

Values of special indefinite quadratic forms

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1. Introduction. Let Q denote a d -dimensional quadratic form. For $a, b \in \mathbb{R}$ we consider the set E of points in the d -dimensional Euclidean space at which Q takes values between a and b . In case the quadratic form $Q[x]$ is positive definite, E is an elliptic shell, but in this paper we will investigate indefinite forms, and then E is a hyperbolic shell.

For a (measurable) set $B \subset \mathbb{R}^d$ the *lattice volume* of B is the number of lattice points in B (formally $\text{vol}_{\mathbb{Z}} B := \#(B \cap \mathbb{Z}^d)$), and $\text{vol} B$ denotes the Lebesgue measure of B . For the hyperbolic shell E we want to approximate its lattice volume by the Lebesgue volume, and estimate a *relative* lattice point rest of *large* parts of the hyperbolic shell E . Therefore we consider for $r > 0$ the d -dimensional cube C_r with edge length r and intersect the cube C_r with the hyperbolic shell E . The *relative* lattice point rest of $E \cap C_r$ is now defined by

$$\Delta := \left| \frac{\text{vol}_{\mathbb{Z}}(E \cap C_r) - \text{vol}(E \cap C_r)}{\text{vol}(E \cap C_r)} \right|.$$

We will show, for special indefinite forms Q , that $\Delta = \mathcal{O}(1)$ as $r \rightarrow \infty$ (Theorem 2.1) and that even $\Delta = o(1)$ as $r \rightarrow \infty$ provided that Q is irrational (Theorem 2.2). Recall that a quadratic form $Q[x]$ and the corresponding operator Q with non-zero matrix $Q = (q_{ij})$, $1 \leq i, j \leq d$, is called *rational* if there exists a real number $\lambda \neq 0$ such that the matrix λQ has integer entries only; otherwise it is called *irrational*.

Similar results for forms Q of signature (p, q) satisfying $\max(p, q) \geq 3$ have been proved by Eskin, Margulis and Mozes in [EMM98]. These are quantitative versions of the well-known Oppenheim problem concerning the distribution of values of $Q[m]$, $m \in \mathbb{Z}^d$. In 1929, Oppenheim ([Opp29], [Opp31]) conjectured that if $d \geq 5$ then for an irrational non-degenerate

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quadratic form Q the quantity $m(Q) := \inf\{|Q[m]| : m \in \mathbb{Z}^d, m \neq 0\}$ equals zero. In the rational case this was known by Meyer's theorem (see [Cas78]). Later it was conjectured that $m(Q) = 0$ even for $d \geq 3$ and Q irrational (for irrational diagonal forms this was suspected in [DH46] and it is not true in dimensions 3 and 4 without the assumption of irrationality). The different approaches to this and related problems involve various mathematical methods from analytic number theory, ergodic theory, representation theory of Lie groups, reduction theory and the geometry of numbers.

In [Mar89] Margulis established the Oppenheim conjecture in dimensions $d \geq 3$, as stated by Davenport and Heilbronn for $d \geq 5$. In his seminal work he proved that the set of values of Q at lattice points is dense in \mathbb{R} . Quantitative versions of this problem were later on developed by Dani and Margulis ([DM93]) and Eskin, Margulis and Mozes ([EMM98]). They consist of *quantitative* bounds on the ratio between the lattice volume and the Lebesgue volume of the set of points in the cube C_r where the quadratic form takes values in a small interval. The quantitative bounds provided in these results yield the asymptotic number of points in these regions as a polynomial in r up to a non-effective error term tending to zero in proportion to the leading term. The estimates thus obtained are implicit, since they do not provide explicit bounds in terms of diophantine approximations of irrational coefficients of the form. For a detailed discussion of results on these problems by Oppenheim, Heilbronn and Davenport and others, see [Mar97]. In [BG99] Bentkus and Götze proved explicit error bounds in the quantitative Oppenheim problem for the elliptic shell as well as for hyperbolic shells for $d \geq 9$ by a common approach. They provide more explicit bounds (in terms of diophantine approximation) for *distribution functions* of the values of the quadratic form on C_r , whereas the direct application of the previous methods seems to be restricted to the case of the concentration in compact intervals.

In [Göt04] Götze showed that in the positive definite case for $d \geq 5$ the lattice point rest is of order $\mathcal{O}(r^{d-2})$ for arbitrary forms, and of order $o(r^{d-2})$ if the form is irrational. These results refine earlier bounds of the same order for dimensions $d \geq 9$ (see also [Göt04] for the history of such estimates and further references).

In the present paper we apply techniques of [Göt04] to special *indefinite* forms and we obtain explicit bounds in terms of certain Minkowski minima of convex bodies related to these quadratic forms. When adapting these techniques, the main problem is to estimate the difference between the lattice volume and the Lebesgue volume by an integral of generalized theta functions. In order to achieve such an estimate, we develop tools, different from those in [Göt04], which involve adjustable smooth approximations of the indicator functions of the hyperboloid and of the cube C_r . The bound

given by an integral of theta functions does *not* use the special structure of the indefinite forms under consideration. Furthermore, a careful modification of the arguments in [Göt04] even leads to a bound in terms of the Minkowski minima mentioned above, which holds for *any* indefinite form. The special structure of the forms is only used when we estimate the appearing functions of Minkowski minima by adapting the techniques of [Göt04] to the indefinite case. As in the positive definite case, we show that in the irrational case the maximal gap between successive values of the quadratic form at lattice points converges to 0 as r tends to infinity (Corollary 2.4). Furthermore, we extend the results of Bentkus and Götze ([BG99]) on distribution functions for values of quadratic forms to dimensions from 5 to 8 (Theorem 2.7). In addition, we obtain a result for multivariate diophantine approximations for these special indefinite forms (Theorem 2.6).

This paper is organized as follows: In the second section, we state the two main results about the asymptotics of the relative lattice point rest and derive two important corollaries concerning gaps between values of the quadratic form and concerning multivariate diophantine approximations. Furthermore, we give explicit quantitative bounds for the relative lattice point rest. In the third section, we prove the results of the second section. In the fourth section, we collect auxiliary results (e.g. from geometry of numbers, metric number theory, theory of theta functions), which are used in the proofs of the theorems.

2. Results. Let \mathbb{R}^d , $1 \leq d < \infty$, denote the d -dimensional Euclidean space with scalar product $\langle \cdot, \cdot \rangle$ and norm $|\cdot|$ defined by $|x|^2 = \langle x, x \rangle = x_1^2 + \dots + x_d^2$ for $x = (x_1, \dots, x_d) \in \mathbb{R}^d$. Let \mathbb{Z}^d denote the standard lattice of points with integer coordinates in \mathbb{R}^d .

Consider the quadratic form

$$Q[x] := \langle Qx, x \rangle \quad \text{for } x \in \mathbb{R}^d,$$

where $Q : \mathbb{R}^d \rightarrow \mathbb{R}^d$ denotes a symmetric linear operator in $\text{GL}(d, \mathbb{R})$ with eigenvalues, say, q_1, \dots, q_d . We write

$$(2.1) \quad q_0 := \min_{1 \leq j \leq d} |q_j|, \quad q := \max_{1 \leq j \leq d} |q_j|, \quad \bar{q} := \max\{q_0^{-1}, q\}.$$

We always assume that the form is non-degenerate, that is, $q_0 > 0$.

We say that a quadratic form Q is of *block-type* if $Q = Q^+ - Q^-$, where Q^+ and Q^- are positive definite quadratic forms, $Q^+[x]$ only depends on the first d_1 coordinates of \mathbb{R}^d and $Q^-[x]$ on the $d - d_1$ remaining ones.

For $a, b \in \mathbb{R}$ with $a \leq b$, and $M \in \mathbb{R}^d$, we define

$$(2.2) \quad E_{a,b;M} := \{x \in \mathbb{R}^d : a \leq Q[x - M] \leq b\}.$$

Note that if $Q[x]$ is positive definite, then $E_{a,b;M}$ is an elliptic shell.

Recall that a quadratic form $Q[x]$ and the corresponding operator Q with non-zero matrix $Q = (q_{ij})$, $1 \leq i, j \leq d$, is called *rational* if there exists a real number $\lambda \neq 0$ such that the matrix λQ has integer entries only; otherwise it is called *irrational*.

For $r > 0$ we set $C_r := \{x \in \mathbb{R}^d : |x|_\infty \leq r\}$, where $|\cdot|_\infty$ denotes the maximum norm on \mathbb{R}^d , and

$$(2.3) \quad H_{r,M} := H_{r,M}^{a,b} := E_{a,b;M} \cap C_r.$$

For any (measurable) set $B \subset \mathbb{R}^d$ let $\text{vol } B$ denote the Lebesgue measure of B and $\text{vol}_{\mathbb{Z}} B$ its *lattice volume*, that is, the number of lattice points in $B \cap \mathbb{Z}^d$. We want to investigate the approximation of the lattice volume of $H_{r,M}$ by the Lebesgue volume. Therefore we estimate the following *relative* lattice point rest of *large* parts of hyperbolic shells $H_{r,M}$, $M \in \mathbb{R}^d$, r large:

$$(2.4) \quad \Delta(r, M) := \left| \frac{\text{vol}_{\mathbb{Z}} H_{r,M} - \text{vol } H_{r,M}}{\text{vol } H_{r,M}} \right|.$$

The two main results of this part of the paper are the following

THEOREM 2.1. *Let Q be a non-degenerate, d -dimensional, block-type form with $d \geq 5$, and let $M \in \mathbb{R}^d$. Then*

$$(2.5) \quad \Delta(r, M) = \mathcal{O}(1) \quad \text{as } r \rightarrow \infty.$$

The estimate of Theorem 2.1 refines an explicit bound of order $\mathcal{O}(1)$ obtained for dimensions $d \geq 9$ in [BG97] for *arbitrary* ellipsoids and in [BG99] for *arbitrary* hyperbolic shells. Since this bound is optimal in the case of positive definite forms ([Göt04], p. 196), the bound in Theorem 2.1 is also optimal for block-type forms.

In case Q is *irrational* Theorem 2.1 can be improved.

THEOREM 2.2. *Let Q be an irrational non-degenerate d -dimensional block-type form with $d \geq 5$, and let $M \in \mathbb{R}^d$. Then*

$$(2.6) \quad \Delta(r, M) = o(1).$$

For *irrational* forms and dimension $d \geq 9$ the bound of Theorem 2.2 has already been proved in [BG99]. We should remark again that the bounds of both theorems are explicit and effective.

REMARK 2.3. *For $M \in \mathbb{Q}^d$ the condition $\Delta(r, M) = o(1)$ implies that Q is irrational.*

Using Theorem 2.2 we can easily derive a corollary about gaps between values of block-type forms:

For a positive definite quadratic form, Davenport and Lewis [DL72] conjectured in 1972 that the distance between successive values v_n of the quadratic form $Q[x]$ on \mathbb{Z}^d converges to zero as $n \rightarrow \infty$, provided that the

dimension d is at least five and Q is irrational. This conjecture was proved by Götze [Göt04]. Now we can derive an analogous result for irrational block-type forms and dimension $d \geq 5$.

For a vector $M \in \mathbb{R}^d$ and a constant $c_0 > 0$ let

$$(2.7) \quad V(r) := \{Q[x - M] : x \in \mathbb{Z}^d \cap C_{r/c_0}\} \cap [-c_0r^2, c_0r^2]$$

denote the set of values of $Q[x - M]$, lying in the interval $[-c_0r^2, c_0r^2]$, for lattice points $x \in \mathbb{Z}^d$ in a box of edge length r/c_0 .

We define the maximal gap between successive values as

$$(2.8) \quad d(r) := \sup_{\substack{u \in V(r) \\ (u, +\infty) \cap V(r) \neq \emptyset}} \inf\{v - u : v > u, v \in V(r)\}.$$

COROLLARY 2.4. *Let Q be a non-degenerate d -dimensional block-type form with $d \geq 5$. Then for c_0 small enough:*

- (1) $\lim_{r \rightarrow \infty} d(r) = 0$ if Q is irrational.
- (2) $\lim_{r \rightarrow \infty} d(r) > 0$ if $M \in \mathbb{Q}^d$ and Q is rational.

Theorems 2.1 and 2.2 follow from Theorem 2.5 below. Furthermore, in Theorem 2.5 (combined with (3.1) in the proof of Theorems 2.1 and 2.2), estimates of the remainder terms in (2.5) and (2.6) in terms of certain diophantine properties of Q will be given.

In order to describe the explicit bounds we need to introduce some more notations. Let $|(x, y)|_\infty$ denote the maximum norm of a vector (x, y) in $\mathbb{R}^d \times \mathbb{R}^d$. For any $t > 0$ and $r \geq 2$ consider the norm F on $\mathbb{R}^d \times \mathbb{R}^d$ given by

$$(2.9) \quad F(x, y) := |(r(x + tQy), yr^{-1})|_\infty.$$

We introduce the so called *Minkowski minima* of the convex body $\{F \leq 1\}$ as

$$(2.10) \quad M_{1,t} = \inf\{F(m, n) : (m, n) \in (\mathbb{Z}^d \times \mathbb{Z}^d) \setminus 0\}$$

and we define in general $M_{k,t}$ to be the infimum of $\lambda > 0$ such that the set of lattice points with norm less than λ , that is,

$$\{(m, n) \in \mathbb{Z}^d \times \mathbb{Z}^d : F(m, n) < \lambda\},$$

contains k linearly independent vectors. By definition we have $rM_{k,t} \geq 1$. For $d > 4$ and $r \geq 2$ we introduce

$$(2.11) \quad \Gamma_{T,r} := \inf\{r^d M_{1,t} \cdots M_{d,t} : T^{-1/(d-4)} \leq |t| \leq T\},$$

$$(2.12) \quad \rho(r, Q, T) := \bar{q}^{d+1} T^{-1/2} + \bar{q}^{3d/2} \max\left\{\frac{2}{\pi r}, \frac{\pi}{2q_0qr}, T^{-1/(d-4)}\right\} \\ + \bar{q}^{d+2} \Gamma_{T,r}^{-1/2+2/d} \log(\bar{q} T^{1/2} \Gamma_{T,r} + 1),$$

$$(2.13) \quad \rho(r, Q) := \inf_{T \geq 1} \{ (b-a)\bar{q}^d q^{-1} T^{-1} + T^{-1/2} \bar{q}^{d/2} q^{-1} (2 + q^{1/2})^{d-2} + (4T)^K (r^{2-d} + q_0^{-d/2} r^{2-d/2} + \bar{q} r^{2-d/2} (1 + \log r) + \rho(r, Q, T)) \},$$

where K is a constant chosen according to (3.8) below.

For any fixed $T > 1$ and irrational Q it is shown in Lemma 4.23 that

$$(2.14) \quad \lim_{r \rightarrow \infty} \Gamma_{T,r} = \infty,$$

with a rate depending on the diophantine properties of Q . This implies that

$$(2.15) \quad \lim_{r \rightarrow \infty} \rho(r, Q) = 0.$$

With these notations we may state a theorem providing quantitative bounds for the difference between the volume and the lattice volume of a hyperbolic shell.

