ACTA ARITHMETICA
169.4 (2015)

Theta series associated with certain positive definite binary quadratic forms

by

Heng Huat Chan and Pee Choon Toh (Singapore)

1. Introduction. A binary quadratic form over $\mathbb{Z}$, denoted by $Q(x,y)$, is an expression of the form

$$ax^2+bxy+cy^2$$

with $ac\neq 0$ and $a,b,c\in\mathbb{Z}$. For every integer $d\equiv 0$ or $1 \pmod 4$, the set

$$\{ax^2+bxy+cy^2 \mid b^2-4ac=d\}$$

is non-empty. We call such an integer d a *discriminant*. In this article, we consider only positive definite binary quadratic forms, in other words $ax^2+bxy+cy^2$ satisfying $a>0$ and with discriminant $d<0$.

We say that two binary quadratic forms $Q_1(x,y)$ and $Q_2(x,y)$ are *properly equivalent* if

$$Q_1(x,y)=Q_2(\alpha x+\beta y,\gamma x+\delta y)$$

for some

$$\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in \mathrm{SL}_2(\mathbb{Z}).$$

The set of equivalence classes forms a finite abelian group, $C(d)$. We follow D. Buell [3, p. 193] and call d an *idoneal* discriminant if $C(d)$ is the trivial group or a direct sum of cyclic groups of order 2. A discriminant d is *fundamental* if it cannot be written in the form st^2 where $t>1$ and s is itself a discriminant.

There are a total of 101 known idoneal discriminants, 65 of which are fundamental (see Tables 1 and 2). Coincidentally, among these 101 idoneal discriminants, there are exactly 65 discriminants d (Tables 2 and 4) such that $|d|/4$ are integers. These integers are collectively known as *Euler's idoneal*

2010 *Mathematics Subject Classification*: Primary 11E16; Secondary 11R29.
Key words and phrases: binary quadratic forms, Lambert series, discriminants.

DOI: 10.4064/aa169-4-3

numbers. We refer the reader to E. Kani's article [7] for a survey of the main results on idoneal numbers.

Table 1. Fundamental idoneal discriminants, $D \equiv 1 \pmod 4$

$\lvert C(D)\rvert$	D
1	$-3, -7, -11, -19, -43, -67, -163$
2	$-15, -35, -51, -91, -115, -123,$ $-187, -235, -267, -403, -427$
4	$-195, -435, -483, -555, -595, -627,$ $-715, -795, -1435$
8	$-1155, -1995, -3003, -3315$

Table 2. Fundamental idoneal discriminants, $D \equiv 0 \pmod 4$

$\lvert C(D)\rvert$	D
1	$-4, -8$
2	$-20, -24, -40, -52, -88, -148, -232$
4	$-84, -120, -132, -168, -228, -280,$ $-312, -340, -372, -408, -520, -532,$ $-708, -760, -1012$
8	$-420, -660, -840, -1092, -1320, -1380,$ $-1428, -1540, -1848$
16	-5460

Table 3. Non-fundamental idoneal discriminants, $d \equiv 1 \pmod 4$

$\lvert C(d)\rvert$	$d = t^2 D$
1	-27
2	$-75, -99, -147$
4	-315

Table 4. Non-fundamental idoneal discriminants, $d \equiv 0 \pmod 4$

$\lvert C(d)\rvert$	$d = t^2 D$
1	$-12, -16, -28$
2	$-32, -36, -48, -60, -64, -72,$ $-100, -112$
4	$-96, -160, -180, -192, -240, -288,$ $-352, -448, -928$
8	$-480, -672, -960, -1120, -1248, -1632,$ $-2080, -3040$
16	$-3360, -5280, -7392$

