\Omega(x)}{\mathcal{K}_\Omega(x+\omega_1/2)}\,\frac{\mathcal{K}_\Omega(x)}{\mathcal{K}_\Omega(x+\omega_2/2)}\\
&= e\Big(\frac{1}{8}(N\alpha-1)E_\Omega(\omega_2,\omega_1)\Big)\frac{\mathcal{K}_{\alpha^{-1}\Omega}((\omega_1+\omega_2)/2)}{\mathcal{K}_\Omega((\omega_1+\omega_2)/2)}\,\frac{\mathcal{K}_\Omega(\omega_1/2)}{\mathcal{K}_{\alpha^{-1}\Omega}(\omega_1/2)}\,\frac{\mathcal{K}_\Omega(\omega_2/2)}{\mathcal{K}_{\alpha^{-1}\Omega}(\omega_2/2)}\\
&= e\Big(\frac{1}{8}(N\alpha-1)E_\Omega(\omega_2,\omega_1)\Big)\alpha\frac{\mathcal{K}_\Omega(\alpha(\omega_1+\omega_2)/2)}{\mathcal{K}_\Omega((\omega_1+\omega_2)/2)}\,\frac{\mathcal{K}_\Omega(\omega_1/2)}{\mathcal{K}_\Omega(\alpha\omega_1/2)}\,\frac{\mathcal{K}_\Omega(\omega_2/2)}{\mathcal{K}_\Omega(\alpha\omega_2/2)}.
\end{aligned}$$

Hence

$$\begin{aligned}
\xi_\Omega(\alpha) &= \alpha\Big(\prod_{\substack{x\in\mathrm{Ker}(\alpha)\\ x\neq 0}} f_\Omega(x)\Big)^{-1}\\
&= e\Big(\frac{1}{8}(N\alpha-1)E_\Omega(\omega_1,\omega_2)\Big)\frac{\mathcal{K}_\Omega((\omega_1+\omega_2)/2)}{\mathcal{K}_\Omega(\alpha(\omega_1+\omega_2)/2)}\,\frac{\mathcal{K}_\Omega(\alpha\omega_1/2)}{\mathcal{K}_\Omega(\omega_1/2)}\,\frac{\mathcal{K}_\Omega(\alpha\omega_2/2)}{\mathcal{K}_\Omega(\omega_2/2)}.
\end{aligned}$$

Here we define ε_Ω by

$$\varepsilon_\Omega(\alpha) := e\Big(\frac{1}{8}(N\alpha-1)E_\Omega(\omega_1,\omega_2)\Big)\prod_\rho\frac{\mathcal{K}_\Omega(\alpha\rho)}{\mathcal{K}_\Omega(\rho)} \tag{4.1}$$

for $\alpha\in J^*(2)$, where ρ runs over the set $\{\omega_1/2,\omega_2/2,(\omega_1+\omega_2)/2\}$. Then we have

$$\xi_\Omega(\alpha) = \varepsilon_\Omega(\alpha)\frac{\mathcal{K}^2_\Omega((\omega_1+\omega_2)/2)}{\mathcal{K}^2_\Omega(\alpha(\omega_1+\omega_2)/2)}. \tag{4.2}$$

From the definition, it is easy to see that $\varepsilon^4_\Omega(\alpha)=1$. Of course, the definition of ε_Ω depends on the basis $\{\omega_1,\omega_2\}$ of Ω. Indeed, by a short calculation, we have the following

LEMMA 4.1. *Any of three substitutions* $(\omega_1,\omega_2)\to(\omega_2,\omega_1)$, $(\omega_1,\omega_2)\to(\omega_2,-\omega_1)$ *and* $(\omega_1,\omega_2)\to(\omega_1,\omega_1+\omega_2)$ *multiplies* $\varepsilon_\Omega(\alpha)$ *by the quantity*

$$\chi_4\circ N(\alpha)=\chi_4(N\alpha)=(-1)^{\frac{1}{2}(N\alpha-1)}.$$

REMARK. $\chi_4\circ N$ is a quadratic character of $(O/4O)^\times$. In particular, when $N(\alpha)\equiv 1 \pmod 4$ for any $\alpha\in J^*(2)$, the definition of ε_Ω does not depend on the choice of a basis $\{\omega_1,\omega_2\}$ of Ω.

In case (a) where $(O/2O)^\times \cong \{1\}$, since $\alpha \equiv 1 \pmod{2O}$ for any $\alpha \in J^*(2)$, we have $\varepsilon_\Omega^2(\alpha) = \xi_\Omega^2(\alpha) = 1$.