THEOREM 2.5. *Let Q be a non-degenerate d -dimensional block-type form with $d \geq 5$, and let $M \in \mathbb{R}^d$. Furthermore, let $c(Q, M) \geq 1$ be defined as in Theorem 3.1 below. Then there exist constants $c_j > 0$, $j = 1, 2$, depending on d only and a constant $r_0 = r(Q, M, a, b) > 0$ such that, for any $r \geq r_0$,*

$$(1) \quad |\text{vol}_{\mathbb{Z}} H_{r,M} - \text{vol } H_{r,M}| \leq c_1 r^{d-2} ((b-a+1)\bar{q}^d q^{-1} + c(Q, M)\bar{q}^{d+1}(\log q + 1) + 1),$$

$$(2) \quad |\text{vol}_{\mathbb{Z}} H_{r,M} - \text{vol } H_{r,M}| \leq c_2 r^{d-2} \times \left((b-a)\bar{q}^{d+1} q^{-1} \left(|M| + 2q^{-1/2} \frac{|a| + |b|}{r} \right) r^{-1 + c(Q, M)\rho(r, Q)} \right),$$

where $\lim_{r \rightarrow \infty} \rho(r, Q) = 0$ if Q is irrational.

Note that the summand $\rho(r, Q)r^{d-2}$ in the bound in Theorem 2.5 is at least of order $\mathcal{O}(r^{d/2} \log r)$. It may be indeed of this order since $rM_{j,t} \ll_d r$ shows that the maximal value of $\Gamma_{T,r}$ is of order $\mathcal{O}(r^d)$ and we may choose $T = \mathcal{O}(r^{\bar{\beta}})$ with $\bar{\beta} > 0$ sufficiently large.

Note that an error bound of order $r^{d/2+\varepsilon}$ has been proved by Jarník [Jar28] for diagonal $Q = \text{diag}(s_1, \dots, s_d)$, $s_j > 0$, for Lebesgue almost all coefficients s_j .

The proof of Theorem 2.5 is based, roughly speaking, on a “continuous” approximation of $|\text{vol}_{\mathbb{Z}} H_{r,M} - \text{vol } H_{r,M}|$ by an integral over generalized theta functions. We will derive bounds for parts of this integral, which use the distribution of the first Minkowski minimum $M_{1,t}$. We investigate this distribution using results from metric number theory. As a consequence of this investigation, we also get a result for multivariate diophantine approximation:

For a vector $x \in \mathbb{R}^d$ let $\|x\| := \inf_{m \in \mathbb{Z}^d} |x - m|_\infty$ denote the error of integer approximation. For real numbers $t > 0$ and $\nu > 1$ we introduce

$$(2.16) \quad D(t, \nu) = \nu \min \{ \|tQn\| : n \in \mathbb{Z}^d, 0 < |n|_\infty \leq \nu \},$$

and let λ denote the Lebesgue measure. Then we have

THEOREM 2.6. *Let Q be a symmetric, non-degenerate block-type form, normalized so that $q_0 = 1$. Then there exists a constant $c(d) > 1$ depending on d only such that for any $r \geq 1$, $0 < \xi - \kappa < 1$ and $\nu \geq \tau_Q$,*

$$(2.17) \quad \lambda\{t \in [\kappa, \xi] : M_{1,t} \leq \tau\} \leq c(d)(q\tau^2(\xi - \kappa) + \tau r^{-1}),$$

$$(2.18) \quad \sup_{t \in [\kappa, \xi]} M_{1,t} \geq \min\{\tau_Q, r(\xi - \kappa)\},$$

$$(2.19) \quad \sup_{t \in [\kappa, \xi]} D(t, \nu) \geq \min\{\tau_Q, \nu(\xi - \kappa)/2\},$$

where

$$\tau_Q := \left(\frac{c(d) + 2}{2c(d)q} \right)^{1/2}.$$

Refining the proofs, we may extend Theorems 2.1 and 2.2 to include the case $a = -\infty$, i.e. the case of distribution functions. This partially extends a result obtained by Bentkus and Götze in [BG99] to dimensions from 5 to 8.

THEOREM 2.7. *Let Q be a non-degenerate, d -dimensional, block-type form with $d \geq 5$, let $M \in \mathbb{R}^d$, and set*

$$F_{r,M}(b) := \{x \in \mathbb{R}^d : Q[x - M] \leq b, |x|_\infty \leq r\}.$$

Then

$$\left| \frac{\text{vol}_{\mathbb{Z}} F_{r,M}(b) - \text{vol } F_{r,M}(b)}{\text{vol } F_{r,M}(b)} \right| = \begin{cases} o(1) & \text{if } Q \text{ is irrational,} \\ \mathcal{O}(1) & \text{otherwise,} \end{cases}$$

as $r \rightarrow \infty$.

3. Proofs. First we deduce Theorems 2.1 and 2.2 from Theorem 2.5:

Proof of Theorems 2.1 and 2.2. By Lemma 4.1, for $M = (M_1, \dots, M_d)$ and r large,

$$(3.1) \quad \begin{aligned} \text{vol } H_{r,M} & \gg_d (b - a)q^{-d/2} (q_0^{d/2} + r^{-1}(|q_1|^{1/2}M_1, \dots, |q_d|^{1/2}M_d))^{d-2} r^{d-2} \\ & \gg_d (b - a)q^{-d/2} q_0^{d(d-2)/2} r^{d-2}. \end{aligned}$$

Dividing the inequalities in Theorem 2.5(1) in the general case (resp. Theorem 2.5(2) in the irrational case) by $\text{vol } H_{r,M}$ and using (3.1) completes the proof of Theorem 2.1 (resp. 2.2). ■

Proof of Corollary 2.4. If Q is irrational, Theorem 2.5(2) and the estimate (3.1) imply that whenever $c_0 > 0$ is small enough, for any $u, v \in [-c_0r^2, c_0r^2]$ we have

$$(3.2) \quad \left| \frac{\text{vol}_{\mathbb{Z}} H_{r,M}^{u,v}}{\text{vol}_{\mathbb{R}} H_{r,M}^{u,v}} - 1 \right| < \frac{1}{2} \quad \text{as } r \rightarrow \infty.$$

Hence, $H_{r,M}^{u,v} \cap \mathbb{Z}^d \neq \emptyset$ for r sufficiently large. This implies that $\lim_{r \rightarrow \infty} d(r) = 0$, proving (1).

If Q is rational, there exists a $\lambda > 0$ such that λQ has integer entries. If $M \in \mathbb{Q}^d$, there exists a $\mu \in \mathbb{Z}$, $\mu \neq 0$, such that $\mu M \in \mathbb{Z}^d$. Hence, $Q[m - M] \in \lambda^{-1}\mu^{-2}\mathbb{Z}^d$ for all $m \in \mathbb{Z}^d$. Therefore $d(r) \geq \lambda^{-1}\mu^{-2} > 0$ for all $r \geq 1$, which proves (2). ■

We remark that by using (3.14) and (3.1) one can obtain explicit bounds for $d(r)$ in terms of r and $\rho(r, Q)$, representing diophantine properties of Q .

Proof of Remark 2.3. Analyzing the proof of Corollary 2.4(1) we see that $\Delta(r, M) = o(1)$ already implies $\lim_{r \rightarrow \infty} d(r) = 0$. If $M \in \mathbb{Q}^d$ and Q is rational, then Corollary 2.4(2) yields $\lim_{r \rightarrow \infty} d(r) > 0$. Thus, the conditions $M \in \mathbb{Q}^d$ and $\Delta(r, M) = o(1)$ force the irrationality of Q . ■

The first step in proving Theorem 2.5 is to analyze smooth approximations of the lattice volume of H_r .

For $a, b \in \mathbb{R}$ and a smoothing parameter $w > 0$ we define $g_{a,b,w} : \mathbb{R} \rightarrow [0, 1]$ by

$$(3.3) \quad g_{a,b,w}(x) := \frac{1}{w} ((b + w - x)_+ - (b - x)_+ - (a - x)_+ + (a - w - x)_+).$$

This is a linear continuous approximation of the indicator function $I_{[a,b]}$ of the interval $[a, b]$. By Lemma 4.8 we may rewrite $g_{a,b,w}$ as

$$(3.4) \quad \begin{aligned} g_{a,b,w}(x) &= \frac{1}{2\pi i} \int_{\beta-i\infty}^{\beta+i\infty} (e^{(b+w-x)z} - e^{(b-x)z} - e^{(a-x)z} + e^{(a-w-x)z}) \frac{dz}{wz^2} \\ &= \frac{1}{2\pi i} \int_{\beta-i\infty}^{\beta+i\infty} \exp\{-xz\} h_{a,b,w}(z) \frac{dz}{z}, \end{aligned}$$

where

$$h_{a,b,w}(z) := \frac{\exp\{wz\} - 1}{wz} (\exp\{bz\} - \exp\{(a-w)z\}).$$

Using $g_{a,b,w}$ we construct a continuous approximation $V_{w,\varepsilon}^{\mathbb{Z}}(r; a, b, M)$ of the (monotone) lattice point counting function $r \mapsto \text{vol}_{\mathbb{Z}}(H_{r,M})$ depending on two smoothing parameters $w > 0$ and $\varepsilon > 0$. Setting $Q_+ := (Q^T Q)^{1/2}$, we

define

$$(3.5) \quad V_{w,\varepsilon}^{\mathbb{Z}}(r; a, b, M) := \sum_{x \in \mathbb{Z}^d} \exp\left\{-\frac{2}{r^2} Q_+[x]\right\} g_{a,b,w}(Q[x - M]) \chi_\varepsilon\left(\frac{x}{r}\right),$$

$$(3.6) \quad V_{w,\varepsilon}^{\mathbb{R}}(r; a, b, M) := \int_{\mathbb{R}^d} \exp\left\{-\frac{2}{r^2} Q_+[x]\right\} g_{a,b,w}(Q[x - M]) \chi_\varepsilon\left(\frac{x}{r}\right) dx,$$

where $\chi_{\pm\varepsilon}$ is a function with the following properties:

- (1) For $u \in \mathbb{R}^d$,

$$\chi_{\pm\varepsilon}(u) = \begin{cases} \exp\{2Q_+[u]\} & \text{if } |u|_\infty \leq \min\{1, 1 \pm \varepsilon\}, \\ 0 & \text{if } |u|_\infty > \max\{1, 1 \pm \varepsilon\}. \end{cases}$$

- (2) There exists a constant $c_1(Q, M) > 0$ such that for

$$(3.7) \quad \bar{\chi}_{\pm\varepsilon}(x) := \chi_{\pm\varepsilon}(x) \exp\{\langle x, 2r^{-1}QM \rangle\}$$

the following estimates hold for an appropriate $K = K(d) \in \mathbb{N}$:

$$(3.8) \quad \begin{aligned} (a) \quad & \int_{\mathbb{R}^d} |\widehat{\bar{\chi}}_{\pm\varepsilon}(v)| dv \ll_d c_1(Q, M) \varepsilon^{-K}, \\ (b) \quad & \int_{\{|v|_\infty > d^{-1/2}r\}} |\widehat{\bar{\chi}}_{\pm\varepsilon}(v)| dv \ll_d c_1(Q, M) \varepsilon^{-K} r^{-1} \quad \text{for all } r \geq 1. \end{aligned}$$

The existence of $\chi_{\pm\varepsilon}$ follows by standard arguments in Fourier analysis (cf. [Els06, p. 27, Lemma 2.4.5]). Note that the function

$$(3.9) \quad \psi_{r,\pm\varepsilon}(x) := \exp\left\{-\frac{2}{r^2} Q_+[x]\right\} \chi_{\pm\varepsilon}\left(\frac{x}{r}\right)$$

approximates the indicator function $I_{\{|x|_\infty \leq r\}}$ and hence the equations $V_{0,0}^{\mathbb{Z}}(r; a, b, M) = \text{vol}_{\mathbb{Z}}(H_{r,M})$ and $V_{0,0}^{\mathbb{R}}(r; a, b, M) = \text{vol}_{\mathbb{R}}(H_{r,M})$ are suggestive.

Proof of Theorem 2.5. Let $0 < \varepsilon \leq 1/4$. By Lemma 4.4 there exists a constant $c = c(d) > 0$ such that

$$(3.10) \quad \begin{aligned} |\text{vol}_{\mathbb{Z}} H_{r,M} - \text{vol} H_{r,M}| &\leq \max\{\Delta_{-\varepsilon}, \Delta_\varepsilon\} + c(b - a) q_0^{-d/2} q^{(d-2)/2} \\ &\quad \times (\varepsilon + q_0^{-1/2} q^{1/2} |M| r^{-1} + 2q_0^{-1/2} (|a| + |b|) r^{-2}) r^{d-2}, \end{aligned}$$

where $\Delta_{\pm\varepsilon}$ is defined by using (3.9) as follows:

$$(3.11) \quad \Delta_{\pm\varepsilon} := \left| \int_{\mathbb{R}^d} I_{H_{r,M}}(x) \psi_{r,\pm\varepsilon}(x) dx - \sum_{x \in \mathbb{Z}^d} I_{H_{r,M}}(x) \psi_{r,\pm\varepsilon}(x) \right|.$$

Hence, by Lemma 4.6,

$$(3.12) \quad \Delta_{\pm\varepsilon} \ll_d \max_{\pm} \sup_{\substack{a' \in [a-w, a+w] \\ b' \in [b-w, b+w]}} |V_{w, \pm\varepsilon}^{\mathbb{R}}(r; a', b', M) - V_{w, \pm\varepsilon}^{\mathbb{Z}}(r; a', b', M)| \\ + 8wq_0^{-d/2} q^{(d-2)/2} \left(1 + \varepsilon + q^{1/2} \frac{|M|}{r}\right)^{d-2} r^{d-2}.$$

Collecting the estimates (3.10) and (3.12), for $w > 0$ and $0 < \varepsilon \leq 1/4$ we obtain

$$(3.13) \quad r^{2-d} |\text{vol}_{\mathbb{Z}} H_{r,M} - \text{vol} H_{r,M}| \ll_d (b-a) \bar{q}^d q^{-1} \varepsilon \\ + (b-a) \bar{q}^{d+1} q^{-1} \left(\frac{|M|}{r} + 2q^{-1/2} \frac{|a| + |b|}{r^2} \right) + w \bar{q}^d q^{-1} \left(1 + \varepsilon + q^{1/2} \frac{|M|}{r}\right)^{d-2} \\ + \max_{\pm} \sup_{a', b'} |V_{w, \pm\varepsilon}^{\mathbb{R}}(r; a', b', M) - V_{w, \pm\varepsilon}^{\mathbb{Z}}(r; a', b', M)| r^{2-d}.$$

If we now choose $w = 1$, $\varepsilon = 1/4$ and $r > r_0$ large enough, (3.13) and the result of the crucial Theorem 3.1(1) below yields (note that $d \geq 5$)

$$|\text{vol}_{\mathbb{Z}} H_{r,M} - \text{vol} H_{r,M}| \\ \ll_d (b-a+1) \bar{q}^d q^{-1} r^{d-2} + c(Q, M) (1 + q_0^{-d/2} r^{d/2} + \bar{q}^{d+1} (\log q + 1) r^{d-2}) \\ \ll_d ((b-a+1) \bar{q}^d q^{-1} + c(Q, M) \bar{q}^{d+1} (\log q + 1) + 1) r^{d-2}$$

for r large enough. This proves Theorem 2.5(1).

