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A further discussion of the Hausdorff dimension in Engel expansions

by

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1. Introduction. In [GA], the Engel expansion transformation of $x \in (0,1]$ was defined as $Tx = d_1(x)x - 1$, where $d_1(x) = [1/x] + 1$, and the partial quotients $\{d_n(x)\}_{n\ge 1}$ of the Engel expansion were defined by $d_n(x) = d_1(T^{n-1}(x))$. By the algorithm, one has $d_{j+1}(x) \ge d_j(x)$ for $j \ge 1$, and any $x \in (0,1]$ can be expanded as

$$x = \frac{1}{d_1(x)} + \frac{1}{d_1(x)d_2(x)} + \cdots + \frac{1}{d_1(x)d_2(x)\cdots d_n(x)} + \cdots, \tag{1.1}$$

which is denoted by $x = [d_1(x), d_2(x), \ldots]$ for short. In [GA], J. Galambos proved that for almost all $x \in (0,1]$,

$$\lim_{n\to\infty} d_n(x)^{1/n} = 1. \tag{1.2}$$

Also, he posed the problem of finding the Hausdorff dimension of the set where (1.2) fails. In [WU], J. Wu proved that this Hausdorff dimension is 1. More generally, he proved that for any $\alpha \ge 1$, the Hausdorff dimension of the set

$$A(\alpha) = \{x \in (0,1] : \lim_{n\to\infty} d(x)^{1/n} = \alpha\} \tag{1.3}$$

is 1. In this paper, we find the Hausdorff dimension of the set

$$E(\alpha) = \left\{x \in (0,1] : \lim_{n\to\infty} \frac{\log d_{n+1}(x)}{\log d_n(x)} = \alpha\right\}. \tag{1.4}$$

Theorem 1.1. *For any* $\alpha \ge 1$, $\dim_{\mathrm{H}} E(\alpha) = 1/\alpha$.

2. The proof of the theorem

2.1. Upper bound. Firstly, a simple but useful lemma is stated.

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Lemma 2.1. *Suppose* $2 \le d_1 \le d_2 \le \cdots \le d_n$ *are given integers, let* $S \subset (0,1]$ *and*

$$S' = \left\{ x' : x' = \frac{1}{d_1} + \frac{1}{d_1 d_2} + \cdots + \frac{x}{d_1 d_2 \cdots d_n},\, x \in S \right\}.$$

Then $\dim_{\mathrm{H}} S = \dim_{\mathrm{H}} S'$.

Proof. Define the map $S \to S'$ by

$$f_{d_1 d_2 \cdots d_n}(x) = \frac{1}{d_1} + \frac{1}{d_1 d_2} + \cdots + \frac{x}{d_1 d_2 \cdots d_n}.$$

Since

$$|x_1' - x_2'| = \frac{|x_1 - x_2|}{d_1 d_2 \cdots d_n} \quad \text{for any } x_1, x_2 \in S,$$

the map f is bi-lipschitz, so $\dim_{\mathrm{H}} S = \dim_{\mathrm{H}} S'$. ∎

Lemma 2.2. $\dim_{\mathrm{H}} E(\alpha) \le 1/\alpha$.

Proof. By the definition of $E(\alpha)$, for any $x \in E(\alpha)$ and $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $[d_n^{\alpha-\epsilon}(x)] \le d_{n+1}(x) \le [d_n^{\alpha+\epsilon}(x)]$ for any $n \ge N$. Take

$$\begin{aligned} E_\epsilon(\alpha) &= \bigcup_{N=1}^{\infty} \{x \in (0,1] : [d_n^{\alpha-\epsilon}(x)] \le d_{n+1}(x) \le [d_n^{\alpha+\epsilon}(x)],\, \forall n \ge N\} \\ &= \bigcup_{N=1}^{\infty} E_\epsilon(N, \epsilon). \end{aligned}$$

Obviously, $E(\alpha) \subset E_\epsilon(\alpha)$ for any $0 < \epsilon < \alpha$. By Lemma 2.1, $\dim_{\mathrm{H}} E_\epsilon(N,\alpha) = \dim_{\mathrm{H}} E_\epsilon(1,\alpha)$ for any $N \ge 1$, thus

$$\dim_{\mathrm{H}} E(\alpha) \le \sup_{N \ge 1} \dim_{\mathrm{H}} E_\epsilon(N,\alpha) = \dim_{\mathrm{H}} E_\epsilon(1,\alpha).$$

In what follows, we use the symbolic space $D = \bigcup_{n=0}^{\infty} D_n$, where $D_0 = \emptyset$ and for any $n \ge 1$,

$$D_n = \{(\sigma_1, \ldots, \sigma_n) \in \mathbb{N}^n : [\sigma_k^{\alpha-\epsilon}] \le \sigma_{k+1} \le [\sigma_k^{\alpha+\epsilon}],\, \forall 1 \le k \le n-1\}.$$