In case (b) where $(O/2O)^\times$ is a group of order 2, we first choose a basis $\{\omega_1, \omega_2\}$ of Ω such that $\omega_1/2$ and $\omega_2/2$ represent two distinct primitive 2-division points of Ω. Then for any $\alpha \in J^*(2)$, we have $\alpha(\omega_1+\omega_2)/2 \equiv (\omega_1+\omega_2)/2 \pmod{\Omega}$ and $\varepsilon_\Omega^2(\alpha) = \xi_\Omega^2(\alpha)$ by (4.2). In particular, if $\alpha \not\equiv 1 \pmod{2O}$, then $\varepsilon_\Omega^2(\alpha) = -1$ (see Sec. 2). Moreover, by Lemma 4.1, the same assertion holds without any restriction on the choice of a basis $\{\omega_1, \omega_2\}$ of Ω.

In case (c) where $(O/2O)^\times$ is a group of order 3, $(\alpha^2+\alpha+1)(\omega_1+\omega_2)/2 \equiv 0 \pmod{\Omega}$ for any $\alpha \in J^*(2)$ such that $\alpha \not\equiv 1 \pmod{2O}$. For simplicity, we let $\tau = (\omega_1+\omega_2)/2$ and $(\alpha^2+\alpha+1)\tau = u$ with some $u \in \Omega$. Then from (3.1),

$$\begin{aligned}
\xi_\Omega^2(\alpha) &= -\frac{D_\Omega^2(\alpha^2\tau;\alpha\tau)}{D_\Omega^2(\alpha^2\tau;\tau)} \\
&= -e(E_\Omega(\alpha^2\tau,\alpha\tau) - E_\Omega(\alpha^2\tau,\tau))\frac{\mathcal{K}_\Omega^2((\alpha^2+\alpha)\tau)}{\mathcal{K}_\Omega^2(\alpha^2\tau)\mathcal{K}_\Omega^2(\alpha\tau)}\,\frac{\mathcal{K}_\Omega^2(\alpha^2\tau)\mathcal{K}_\Omega^2(\tau)}{\mathcal{K}_\Omega^2((\alpha^2+1)\tau)} \\
&= -e(E_\Omega(\alpha^2\tau,(\alpha-1)\tau))\frac{\mathcal{K}_\Omega^2(-\tau+u)\mathcal{K}_\Omega^2(\tau)}{\mathcal{K}_\Omega^2(\alpha\tau)\mathcal{K}_\Omega^2(-\alpha\tau+u)} \\
&= -e(2E_\Omega(\alpha\tau,\tau))\frac{\mathcal{K}_\Omega^4(\tau)}{\mathcal{K}_\Omega^4(\alpha\tau)}.
\end{aligned}$$

Here

$$\begin{aligned}
2E_\Omega(\alpha\tau,\tau) &= E_\Omega(\alpha(\omega_1+\omega_2)/2, \omega_1+\omega_2) \\
&\equiv E_\Omega(\omega_1/2, \omega_1+\omega_2) \text{ or } E_\Omega(\omega_2/2, \omega_1+\omega_2) \pmod{\mathbb{Z}} \\
&\equiv \frac{1}{2} \pmod{\mathbb{Z}},
\end{aligned}$$

and hence $e(2E_\Omega(\alpha\tau,\tau)) = -1$. Then

$$\xi_\Omega^2(\alpha) = \frac{\mathcal{K}_\Omega^4((\omega_1+\omega_2)/2)}{\mathcal{K}_\Omega^4(\alpha(\omega_1+\omega_2)/2)},$$

and by (4.2), we have $\varepsilon_\Omega^2(\alpha) = 1$.

Consequently, in both cases (a) and (c) the values of ε_Ω are ± 1, and in case (b) they are ± 1 and $\pm\sqrt{-1}$. Moreover we have the following

Proposition 4.2. *$\varepsilon_\Omega(\alpha)$ depends only on the class of α modulo $4O$.*

Proof. Assume that $\alpha_1 \equiv \alpha \pmod{4O}$, i.e. $\alpha_1 = \alpha + 4u$ with some $u \in O$. Then, on one hand, since

$$N\alpha_1 = N\alpha + 4\,\mathrm{Tr}(\overline{\alpha}u) + 16Nu,$$

we have

$$e\Big(\frac{1}{8}(N\alpha_1 - 1)E_\Omega(\omega_1,\omega_2)\Big)$$
$$= e\Big(\frac{1}{8}(N\alpha - 1)E_\Omega(\omega_1,\omega_2)\Big)\cdot e\Big(\frac{1}{2}\operatorname{Tr}(\overline{\alpha}u)E_\Omega(\omega_1,\omega_2)\Big),$$

where Tr is the usual trace. On the other hand, by (K1),

$$\prod_\rho \mathcal{K}_\Omega(\alpha_1\rho) = \prod_\rho \mathcal{K}_\Omega(\alpha\rho + 4u\rho) = e(M)\prod_\rho \mathcal{K}_\Omega(\alpha\rho),$$