To prove Theorem 2.5(2), for $T \geq 1$ we choose

$$w := T^{-1/2} \quad \text{and} \quad \varepsilon := (4T)^{-1},$$

and by (3.13) and Theorem 3.1(2) below we deduce that for r sufficiently large,

$$r^{2-d} |\text{vol}_{\mathbb{Z}} H_{r,M} - \text{vol} H_{r,M}| \ll_d (b-a) \bar{q}^d q^{-1} (4T)^{-1} \\ + (b-a) \bar{q}^{d+1} q^{-1} \left(|M| + 2q^{-1/2} \frac{|a| + |b|}{r} \right) r^{-1} + T^{-1/2} \bar{q}^{d/2} q^{-1} (2 + q^{1/2})^{d-2} \\ + c(Q, M) (4T)^K (1 + q_0^{-d/2} r^{d/2} + \bar{q}^d r^{d/2} (1 + \log r) + r^{d-2} \rho(r, Q, T)) r^{2-d}.$$

By taking the infimum over all $T \geq 1$ we get, with (2.13),

$$(3.14) \quad r^{2-d} |\text{vol}_{\mathbb{Z}} H_{r,M} - \text{vol} H_{r,M}| \\ \ll_d (b-a) \bar{q}^{d+1} q^{-1} \left(|M| + 2q^{-1/2} \frac{|a| + |b|}{r} \right) r^{-1} + c(Q, M) \rho(r, Q),$$

which proves Theorem 2.5(2) for an appropriate choice of r_0 . ■

The key tool in the previous proofs is the following

THEOREM 3.1. *Let Q denote a non-degenerate d -dimensional quadratic form of block-type with $d \geq 5$. Then for all $M \in \mathbb{R}^d$ there exist constants $c(Q, M)$, $r_0 > 0$ such that for any $r \geq r_0$ and any $T \geq 1$:*

$$\begin{aligned}
 (1) \quad & |V_{1, \pm \varepsilon}^{\mathbb{Z}}(r; a, b, M) - V_{1, \pm \varepsilon}^{\mathbb{R}}(r; a, b, M)| \\
 & \ll_d c(Q, M) \varepsilon^{-K} (1 + 2q_0^{-d/2} r^{d/2} + \bar{q}^{d+1} (\log q + 1) r^{d-2}). \\
 (2) \quad & |V_{T^{-1/2}, \pm \varepsilon}^{\mathbb{Z}}(r; a, b, M) - V_{T^{-1/2}, \pm \varepsilon}^{\mathbb{R}}(r; a, b, M)| \ll_d c(Q, M) \varepsilon^{-K} \\
 & \times (1 + q_0^{-d/2} r^{d/2} + \bar{q}^d r^{d/2} (1 + \log r) + r^{d-2} \rho(r, Q, T)),
 \end{aligned}$$

where $\rho(r, Q, T)$ is defined in (2.12).

In both inequalities $K = K(d)$ is chosen according to (3.8).

Proof. We want to estimate the difference between these two approximations by integrals of theta functions. By (3.4)–(3.6) we have

$$\begin{aligned}
 & |V_{w, \pm \varepsilon}^{\mathbb{Z}}(r; a, b, M) - V_{w, \pm \varepsilon}^{\mathbb{R}}(r; a, b, M)| \\
 & = \left| \sum_{x \in \mathbb{Z}^d} \exp\left\{-\frac{2}{r^2} Q_+[x]\right\} \frac{1}{2\pi i} \int_{\beta-i\infty}^{\beta+i\infty} \exp\{-zQ[x-M]\} h_{a,b,w}(z) \frac{dz}{z} \chi_{\pm \varepsilon}\left(\frac{x}{r}\right) \right. \\
 & \quad \left. - \int_{\mathbb{R}^d} \exp\left\{-\frac{2}{r^2} Q_+[x]\right\} \frac{1}{2\pi i} \int_{\beta-i\infty}^{\beta+i\infty} \exp\{-zQ[x-M]\} h_{a,b,w}(z) \frac{dz}{z} \chi_{\pm \varepsilon}\left(\frac{x}{r}\right) dx \right|.
 \end{aligned}$$

Choosing $\beta = r^{-2}$, decomposing $Q[x-M] = Q[x] + Q[M] - 2\langle x, QM \rangle$ (recall that Q is self-adjoint), and using Fubini’s theorem, we get

$$\begin{aligned}
 & |V_{w, \pm \varepsilon}^{\mathbb{Z}}(r; a, b, M) - V_{w, \pm \varepsilon}^{\mathbb{R}}(r; a, b, M)| \\
 & = \left| \int_{r^{-2}-i\infty}^{r^{-2}+i\infty} \exp\{-zQ[M]\} h_{a,b,w}(z) \right. \\
 & \quad \times \left\{ \sum_{x \in \mathbb{Z}^d} \exp\left\{-\frac{2}{r^2} Q_+[x] - zQ[x] + i\langle x, 2 \operatorname{Im}(z)QM \rangle\right\} \bar{\chi}_{\pm \varepsilon}\left(\frac{x}{r}\right) \right. \\
 & \quad \left. \left. - \int_{\mathbb{R}^d} \exp\left\{-\frac{2}{r^2} Q_+[x] - zQ[x] + i\langle x, 2 \operatorname{Im}(z)QM \rangle\right\} \bar{\chi}_{\pm \varepsilon}\left(\frac{x}{r}\right) dx \right\} \frac{dz}{z} \right|,
 \end{aligned}$$

where $\bar{\chi}_{\pm \varepsilon}$ is defined as in (3.7).

Since $\bar{\chi}_{\pm \varepsilon}(x) = (2\pi)^{-d} \int_{\mathbb{R}^d} \widehat{\bar{\chi}}_{\pm \varepsilon}(v) \exp\{-i\langle x, v \rangle\} dv$ by the Fourier inversion theorem, we obtain

$$\begin{aligned}
 (3.15) \quad & |V_{w,\pm\epsilon}^{\mathbb{Z}}(r; a, b, M) - V_{w,\pm\epsilon}^{\mathbb{R}}(r; a, b, M)| \\
 &= \left| \int_{r^{-2}-i\infty}^{r^{-2}+i\infty} \exp\{-zQ[M]\} h_{a,b,w}(z) \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \widehat{\chi}_{\pm\epsilon}(v) \right. \\
 &\quad \times \left\{ \sum_{x \in \mathbb{Z}^d} \exp\left\{-\frac{2}{r^2} Q_+[x] - zQ[x] + i \left\langle x, 2 \operatorname{Im}(z)QM - \frac{v}{r} \right\rangle\right\} \right. \\
 &\quad \left. \left. - \int_{\mathbb{R}^d} \exp\left\{-\frac{2}{r^2} Q_+[x] - zQ[x] + i \left\langle x, 2 \operatorname{Im}(z)QM - \frac{v}{r} \right\rangle\right\} dx \right\} dv \frac{dz}{z} \right|.
 \end{aligned}$$

For $v \in \mathbb{C}^d$ we introduce the following theta sum and theta integral:

$$(3.16) \quad \theta_v(z) := \exp\{-zQ[M]\} \sum_{x \in \mathbb{Z}^d} \exp\{-\Theta_{Q,M,r,v}(z, x)\},$$

$$(3.17) \quad \theta_{0,v}(z) := \exp\{-zQ[M]\} \int_{\mathbb{R}^d} \exp\{-\Theta_{Q,M,r,v}(z, x)\} dx$$

where

$$\Theta_{Q,M,r,v}(z, x) := \frac{2}{r^2} Q_+[x] - zQ[x] - i \left\langle x, \frac{v}{r} - 2 \operatorname{Im}(z)QM \right\rangle.$$

Then we can rewrite (3.15) as follows:

$$\begin{aligned}
 & |V_{w,\pm\epsilon}^{\mathbb{Z}}(r; a, b, M) - V_{w,\pm\epsilon}^{\mathbb{R}}(r; a, b, M)| \\
 &= \left| \int_{r^{-2}-i\infty}^{r^{-2}+i\infty} h_{a,b,w}(z) \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \widehat{\chi}_{\pm\epsilon}(v) \{\theta_v(z) - \theta_{0,v}(z)\} dv \frac{dz}{z} \right|.
 \end{aligned}$$

Consider the segments $J_0 := [r^{-2} - i/r, r^{-2} + i/r]$ and $J_1 := (r^{-2} + i\mathbb{R}) \setminus J_0$. Then we may split

$$\begin{aligned}
 (3.18) \quad & |V_{w,\pm\epsilon}^{\mathbb{Z}}(r; a, b, M) - V_{w,\pm\epsilon}^{\mathbb{R}}(r; a, b, M)| \\
 &\ll_d \left| \int_{J_0} h_{a,b,w}(z) \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \widehat{\chi}_{\pm\epsilon}(v) \{\theta_v(z) - \theta_{0,v}(z)\} dv \frac{dz}{z} \right. \\
 &\quad \left. - \int_{J_1} h_{a,b,w}(z) \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \widehat{\chi}_{\pm\epsilon}(v) \theta_{0,v}(z) dv \frac{dz}{z} \right. \\
 &\quad \left. + \int_{J_1} h_{a,b,w}(z) \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \widehat{\chi}_{\pm\epsilon}(v) \theta_v(z) dv \frac{dz}{z} \right| \\
 &= |I_0 - I_1 + I_2|, \quad \text{say.}
 \end{aligned}$$

Before estimating these integrals we derive a bound for $h_{a,b,w}(r^{-2} + it)$, $t \in \mathbb{R}$. Using

$$(3.19) \quad \left| \frac{\exp\{w(r^{-2} + it)\} - 1}{w} \right| \leq \min \left\{ e|r^{-2} + it|, \frac{e + 1}{w} \right\}$$

for $r^2 \geq \max(w, b) > 0$, $r \geq 1$, we obtain

$$(3.20) \quad \left| \frac{h_{a,b,w}(r^{-2} + it)}{r^{-2} + it} \right| \ll (e^{br^{-2}} + e^{ar^{-2}}) \frac{1}{w|r^{-2} + it|^2} \ll \frac{1}{w|r^{-2} + it|^2},$$

as well as

$$(3.21) \quad \left| \frac{h_{a,b,w}(r^{-2} + it)}{r^{-2} + it} \right| \ll (e^{br^{-2}} + e^{ar^{-2}})|r^{-2} + it|^{-1} \ll |r^{-2} + it|^{-1}.$$

Estimation of I_0 . Inequality (3.21) and Lemma 4.12 for $t \in J_0$ yield

$$\begin{aligned} \Theta_t &:= |(r^{-2} + it)^{-1} h_{a,b,w}(r^{-2} + it)| \\ &\quad \times \left| \int_{\mathbb{R}^d} \widehat{\chi}_{\pm\varepsilon}(v) \{\theta_v(r^{-2} + it) - \theta_{0,v}(r^{-2} + it)\} dv \right| \\ &\ll_d q_0^{-d/2} |r^{-2} + it|^{-(d+2)/2} \exp\{-c(Q) \operatorname{Re}((r^{-2} + it)^{-1})\} \int_{\mathbb{R}^d} |\widehat{\chi}_{\pm\varepsilon}(v)| dv \\ &\quad + 2|r^{-2} + it|^{-1} \int_{\mathbb{R}^d} |\widehat{\chi}_{\pm\varepsilon}(v)| I_{(r,\infty)}(|v|) dv, \end{aligned}$$

where $c(Q)$ is described in Lemma 4.12. Writing

$$|r^{-2} + it| = r^{-2}(1 + r^4 t^2)^{1/2} \quad \text{and} \quad \operatorname{Re}((r^{-2} + it)^{-1}) = r^2/(1 + r^4 t^2),$$

we may introduce the variable $s = (1 + r^4 t^2)^{-1}$ and the function $h(s) := s^{(d+2)/4} \exp\{-c(Q)sr^2\}$. The maximal value of h on $[0, \infty)$ is attained at $s_0 = (d + 2)/(4c(Q)r^2)$ and it is bounded by $(c(Q)r^2)^{-(d+2)/4}$ up to a constant depending on d only.

Using the properties of $\chi_{\pm\varepsilon}$ (see p. 209) and the fact that $|v| \geq r$ implies $|v|_\infty \geq d^{-1/2}r$ we now obtain

$$\begin{aligned} \sup_{t \in J_0} \Theta_t &\ll_d q_0^{-d/2} r^{d+2} \sup_{s \geq 0} h(s) \int_{\mathbb{R}^d} |\widehat{\chi}_{\pm\varepsilon}(v)| dv + 2r^2 \int_{\mathbb{R}^d} |\widehat{\chi}_{\pm\varepsilon}(v)| I_{(r,\infty)}(|v|) dv \\ &\ll_d q_0^{-d/2} r^{d+2} (c(Q)r^2)^{-(d+2)/4} \int_{\mathbb{R}^d} |\widehat{\chi}_{\pm\varepsilon}(v)| dv \\ &\quad + 2r^2 \int_{\mathbb{R}^d} |\widehat{\chi}_{\pm\varepsilon}(v)| I_{(d^{-1/2}r,\infty)}(|v|_\infty) dv \\ &\ll_d q_0^{-d/2} r^{d+2} (c(Q)r^2)^{-(d+2)/4} c_1(Q, M) \varepsilon^{-K} + c_1(Q, M) \varepsilon^{-K} r. \end{aligned}$$

Integrating this bound over J_0 , for a suitably chosen constant $c_2(Q, M) > 0$ we get

$$(3.22) \quad |I_0| \leq \int_{-1/r}^{1/r} \Theta_t dt \ll_d c_2(Q, M) \varepsilon^{-K} q_0^{-d/2} r^{d/2} + c_1(Q, M) \varepsilon^{-K}.$$

Estimation of I_1 . Using Lemma 4.11, (4.13) and (4.15), we have

$$(3.23) \quad |\theta_{0,v}(z)| \ll_d q_0^{-d/2} |z|^{-d/2}.$$

Therefore, by the properties of $\chi_{\pm\varepsilon}$ (p. 209) and (3.21) for $r^2 \geq \max\{w, b, 1\}$ we get

$$(3.24) \quad \begin{aligned} |I_1| &\ll_d q_0^{-d/2} c_1(Q, M) \varepsilon^{-K} \int_{J_1} |(r^{-2} + it)^{-(1+d/2)}| dt \\ &\ll_d q_0^{-d/2} c_1(Q, M) \varepsilon^{-K} \int_{1/r}^{\infty} t^{-(1+d/2)} dt \\ &\ll_d q_0^{-d/2} c_1(Q, M) \varepsilon^{-K} r^{d/2}, \end{aligned}$$

using the symmetry in t around 0.