For any $\sigma = (\sigma_1, \ldots, \sigma_n) \in D_n$, we call the set

$$I(\sigma_1, \ldots, \sigma_n) := \mathrm{cl}\{x \in (0,1] : d_1(x) = \sigma_1, \ldots, d_n(x) = \sigma_n\}$$

an nth *basic cylinder*, and

$$J_\sigma := \bigcup_{[\sigma_n^{\alpha-\epsilon}] \le \sigma_{n+1} \le [\sigma_n^{\alpha+\epsilon}]} I(\sigma_1, \ldots, \sigma_n, \sigma_{n+1})$$

an nth *basic interval*. Then

$$|J_\sigma| = \left|\frac{1}{\sigma_1\cdots\sigma_n[\sigma_n^{\alpha-\epsilon}]} - \frac{1}{\sigma_1\cdots\sigma_n[\sigma_n^{\alpha+\epsilon}]}\right| \le \frac{1}{\sigma_1\cdots\sigma_{n-1}}\left(\frac{2}{\sigma_n\sigma_n^{\alpha-\epsilon}} - \frac{1}{\sigma_n\sigma_n^{\alpha+\epsilon}}\right) \le \frac{4}{\sigma_1\cdots\sigma_{n-1}\sigma_n^{1+\alpha-\epsilon}}.$$

Notice that

$$E_\epsilon(1,\alpha) = \{x\in(0,1] : [\sigma_n^{\alpha-\epsilon}] \le \sigma_{n+1} \le [\sigma_n^{\alpha+\epsilon}],\ \forall n\ge 1\} = \bigcap_{n=1}^{\infty}\bigcup_{\sigma\in D_n} J_\sigma.$$

Take $\epsilon>0$ so small that for any $n\ge 1$,

$$\sum_{[\sigma_{n-1}^{\alpha-\epsilon}]\le\sigma_n\le[\sigma_{n-1}^{\alpha+\epsilon}]}\left(\frac{2}{\sigma_n^{\alpha-\epsilon}}\right)^{1/\alpha} \le 1.$$

Then

$$\mathcal{H}^{1/\alpha}(E_\epsilon(1,\alpha)) \le \liminf_{n\to\infty}\sum_{\sigma\in D_n}|J_\sigma|^{1/\alpha} \le \liminf_{n\to\infty}\sum_{\sigma\in D_n}\left(\frac{4}{\sigma_1\cdots\sigma_{n-1}\sigma_n^{1+\alpha-\epsilon}}\right)^{1/\alpha}$$

$$= \liminf_{n\to\infty}\sum_{(\sigma_1,\dots,\sigma_{n-1})\in D_{n-1}}\left(\frac{4}{\sigma_1\cdots\sigma_{n-1}^{1+\alpha-\epsilon}}\right)^{1/\alpha}\sum_{[\sigma_{n-1}^{\alpha-\epsilon}]\le\sigma_n\le[\sigma_{n-1}^{\alpha+\epsilon}]}\left(\frac{\sigma_{n-1}^{\alpha-\epsilon}}{\sigma_n^{1+\alpha-\epsilon}}\right)^{1/\alpha}$$

$$\le \liminf_{n\to\infty}\sum_{(\sigma_1,\dots,\sigma_{n-1})\in D_{n-1}}\left(\frac{4}{\sigma_1\cdots\sigma_{n-1}^{1+\alpha-\epsilon}}\right)^{1/\alpha}\sum_{[\sigma_{n-1}^{\alpha-\epsilon}]\le\sigma_n\le[\sigma_{n-1}^{\alpha+\epsilon}]}\left(\frac{2}{\sigma_n^{\alpha-\epsilon}}\right)^{1/\alpha}$$

$$\le \liminf_{n\to\infty}\sum_{(\sigma_1,\dots,\sigma_{n-1})\in D_{n-1}}\left(\frac{4}{\sigma_1\cdots\sigma_{n-2}\sigma_{n-1}^{1+\alpha-\epsilon}}\right)^{1/\alpha} < \infty,$$

which implies $\dim_{\mathrm{H}} E(\alpha)\le 1/\alpha$. ■

2.2. Lower bound. We state the mass distribution principle first.

LEMMA 2.3 (Distribution Principle, see also [FA, Proposition 2.3]). *Let E be a Borel set, and μ a measure with $\mu(E)>0$. If for any $x\in E$,*

$$\liminf_{r\to 0}\frac{\log\mu(B(x,r))}{\log r}\ge s,$$

where $B(x,r)$ denotes the open ball with center at x and radius r, then $\dim_H E\ge s$.