where ρ runs over $\{\omega_1/2, \omega_2/2, (\omega_1+\omega_2)/2\}$ and M is given by

$$M = 2\Big(\sum_\rho N\rho\Big)\cdot E_\Omega(u,\alpha).$$

Moreover a short calculation shows that

$$\begin{aligned} M &\equiv \frac{1}{2}(\overline{\omega_1}\omega_2 + \omega_1\overline{\omega_2})E_\Omega(u,\alpha) \pmod{\mathbb{Z}} \\ &\equiv \frac{1}{2}(\overline{u}\alpha + u\overline{\alpha})E_\Omega(\omega_1,\omega_2) \pmod{\mathbb{Z}} \\ &= \frac{1}{2}\operatorname{Tr}(\overline{u}\alpha)E_\Omega(\omega_1,\omega_2). \end{aligned}$$

Hence $\varepsilon_\Omega(\alpha_1) = \varepsilon_\Omega(\alpha)$.

Remark. In the same way as in the proof of Proposition 4.2, we find that $\varepsilon_\Omega^2(\alpha)$ depends only on the class of α modulo $2O$.

Proposition 4.2 suggests that ε_Ω could be a character of $(O/4O)^\times$. However, that is not true in general. Namely, in the next section, we shall prove

Theorem 4.3. *$\varepsilon_\Omega(\alpha\beta) = \varepsilon_\Omega(\alpha)^{N\beta}\varepsilon_\Omega(\beta) = \varepsilon_\Omega(\alpha)\varepsilon_\Omega(\beta)^{N\alpha}$ for any α,β in $J^*(2)$.*

Theorem 4.3 illustrates an action of $\mathrm{Gal}(\Sigma^{\mathrm{ab}}/\mathcal{H})$ on $\varepsilon_\Omega(\alpha)$, where $\mathcal{H}$ is the ring class field over Σ corresponding to the order O. Namely, let $\sigma(\beta) := (\beta O, \Sigma^{\mathrm{ab}}/\mathcal{H})$ be the Artin automorphism belonging to the principal O-ideal βO. Then

$$\varepsilon_\Omega(\alpha)^{\sigma(\beta)} = \varepsilon_\Omega(\alpha)^{N\beta} = \frac{\varepsilon_\Omega(\alpha\beta)}{\varepsilon_\Omega(\beta)}.$$

At any rate, as a consequence of Theorem 4.3, we conclude that in both cases (a) and (c), ε_Ω defines a character of $(O/4O)^\times$ of order 2. Also in case (b), if $N\alpha \equiv 1 \pmod 4$ for any $\alpha \in J^*(2)$, then ε_Ω defines a character of $(O/4O)^\times$ of order 4. We are in case (b) if and only if $4 \mid d$. Moreover, if $d = 4d_0$ and $d_0 \equiv 0, 1 \pmod 4$, then always $N\alpha \equiv 1 \pmod 4$ for any $\alpha \in J^*(2)$. However in the cases where $d = 4d_0$ and $d_0 \equiv 2, 3 \pmod 4$, we see that $N\alpha \equiv -1 \pmod 4$ for any $\alpha \in J^*(2)$ such that $\alpha \not\equiv 1 \pmod{2O}$. Hence in this case, ε_Ω cannot be a character of $(O/4O)^\times$. Indeed, by Theorem 4.3,

$$\frac{\varepsilon_\Omega(\alpha\beta)}{\varepsilon_\Omega(\alpha)\varepsilon_\Omega(\beta)} = \varepsilon_\Omega(\alpha)^{N\beta-1} = -1$$

for any $\alpha, \beta \in J^*(2)$ such that $\alpha \not\equiv 1$, $\beta \not\equiv 1 \pmod{2O}$.

Here we make a slight modification. In the case where $4 \mid d$ and $d_0 \equiv 2, 3 \pmod 4$, in place of ε_Ω, we consider $\tilde{\varepsilon}_\Omega$ defined by

$$\tilde{\varepsilon}_\Omega(\alpha) := e\left(\frac{1}{8}(N\alpha - 1)\right)\varepsilon_\Omega(\alpha).$$

Then $\tilde{\varepsilon}_\Omega$ also has the cocycle property, i.e.

$$\tilde{\varepsilon}_\Omega(\alpha\beta) = \tilde{\varepsilon}_\Omega(\alpha)^{N\beta}\tilde{\varepsilon}_\Omega(\beta) = \tilde{\varepsilon}_\Omega(\alpha)\tilde{\varepsilon}_\Omega(\beta)^{N\alpha}$$

for any $\alpha, \beta \in J^*(2)$. Moreover $\tilde{\varepsilon}_\Omega(\alpha)$ depends only on the class of α modulo $4O$ and further $\tilde{\varepsilon}_\Omega^2(\alpha) = 1$ for any $\alpha \in J^*(2)$. Indeed, by Remark to Proposition 4.2, $\varepsilon_\Omega^2(\alpha) = -1$ if and only if $\alpha \not\equiv 1 \pmod{2O}$ and equivalently $N\alpha \not\equiv 1 \pmod 4$. Thus $\tilde{\varepsilon}_\Omega$ defines a character of $(O/4O)^\times$ of order 2. We will use this modified character in Section 6.