Estimation of I_2 . From (3.20) and (3.21) we obtain $|h_{a,b,w}(r^{-2} + it)| \ll_d \min\{1, (|r^{-2} + it|w)^{-1}\}$, which implies

$$(3.25) \quad \begin{aligned} |I_2| &\ll_d \int_{\mathbb{R}^d} \int_{|t|>1/r} \left| \theta_v \left(\frac{1}{r^2} + it \right) \right| \min \left\{ 1, \frac{1}{w|r^{-2} + it|} \right\} \frac{dt}{|r^{-2} + it|} |\widehat{\chi}_{\pm\varepsilon}(v)| dv \\ &\ll_d \int_{\mathbb{R}^d} \int_{|u|>2/\pi r} |\theta_v(r^{-2} + i\pi u/2)| g(u) du |\widehat{\chi}_{\pm\varepsilon}(v)| dv, \end{aligned}$$

where

$$(3.26) \quad g(u) = \min\{1, (w|u|)^{-1}\} |u|^{-1}.$$

Using Lemma 4.15 and the properties of $\chi_{\pm\varepsilon}$ (see p. 209), we have

$$(3.27) \quad \begin{aligned} |I_2| &\ll_d \bar{q}^d r^{d/2} \int_{\mathbb{R}^d} \int_{|u|>2/\pi r} (M_{1,t} \cdots M_{d,t})^{-1/2} g(u) du |\widehat{\chi}_{\pm\varepsilon}(v)| dv \\ &\ll_d \bar{q}^d r^{d/2} c_1(Q, M) \varepsilon^{-K} \int_{|u|>2/\pi r} (M_{1,t} \cdots M_{d,t})^{-1/2} g(u) du, \end{aligned}$$

where $M_{j,t}$ denote Minkowski's successive minima for the norm on \mathbb{R}^{2d} related to Q , defined by (4.27) and (4.30), and $c_1(Q, M) > 0$ is a constant

chosen according to (3.7). Define

$$(3.28) \quad G(\kappa, \xi) := \int_{\kappa}^{\xi} g(t) dt \quad \text{for } 0 < \kappa < \xi \leq \infty.$$

For $\kappa \geq \xi > 0$ we define $G(\kappa, \xi) = 0$. Note that

$$(3.29) \quad G(\kappa, \xi) = \begin{cases} \log(\xi/\kappa) & \text{for } \kappa \leq \xi \leq w^{-1}, \\ -\log(w\kappa) + 1 - (w\xi)^{-1} & \text{for } \kappa \leq w^{-1} \leq \xi, \\ (w\kappa)^{-1} - (w\xi)^{-1} & \text{for } w^{-1} \leq \kappa \leq \xi. \end{cases}$$

The equality (3.29) and the definition of G imply

$$(3.30) \quad G(\kappa, \xi) \leq \min\{|\log(w\kappa)| + 1, |\log(\xi/\kappa)|, (w\kappa)^{-1}\} \quad \text{for } \kappa, \xi > 0.$$

If we write $M(t) = M_{1,t} \cdots M_{d,t}$, the upper bound for $|I_2|$ in (3.27) in terms of Minkowski's successive minima yields

$$(3.31) \quad \begin{aligned} |I_2| &\ll_d \bar{q}^d r^{d/2} c_1(Q, M) \varepsilon^{-K} \int_{|t| > 2/\pi r} \frac{g(t)}{M(t)^{1/2}} dt \\ &= 2\bar{q}^d r^{d/2} c_1(Q, M) \varepsilon^{-K} I_3, \end{aligned}$$

where

$$(3.32) \quad I_3 = \int_{2/\pi r}^{\infty} \frac{g(t)}{M(t)^{1/2}} dt.$$

The last equality in (3.31) follows from the fact that the functions $g(\cdot)$ and $M(\cdot)$ are even (see (4.33)).

After these preparations, we may now complete the proof of Theorem 3.1:

Proof of Theorem 3.1(1). Let

$$(3.33) \quad \gamma(\kappa, \xi) = r^d \inf_{\kappa \leq t \leq \xi} M(t) \quad \text{for } \kappa, \xi \in \mathbb{R}.$$

Applying Lemma 4.22 for the interval with endpoints $\kappa = 2/\pi r$ and $\xi = \infty$, we get

$$(3.34) \quad \begin{aligned} I_3 &\ll_d q_0^{-1} r^{d/2-2} \int_{\gamma_0}^{D_0} v^{-1/2+1/d} (qv^{1/d} G(\kappa_0(v^{1/d}), \infty) + g(\kappa_0(v^{1/d}))) \frac{dv}{v} \\ &\quad + G\left(\frac{2}{\pi r}, \infty\right) \end{aligned}$$

with

$$(3.35) \quad \begin{aligned} \gamma_0 &= \gamma\left(\frac{2}{\pi r}, \infty\right), \quad D_0 = \max\left\{\left(\frac{r}{2d}\right)^d, \gamma_0\right\}, \\ \kappa_0(v) &= \max\left\{\frac{2}{\pi r}, \frac{1}{2qvd^{1/2}}\right\}. \end{aligned}$$

Note that $\gamma_0 \geq 1$ by (4.29). In the following we choose $w = 1$. Using (3.26), (3.30), (3.31), (3.34), (3.35) and hence $g(\kappa_0(v^{1/d})) \ll_d qv^{1/d}$, we obtain for $d > 4$ and $r \geq \max\{2/\pi q, 2/\pi\}$,

$$(3.36) \quad \begin{aligned} |I_2| &\ll_d c_1(Q, M)\varepsilon^{-K}\bar{q}^{-d+1}r^{d/2}r^{d/2-2} \int_1^{D_0} v^{-1/2+2/d}(\log(qv^{1/d}) + 2) \frac{dv}{v} \\ &\quad + c_1(Q, M)\varepsilon^{-K}\bar{q}^{-d}r^{d/2}(\log r + 1) \\ &\ll_d c_1(Q, M)\varepsilon^{-K}\bar{q}^{-d+1}(\log q + 1)r^{d-2}. \end{aligned}$$

For $r \geq r_0 := \max\{2/\pi q, 2/\pi, r_0(Q, M)\}$, where $r_0(Q, M)$ is a constant chosen as in Lemmas 4.12 and 4.13, this bound for I_2 implies, in view of (3.18), (3.22) and (3.24), that

$$\begin{aligned} |V_{1,\pm\varepsilon}^{\mathbb{Z}}(r; a, b, M) - V_{1,\pm\varepsilon}^{\mathbb{R}}(r; a, b, M)| &\ll_d c_2(Q, M)\varepsilon^{-K}q_0^{-d/2}r^{d/2} \\ &\quad + c_1(Q, M)\varepsilon^{-K}(1 + q_0^{-d/2}r^{d/2} + \bar{q}^{-d+1}(\log q + 1)r^{d-2}), \end{aligned}$$

where the constants $c_1(Q, M)$ and $c_2(Q, M)$ are chosen according to Lemma 3.7 and (3.22). Setting $c(Q, M) := \max\{c_1(Q, M), c_2(Q, M)\}$ proves Theorem 3.1(1).

Proof of Theorem 3.1(2). In order to use non-trivial bounds for $\gamma(\kappa, \xi)$ in the irrational case we introduce further auxiliary parameters η, T such that $2/\pi r \leq \eta \leq T$ with $T \geq 1$ which will be determined and optimized later. Thus we may split the integral I_3 in (3.32) which bounds $|I_2|$ in (3.31) into the parts

$$(3.37) \quad \begin{aligned} I_3 &= \int_{2/\pi r}^{\eta} \frac{g(t)}{M(t)^{1/2}} dt + \int_{\eta}^T \frac{g(t)}{M(t)^{1/2}} dt + \int_T^{\infty} \frac{g(t)}{M(t)^{1/2}} dt \\ &= I_4 + I_5 + I_6, \quad \text{say.} \end{aligned}$$

We define, similarly to (3.35),

$$(3.38) \quad \gamma_1 = \gamma(2/\pi r, \eta), \quad \gamma_2 = \gamma(\eta, T), \quad \gamma_3 = \gamma(T, \infty),$$

$$(3.39) \quad D_j = \max\{(2d)^{-d}r^d, \gamma_j\}, \quad j = 1, 2, 3,$$

$$(3.40) \quad \begin{aligned} \kappa_1(v) &= \max\{2/\pi r, f(v)\}, \quad \kappa_2(v) = \max\{\eta, f(v)\}, \\ \kappa_3(v) &= \max\{T, f(v)\}, \end{aligned}$$

where $f(v) = (2qvd^{1/2})^{-1}$, $v > 0$. By (4.29) we have again

$$(3.41) \quad \gamma_j \geq 1, \quad j = 1, 2, 3.$$

Using (3.26) and (3.40), we see that

$$(3.42) \quad g(\kappa_j(v)) \leq 2qvd^{1/2}, \quad j = 1, 2, 3.$$

First, we apply Lemma 4.22 as above to the interval with endpoints $\kappa = 2/\pi r$ and $\xi = \eta$. Corollary 4.17 implies that if $\eta \geq \pi/2q_0qr$ the quantity γ_1 (defined by (3.33) and (3.38)) satisfies

$$(3.43) \quad \gamma_1 \geq \delta := (dq\eta)^{-d},$$

since $d \geq 5$ and

$$\inf_{t \in [2/\pi r, \eta]} \left\{ \frac{q_0|t|r}{2}, \frac{1}{q|t|r} \right\} = \frac{1}{q\eta r} \quad \text{whenever} \quad \eta \geq \frac{\pi}{2q_0qr}.$$

In view of (3.29), (3.30), (3.42) and (3.43) Lemma 4.22 yields the estimate

$$(3.44) \quad \begin{aligned} I_4 &\ll_d q_0^{-1} r^{d/2-2} \int_{\gamma_1}^{D_1} v^{-1/2+1/d} (v^{1/d} q G(\kappa_1(v^{1/d}), \eta) \\ &\quad + g(\kappa_1(v^{1/d}))) \frac{dv}{v} + G(2/\pi r, \eta) \\ &\ll_d q_0^{-1} q r^{d/2-2} \int_{\delta}^{D_1} v^{-1/2+2/d} (|\log(qv^{1/d}\eta)| + 1) \frac{dv}{v} + G(2/\pi r, \eta) \\ &\ll_d q_0^{-1} q^{d/2-1} r^{d/2-2} \eta^{d/2-2} + G(2/\pi r, \eta), \end{aligned}$$

provided that $d > 4$, using the change of variables $v = \delta u$ in the last inequality.

In order to estimate I_5 we choose $\kappa = \eta$ and $\xi = T$. By Lemma 4.22 we obtain as above

$$(3.45) \quad \begin{aligned} I_5 &\ll_d q_0^{-1} r^{d/2-2} \int_{\gamma_2}^{D_2} v^{-1/2+1/d} (v^{1/d} q G(\kappa_2(v^{1/d}), T) \\ &\quad + g(\kappa_2(v^{1/d}))) \frac{dv}{v} + G(\eta, T) \\ &\ll_d q_0^{-1} q r^{d/2-2} \int_{\gamma_2}^{D_2} v^{-1/2+2/d} (|\log(qv^{1/d}/w)| + 1) \frac{dv}{v} + G(\eta, T) \\ &\ll_d q_0^{-1} q r^{d/2-2} \gamma_2^{-1/2+2/d} (|\log(q\gamma_2)| + |\log w| + 1) + G(\eta, T). \end{aligned}$$

Finally, for the term I_6 choose $\kappa = T$ and $\xi = \infty$ and use (3.41) for $j = 3$. Recall that we choose $T \geq 1$. Thus, as above, using Lemma 4.22 and the fact that $G(\kappa_3(v^{1/d}), \infty) \leq G(T, \infty) \leq T^{-1}w^{-1}$ and $g(\kappa_3(v^{1/d})) \leq T^{-2}w^{-1}$,

we obtain (see (3.26), (3.30) and (3.40))

$$(3.46) \quad I_6 \ll_d q_0^{-1} r^{d/2-2} \int_1^{D_3} v^{-1/2+1/d} (v^{1/d} q G(\kappa_3(v^{1/d}), \infty) + g(\kappa_3(v^{1/d}))) \frac{dv}{v} + G(T, \infty) \\ \ll_d q_0^{-1} q r^{d/2-2} T^{-1} w^{-1} + G(T, \infty).$$

Collecting (3.44)–(3.46), by combining the terms $G(\kappa, \xi)$ and using (3.37) and the estimates (3.41) we get

$$(3.47) \quad I_3 \ll_d q_0^{-1} r^{d/2-2} \left\{ q^{d/2-1} \eta^{d/2-2} + q \gamma_2^{-1/2+2/d} (\log(q \gamma_2) + |\log w| + 1) + \frac{q}{T w} \right\} + G(2/\pi r, \infty).$$

In view of (3.31) this bound for I_3 yields

$$(3.48) \quad |I_2| \ll_d c_1(Q, M) \varepsilon^{-K} q^d r^{d/2} (1 + \log r) + c_1(Q, M) \varepsilon^{-K} \bar{q}^d r^{d-2} \\ \times \{ (T w)^{-1} + q^{d/2-1} \eta^{d/2-2} + q \gamma_2^{-1/2+2/d} (\log(q \gamma_2) + |\log w| + 1) \} \\ \ll_d c(Q, M) \varepsilon^{-K} \bar{q}^d r^{d/2} (1 + \log r) + c(Q, M) \varepsilon^{-K} r^{d-2} \\ \times \left\{ \frac{\bar{q}^{d+1}}{T w} + \bar{q}^{3d/2} \eta^{d/2-2} + \bar{q}^{d+2} \gamma_2^{-1/2+2/d} (\log(\bar{q} \gamma_2) + |\log w| + 1) \right\},$$

where $c(Q, M) := \max\{c_1(Q, M), c_2(Q, M)\}$. By Lemma 4.23 for η, T fixed, we have $\gamma_2 \rightarrow \infty$ as $r \rightarrow \infty$ and we may now choose the auxiliary parameters η, w and T to minimize the right hand side of (3.48) as follows. Let

$$(3.49) \quad T \geq 1, \quad w = T^{-1/2}, \quad \eta = \max \left\{ \frac{2}{\pi r}, \frac{\pi}{2q_0 q r}, T^{-1/(d-4)} \right\},$$

provided that $d \geq 5$.

For $r \geq r_0 := \max\{2/\pi, \pi/2q_0q, r_0(Q, M)\}$, where $r_0(Q, M)$ is a constant chosen as in Lemmas 4.12 and 4.13, in view of (3.18), (3.22), (3.24), (3.38), (3.41), (3.48) and (3.49) we obtain the following bound:

$$|V_{T^{-1/2}, \varepsilon}^{\mathbb{Z}}(r; a, b, M) - V_{T^{-1/2}, \varepsilon}^{\mathbb{R}}(r; a, b, M)| \\ \ll_d c(Q, M) \varepsilon^{-K} (1 + q_0^{-d/2} r^{d/2} + \bar{q}^d r^{d/2} (1 + \log r) + r^{d-2} \rho(r, Q, T)),$$

where $\rho(r, Q, T)$ is defined as in (2.12). This completes the proof of Theorem 3.1(2). ■

Proof of Theorem 2.6. The estimate (2.17) immediately follows from Corollary 4.21. This inequality ensures that there exists a $t \in [\kappa, \xi]$ such

that $M_{1,t} > \tau$ whenever $c(d)(q\tau^2(\xi - \kappa) + \tau r^{-1}) < \xi - \kappa$. This condition is equivalent to

$$(3.50) \quad \tau < \left(\frac{1}{c(d)} - q\tau^2 \right) (\xi - \kappa)r.$$

Since $\tau \leq \tau_Q$, where $\tau_Q := ((c(d) + 2)/2c(d)q)^{1/2}$, implies $1/c(d) - q\tau^2 \geq 1/2$, we may conclude that the condition (3.50) (and hence $M_{1,t} > \tau$) follows from the inequality $\tau \leq \min\{\tau_Q, r(\xi - \kappa)/2\}$, which proves (2.18).