We define a *generalized Lambert series* as a series of the form

$$\sum_{n=1}^{\infty}(\mu * \nu)(n)q^n$$

where μ and ν are Dirichlet characters, and $*$ is the Dirichlet convolution given by

$$(F * G)(n) = \sum_{\ell | n} F(\ell)G(n/\ell).$$

In [16], P. C. Toh showed that if d is one of the 65 fundamental idoneal discriminants, then the theta series

$$\sum_{m,n\in\mathbb{Z}} q^{am^2+bmn+cn^2},$$

with $b^2 - 4ac = d$, can be expressed in terms of generalized Lambert series. In the case where $C(d)$ is trivial, this is well known [9, Theorem 204], and explicit examples can be found in [12, pp. 121–123]. For example, when $d = -7$, we have

$$\text{(1.1)} \quad \sum_{m,n\in\mathbb{Z}} q^{m^2+mn+2n^2} = 1 + 2\sum_{k=1}^{\infty}\sum_{\ell|k}\left(\frac{-7}{\ell}\right)q^k = 1 + 2\sum_{n=1}^{\infty}\left(\frac{-7}{n}\right)\frac{q^n}{1-q^n},$$

where $\left(\frac{D}{\cdot}\right)$ is the Kronecker symbol modulo $|D|$.

The purpose of this article is to extend the results of [16] to the 36 non-fundamental idoneal discriminants (Tables 3 and 4). At this point, we would like to mention that K. S. Williams and his collaborators have done extensive work on studying the number of representations of n by binary quadratic forms. For example, one may look at [5, 17, 15] and the references therein. A recent work by F. Patane [11] also considers the problem of extending results for fundamental discriminants to non-fundamental discriminants. Some of the identities we will prove in this article have already been proved in these works, but we present a different approach. We briefly illustrate below how this is done. Using elementary series manipulations, one can show that

$$\text{(1.2)} \quad \begin{aligned} \sum_{m,n\in\mathbb{Z}} q^{m^2+mn+2n^2} &= \sum_{m,n\in\mathbb{Z}} q^{m^2+7n^2} + \sum_{m,n\in\mathbb{Z}} q^{(m+1/2)^2+7(n+1/2)^2} \\ &= \sum_{m,n\in\mathbb{Z}} q^{m^2+7n^2} + 4 \sum_{\substack{m,n\in\mathbb{Z} \\ m\equiv 1 \ (\mathrm{mod}\ 4)}} q^{2(m^2+mn+2n^2)}. \end{aligned}$$

Note that the second theta series above is essentially identical to that of (1.1) except for the congruence condition on m. In Section 2, we will prove the following analogue of (1.1):

$$\sum_{\substack{m,n\in\mathbb{Z}\\ m\equiv 1\ (\mathrm{mod}\ 4)}} q^{m^2+mn+2n^2} = \sum_{n=1}^{\infty}\left(\frac{-28}{n}\right)\frac{q^n}{1-q^n}. \tag{1.3}$$

Combining (1.1)–(1.3) yields the identity associated with the non-fundamental discriminant $d = -28$:

$$\sum_{m,n\in\mathbb{Z}} q^{m^2+7n^2} = 1 + 2\sum_{n=1}^{\infty}\left(\frac{-7}{n}\right)\frac{q^n}{1-q^n} - 4\sum_{n=1}^{\infty}\left(\frac{-28}{n}\right)\frac{q^{2n}}{1-q^{2n}}. \tag{1.4}$$

Obtaining an analogous Lambert series representation of

$$\sum_{\substack{m,n\in\mathbb{Z}\\ m\equiv m_1\ (\mathrm{mod}\ t)\\ n\equiv n_1\ (\mathrm{mod}\ t)}} q^{am^2+bmn+cn^2}$$

from the representation of

$$\sum_{m,n\in\mathbb{Z}} q^{am^2+bmn+cn^2}$$

turns out to be a key ingredient in deriving identities associated with non-fundamental discriminants.

In Section 2, detailed proofs of (1.4) and other similar identities associated with $d = -12$ and $d = -60$ will be given. The discriminants -28, -12 and -60 are the only idoneal discriminants of the form $4D$ where D is an odd fundamental discriminant. There are 16 discriminants of the form $4D$ where D is an even fundamental idoneal discriminant. These will be discussed in Section 3. In Section 4, we study the nine discriminants of the form p^2D for some odd prime p. The remaining eight discriminants will be discussed in Section 5.