5. Proof of Theorem 4.3. Let the notation be as in Section 4. For a complete proof of Theorem 4.3, it suffices to prove

$$\varepsilon_\Omega(\alpha\beta) = \varepsilon_\Omega(\alpha)\varepsilon_\Omega(\beta)^{N\alpha},$$

which is equivalent to

$$\xi_\Omega(\alpha\beta) = \xi_\Omega(\alpha)\xi_\Omega(\beta)^{N\alpha}\left(\frac{\mathcal{K}_\Omega^2(\beta(\omega_1+\omega_2)/2)}{\mathcal{K}_\Omega^2((\omega_1+\omega_2)/2)}\right)^{N\alpha}\frac{\mathcal{K}_\Omega^2(\alpha(\omega_1+\omega_2)/2)}{\mathcal{K}_\Omega^2(\alpha\beta(\omega_1+\omega_2)/2)}.$$

For this purpose we can apply the product formula for f_Ω in Theorem 3.1. For simplicity, we let $\tau = (\omega_1+\omega_2)/2$ again. Then on one hand,

$$\begin{aligned} f_\Omega(\alpha\beta z)\frac{D_\Omega^2(\alpha\beta z;\alpha\beta\tau)}{D_\Omega^2(\alpha\beta z;\tau)} &= \xi_\Omega(\alpha\beta)\prod_{x\in\operatorname{Ker}(\alpha\beta)} f_\Omega(z+x) \\ &= \xi_\Omega(\alpha\beta)\prod_{\substack{\tilde{r} \bmod \alpha\beta O \\ \tilde{r}\in O}} f_\Omega(z+\tilde{r}x_{\alpha\beta}), \end{aligned}$$

where $x_{\alpha\beta}$ is a fixed primitive $\alpha\beta$-division point of Ω. On the other hand,

$$\begin{aligned} f_\Omega(\alpha\beta z)\frac{D_\Omega^2(\alpha\beta z;\alpha\beta\tau)}{D_\Omega^2(\alpha\beta z;\tau)} &= \frac{D_\Omega^2(\alpha\beta z;\alpha\beta\tau)}{D_\Omega^2(\alpha\beta z;\alpha\tau)}\cdot f_\Omega(\alpha(\beta z))\frac{D_\Omega^2(\alpha(\beta z);\alpha\tau)}{D_\Omega^2(\alpha(\beta z);\tau)} \\ &= \frac{D_\Omega^2(\alpha\beta z;\alpha\beta\tau)}{D_\Omega^2(\alpha\beta z;\alpha\tau)}\cdot \xi_\Omega(\alpha)\prod_{\substack{r_1 \bmod \alpha O \\ r_1\in O}} f_\Omega(\beta z + r_1 x_\alpha), \end{aligned}$$

where $x_\alpha := \beta x_{\alpha\beta}$ and this gives a primitive α-division point of Ω. Moreover, in the above equality,

$$\begin{aligned}
&\prod_{\substack{r_1 \bmod \alpha O \\ r_1 \in O}} f_\Omega(\beta z + r_1 x_\alpha) \\
&\quad = \prod_{\substack{r_1 \bmod \alpha O \\ r_1 \in O}} f_\Omega(\beta(z + r_1 x_{\alpha\beta})) \frac{D_\Omega^2(\beta(z + r_1 x_{\alpha\beta}); \beta\tau)}{D_\Omega^2(\beta(z + r_1 x_{\alpha\beta}); \tau)} \\
&\qquad \times \prod_{\substack{r_1 \bmod \alpha O \\ r_1 \in O}} \frac{D_\Omega^2(\beta z + r_1 x_\alpha; \tau)}{D_\Omega^2(\beta z + r_1 x_\alpha; \beta\tau)} \\
&\quad = \xi_\Omega(\beta)^{N\alpha} \prod_{\substack{r_1 \bmod \alpha O \\ r_1 \in O}} \frac{D_\Omega^2(\beta z + r_1 x_\alpha; \tau)}{D_\Omega^2(\beta z + r_1 x_\alpha; \beta\tau)} \cdot \prod_{\substack{r_1 \bmod \alpha O \\ r_2 \bmod \beta O}} f_\Omega(z + r_1 x_{\alpha\beta} + r_2 x_\beta) \\
&\quad = \xi_\Omega(\beta)^{N\alpha} \prod_{\substack{r_1 \bmod \alpha O \\ r_1 \in O}} \frac{D_\Omega^2(\beta z + r_1 x_\alpha; \tau)}{D_\Omega^2(\beta z + r_1 x_\alpha; \beta\tau)} \cdot \prod_{\tilde{r} \bmod \alpha\beta O} f_\Omega(z + \tilde{r} x_{\alpha\beta}).
\end{aligned}$$