By definition of $M_{1,t}$ the inequality $M_{1,t} > \bar{\tau} := \min\{\tau_Q, r(\xi - \kappa)/2\}$ implies that if $0 < |n|_\infty < \bar{\tau}r$ then $\bar{\tau}r \|tQn\| > \bar{\tau}^2$. For $\nu > \tau_Q$ there exists an $r \geq 1$ such that $\nu = \bar{\tau}r$. Therefore, by (2.16) we get $D(t, \nu) \geq \bar{\tau}^2$. Furthermore, we have $\bar{\tau}^2 = \min\{\tau_Q^2, \nu(\xi - \kappa)/2\}$, since either $r(\xi - \kappa)/2 > \tau_Q$ and $\bar{\tau} = \tau_Q$, or $\bar{\tau} = r(\xi - \kappa)/2$. This proves (2.19). ■

Proof of Theorem 2.7. Since the cube C_r is compact the quantity

$$(3.51) \quad a_r := \min\{Q[x - M] : x \in C_r\}$$

is a well-defined real number and we obviously get

$$(3.52) \quad F_{r,M}(b) = H_{r,M}^{a_r,b},$$

where $H_{r,M}^{a_r,b}$ is defined as in (2.3).

A careful analysis of the proof shows that Theorem 3.1 also holds for $a = a_r$, $r \geq r_0$. This, together with Lemma 4.7 for $\varepsilon = (4T)^{-1}$, $T \geq 1$, implies that there exist constants $c_j > 0$, $j = 1, 2, 3$, depending on Q and d only, and a constant $r_0 = r_0(Q, M, b) > 0$ such that, for any $r \geq r_0$ (cf. proof of Theorem 2.5):

- (1) $|\text{vol}_{\mathbb{Z}} F_{r,M}(b) - \text{vol } F_{r,M}(b)| \leq r^{d-2}(c_1(b - a_r + 1) + c_2)$.
- (2) $|\text{vol}_{\mathbb{Z}} F_{r,M}(b) - \text{vol } F_{r,M}(b)| \leq r^{d-2}(c_3(b - a_r)\tilde{\rho}(r, Q))$,

where

$$\tilde{\rho}(r, Q) := \inf_{T \geq 1} \left\{ T^{-1} + \frac{T^{-1/2}(1 + T^{-1})^{d-2}}{b - a_r} + \frac{(4T)^K}{b - a_r} (r^{2-d} + q_0^{-d/2}r^{2-d/2} + \bar{q}r^{2-d/2}(1 + \log r) + \rho(r, Q, T)) \right\},$$

and $\lim_{r \rightarrow \infty} \tilde{\rho}(r, Q) = 0$ if Q is irrational.

In the definition of $\tilde{\rho}(r, Q)$ the constant K is chosen according to (3.8).

Dividing these inequalities by the inequality in Lemma 4.3(2) for $\xi = 1$ completes the proof of Theorem 2.7. ■

4. Lemmas. Let $I = [a, b]$, $a, b \in \mathbb{R}$, and I_0 denote finite intervals. For $M \in \mathbb{R}^d$ we consider

$$(4.1) \quad H(r) := H(r, I_0, I, M) := \{x \in \mathbb{R}^d : r^{-1}|x|_\infty \in I_0, Q[x - M] \in I\}.$$

The diagonal matrix $D(Q)$ is defined by

$$(D(Q))_{i,i} := \sqrt{|q_i|}, \quad 1 \leq i \leq d.$$

LEMMA 4.1. *Let $I_0 = [0, \xi]$ and $\tau = \xi + |D(Q)M|/r$, $\sigma = q_0^{d/2}\xi - |D(Q)M|/r$. Then*

$$\text{vol } H(r) \ll_d (b-a)q_0^{-d/2}q^{(d-2)/2}\tau^{d-2}r^{d-2}.$$

If $\sigma > 0$ and $|a| + |b| \leq \sigma^2 r^2/5$ then

$$\text{vol } H(r) \gg_d (b-a)q^{-d/2}\sigma^{d-2}r^{d-2}.$$

Proof. See [BG99, p. 1023, Lemma 8.2], or [Els06, p. 24, Lemma 2.4.3]. ■

LEMMA 4.2. *Let $I_0 = [1 - \delta, 1 + \delta]$, $0 \leq \delta \leq 1/4$. Assume that r is so large that*

$$(4.2) \quad \varepsilon_1 := r^{-1}|D(Q)M| \leq q_0^{1/2}/4, \quad \varepsilon_2 := r^{-2}(|a| + |b|) \leq q_0/8.$$

Then

$$\text{vol } H(r) \ll_d (b-a)(\delta + q_0^{-1/2}\varepsilon_1 + 2q_0^{-1/2}\varepsilon_2)r^{d-2}q_0^{-d/2}q^{(d-2)/2}.$$

Proof. See [BG99, p. 1025, Lemma 8.3], or [Els06, p. 26, Lemma 2.4.4]. ■

Since a_r defined in (3.51) satisfies

$$(4.3) \quad |a_r|/r^2 \leq q$$

for r large enough, in the case $a = a_r$ we obtain the following lemma by slightly modifying the proof of Lemma 4.1 given in [BG99] resp. [Els06]. Using these modifications we also get an analogue of Lemma 4.2.

LEMMA 4.3. *Let $I_r := [a_r, b]$ and let $F(r) := H(r, I_0, I_r, M)$ be defined as in (4.1). There exist constants $C_{Q,1}, C_{Q,2}, C_{Q,3} \geq 1$ depending on d and Q only and a constant $r_0 = r_0(Q, M, b) \geq 1$ such that for $r \geq r_0$:*

- (1) $\text{vol } F(r) \leq (b - a_r)C_{Q,1}\xi^{d-2}r^{d-2}$ for $I_0 = [0, \xi]$.
- (2) $\text{vol } F(r) \geq (b - a_r)C_{Q,2}\xi^{d-2}r^{d-2}$ for $I_0 = [0, \xi]$.
- (3) $\text{vol } F(r) \leq (b - a_r)C_{Q,3}\delta r^{d-2}$ for $I_0 = [1 - \delta, 1 + \delta]$, $0 \leq \delta \leq 1/4$.

The constants $C_{Q,1}, C_{Q,2}, C_{Q,3}$ can be computed explicitly.

We want to estimate the error terms caused by the approximations of the (lattice point) volumes of the hyperbolic shell $H_{r,M}$. In the notation of (3.5)–(3.6), considering, for $\varepsilon > 0$,

$$\psi_{r,\pm\varepsilon}(x) = \exp\left\{-\frac{2}{r^2}Q_+[x]\right\}\chi_{\pm\varepsilon}\left(\frac{x}{r}\right)$$

and

$$\Delta_{\pm\varepsilon} = \left| \int_{\mathbb{R}^d} I_{H_{r,M}}(x)\psi_{r,\pm\varepsilon}(x) dx - \sum_{x \in \mathbb{Z}^d} I_{H_{r,M}}(x)\psi_{r,\pm\varepsilon}(x) \right|$$

(see (3.9) and (3.11)), we define additionally

$$(4.4) \quad v_\varepsilon := \text{vol}(H_{r,M} \cap \{x \in \mathbb{R}^d : r(1 - \varepsilon) \leq |x|_\infty \leq r(1 + \varepsilon)\})$$

and get the following estimate:

LEMMA 4.4. *For $0 < \varepsilon \leq 1/4$ there exists a constant $c = c(d) > 0$ such that*

$$(4.5) \quad |\text{vol}_{\mathbb{Z}} H_{r,M} - \text{vol } H_{r,M}| \leq \max\{\Delta_{-\varepsilon}, \Delta_\varepsilon\} + c(b - a)q_0^{-d/2}q^{(d-2)/2} \\ \times (\varepsilon + q_0^{-1/2}q^{1/2}|M|r^{-1} + 2q_0^{-1/2}(|a| + |b|)r^{-2})r^{d-2}.$$

Proof. Obviously, we can estimate

$$\text{vol}_{\mathbb{Z}} H_{r,M} \leq \sum_{x \in \mathbb{Z}^d} I_{H_{r,M}}(x)\psi_{r,\varepsilon}(x), \quad \text{vol } H_{r,M} \leq \int_{\mathbb{R}^d} I_{H_{r,M}}(x)\psi_{r,-\varepsilon}(x) dx + v_\varepsilon, \\ \text{vol}_{\mathbb{Z}} H_{r,M} \geq \sum_{x \in \mathbb{Z}^d} I_{H_{r,M}}(x)\psi_{r,-\varepsilon}(x), \quad \text{vol } H_{r,M} \geq \int_{\mathbb{R}^d} I_{H_{r,M}}(x)\psi_{r,\varepsilon}(x) dx - v_\varepsilon.$$

If $\text{vol}_{\mathbb{Z}} H_{r,M} - \text{vol } H_{r,M} \geq 0$ these estimates imply

$$|\text{vol}_{\mathbb{Z}} H_{r,M} - \text{vol } H_{r,M}| \leq \Delta_{+\varepsilon} + v_\varepsilon,$$

and otherwise we obtain

$$|\text{vol}_{\mathbb{Z}} H_{r,M} - \text{vol } H_{r,M}| \leq \Delta_{-\varepsilon} + v_\varepsilon.$$

Using Lemma 4.2 for $I_0 = [1 - \varepsilon, 1 + \varepsilon]$, since $|D(Q)M| \leq q^{1/2}|M|$ we get $v_\varepsilon \ll_d (b - a)(\varepsilon + q_0^{-1/2}q^{1/2}|M|r^{-1} + 2q_0^{-1/2}(|a| + |b|)r^{-2})r^{d-2}q_0^{-d/2}q^{(d-2)/2}$, which proves (4.5). ■

LEMMA 4.5. *Fix $a, b \in \mathbb{R}$, $w > 0$ and let $g_{a,b,w}$ be defined as in (3.3). Then:*

- (1) *There exist $a' \in [a - w, a + w]$ and $b' \in [b - w, b + w]$ such that*

$$\sum_{x \in \mathbb{Z}^d} (I_{[a,b]} - g_{a',b',w})(Q[x - M])\psi_{r,\pm\varepsilon}(x) = 0.$$

- (2) $\sup_{\substack{a' \in [a-w, a+w] \\ b' \in [b-w, b+w]}} \left| \int_{\mathbb{R}^d} (I_{[a,b]} - g_{a',b',w})(Q[x - M])\psi_{r,\pm\varepsilon}(x) dx \right| \\ \ll_d 8wq_0^{-d/2}q^{(d-2)/2} \left(1 + \varepsilon + q^{1/2} \frac{|M|}{r} \right)^{d-2} r^{d-2}.$

Proof. The sum in (1) is finite, since $\psi_{r,\pm\varepsilon}$ has bounded support. Hence, the map $G : (a', b') \mapsto \sum_{x \in \mathbb{Z}^d} (I_{[a,b]} - g_{a',b',w})(Q[x - M])\psi_{r,\pm\varepsilon}(x)$ is continuous and (1) follows by the intermediate value theorem.

For all $a' \in [a - w, a + w]$ and all $b' \in [b - w, b + w]$ we can estimate

$$(4.6) \quad |(I_{[a,b]} - g_{a',b',w})(Q[x - M])| \leq I_{[a-2w, a+2w] \cup [b-2w, b+2w]}(Q[x - M]).$$

This implies

$$\begin{aligned}
 (4.7) \quad & \sup_{a',b'} \left| \int_{\mathbb{R}^d} (I_{[a,b]} - g_{a',b',w})(Q[x - M])\psi_{r,\pm\varepsilon}(x) dx \right| \\
 & \leq \int_{I_{[a-2w,a+2w]} \cup I_{[b-2w,b+2w]}} (Q[x - M])\psi_{r,\pm\varepsilon}(x) dx \\
 & \leq \int (I_{[a-2w,a+2w]} + I_{[b-2w,b+2w]})(Q[x - M])I_{[0,r(1+\varepsilon)]}(|x|_\infty) dx,
 \end{aligned}$$

since $\psi_{r,\pm\varepsilon}(x) \leq I_{[0,r(1+\varepsilon)]}(|x|_\infty)$.

Using Lemma 4.1 with $I_0 = [0, 1 + \varepsilon]$, by (4.7) we get

$$\begin{aligned}
 (4.8) \quad & \sup_{a',b'} \left| \int_{\mathbb{R}^d} (I_{[a,b]} - g_{a',b',w})(Q[x - M])\psi_{r,\pm\varepsilon}(x) dx \right| \\
 & \ll_d 8wq_0^{-d/2} q^{(d-2)/2} (1 + \varepsilon + r^{-1}|D(Q)M|)^{d-2} r^{d-2} \\
 & \leq 8wq_0^{-d/2} q^{(d-2)/2} (1 + \varepsilon + r^{-1}q^{1/2}|M|)^{d-2} r^{d-2},
 \end{aligned}$$

which proves (2). ■

LEMMA 4.6. Consider $\Delta_{\pm\varepsilon}$, $\varepsilon > 0$, defined in (3.11). Then

$$\begin{aligned}
 \Delta_{\pm\varepsilon} \ll_d & \sup_{\substack{a' \in [a-w, a+w] \\ b' \in [b-w, b+w]}} |V_{w,\pm\varepsilon}^{\mathbb{R}}(r; a', b', M) - V_{w,\pm\varepsilon}^{\mathbb{Z}}(r; a', b', M)| \\
 & + 8wq_0^{-d/2} q^{(d-2)/2} \left(1 + \varepsilon + q^{1/2} \frac{|M|}{r} \right)^{d-2} r^{d-2},
 \end{aligned}$$

where $V_{w,\pm\varepsilon}^{\mathbb{R}}(r; a', b', M)$ and $V_{w,\pm\varepsilon}^{\mathbb{Z}}(r; a', b', M)$ are defined as in (3.6) and (3.5) respectively.

Proof. Using functions $g_{a,b,w}$ defined in (3.3) by the triangle inequality (recall the definition of $\psi_{r,\pm\varepsilon}$ in (3.9)) we obtain

$$\begin{aligned}
 (4.9) \quad \Delta_{\pm\varepsilon} & = \left| \int_{\mathbb{R}^d} I_{H_{r,M}}(x)\psi_{r,\pm\varepsilon}(x) dx - \sum_{x \in \mathbb{Z}^d} I_{H_{r,M}}(x)\psi_{r,\pm\varepsilon}(x) \right| \\
 & \leq \left| \int_{\mathbb{R}^d} (I_{[a,b]} - g_{a',b',w})(Q[x - M])\psi_{r,\pm\varepsilon}(x) dx \right| \\
 & \quad + |V_{w,\pm\varepsilon}^{\mathbb{R}}(r; a', b', M) - V_{w,\pm\varepsilon}^{\mathbb{Z}}(r; a', b', M)| \\
 & \quad + \left| \sum_{x \in \mathbb{Z}^d} (I_{[a,b]} - g_{a',b',w})(Q[x - M])\psi_{r,\pm\varepsilon}(x) \right|.
 \end{aligned}$$

Choosing a', b' according to Lemma 4.5(1) and estimating the first summand by taking the supremum, we obtain

$$(4.10) \quad \Delta_{\pm\varepsilon} \leq \sup_{\substack{a' \in [a-w, a+w] \\ b' \in [b-w, b+w]}} \int_{\mathbb{R}^d} | (I_{[a,b]} - g_{a',b',w})(Q[x - M])\psi_{r,\pm\varepsilon}(x) dx | \\ + \sup_{\substack{a' \in [a-w, a+w] \\ b' \in [b-w, b+w]}} |V_{w,\pm\varepsilon}^{\mathbb{R}}(r; a', b', M) - V_{w,\pm\varepsilon}^{\mathbb{Z}}(r; a', b', M)|.$$

The application of Lemma 4.5(2) completes the proof. ■

Repeating the proofs of Lemmas 4.4–4.6 in the case $a = a_r$ with the use of Lemma 4.3 instead of Lemmas 4.1 and 4.2 we get immediately

LEMMA 4.7. *Let $F_{r,M}(b)$ be defined as in (3.52). Then there exist constants $r_0 = r_0(Q, M, b) \geq 1$ and $c_{Q,1}, c_{Q,2} \geq 1$ depending on Q and d only such that for $w > 0$ and $0 < \varepsilon < 1/4$,*

$$|\text{vol}_{\mathbb{Z}} F_{r,M}(b) - \text{vol} F_{r,M}(b)| \leq (c_{Q,1}(b - a_r)\varepsilon + c_{Q,2}w(1 + \varepsilon)^{d-2})r^{d-2} \\ + \sup_{\substack{a' \in [a_r-w, a_r+w] \\ b' \in [b-w, b+w]}} |V_{w,\pm\varepsilon}^{\mathbb{R}}(r; a', b', M) - V_{w,\pm\varepsilon}^{\mathbb{Z}}(r; a', b', M)|,$$

where $V_{w,\pm\varepsilon}^{\mathbb{R}}(r; a', b', M)$ and $V_{w,\pm\varepsilon}^{\mathbb{Z}}(r; a', b', M)$ are defined as in (3.6) and (3.5) respectively.