In the rest of this article, we will use D to denote fundamental idoneal discriminants, and d or t^2D to denote non-fundamental discriminants.

2. Discriminants of the form $4D$ where D is odd. There are only three known idoneal discriminants of the form $d = 4D$ where D is an odd fundamental idoneal discriminant. These are $d = -12, -28$ and -60. In these cases, $|C(4D)| = |C(D)|$. The main results of this section are the following four identities.

THEOREM 2.1. *We have*

$$\sum_{m,n\in\mathbb{Z}} q^{m^2+3n^2} = 1 + 2\sum_{n=1}^{\infty}\left(\frac{-3}{n}\right)\frac{q^n}{1-q^n} + 4\sum_{n=1}^{\infty}\left(\frac{-3}{n}\right)\frac{q^{4n}}{1-q^{4n}}, \tag{2.1a}$$

$$(2.1b)\quad \sum_{m,n\in\mathbb{Z}} q^{m^2+7n^2} = 1 + 2\sum_{n=1}^{\infty}\left(\frac{-7}{n}\right)\frac{q^n}{1-q^n} - 4\sum_{n=1}^{\infty}\left(\frac{-28}{n}\right)\frac{q^{2n}}{1-q^{2n}},$$

$$(2.1c)\quad \sum_{m,n\in\mathbb{Z}} q^{m^2+15n^2} + \sum_{m,n\in\mathbb{Z}} q^{3m^2+5n^2} = 2 + 2\sum_{n=1}^{\infty}\left(\frac{-15}{n}\right)\frac{q^n}{1-q^n} - 4\sum_{n=1}^{\infty}\left(\frac{-60}{n}\right)\frac{q^{2n}}{1-q^{2n}},$$

$$(2.1d)\quad \sum_{m,n\in\mathbb{Z}} q^{m^2+15n^2} - \sum_{m,n\in\mathbb{Z}} q^{3m^2+5n^2} = 2\sum_{k,n=1}^{\infty}\left(\frac{-3}{k}\right)\left(\frac{5}{n}\right)q^{kn} + 4\sum_{k,n=1}^{\infty}\left(\frac{-3}{k}\right)\left(\frac{20}{n}\right)q^{2kn}.$$

For ease of comparison, we state the identities for discriminants -3, -7 and -15 (see [16, p. 232]):

$$(2.2a)\quad \sum_{m,n\in\mathbb{Z}} q^{m^2+mn+n^2} = 1 + 6\sum_{n=1}^{\infty}\left(\frac{-3}{n}\right)\frac{q^n}{1-q^n},$$

$$(2.2b)\quad \sum_{m,n\in\mathbb{Z}} q^{m^2+mn+2n^2} = 1 + 2\sum_{n=1}^{\infty}\left(\frac{-7}{n}\right)\frac{q^n}{1-q^n},$$

$$(2.2c)\quad \sum_{m,n\in\mathbb{Z}} q^{m^2+mn+4n^2} + \sum_{m,n\in\mathbb{Z}} q^{2m^2+mn+2n^2} = 2 + 2\sum_{n=1}^{\infty}\left(\frac{-15}{n}\right)\frac{q^n}{1-q^n},$$

$$(2.2d)\quad \sum_{m,n\in\mathbb{Z}} q^{m^2+mn+4n^2} - \sum_{m,n\in\mathbb{Z}} q^{2m^2+mn+2n^2} = 2\sum_{k,n=1}^{\infty}\left(\frac{-3}{k}\right)\left(\frac{5}{n}\right)q^{kn}.$$

As mentioned in Section 1, the key to proving Theorem 2.1 lies in the following lemma.