Here $x_\beta := \alpha x_{\alpha\beta}$ and this gives a primitive β-division point of Ω. Note that the set $\{r_1 + \alpha r_2 \mid r_1 \bmod \alpha O,\ r_2 \bmod \beta O\}$ is a complete set of representatives of $O/\alpha\beta O$. Hence we obtain

$$\xi_\Omega(\alpha\beta) = \xi_\Omega(\alpha)\xi_\Omega(\beta)^{N\alpha} \cdot F_\Omega(z; \alpha, \beta)$$

where

$$F_\Omega(z; \alpha, \beta) = \frac{D_\Omega^2(\alpha\beta z; \alpha\beta\tau)}{D_\Omega^2(\alpha\beta z; \alpha\tau)} \cdot \prod_{\substack{r_1 \bmod \alpha O \\ r_1 \in O}} \frac{D_\Omega^2(\beta z + r_1 x_\alpha; \tau)}{D_\Omega^2(\beta z + r_1 x_\alpha; \beta\tau)}.$$

Moreover, by (D6), we have

$$\begin{aligned}
F_\Omega(z; \alpha, \beta) &= \frac{D_\Omega^2(\alpha\beta z; \alpha\beta\tau)}{D_\Omega^2(\alpha\beta z; \alpha\tau)} \cdot \frac{D_{\alpha^{-1}\Omega}^2(\beta z; \tau)}{D_{\alpha^{-1}\Omega}^2(\beta z; \beta\tau)} \cdot \frac{\mathcal{K}_\Omega^2(\beta\tau)^{N\alpha}}{K_{\alpha^{-1}\Omega}^2(\beta\tau)} \cdot \frac{K_{\alpha^{-1}\Omega}^2(\tau)}{\mathcal{K}_\Omega^2(\tau)^{N\alpha}} \\
&\qquad \times e\Big(E_\Omega\Big(\sum_{r_1 \bmod \alpha O} r_1 x_\alpha, 2(1-\beta)\tau\Big)\Big) \\
&= \frac{\mathcal{K}_\Omega^2(\beta\tau)^{N\alpha}}{\mathcal{K}_\Omega^2(\alpha\beta\tau)} \cdot \frac{\mathcal{K}_\Omega^2(\alpha\tau)}{\mathcal{K}_\Omega^2(\tau)^{N\alpha}} \\
&= \left(\frac{\mathcal{K}_\Omega^2(\beta(\omega_1 + \omega_2)/2)}{\mathcal{K}_\Omega^2((\omega_1 + \omega_2)/2)}\right)^{N\alpha} \frac{\mathcal{K}_\Omega^2(\alpha(\omega_1 + \omega_2)/2)}{\mathcal{K}_\Omega^2(\alpha\beta(\omega_1 + \omega_2)/2)},
\end{aligned}$$

and this proves Theorem 4.3.

6. Quadratic reciprocity law. Let the notation be as in the preceding sections. For any α, β in $J^*(2)$ such that $(\alpha, \beta) = 1$, we consider the quadratic symbol $\left(\frac{\alpha}{\beta}\right)_2$ given by

$$\left(\frac{\alpha}{\beta}\right)_2 = \prod_{x \in S_\beta} \varepsilon(\alpha, x),$$

where S_β is a subset of $\mathrm{Ker}(\beta)$ such that $\mathrm{Ker}(\beta)$ is the disjoint union of $\{0\}$, S_β and $-S_\beta$, and $\varepsilon(\alpha, x) \in \{\pm 1\}$ is so determined that $\alpha x = \varepsilon(\alpha, x)\gamma(x)$ with a unique $\gamma(x)$ in S_β. As in the classical results of Eisenstein ([4], [5]), but using our product formula of Theorem 3.1, we obtain the following explicit quadratic reciprocity law, which refines the formula of Bayad ([1, Theorem 1.10]).