LEMMA 4.8. *For any $\beta > 0$ and $T \in \mathbb{R}$,*

$$(4.11) \quad \frac{1}{2\pi i} \int_{\beta-i\infty}^{\beta+i\infty} \exp\{zT\} \frac{dz}{z^2} = \max\{T, 0\} = T_+.$$

Proof. Complement the interval $(\beta - i\infty, \beta + i\infty)$ by an infinite half circle in $\text{Re } z \geq 0$ (resp. $\text{Re } z \leq 0$) for $T < 0$ (resp. $T \geq 0$) and apply standard residue calculus. ■

LEMMA 4.9. *Let Ω be a symmetric, $d \times d$ complex matrix whose imaginary part is positive definite. Then*

$$\sum_{m \in \mathbb{Z}^d} \exp\{\pi i \Omega[m] + 2\pi i \langle m, v \rangle\} = (\det(\Omega/i))^{-1/2} \exp\{-\pi i \Omega^{-1}[v]\} \\ \times \sum_{n \in \mathbb{Z}^d} \exp\{-\pi i \Omega^{-1}[n] + 2\pi i \langle n, \Omega^{-1}v \rangle\}$$

and

$$\int_{\mathbb{R}^d} \exp\{\pi i \Omega[x] + 2\pi i \langle x, v \rangle\} dx = (\det(\Omega/i))^{-1/2} \exp\{-\pi i \Omega^{-1}[v]\},$$

where $\Omega^{-1}[x]$ denotes the quadratic form $\langle \Omega^{-1}x, x \rangle$, defined by the inverse operator $\Omega^{-1} : \mathbb{C}^d \rightarrow \mathbb{C}^d$ (which exists since Ω is an element of Siegel's upper half plane).

Proof. See [Mum83, p. 195 (5.6) and Lemma 5.8]. ■

COROLLARY 4.10. *Let $z \in \mathbb{C}^d$ with $\operatorname{Re} z > 0$, $v \in \mathbb{C}^d$, and let Ω be a positive definite, symmetric $d \times d$ matrix. Then*

$$\begin{aligned} \sum_{m \in \mathbb{Z}^d} \exp\{-z\Omega[m] + 2\pi i \langle m, v \rangle\} \\ = \left(\det \left(z \frac{\Omega}{\pi} \right) \right)^{-1/2} \sum_{n \in \mathbb{Z}^d} \exp\left\{ -\frac{\pi^2}{z} \Omega^{-1}[n + v] \right\}. \end{aligned}$$

Proof. Apply Lemma 4.9 to the matrix $(i/\pi)z\Omega$. ■

LEMMA 4.11. *Let $z = 1/r^2 + it$, $r > 0$, $t \in \mathbb{R}$ and $v \in \mathbb{C}^d$. Then*

$$\begin{aligned} (4.12) \quad & \sum_{m \in \mathbb{Z}^d} \exp\left\{ -\frac{2}{r^2} Q_+[m] - zQ[m] + 2\pi i \langle m, v \rangle \right\} \\ & = \det \left(\frac{1}{\pi} \left(\frac{2}{r^2} Q_+ + zQ \right) \right)^{-1/2} \exp\left\{ -\pi^2 \left(\frac{2}{r^2} Q_+ + zQ \right)^{-1} [v] \right\} \\ & \quad \times \sum_{n \in \mathbb{Z}^d} \exp\left\{ -\pi^2 \left(\frac{2}{r^2} Q_+ + zQ \right)^{-1} [n] - 2\pi^2 \left\langle \left(\frac{2}{r^2} Q_+ + zQ \right)^{-1} n, v \right\rangle \right\}, \end{aligned}$$

and

$$\begin{aligned} (4.13) \quad & \int_{\mathbb{R}^d} \exp\left\{ -\frac{2}{r^2} Q_+[x] - zQ[x] + 2\pi i \langle x, v \rangle \right\} dx \\ & = \det \left(\frac{1}{\pi} \left(\frac{2}{r^2} Q_+ + zQ \right) \right)^{-1/2} \exp\left\{ -\pi^2 \left(\frac{2}{r^2} Q_+ + zQ \right)^{-1} [v] \right\}, \end{aligned}$$

where $\left(\frac{2}{r^2} Q_+ + zQ \right)^{-1}[x]$ denotes the quadratic form $\langle \left(\frac{2}{r^2} Q_+ + zQ \right)^{-1} x, x \rangle$, defined by means of the positive definite operator $\left(\frac{2}{r^2} Q_+ + zQ \right)^{-1} : \mathbb{R}^d \rightarrow \mathbb{R}^d$.

Proof. For $\Omega := \frac{i}{\pi} \left(\frac{2}{r^2} Q_+ + zQ \right)$ and $z = 1/r^2 + it$, $t \in \mathbb{R}$, the imaginary part $\operatorname{Im} \Omega$ is positive definite. The application of Lemma 4.9 to Ω completes the proof. ■

LEMMA 4.12. *Let $\theta_v(z)$ and $\theta_{0,v}(z)$ denote the theta sum and theta integral in (3.16) and (3.17) respectively. Then there is a constant $c = c(Q) > 0$ such that for $r \geq r_0 = r_0(Q, M) \geq 1$ and $t \in \mathbb{R}$ with $|t| < 1/r$,*

$$\begin{aligned} |(\theta_v - \theta_{0,v})(r^{-2} + it)| \ll_d q_0^{-d/2} |r^{-2} + it|^{-d/2} \exp\{-c \operatorname{Re}((r^{-2} + it)^{-1})\} \\ + 2I_{(r,\infty)}(|v|). \end{aligned}$$

Proof. Using Lemma 4.11, by (3.16), (3.17) and the self-adjointness of the matrix $(\frac{2}{r^2}Q_+ + zQ)^{-1}$ we obtain

$$\begin{aligned}
 (4.14) \quad & (\theta_v - \theta_{0,v})(z) \\
 &= \exp\{-zQ[M]\} \det\left(\frac{1}{\pi} \Omega\right)^{-1/2} \exp\left\{-\pi^2 \Omega^{-1} \left[-\frac{\tilde{v}}{2\pi r}\right]\right\} \\
 &\quad \times \sum_{n \in \mathbb{Z}^d \setminus \{0\}} \exp\left\{-\pi^2 \Omega^{-1}[n] - 2\pi^2 \left\langle \Omega^{-1}n, -\frac{\tilde{v}}{2\pi r} \right\rangle\right\} \\
 &= \exp\{-zQ[M]\} \det\left(\frac{1}{\pi} \Omega\right)^{-1/2} \sum_{n \in \mathbb{Z}^d \setminus \{0\}} \exp\left\{-\Omega^{-1} \left[\pi n - \frac{\tilde{v}}{2r}\right]\right\},
 \end{aligned}$$

where $\Omega := \frac{2}{r^2}Q_+ + zQ$ and $\tilde{v} := 2r \operatorname{Im}(z)QM - v$. Note that for $z = r^{-2} + it$ and $t \leq 1/r$ there exists a constant $c_0 = c_0(Q, M) > 0$ such that $|2r \operatorname{Im}(z)QM| \leq c_0$ uniformly in r . Using

$$\det\left(\frac{1}{\pi} \Omega\right) = \frac{1}{\pi^d} \prod_{1 \leq j \leq d} \left(\frac{2}{r^2} |q_j| + zq_j\right) \quad \text{and} \quad \left|\frac{2}{r^2} |q_j| + zq_j\right| \geq |q_j| \cdot |r^{-2} + it|$$

for $z = r^{-2} + it$ and all $1 \leq j \leq d$, we have

$$(4.15) \quad \left| \det\left(\frac{1}{\pi} \Omega\right)^{-1/2} \right| \leq \pi^{d/2} q_0^{-d/2} |z|^{-d/2}.$$

Since Ω can be orthogonal diagonalized, the matrix $\operatorname{Re}(\Omega^{-1})$ has eigenvalues $\operatorname{Re}\left(\left(\frac{2}{r^2} |q_j| + zq_j\right)^{-1}\right)$, $1 \leq j \leq d$. For $t \leq 1/r$ we have

$$\operatorname{Re}\left(\left(\frac{2}{r^2} |q_j| + zq_j\right)^{-1}\right) \geq \frac{1}{|q_j|} \operatorname{Re}(z^{-1}) \geq \frac{1}{q} \operatorname{Re}(z^{-1}), \quad 1 \leq j \leq d.$$

Hence,

$$\begin{aligned}
 (4.16) \quad & \left| \exp\left\{-\pi^2 \Omega^{-1} \left[n - \frac{\tilde{v}}{2\pi r}\right]\right\} \right| = \exp\left\{-\operatorname{Re}\left(\Omega^{-1} \left[\pi n - \frac{\tilde{v}}{2r}\right]\right)\right\} \\
 &= \exp\left\{-\operatorname{Re}(\Omega^{-1}) \left[\pi n - \frac{\tilde{v}}{2r}\right]\right\} \\
 &\leq \exp\left\{-\frac{1}{q} \operatorname{Re}(z^{-1}) \left|\pi n - \frac{\tilde{v}}{2r}\right|^2\right\}.
 \end{aligned}$$

Using (4.14)–(4.16) we get

$$(4.17) \quad |(\theta_v - \theta_{0,v})(r^{-2} + it)| \ll_d \exp\left\{-\frac{1}{r^2} Q[M]\right\} q_0^{-d/2} |r^{-2} + it|^{-d/2} \\ \times \sum_{n \in \mathbb{Z}^d \setminus \{0\}} \exp\left\{-\frac{1}{q} \operatorname{Re}((r^{-2} + it)^{-1}) \left|\pi n - \frac{\tilde{v}}{2r}\right|^2\right\}.$$

For $|\tilde{v}| \leq \pi r$ we obtain

$$\exp\left\{-\frac{1}{q} \operatorname{Re}((r^{-2} + it)^{-1}) \left|\pi n - \frac{\tilde{v}}{2r}\right|^2\right\} \leq \exp\left\{-\frac{1}{q} \operatorname{Re}((r^{-2} + it)^{-1}) \frac{|\pi n|^2}{2}\right\}$$

and hence, for an appropriate constant $c = c(Q) > 0$,

$$(4.18) \quad \sum_{n \in \mathbb{Z}^d \setminus \{0\}} \exp\left\{-\frac{1}{q} \operatorname{Re}((r^{-2} + it)^{-1}) \left|\pi n - \frac{\tilde{v}}{2r}\right|^2\right\} \\ \leq \sum_{n \in \mathbb{Z}^d \setminus \{0\}} \exp\left\{-\frac{1}{q} \operatorname{Re}((r^{-2} + it)^{-1}) \frac{|\pi n|^2}{2}\right\} \\ \ll \exp\{-c \operatorname{Re}((r^{-2} + it)^{-1})\}.$$

For $|\tilde{v}| > \pi r$ set $\tilde{v} = L\pi r + w$ with $L \in \mathbb{Z}$, $|w| \leq \pi r$. Then $w = \tilde{v}'$ for $v' := v + L\pi r$. By (4.14) we have obviously $\theta_v = \theta_{v'}$, and therefore by (4.18) and (4.17) we get

$$(4.19) \quad |(\theta_v - \theta_{0,v})(r^{-2} + it)| \\ \leq |(\theta_{v'} - \theta_{0,v'})(r^{-2} + it)| + |(\theta_{0,v'} - \theta_{0,v})(r^{-2} + it)| \\ \ll_d \exp\left\{-\frac{1}{r^2} Q[M]\right\} q_0^{-d/2} |r^{-2} + it|^{-d/2} \exp\{-c \operatorname{Re}((r^{-2} + it)^{-1})\} \\ + |\theta_{0,v'}(r^{-2} + it)| + |\theta_{0,v}(r^{-2} + it)| \\ \leq \exp\left\{-\frac{1}{r^2} Q[M]\right\} q_0^{-d/2} |r^{-2} + it|^{-d/2} \exp\{-c \operatorname{Re}((r^{-2} + it)^{-1})\} + 2.$$

The result now follows by (4.17)–(4.19) for $r \geq r_0$, $r_0 \geq 1$ large enough, since $|\tilde{v}| > \pi r$ implies $|v| \geq \pi r - c_0(Q, M) \geq r$ for r large enough. ■

LEMMA 4.13. *Let $\theta_v(z)$ denote the theta function in (3.16) depending on Q and $v \in \mathbb{C}^d$. For $r \geq r_0 = r_0(Q, M) \geq 1$ and $t \in \mathbb{R}$,*

$$(4.20) \quad |\theta_v(r^{-2} + it)| \ll_d (\det \Omega)^{-1/4} r^{d/2} \psi(r, t)^{1/2},$$

where

$$\psi(r, t) = \sum_{m, n \in \mathbb{Z}^d} \exp\left\{-\frac{r^2}{2} \Omega^{-1} [\pi m - 2tQn] - \frac{2}{r^2} \Omega[n]\right\},$$

with $\Omega := 2Q_+ + Q$.

Note that the right hand side of this inequality is independent of $v \in \mathbb{C}^d$.