Recall that for $\alpha\ge 1$,

$$E(\alpha) = \left\{x\in(0,1] : \lim_{n\to\infty}\frac{\log d_{n+1}(x)}{\log d_n(x)} = \alpha\right\}.$$

To give a lower bound of $\dim_{\mathrm{H}} E(\alpha)$, we define a subset of $E(\alpha)$ by

$$F = \{x \in (0,1] : 2[d_n^\alpha(x)] + 1 \le d_{n+1}(x) \le 3[d_n^\alpha(x)] \text{ for all } n \ge 1\}.$$

To exhibit the structure of the set F, we use the symbolic space $D = \bigcup_{n=0}^\infty D_n$, where $D_0 = \emptyset$ and for any $n \ge 1$,

$$D_n = \{(\sigma_1, \ldots, \sigma_n) \in \mathbb{N}^n : 2[\sigma_k^\alpha] + 1 \le \sigma_{k+1} \le 3[\sigma_k^\alpha],\ 1 \le k < n\}.$$

As in Section 2.1, for each $\sigma = (\sigma_1, \ldots, \sigma_n) \in D_n$, we call

$$I(\sigma_1, \ldots, \sigma_n) := \mathrm{cl}\{x \in [0,1] : d_1(x) = \sigma_1, \ldots, d_n(x) = \sigma_n\},$$
$$J(\sigma_1, \ldots, \sigma_n) := \bigcup_{2[\sigma_n^\alpha]+1 \le \sigma_{n+1} \le 3[\sigma_n^\alpha]} I(\sigma_1, \ldots, \sigma_n, \sigma_{n+1})$$

an nth basic cylinder and nth basic interval respectively. Then by a simple computation, we have

$$|I(\sigma_1, \ldots, \sigma_n)| = \frac{1}{\sigma_1 \cdots \sigma_{n-1}(\sigma_n - 1)\sigma_n},$$

and

$$\frac{1}{6}\,\frac{1}{\sigma_1 \cdots \sigma_{n-1}\sigma_n^{\alpha+1}} \le |J(\sigma_1, \ldots, \sigma_n)| \le \frac{1}{3}\,\frac{1}{\sigma_1 \cdots \sigma_{n-1}\sigma_n^{\alpha+1}},$$

and one can observe that

$$F = \bigcap_{n=1}^\infty \bigcup_{(\sigma_1, \ldots, \sigma_n) \in D_n} J(\sigma_1, \ldots, \sigma_n).$$

Lemma 2.4. $\dim_{\mathrm{H}} E(\alpha) \ge 1/\alpha$.

Proof. For each $n \ge 1$ and $\sigma = (\sigma_1, \ldots, \sigma_n) \in D_n$, denote by $g^\ell(\sigma_1, \ldots, \sigma_n)$ the length of the gap between the left endpoint of $I(\sigma_1, \ldots, \sigma_n)$ and the left endpoint of $J(\sigma_1, \ldots, \sigma_n)$ and by $g^r(\sigma_1, \ldots, \sigma_n)$ the length of the gap between the right endpoint of $I(\sigma_1, \ldots, \sigma_n)$ and the right endpoint of $J(\sigma_1, \ldots, \sigma_n)$. Finally, let

$$G(\sigma_1, \ldots, \sigma_n) := \min\{g^\ell(\sigma_1, \ldots, \sigma_n), g^r(\sigma_1, \ldots, \sigma_n)\}.$$

Then

$$\frac{1}{3}\,\frac{1}{\sigma_1 \cdots \sigma_{n-1}\sigma_n^{\alpha+1}} \le G(\sigma_1, \ldots, \sigma_n) \le \frac{2}{3}\,\frac{1}{\sigma_1 \cdots \sigma_{n-1}\sigma_n^{\alpha+1}}.$$

Now, we define a probability measure on F. The set function $\mu : \{J(\sigma) : \sigma \in D \setminus D_0\} \to \mathbb{R}^+$ is given by

$$\mu(J(\sigma_1, \ldots, \sigma_n)) = \frac{1}{[\sigma_{n-1}^\alpha]}\,\mu(J(\sigma_1, \ldots, \sigma_{n-1})) = \prod_{k=1}^{n-1} \frac{1}{[\sigma_k^\alpha]},$$

and

$$\mu(J(\sigma_1)) = \frac{1}{\sharp D_1},$$

where $\sharp$ denotes cardinality. It can be easily verified that

$$\mu(J(\sigma_1,\ldots,\sigma_{n-1})) = \sum_{\sigma_n} \mu(J(\sigma_1,\ldots,\sigma_n)),$$

where the summation is taken over all σ_n such that $(\sigma_1,\ldots,\sigma_{n-1},\sigma_n) \in D_n$. Notice that $\sum_{\sigma_1\in D_1}\mu(J(\sigma_1)) = 1$, so, by Kolmogorov's extension theorem, the function μ can be extended to a probability measure supported on F, which is still denoted by μ.