THEOREM 6.1. *For $\alpha, \beta \in J^*(2)$ such that $(\alpha, \beta) = 1$,*

$$\left(\frac{\alpha}{\beta}\right)_2 \left(\frac{\beta}{\alpha}\right)_2 = (-1)^{\frac{1}{4}(N\alpha - 1)(N\beta - 1)} \frac{\varepsilon_\Omega(\alpha)^{\frac{1}{2}(N\beta - 1)}}{\varepsilon_\Omega(\beta)^{\frac{1}{2}(N\alpha - 1)}}.$$

Proof. Since f_Ω is an odd function, $f_\Omega(\alpha x) = \varepsilon(\alpha, x) f_\Omega(\gamma(x))$ for any $x \in S_\beta$. Thus

$$\left(\frac{\alpha}{\beta}\right)_2 = \prod_{x \in S_\beta} \varepsilon(\alpha, x) = \prod_{x \in S_\beta} \frac{f_\Omega(\alpha x)}{f_\Omega(\gamma(x))} = \prod_{x \in S_\beta} \frac{f_\Omega(\alpha x)}{f_\Omega(x)}.$$

Moreover, using the product formula of Theorem 3.1, we have

$$\begin{aligned}\left(\frac{\alpha}{\beta}\right)_2 &= \prod_{x \in S_\beta} \left(\frac{D_\Omega^2(\alpha x; (\omega_1 + \omega_2)/2)}{D_\Omega^2(\alpha x; \alpha(\omega_1 + \omega_2)/2)} \xi_\Omega(\alpha) \prod_{\substack{x' \in \mathrm{Ker}(\alpha) \\ x' \neq 0}} f_\Omega(x + x') \right) \\ &= \xi_\Omega(\alpha)^{(N\beta - 1)/2} A_\beta^{(\alpha)} \prod_{x \in S_\beta} \prod_{x' \in S_\alpha} f_\Omega(x + x') f_\Omega(x - x'),\end{aligned}$$

where

$$A_\beta^{(\alpha)} = \prod_{x \in S_\beta} \frac{D_\Omega^2(\alpha x; \tau)}{D_\Omega^2(\alpha x; \alpha\tau)} \quad \text{with} \quad \tau = \frac{\omega_1 + \omega_2}{2}.$$

By (D8),

$$\begin{aligned}A_\beta^{(\alpha)} &= \prod_{x \in S_\beta} \frac{\wp_\Omega(\alpha x) - \wp_\Omega(\tau)}{\wp_\Omega(\alpha x) - \wp_\Omega(\alpha\tau)} \\ &= \prod_{x \in S_\beta} \frac{\wp_\Omega(\tau) - \wp_\Omega(x)}{\wp_\Omega(\alpha\tau) - \wp_\Omega(x)} \qquad (\wp_\Omega \text{ is even and } \Omega \text{ elliptic}) \\ &= \prod_{x \in S_\beta} \frac{D_\Omega(\tau; x) D_\Omega(\tau; -x)}{D_\Omega(\alpha\tau; x) D_\Omega(\alpha\tau; -x)} = \prod_{\substack{x \in \mathrm{Ker}(\beta) \\ x \neq 0}} \frac{D_\Omega(\tau; x)}{D_\Omega(\alpha\tau; x)},\end{aligned}$$

and then using (D4) and (D7) (with $\Lambda = \beta^{-1}\Omega$)

$$A_\beta^{(\alpha)} = \frac{\mathcal{K}_\Omega(\alpha\tau)^{N\beta}}{\mathcal{K}_{\beta^{-1}\Omega}(\alpha\tau)} \cdot \frac{\mathcal{K}_{\beta^{-1}\Omega}(\tau)}{\mathcal{K}_\Omega(\tau)^{N\beta}} = \left\{\frac{\mathcal{K}_\Omega(\alpha\tau)}{\mathcal{K}_\Omega(\tau)}\right\}^{N\beta} \cdot \frac{\mathcal{K}_\Omega(\beta\tau)}{\mathcal{K}_\Omega(\alpha\beta\tau)}$$
$$= H(\alpha,\beta;\tau)\left\{\frac{\mathcal{K}_\Omega(\alpha\tau)}{\mathcal{K}_\Omega(\tau)}\right\}^{N\beta-1}.$$

Here

$$H(\alpha,\beta;\tau) := \frac{\mathcal{K}_\Omega(\alpha\tau)\mathcal{K}_\Omega(\beta\tau)}{\mathcal{K}_\Omega(\tau)\mathcal{K}_\Omega(\alpha\beta\tau)}.$$

Note that $H(\alpha,\beta;\tau) = H(\beta,\alpha;\tau)$. Moreover, since

$$\xi_\Omega(\alpha) = \varepsilon_\Omega(\alpha)\frac{\mathcal{K}^2_\Omega(\tau)}{\mathcal{K}^2_\Omega(\alpha\tau)},$$

we have

$$\xi_\Omega(\alpha)^{\frac{1}{2}(N\beta-1)}A_\beta^{(\alpha)} = H(\alpha,\beta;\tau)\varepsilon_\Omega(\alpha)^{\frac{1}{2}(N\beta-1)},$$

and hence

$$\left(\frac{\alpha}{\beta}\right)_2 = H(\alpha,\beta;\tau)\varepsilon_\Omega(\alpha)^{\frac{1}{2}(N\beta-1)}\prod_{x\in S_\beta}\prod_{x'\in S_\alpha} f_\Omega(x+x')f_\Omega(x-x').$$