Proof. For any $x, y \in \mathbb{R}^d$ we have

$$(4.21) \quad 2(\Omega[x] + \Omega[y]) = \Omega[x + y] + \Omega[x - y],$$

$$(4.22) \quad \langle \Omega(x + y), x - y \rangle = \Omega[x] - Q[y].$$

Rearranging $\theta_v(z)\overline{\theta_v(z)}$ and using (4.22), we would like to use $m + n$ and $m - n$ as new summation variables on a lattice. But both vectors have the same parity, i.e., $m + n \equiv m - n \pmod{2}$. Since they are dependent one has to consider the 2^d sublattices indexed by $\alpha = (\alpha_1, \dots, \alpha_d)$ with $\alpha_j = 0, 1$, for $1 \leq j \leq d$:

$$\mathbb{Z}_\alpha^d := \{m \in \mathbb{Z}^d : m \equiv \alpha \pmod{2}\},$$

where, for $m = (m_1, \dots, m_d)$, $m \equiv \alpha \pmod{2}$ means $m_j \equiv \alpha_j \pmod{2}$ for $1 \leq j \leq d$. Thus writing

$$\theta_{v,\alpha}(z) := \sum_{m \in \mathbb{Z}_\alpha^d} \exp\left\{-\frac{2}{r^2} Q_+[m] - zQ[m] - i\left\langle m, \frac{v}{r} - 2tQM \right\rangle\right\},$$

we obtain $\theta_v(z) = \exp\{-zQ[M]\} \sum_\alpha \theta_{v,\alpha}(z)$ and hence by the Cauchy-Schwarz inequality

$$(4.23) \quad |\theta_v(z)|^2 \leq 2^d \exp\left\{-\frac{2}{r^2} Q[M]\right\} \sum_\alpha |\theta_{v,\alpha}(z)|^2.$$

Using (4.22) and the absolute convergence of $\theta_\alpha(z)$, we may rewrite the quantity $\theta_{v,\alpha}(z)\overline{\theta_{v,\alpha}(z)}$ for $z = 1/r^2 + it$ and $\tilde{v} := v - 2trQM$ as

$$(4.24) \quad \begin{aligned} & \theta_{v,\alpha}(z)\overline{\theta_{v,\alpha}(z)} \\ &= \sum_{m,n \in \mathbb{Z}_\alpha^d} \exp\left\{-\frac{1}{r^2} (\Omega[m] + \Omega[n]) - it(Q[m] - Q[n]) - i\left\langle m - n, \frac{\tilde{v}}{r} \right\rangle\right\} \\ &= \sum_{m,n \in \mathbb{Z}_\alpha^d} \exp\left\{-\frac{2}{r^2} (\Omega[\bar{m}] + \Omega[\bar{n}]) - 2i\left\langle 2tQ\bar{m} + \frac{\tilde{v}}{r}, \bar{n} \right\rangle\right\} \end{aligned}$$

where $\bar{m} = (m + n)/2$ and $\bar{n} = (m - n)/2$. Note that the map

$$H : \bigcup_\alpha \mathbb{Z}_\alpha^d \times \mathbb{Z}_\alpha^d \rightarrow \mathbb{Z}^d \times \mathbb{Z}^d, \quad (m, n) \mapsto ((m + n)/2, (m - n)/2),$$

is a bijection. Therefore by (4.23) we get

$$\begin{aligned}
 (4.25) \quad & \exp\left\{\frac{2}{r^2} Q[M]\right\} |\theta_v(z)|^2 \\
 & \ll_d \sum_{\alpha \in \{0,1\}^d} \sum_{\bar{m}, \bar{n} \in \mathbb{Z}_\alpha^d} \exp\left\{-\frac{2}{r^2} (\Omega[\bar{m}] + \Omega[\bar{n}]) - 2i \left\langle 2tQ\bar{m} + \frac{\tilde{v}}{r}, \bar{n} \right\rangle\right\} \\
 & = \sum_{\bar{m}, \bar{n} \in \mathbb{Z}^d} \exp\left\{-\frac{2}{r^2} (\Omega[\bar{m}] + \Omega[\bar{n}]) - 2i \left\langle 2tQ\bar{m} + \frac{\tilde{v}}{r}, \bar{n} \right\rangle\right\}.
 \end{aligned}$$

In this double sum fix \bar{n} and sum over $\bar{m} \in \mathbb{Z}^d$ first. Using Corollary 4.10 for $z = 2/r^2$, we get for $\delta := (\det(\frac{2}{\pi r^2} \Omega))^{-1/2}$, by the symmetry of Q ,

$$\begin{aligned}
 \theta_v(z, \bar{n}) & := \sum_{\bar{m} \in \mathbb{Z}^d} \exp\left\{-\frac{2}{r^2} (\Omega[\bar{m}] + \Omega[\bar{n}]) - 2i \left\langle 2tQ\bar{m} + \frac{\tilde{v}}{r}, \bar{n} \right\rangle\right\} \\
 & = \delta \sum_{m \in \mathbb{Z}^d} \exp\left\{-\frac{r^2}{2} \Omega^{-1}[\pi m - 2tQ\bar{n}] - \frac{2}{r^2} \Omega[\bar{n}] - 2i \left\langle \frac{\tilde{v}}{r}, \bar{n} \right\rangle\right\}.
 \end{aligned}$$

Thus,

$$(4.26) \quad |\theta_v(z, \bar{n})| \leq \delta \sum_{m \in \mathbb{Z}^d} \exp\left\{-\frac{r^2}{2} \Omega^{-1}[\pi m - 2tQ\bar{n}] - \frac{2}{r^2} \Omega[\bar{n}]\right\}.$$

Hence, by (4.25) and (4.26) we obtain

$$\begin{aligned}
 |\theta_v(z)|^2 & \ll_d \exp\left\{-\frac{2}{r^2} Q[M]\right\} (\det \Omega)^{-1/2} r^d \\
 & \quad \times \sum_{m, \bar{n} \in \mathbb{Z}^d} \exp\left\{-\frac{r^2}{2} \Omega^{-1}[\pi m - 2tQ\bar{n}] - \frac{2}{r^2} \Omega[\bar{n}]\right\},
 \end{aligned}$$

which proves Lemma 4.13 for $r > r_0 = r_0(Q, M) := |Q[M]|^{1/2} + 1$. ■

In the following we shall use some facts from the geometry of numbers (see [Dav58]).

Let $F : \mathbb{R}^d \rightarrow [0, \infty)$ denote a norm on \mathbb{R}^d , that is, $F(\alpha x) = |\alpha|F(x)$ for $\alpha \in \mathbb{R}$, and $F(x + y) \leq F(x) + F(y)$. The *successive minima* $M_1 \leq \dots \leq M_d$ of F with respect to the lattice \mathbb{Z}^d are defined as follows: Let $M_1 = \inf\{F(m) : m \neq 0, m \in \mathbb{Z}^d\}$ and define M_k as the infimum of $\lambda > 0$ such that the set $\{m \in \mathbb{Z}^d : F(m) < \lambda\}$ contains k linearly independent vectors. It is easy to see that these infima are attained, that is, there exist linearly independent vectors $a_1, \dots, a_d \in \mathbb{Z}^d$ such that $F(a_j) = M_j$.

LEMMA 4.14. *Let $L_j(x) = \sum_{k=1}^d q_{jk}x_k$, $1 \leq j \leq d$, denote linear forms on \mathbb{R}^d such that $q_{jk} = q_{kj}$, $j, k = 1, \dots, d$. Assume that $r \geq 1$ and let $\|v\|$*

denote the distance of the number v to the nearest integer. Then the number of $m = (m_1, \dots, m_d) \in \mathbb{Z}^d$ such that

$$\|L_j(m)\| < r^{-1}, \quad |m_j| < r, \quad \text{for all } 1 \leq j \leq d,$$

is bounded from above by $c_d(M_1 \cdots M_d)^{-1}$, where $c_d > 0$ denotes a constant depending on d only, $M_1 \leq \cdots \leq M_d$ are the first d of the $2d$ successive minima $M_1 \leq \cdots \leq M_{2d}$ of the norm $F : \mathbb{R}^{2d} \rightarrow [0, \infty)$ defined for $y = (x, \bar{x}) \in \mathbb{R}^{2d}$, $x, \bar{x} \in \mathbb{R}^d$, $\bar{x} = (\bar{x}_1, \dots, \bar{x}_d)$, as

$$(4.27) \quad F(y) := \max\{r|L_1(x) - \bar{x}_1|, \dots, r|L_d(x) - \bar{x}_d|, r^{-1}|x|_\infty\}.$$

Moreover,

$$(4.28) \quad \frac{1}{2d} \leq M_k M_{2d+1-k} \leq (2d)^{2d-1}, \quad 1 \leq k \leq 2d.$$

Proof. See [Dav58, (20), p. 113, Lemma 3]. ■

Note that for some constant, say $c(d) > 0$, depending on d only,

$$(4.29) \quad r^{-1} \leq M_1 \leq \cdots \leq M_d \leq c(d),$$

where the first inequality is obvious by $F(m, \bar{m}) \geq r^{-1}|m|_\infty$. If here $m = 0$ then $\bar{m} \neq 0$ and $F(m, \bar{m}) = r|\bar{m}|_\infty \geq r^{-1}|\bar{m}|_\infty \geq r^{-1}$. Finally, $M_d \ll_d 1$ follows from (4.28) for $k = d$.

In the following we shall consider linear forms

$$(4.30) \quad L_j(x) = \sum_{k=1}^d tq_{jk}x_k, \quad 1 \leq j \leq d,$$

where $Q = (q_{ij})$, $i, j = 1, \dots, d$, and $t \in \mathbb{R}$ is arbitrary. We denote the corresponding successive minima of the norm $F(\cdot)$ defined by (4.27) and (4.30) for fixed t by $M_{j,t}$, $j = 1, \dots, d$. Thus, we can write

$$(4.31) \quad M_{j,t} = |L(m, n, t)|_\infty$$

for some $m, n \in \mathbb{Z}^d$, where

$$(4.32) \quad \begin{aligned} L(m, n, t) &= (r(m_1 - t(Qn)_1), \dots, r(m_d - t(Qn)_d), r^{-1}n_1, \dots, r^{-1}n_d). \end{aligned}$$

It is easy to see from the definition that

$$(4.33) \quad M_{j,t} = M_{j,-t}, \quad j = 1, \dots, d, \quad t \in \mathbb{R}.$$

LEMMA 4.15. *Let $r \geq 1$. Then*

$$|\theta(r^{-2} + it\pi/2)| \ll_d q_0^{-3d/4} r^{d/2} (M_{1,t} \cdots M_{d,t})^{-1/2}.$$

Proof. By Lemma 4.13 we need to estimate the theta series $\psi(r, t\pi/2)$. Since the matrix $\Omega = 2Q_+ + Q$ is positive definite we may use the inequalities

$\Omega^{-1}[x] \geq \frac{1}{3q}|x|_\infty^2$ and $\Omega[x] \geq q_0|x|_\infty^2$ to get, with $c_Q = \min\{\frac{\pi^2}{6q}, 2q_0\}$,

$$(4.34) \quad \psi(r, t\pi/2) \ll_d \sum_{m,n \in \mathbb{Z}^d} \exp\{-c_Q|L(m, n, t)|_\infty^2\},$$

where $L(m, n, t)$ is defined in (4.32). Let

$$H := \{(m, n) \in \mathbb{Z}^{2d} : |L(m, n, t)|_\infty < 1\}.$$

Now, Lemma 4.14 may be restated for the forms (4.30) as

$$(4.35) \quad \#H \ll_d (M_{1,t} \cdots M_{d,t})^{-1}.$$

In order to bound $\psi(r, t\pi/2)$, for $k := (k_1, \dots, k_{2d}) \in \mathbb{Z}^{2d}$ we introduce the sets

$$B_k := [k_1 - 1/2, k_1 + 1/2) \times \cdots \times [k_{2d} - 1/2, k_{2d} + 1/2),$$

$$H_k := \{(m, n) \in \mathbb{Z}^{2d} : L(m, n, t) \in B_k\}$$

such that $\mathbb{R}^{2d} = \bigcup_k B_k$. For any fixed $(m^*, n^*) \in H_k$ we have

$$(m - m^*, n - n^*) \in H \quad \text{for any } (m, n) \in H_k.$$

Hence, for any $k \in \mathbb{Z}^{2d}$,

$$(4.36) \quad \#H_k \leq \#H \ll_d (M_{1,t} \cdots M_{d,t})^{-1}.$$

Since $x \in B_k$ implies $|x|_\infty \geq |k|_\infty/2$, by (4.34) and (4.36) we obtain

$$\begin{aligned} \psi(r, t\pi/2) &\ll_d \#H_0 + \sum_{k \in \mathbb{Z}^{2d} \setminus 0} \sum_{m,n \in \mathbb{Z}^{2d}} I_{\{L(m,n,t) \in B_k\}} \exp\{-c_Q|k|_\infty^2/4\} \\ &\ll_d (M_{1,t} \cdots M_{d,t})^{-1} \sum_{k \in \mathbb{Z}^{2d}} \exp\{-c_Q|k|_\infty^2/4\} \\ &\ll_d (M_{1,t} \cdots M_{d,t})^{-1} (c_Q^{-1/2} + 1)^{2d}, \end{aligned}$$

using bounds similar to (4.18). Some simple bounds together with Lemma 4.13 finally conclude the proof of Lemma 4.15. ■

In the following we consider an arbitrary, real, symmetric, non-degenerate $d^* \times d^*$ matrix Q^* . The norm on \mathbb{R}^{d^*} , associated to (4.32), and the associated successive minima are denoted by $|L^*|_\infty$ and $M_{j,t}^*$, $1 \leq j \leq d^*$, respectively.

LEMMA 4.16. *Let $(m, n), (m', n') \in \mathbb{Z}^{2d^*} \setminus 0$, $t, t' \in \mathbb{R}$ and $r \geq 1$. Let $M := |L^*(m, n, t)|_\infty$ and $M' := |L^*(m', n', t')|_\infty$. Assume that $\langle Q^*n, n' \rangle > 0$ and*

$$(4.37) \quad \max\{M, M'\} \leq (4d^*)^{-1/2}.$$

Set

$$(4.38) \quad \Delta = \Delta(m, n; m', n') := |\langle n', m \rangle - \langle m', n \rangle|.$$

Then

$$(4.39) \quad \begin{aligned} \text{(i)} \quad \Delta = 0 &\Rightarrow |t - t'| \leq \frac{(d^*)^{1/2} \max\{M, M'\}(|n| + |n'|)}{r \langle Q^*n, n' \rangle}, \\ \text{(ii)} \quad \Delta \neq 0 &\Rightarrow |t - t'| \geq \langle Q^*n, n' \rangle^{-1/2}. \end{aligned}$$

In particular, if we assume $n = n'$ and (4.37) then alternative (i) in (4.39) holds.

Furthermore, assuming $(m, n) \in \mathbb{Z}^{2d^*} \setminus 0$ and $M = |L^*(m, n, t)|_\infty \leq (4d^*)^{-1/2}$ we have either

$$(4.40) \quad \text{(i)} \quad |t| \leq \frac{2d^*M|n|}{r|Q^*n|} \quad \text{or} \quad \text{(ii)} \quad |t| \geq \frac{1}{2|Q^*n|}.$$

This means t, t' resp. $t, 0$ have to be either “near” to each other or “far” apart.

Proof. See [Göt04, p. 217, Lemma 3.6] or [Els06, p. 38, Lemma 2.4.17]. ■

The application of Lemma 4.16 to $Q^* = Q$ yields the following

COROLLARY 4.17. *Let $r \geq 1$ and $d \geq 4$. Then*

$$(4.41) \quad M_{1,t} \cdots M_{d,t} \geq d^{-d} \left(\min \left\{ \frac{q_0|t|r}{2}, \frac{1}{q|t|r} \right\} \right)^d.$$

Proof. Since $|Qn| = |Q_+n|$ we have $|Qn| \geq q_0|n|$, and $|n| \geq q^{-1}|Qn|$. In the case where $M_{j,t} \leq (4d)^{-1/2}$, from (4.40), $|n|_\infty \leq rM_{j,t}$ and $2d^{1/2} \leq d$ we obtain either

$$(4.42) \quad \begin{aligned} \text{(i)} \quad &|t|rd^{-1}q_0 \leq |t|rd^{-1} \frac{|Qn|}{|n|} \leq 2M_{j,t}, \quad \text{or} \\ \text{(ii)} \quad &\frac{1}{|t|} \leq 2|Qn| \leq 2q|n| \leq 2d^{1/2}q|n|_\infty \leq qdrM_{j,t}, \end{aligned}$$

for appropriate $(m, n) \in \mathbb{Z}^{2d}$ depending on j such that $M_{j,t} = |L(m, n, t)|_\infty$. Note that if $M_{j,t} \geq (4d)^{-1/2}$, then $M_{j,t} \geq d^{-1}$ since $d \geq 4$. Combined with (4.42), this proves Corollary 4.17 since

$$\min \left\{ \frac{q_0|t|r}{2}, \frac{1}{q|t|r} \right\} \leq 1$$

(recall that $q_0 \leq q$). ■

In the following two lemmas we will additionally assume that the matrix Q^* is positive definite. The smallest and largest eigenvalues of Q^* are denoted by q_0^* and q^* respectively.