Lemma 2.2. *We have*

$$(2.3a)\quad \sum_{\substack{m,n\in\mathbb{Z}\\ m\equiv 1\,(\mathrm{mod}\,4)}} q^{m^2+mn+n^2} = 2\sum_{n=1}^{\infty}\left(\frac{-12}{n}\right)\frac{q^n}{1-q^{2n}},$$

$$(2.3b)\quad \sum_{\substack{m,n\in\mathbb{Z}\\ m\equiv 1\,(\mathrm{mod}\,4)}} q^{m^2+mn+2n^2} = \sum_{n=1}^{\infty}\left(\frac{-28}{n}\right)\frac{q^n}{1-q^n},$$

$$(2.3c)\quad \sum_{\substack{m,n\in\mathbb{Z}\\ m\equiv 1\,(\mathrm{mod}\,4)}} q^{m^2+mn+4n^2} + \sum_{\substack{m,n\in\mathbb{Z}\\ m\equiv 1\,(\mathrm{mod}\,4)}} q^{2m^2+mn+2n^2} = \sum_{n=1}^{\infty}\left(\frac{-60}{n}\right)\frac{q^n}{1-q^n},$$

$$\sum_{\substack{m,n\in\mathbb{Z}\\ m\equiv 1 \pmod 4}} q^{m^2+mn+4n^2} - \sum_{\substack{m,n\in\mathbb{Z}\\ m\equiv 1 \pmod 4}} q^{2m^2+mn+2n^2} = \sum_{k,n=1}^{\infty}\left(\frac{-3}{k}\right)\left(\frac{20}{n}\right)q^{kn}. \tag{2.3d}$$

Before we proceed with the proof of Lemma 2.2, we note that identities (2.3x) and (2.2x) where x = a, b, c, d are very similar. We emphasize that by inserting the condition "$m \equiv 1 \pmod 4$" on the left hand side of (2.2x) and multiplying 4 into the numerator of the Kronecker symbol (or one of the Kronecker symbols in the case of (2.3d)) on the right hand side of (2.2x), we obtain expressions that appear in (2.3x). Thus, it is easy to record the identities (2.3x) if we have the knowledge of (2.2x). It is also this observation that gives us a hint as to the type of generalized Lambert series we could use to represent series associated with binary quadratic forms with non-fundamental discriminants.

Proof. For the proof of (2.3a), we note that $2\mathfrak{O}_3$ is inert in

$$\mathfrak{O}_3 = \mathbb{Z}[(1+\sqrt{-3})/2].$$

The series on the left hand side of (2.3a) is essentially

$$\sum_{\substack{\mathfrak{a}\subset\mathfrak{O}_3\\ (2,\mathfrak{a})=1}} q^{N(\mathfrak{a})}. \tag{2.4}$$

For if

$$\mathfrak{a} = \left(m + n\frac{1+\sqrt{-3}}{2}\right) \quad \text{and} \quad (2,\mathfrak{a}) = 1,$$

then at least one of m or n must be odd. Now using the transformation $(m,n) \mapsto (-m, m+n)$ we see that

$$\sum_{\substack{m,n\in\mathbb{Z}\\ m\,\text{odd},\,n\,\text{odd}}} q^{m^2+mn+n^2} = \sum_{\substack{m,n\in\mathbb{Z}\\ m\,\text{odd},\,n\,\text{even}}} q^{m^2+mn+n^2} = \frac{1}{2}\sum_{\substack{m,n\in\mathbb{Z}\\ m\,\text{odd}}} q^{m^2+mn+n^2}$$
$$= \sum_{\substack{m,n\in\mathbb{Z}\\ m\equiv 1 \pmod 4}} q^{m^2+mn+n^2}.$$

The corresponding Dirichlet series of (2.4) has Euler product

$$\prod_{\substack{\mathfrak{p}\\ \mathfrak{p}\neq 2\mathfrak{O}_3}} \left(1 - \frac{1}{N(\mathfrak{p})^s}\right)^{-1}.$$