Symmetrically we have

$$\left(\frac{\beta}{\alpha}\right)_2 = H(\beta,\alpha;\tau)\varepsilon_\Omega(\beta)^{\frac{1}{2}(N\alpha-1)}\prod_{v'\in S_\alpha}\prod_{v\in S_\beta} f_\Omega(v'+v)f_\Omega(v'-v)$$
$$= H(\alpha,\beta;\tau)\varepsilon_\Omega(\beta)^{\frac{1}{2}(N\alpha-1)}(-1)^{\frac{1}{4}(N\alpha-1)(N\beta-1)}$$
$$\times \prod_{x\in S_\beta}\prod_{x'\in S_\alpha} f_\Omega(x+x')f_\Omega(x-x'),$$

and hence

$$\left(\frac{\alpha}{\beta}\right)_2\left(\frac{\beta}{\alpha}\right)_2 = \left(\frac{\alpha}{\beta}\right)_2\left(\frac{\beta}{\alpha}\right)_2^{-1} = (-1)^{\frac{1}{4}(N\alpha-1)(N\beta-1)}\frac{\varepsilon_\Omega(\alpha)^{\frac{1}{2}(N\beta-1)}}{\varepsilon_\Omega(\beta)^{\frac{1}{2}(N\alpha-1)}}.$$

This proves Theorem 6.1.

Remark. As is explained in Section 4, ε_Ω in Theorem 6.1 is a character of $(O/4O)^\times$ except for the case where $d = 4d_0$ with $d_0 \equiv 2, 3 \pmod 4$. In the exceptional case, we may replace ε_Ω by $\tilde{\varepsilon}_\Omega$, because

$$\frac{\varepsilon_\Omega(\alpha)^{\frac{1}{2}(N\beta-1)}}{\varepsilon_\Omega(\beta)^{\frac{1}{2}(N\alpha-1)}} = \frac{e\big(\frac{1}{8}(N\alpha-1)\big)^{\frac{1}{2}(N\beta-1)}\cdot\varepsilon_\Omega(\alpha)^{\frac{1}{2}(N\beta-1)}}{e\big(\frac{1}{8}(N\beta-1)\big)^{\frac{1}{2}(N\alpha-1)}\cdot\varepsilon_\Omega(\beta)^{\frac{1}{2}(N\alpha-1)}} = \frac{\tilde{\varepsilon}_\Omega(\alpha)^{\frac{1}{2}(N\beta-1)}}{\tilde{\varepsilon}_\Omega(\beta)^{\frac{1}{2}(N\alpha-1)}},$$

and $\tilde{\varepsilon}_\Omega$ is a character of $(O/4O)^\times$ of order 2.

In the case where d is odd, since ε_Ω is a character of order 2, we have

$$\frac{\varepsilon_\Omega(\alpha)^{\frac{1}{2}(N\beta-1)}}{\varepsilon_\Omega(\beta)^{\frac{1}{2}(N\alpha-1)}} = \varepsilon_\Omega(\alpha)^{\frac{1}{2}(N\beta-1)}\varepsilon_\Omega(\beta)^{\frac{1}{2}(N\alpha-1)}.$$

In the case where $d = 4d_0$ with $d_0 \equiv 0, 1 \pmod 4$, $N\alpha \equiv 1 \pmod 4$ for any $\alpha \in J^*(2)$, and hence we have the same equality as above. In the remaining case where $d = 4d_0$ with $d_0 \equiv 2, 3 \pmod 4$, since $\tilde{\varepsilon}_\Omega$ is a character of order 2, we have

$$\frac{\tilde{\varepsilon}_\Omega(\alpha)^{\frac{1}{2}(N\beta-1)}}{\tilde{\varepsilon}_\Omega(\beta)^{\frac{1}{2}(N\alpha-1)}} = \tilde{\varepsilon}_\Omega(\alpha)^{\frac{1}{2}(N\beta-1)}\tilde{\varepsilon}_\Omega(\beta)^{\frac{1}{2}(N\alpha-1)}.$$

Next, we wish to compare our theorem with the following reciprocity law proved by Hajir and Villegas in [4].

Theorem 6.2 ([6, Theorem 21]). *For $\alpha, \beta \in J^*(2)$ such that $(\alpha, \beta) = 1$,*

$$\left(\frac{\alpha}{\beta}\right)_2\left(\frac{\beta}{\alpha}\right)_2 = (-1)^{\frac{1}{4}(N\alpha-1)(N\beta-1)}\kappa_4(\alpha)^{\frac{1}{2}(N\beta-1)}\kappa_4(\beta)^{\frac{1}{2}(N\alpha-1)}.$$

In Theorem 6.2, κ_4 is a certain character of $(O/4O)^\times$ defined with the use of the Galois action on the quotient of Dedekind η-values. For a precise definition of κ_4, one should refer to [6]. Especially Lemma 12 in [6] is useful for the computation of $\kappa_4(\alpha)$. Comparing Theorem 6.1 with Theorem 6.2, the following questions arise:

Q1. What is the precise relation between κ_4 and ε_Ω (or $\tilde{\varepsilon}_\Omega$)?