LEMMA 4.18. *Let $0 < \kappa < \xi < \infty$. For $g \in C^1[\kappa, \xi]$ such that $g \geq 0$ and $g' \leq 0$ on $[\kappa, \xi]$, define*

$$(4.43) \quad H_{\kappa, \xi}(\tau) := H_{\kappa, \xi, Q^*}(\tau) := \int_{\kappa}^{\xi} I_{\{M_{1,t}^* \leq \tau\}} g(t) dt.$$

Then, for all

$$(4.44) \quad \kappa > (q_0^* r)^{-1}, \quad r^{-1} \leq \tau \leq (2d^*)^{-1}$$

with $\kappa(\tau r) \leq \xi$, where $\kappa(v) = \max\{\kappa, (2q^* v d^{1/2})^{-1}\}$, we have

$$(4.45) \quad H_{\kappa, \xi}(\tau) \ll_{d^*} \bar{H}_{\kappa, \xi}(\tau) := \frac{q^*}{q_0^*} \tau^2 \int_{\kappa(\tau r)}^{\xi} g(t) dt + \frac{1}{q_0^*} \frac{\tau}{r} g(\kappa(\tau r)).$$

In the case where $\kappa(\tau r) > \xi$, we have $H_{\kappa, \xi}(\tau) = 0$.

Proof. See [Göt04, p. 219, Lemma 3.8]. ■

For indicator functions g Lemma 4.18 reads as follows.

LEMMA 4.19. *Let λ denote the Lebesgue measure. There exists a constant $c(d^*)$ depending on d^* only such that for any $r \geq 1$, $\tau > 0$ and $\xi > \kappa$,*

$$I(\tau) := \lambda\{t \in [\kappa, \xi] : M_{1,t}^* \leq \tau\} \leq c(d^*) \left(\frac{q^*}{q_0^*} \tau^2 (\xi - \kappa) + \frac{1}{q_0^*} \tau r^{-1} \right).$$

Proof. See [Göt04, p. 222, Lemma 3.9]. ■

We now return to a general (not necessarily positive definite) non-degenerate, symmetric, real $d \times d$ matrix Q , to the corresponding norm $|L|_\infty$ (see (4.32)) and the associated successive minima $M_{j,t}$ (see (4.31)).

In the following we assume that Q is a *block-type* matrix, that is, there exist positive definite matrices $Q^+ \in \text{GL}(\mathbb{R}^{d^+})$ and $Q^- \in \text{GL}(\mathbb{R}^{d^-})$ with $d^+ + d^- \geq 5$ such that

$$Q = \begin{pmatrix} Q^+ & 0 \\ 0 & -Q^- \end{pmatrix}.$$

We denote the corresponding successive minima of the norm $F^\pm(\cdot)$, defined by the analogue of (4.27) and (4.30) for Q^\pm , for a fixed t , by $M_{j,t}^\pm$, $j = 1, \dots, d^\pm$. Thus, we can write

$$(4.46) \quad M_{j,t}^\pm = |L^\pm(m, n, t)|_\infty$$

for some $m, n \in \mathbb{Z}^{d^\pm}$, where

$$L^\pm(m, n, t) = \left(r(m_1 - t(Q^\pm n)_1), \dots, r(m_{d^\pm} - t(Q^\pm n)_{d^\pm}), \frac{1}{r} n_1, \dots, \frac{1}{r} n_{d^\pm} \right).$$

As in (4.33) we have

$$(4.47) \quad M_{j,t}^\pm = M_{j,-t}^\pm, \quad j = 1, \dots, d^\pm, t \in \mathbb{R}.$$

In this special case there is a simple relation between the first successive minimum of Q and those of Q^+ and Q^- .

LEMMA 4.20. For $t \in \mathbb{R}$,

$$(4.48) \quad M_{1,t} \geq \min\{M_{1,t}^+, M_{1,t}^-\}.$$

In particular, for $\tau \in \mathbb{R}$,

$$I_{\{M_{1,t} \leq \tau\}} \leq I_{\{M_{1,t}^+ \leq \tau\}} + I_{\{M_{1,t}^- \leq \tau\}}.$$

Proof. Choose $(m, n) = \left(\begin{pmatrix} m_+ \\ m_- \end{pmatrix}, \begin{pmatrix} n_+ \\ n_- \end{pmatrix}\right) \in \mathbb{Z}^d \setminus 0$ such that $M_{1,t} = |L(m, n, t)|_\infty$. It is easy to see that

$$M_{1,t} = |L(m, n, t)|_\infty = \max\{|L^+(m_+, n_+, t)|_\infty, |L^-(m_-, n_-, -t)|_\infty\}.$$

Since $(m, n) \neq 0$, it follows that $(m_+, n_+) \neq 0$ or $(m_-, n_-) \neq 0$ and hence by (4.47),

$$|L^+(m_+, n_+, t)|_\infty \geq M_{1,t}^+ \quad \text{or} \quad |L^-(m_-, n_-, -t)|_\infty \geq M_{1,-t}^- = M_{1,t}^-.$$

This proves (4.48). ■

COROLLARY 4.21. Again, let λ denote the Lebesgue measure. Then there exists a constant $c = c(d) > 1$ depending on d only such that for any $r \geq 1$, $\tau > 0$ and $\xi > \kappa$,

$$I(\tau) := \lambda\{t \in [\kappa, \xi] : M_{1,t} \leq \tau\} \leq c \left(\frac{q}{q_0} \tau^2 (\xi - \kappa) + \frac{1}{q_0} \tau r^{-1} \right).$$

Proof. Using Lemmas 4.19 and 4.20 we obtain

$$\begin{aligned} I(\tau) &\leq \int (I_{\{M_{1,t}^+ \leq \tau\}} + I_{\{M_{1,t}^- \leq \tau\}}) \lambda(dt) \\ &\leq (c(d^+) + c(d^-)) \left(\frac{q}{q_0} \tau^2 (\xi - \kappa) + \frac{1}{q_0} \tau r^{-1} \right), \end{aligned}$$

where we have used the fact that q (resp. q_0) is greater (resp. smaller) than the largest (resp. smallest) eigenvalue of Q^+ and Q^- . Taking $c := \max_{d^+, d^- \in \mathbb{N}, d^+ + d^- = d} (c(d^+) + c(d^-))$ completes the proof. ■

LEMMA 4.22. Let $M(t) = M_{1,t} \cdots M_{d,t}$, $\gamma = \gamma(\kappa, \xi) = r^d \inf_{\kappa \leq t \leq \xi} M(t)$. Set

$$D = \max\{(2d)^{-d} r^d, \gamma\} \quad \text{and} \quad G(\kappa, \xi) = \int_{\kappa}^{\xi} g(t) dt$$

for $0 < \kappa < \xi \leq \infty$ and let $g(t)$ and $\kappa(v)$ be as in Lemma 4.18. For $\kappa > \xi$ define $G(\kappa, \xi) = 0$. Then

$$\begin{aligned}
 (4.49) \quad I_{\kappa,\xi} &:= \int_{\kappa}^{\xi} \frac{g(t)}{M(t)^{1/2}} dt \\
 &\ll_d q_0^{-1} r^{d/2-2} \int_{\gamma}^D v^{-1/2+1/d} (qv^{1/d} G(\kappa(v^{1/d}), \xi) + g(\kappa(v^{1/d}))) \frac{dv}{v} \\
 &\quad + G(\kappa, \xi).
 \end{aligned}$$

Proof. We generalize the proof in [Göt04, p. 222, Lemma 3.10]. Write $\bar{\gamma} := \inf_{\kappa \leq t \leq \xi} M(t)$ and $c_d = (2d)^{-d}$. If $\bar{\gamma} \geq c_d$, then $I_{\kappa,\xi} \ll_d G(\kappa, \xi)$ and (4.49) is obvious. In the case

$$(4.50) \quad \bar{\gamma} < c_d$$

we define

$$(4.51) \quad J_{\kappa,\xi}(v) := \int_{\kappa}^{\xi} g(t) I_{\{M(t) \leq v\}} dt$$

for $0 < \kappa < \xi$. Since $M_{j,t} \leq M_{d,t} \ll_d 1$, for $j = 1, \dots, d$, by Lemma 4.14, there exists a constant \bar{M} depending on d only such that $M(t) \leq \bar{M}$ for all t . Therefore we have, for all $t \in [\kappa, \xi]$,

$$M(t)^{-1/2} = \int_{\bar{\gamma}}^{\bar{M}} \varepsilon^{-1/2} dI_{\{M(t) \leq \varepsilon\}}.$$

Hence, Fubini's theorem implies

$$I_{\kappa,\xi} = \int_{\bar{\gamma}}^{\bar{M}} \varepsilon^{-1/2} dJ_{\kappa,\xi}(\varepsilon).$$

Splitting the integral $I_{\kappa,\xi}$ into the part where $\varepsilon \leq c_d$ and its complement, we obtain

$$I_{\kappa,\xi} \leq \int_{\bar{\gamma}}^{c_d} \varepsilon^{-1/2} dJ_{\kappa,\xi}(\varepsilon) + c_d^{-1/2} \int_{\kappa}^{\xi} g(t) dt.$$

Using partial integration we have, by (4.50) and the definition of $\bar{\gamma}$,

$$\begin{aligned}
 (4.52) \quad I_{\kappa,\xi} &\leq c_d^{-1/2} \underbrace{J_{\kappa,\xi}(c_d)}_{=G(\kappa,\xi)} - \bar{\gamma}^{-1/2} \underbrace{J_{\kappa,\xi}(\bar{\gamma})}_{=0} + \frac{1}{2} \int_{\bar{\gamma}}^{c_d} \varepsilon^{-3/2} J_{a,b}(\varepsilon) d\varepsilon \\
 &\quad + c_d^{-1/2} G(\kappa, \xi) \\
 &= \frac{1}{2} \int_{\bar{\gamma}}^{c_d} \varepsilon^{-3/2} J_{a,b}(\varepsilon) d\varepsilon + 2c_d^{-1/2} G(\kappa, \xi).
 \end{aligned}$$

Furthermore, $M(t) \geq (M_{1,t})^d \geq r^{-d}$ (see (4.29)) implies, together with Lemma 4.20,

$$\begin{aligned}
 (4.53) \quad J_{\kappa,\xi}(\varepsilon) &\leq \int_{\kappa}^{\xi} g(t) I_{\{(M_{1,t})^d \leq \varepsilon\}} dt = \int_{\kappa}^{\xi} g(t) I_{\{M_{1,t} \leq \varepsilon^{1/d}\}} dt \\
 &\leq \int_{\kappa}^{\xi} g(t) I_{\{M_{1,t}^+ \leq \varepsilon^{1/d}\}} dt + \int_{\kappa}^{\xi} g(t) I_{\{M_{1,t}^- \leq \varepsilon^{1/d}\}} dt \\
 &= H_{\kappa,\xi,Q^+}(\varepsilon^{1/d}) + H_{\kappa,\xi,Q^-}(\varepsilon^{1/d}),
 \end{aligned}$$

where H_{κ,ξ,Q^\pm} is defined as in (4.43) in Lemma 4.18. The smallest and largest eigenvalues of Q^\pm are denoted by q_0^\pm and q^\pm , respectively.

Since $r^{-d} \leq \varepsilon \leq c_d$ and hence $r^{-1} \leq \varepsilon^{1/d} \leq (2d)^{-1} \leq (2d^\pm)^{-1}$, Lemma 4.18 can be applied and by changing the variable $v = r^d \varepsilon$ we obtain

$$\begin{aligned}
 &\int_{\bar{\gamma}}^{c_d} \varepsilon^{-3/2} H_{\kappa,\xi,Q^\pm}(\varepsilon^{1/d}) d\varepsilon \\
 &\leq c(d^\pm) \int_{\bar{\gamma}}^{c_d} \varepsilon^{-3/2} \left(\frac{q^\pm}{q_0^\pm} \varepsilon^{2/d} G(\kappa(\varepsilon^{1/d} r), \xi) + \frac{1}{q_0^\pm} \frac{\varepsilon^{1/d}}{r} g(\kappa(\varepsilon^{1/d} r)) \right) d\varepsilon \\
 &\leq c(d^\pm) \int_{\gamma}^D r^{3d/2-2} v^{-3/2+1/d} \left(\frac{q^\pm}{q_0^\pm} v^{1/d} G(\kappa(v^{1/d}), \xi) + \frac{1}{q_0^\pm} g(\kappa(v^{1/d})) \right) r^{-d} dv \\
 &= r^{d/2-2} c(d^\pm) \int_{\gamma}^D v^{-1/2+1/d} \left(\frac{q^\pm}{q_0^\pm} v^{1/d} G(\kappa(v^{1/d}), \xi) + \frac{1}{q_0^\pm} g(\kappa(v^{1/d})) \right) \frac{dv}{v}.
 \end{aligned}$$

Analyzing the proof of Lemma 4.18 we may assume that the constant $c(d)$ is increasing in d . Since $q_0 = \min\{q_0^+, q_0^-\}$ and $q = \max\{q^+, q^-\}$, we have

$$\begin{aligned}
 (4.54) \quad &\int_{\bar{\gamma}}^{c_d} \varepsilon^{-3/2} H_{\kappa,\xi,Q^\pm}(\varepsilon^{1/d}) d\varepsilon \\
 &\leq r^{d/2-2} c(d) \int_{\gamma}^D v^{-1/2+1/d} \left(\frac{q}{q_0} v^{1/d} G(\kappa(v^{1/d}), \xi) + \frac{1}{q_0} g(\kappa(v^{1/d})) \right) \frac{dv}{v}.
 \end{aligned}$$

Thus we conclude, by using (4.52)–(4.54), that

$$\begin{aligned}
 I_{\kappa,\xi} &\ll_d r^{d/2-2} \int_{\gamma}^D v^{-1/2+1/d} \left(\frac{q}{q_0} v^{1/d} G(\kappa(v^{1/d}), \xi) + \frac{1}{q_0} g(\kappa(v^{1/d})) \right) \frac{dv}{v} \\
 &\quad + G(\kappa, \xi),
 \end{aligned}$$

which proves (4.49). This completes the proof of Lemma 4.22. ■

LEMMA 4.23. *Let $0 < \kappa < \xi < \infty$ and assume that Q is irrational. Then*

$$\lim_{r \rightarrow \infty} \inf_{t \in [\kappa, \xi]} (rM_{1,t}) \cdots (rM_{d,t}) = \infty.$$

Proof. See [Göt04, p. 224, Lemma 3.11] or [Els06, p. 47, Lemma 2.4.24]. ■

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