For each $x \in F$, there exists a sequence $(\sigma_n)_{n\geq 1}$ such that $(\sigma_1,\ldots,\sigma_n) \in D_n$ and $x \in J(\sigma_1,\ldots,\sigma_n)$, for each $n \geq 1$. Assume that $r > 0$ is small enough and let n be the integer such that

$$G(\sigma_1,\ldots,\sigma_{n+1}) \leq r < G(\sigma_1,\ldots,\sigma_n).$$

By the definition of G, it follows that the ball $B(x,r)$ can only intersect one nth basic cylinder $I(\sigma_1,\ldots,\sigma_n)$.

The following relationship can be verified easily. For any $\epsilon > 0$, there exists $n \geq 1$ such that for each $(\sigma_1,\ldots,\sigma_n) \in D_n$,

$$\prod_{k=1}^{n-1}\frac{1}{[\sigma_k^\alpha]} \leq \left(\frac{1}{\sigma_1\cdots\sigma_{n-1}\sigma_n^{\alpha+1}}\right)^{1/(\alpha+\epsilon)}. \tag{2.1}$$

Two cases will be distinguished.

(i) $G(\sigma_1,\ldots,\sigma_{n+1}) \leq r < |I(\sigma_1,\ldots,\sigma_{n+1})|$. In this case, the ball $B(x,r)$ can intersect at most eight $(n+1)$th basic cylinders contained in $I(\sigma_1,\ldots,\sigma_n)$. So,

$$\mu(B(x,r)) \leq 8\mu(J(\sigma_1,\ldots,\sigma_{n+1})) = 8\prod_{k=1}^{n}\frac{1}{[\sigma_k^\alpha]} \leq 24r^{1/(\alpha+\epsilon)}.$$

(ii) $|I(\sigma_1,\ldots,\sigma_{n+1})| \leq r < G(\sigma_1,\ldots,\sigma_n)$. Notice that

$$\min\{|I(\sigma_1,\ldots,\sigma_n,\bar\sigma_{n+1})| : 2[\sigma_n^\alpha]+1 \leq \bar\sigma_{n+1} \leq 3[\sigma_n^\alpha]\} \geq \frac{1}{9\sigma_1\cdots\sigma_{n-1}\sigma_n^{2\alpha+1}}.$$

In this case, $B(x,r)$ can intersect at most

$$18r\sigma_1\cdots\sigma_{n-1}\sigma_n^{2\alpha+1} + 2 \leq 54r\sigma_1\cdots\sigma_{n-1}\sigma_n^{2\alpha+1} \leq [72r\sigma_1\cdots\sigma_{n-1}\sigma_n^{2\alpha+1}]$$

$(n+1)$th basic cylinders contained in $I(\sigma_1,\ldots,\sigma_n)$. So,

$$\begin{aligned}
&\mu(B(x,r))\\
&\quad\leq \min\{\mu(J(\sigma_1,\ldots,\sigma_n)), [72r\sigma_1\cdots\sigma_{n-1}\sigma_n^{2\alpha+1}]\mu(J(\sigma_1,\ldots,\sigma_{n+1}))\}\\
&\quad\leq \prod_{k=1}^{n-1}\frac{1}{[\sigma_k^\alpha]}\min\left\{1, 72r\sigma_1\cdots\sigma_{n-1}\sigma_n^{2\alpha+1}\frac{1}{[\sigma_n^\alpha]}\right\}
\end{aligned}$$

$$\leq \prod_{k=1}^{n-1} \frac{1}{[\sigma_k^\alpha]} \min\{1, 114r\sigma_1 \cdots \sigma_{n-1}\sigma_n^{\alpha+1}\} \quad (\text{as } \min\{a,b\} \leq a^s b^{1-s},\ 0<s<1)$$
$$\leq \prod_{k=1}^{n-1} \frac{1}{[\sigma_k^\alpha]} (114r\sigma_1 \cdots \sigma_{n-1}\sigma_n^{\alpha+1})^{1/(\alpha+\epsilon)} \qquad (\text{by } (2.1))$$
$$\leq 114^{1/(\alpha+\epsilon)} r^{1/(\alpha+\epsilon)}.$$

By Lemma 2.3, $\dim_{\mathrm{H}} F \geq 1/(\alpha+\epsilon)$. Since ϵ is arbitrary, $\dim_{\mathrm{H}} E(\alpha) \geq \dim_{\mathrm{H}} F \geq 1/\alpha$. Combining this with Lemmas 2.2 and 2.4, we obtain Theorem 1.1. ■

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