Q2. Does the definition of ε_Ω depend essentially on the O-ideal Ω?

In the rest of this note we attempt to answer these questions. We let $\{\omega, 1\}$ be the basis of O where

$$\omega = \begin{cases} (-1+\sqrt{-d})/2 & \text{when } d \text{ is odd,} \\ \sqrt{-d_0} & \text{when } d = 4d_0. \end{cases}$$

We take O itself for Ω, and we write ε_1 for ε_Ω defined using the basis $\{\omega, 1\}$. Then by an explicit computation using Lemma 12 of [6], we obtain

Theorem 6.3.

(i) *If d is odd, both ε_1 and κ_4 are characters of order 2, and $\varepsilon_1 = \kappa_4$.*

(ii) *If $d = 4d_0$ and $d_0 \equiv 0, 1 \pmod 4$, both ε_1 and κ_4 are characters of order 4, and $\varepsilon_1 = \kappa_4^3$.*

(iii) *If $d = 4d_0$ and $d_0 \equiv 2, 3 \pmod 4$, both $\tilde{\varepsilon}_1$ and κ_4 are characters of order 2, and $\tilde{\varepsilon}_1 = \kappa_4$. Here $\tilde{\varepsilon}_1$ is defined by*

$$\tilde{\varepsilon}_1(\alpha) = e\big(\tfrac{1}{8}(N\alpha-1)\big)\varepsilon_1(\alpha) \quad \text{for } \alpha \in J^*(2).$$

Now let Ω and Ω_1 be two proper O-ideals which are similar to each other, i.e. $\Omega_1 = \mu\Omega$ with some $\mu \in \Sigma$. We fix a basis $\{\omega_1, \omega_2\}$ of Ω and take

$\{\mu\omega_1, \mu\omega_2\}$ for a basis of $\Omega_1 = \mu\Omega$. Then, from the definition (4.1) and the homogeneity of $\mathcal{K}_\Omega(z)$, we see that

$$\varepsilon_{\Omega_1}(\alpha) = \varepsilon_{\mu\Omega}(\alpha) = \varepsilon_\Omega(\alpha) \quad \text{for any } \alpha \in J^*(2).$$

In each O-ideal class, there exists a prime ideal $\mathfrak{p}$ such that

$$N\mathfrak{p} = p \equiv \begin{cases} 1 \pmod 4 & \text{when } d = 4d_0 \text{ and } d_0 \equiv 0, 1 \pmod 4, \\ 1 \pmod 2 & \text{otherwise.} \end{cases}$$

We know that

$$(w_{\mathcal{H}}, 4) = \begin{cases} 4 & \text{when } d = 4d_0 \text{ and } d_0 \equiv 0, 1 \pmod 4, \\ 2 & \text{otherwise,} \end{cases}$$

where $w_{\mathcal{H}}$ is the number of roots of unity contained in the ring class field $\mathcal{H}$ over Σ corresponding to the ring O. We let $\{\omega + \nu, p\}$ be a canonical basis of $\mathfrak{p}$. Then ν is uniquely determined modulo p. Here we may assume additionally that

$$\nu \equiv \begin{cases} 0 \pmod 8 & \text{when } d \text{ is odd,} \\ 1 \pmod 8 & \text{when } d \text{ is even.} \end{cases}$$

Then, by an explicit computation with the use of the basis $\{\omega+\nu, p\}$ of $\mathfrak{p}$, we can confirm that $\varepsilon_{\mathfrak{p}}(\alpha) = \varepsilon_1(\alpha)$ for any $\alpha \in J^*(2)$. Finally we can summarize our arguments as follows:

Theorem 6.4.

(i) *When $d = 4d_0$ with $d_0 \equiv 0, 1 \pmod 4$, the definition of ε_Ω depends neither on the choice of a basis of Ω nor on Ω itself, and $\varepsilon_\Omega = \varepsilon_1 = \kappa_4^3$.*

(ii) *When d is odd or $d = 4d_0$ with $d_0 \equiv 2, 3 \pmod 4$, for any Ω, ε_Ω is one of $\{\varepsilon_1, \varepsilon_1 \cdot \chi_4 \circ N\}$.*

Acknowledgements. The author would like to thank Professor Gilles Robert for enlightening discussions and many useful suggestions concerning this study. He also thanks the referee for helpful suggestions.

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Received on 17.9.2009
and in revised form on 31.5.2010 (6152)