Note that this last product can be written as

$$\left(1-\frac{1}{3^s}\right)^{-1}\prod_{\substack{p \text{ is inert}\\ p\neq 2}}\left(1-\frac{1}{p^{2s}}\right)^{-1}\prod_{p \text{ splits}}\left(1-\frac{1}{p^s}\right)^{-2}$$
$$=\left(1-\frac{1}{2^s}\right)\zeta(s)L\left(\left(\frac{-12}{\cdot}\right),s\right).$$

The inverse Mellin transform of the above Dirichlet series is a constant multiple of the right hand side of (2.3a). The absence of the prime 2 in the product is indicated by the Kronecker symbol $\left(\frac{-12}{\cdot}\right)$.

The proof of (2.3b) is essentially the same as that of (2.3a) except that in this case, $2\mathfrak{O}_7$ splits completely, where $\mathfrak{O}_7=\mathbb{Z}[(1+\sqrt{-7})/2]$. Let $2\mathfrak{O}_7=\mathfrak{s}\mathfrak{s}'$. The left hand side of (2.3b) is then a constant multiple of

$$\sum_{\substack{\mathfrak{a}\subset\mathfrak{O}_7\\ (\mathfrak{s},\mathfrak{a})=1}} q^{N(\mathfrak{a})}.$$

Now, the corresponding Dirichlet series is

$$\left(1-\frac{1}{2^s}\right)^{-1}\left(1-\frac{1}{7^s}\right)^{-1}\prod_{p \text{ is inert}}\left(1-\frac{1}{p^{2s}}\right)^{-1}\prod_{\substack{p \text{ splits}\\ p\neq 2}}\left(1-\frac{1}{p^s}\right)^{-2}$$
$$=\zeta(s)L\left(\left(\frac{-28}{\cdot}\right),s\right).$$

The inverse Mellin transform of the above is a constant multiple of the right hand side of (2.3b). Note that identities (2.3a) and (2.3b) differ in the denominator of the Lambert series and this is due to the extra factor $(1-1/2^s)$, which results in

$$\sum_{n=1}^{\infty}\left(\frac{-12}{n}\right)\frac{q^n}{1-q^n}-\sum_{n=1}^{\infty}\left(\frac{-12}{n}\right)\frac{q^{2n}}{1-q^{2n}}=\sum_{n=1}^{\infty}\left(\frac{-12}{n}\right)\frac{q^n}{1-q^{2n}}.$$

The proof of (2.3c) and (2.3d) is the same as that of (2.3b), since the prime ideal $2\mathfrak{O}_{15}$ splits completely. The computations are similar to those in [16] except that the Euler product is short of one copy of $(1-1/2^s)^{-1}$, and this results in the Dirichlet series

$$L\left(\left(\frac{-d_1}{\cdot}\right),s\right)L\left(\left(\frac{-4d_2}{\cdot}\right),s\right)$$

instead of

$$L\left(\left(\frac{-d_1}{\cdot}\right),s\right)L\left(\left(\frac{-d_2}{\cdot}\right),s\right). \blacksquare$$

We establish another three lemmas that are necessary for the proof of Theorem 2.1.

Lemma 2.3. *If $\ell = (|D|+1)/4$, then*

$$\sum_{m,n\in\mathbb{Z}} q^{m^2+mn+\ell n^2} = \sum_{m,n\in\mathbb{Z}} q^{m^2+|D|n^2} + \sum_{m,n\in\mathbb{Z}} q^{(m+1/2)^2+|D|(n+1/2)^2}.$$

Proof. Summing over even and odd values of n, we get

$$\begin{aligned}\sum_{m,n\in\mathbb{Z}} q^{m^2+mn+\ell n^2} &= \sum_{m,k\in\mathbb{Z}} q^{m^2+2mk+4\ell k^2} + \sum_{m,k\in\mathbb{Z}} q^{m^2+m+2mk+\ell(4k^2+4k+1)} \\ &= \sum_{m,k\in\mathbb{Z}} q^{(m+k)^2+|D|k^2} + \sum_{m,k\in\mathbb{Z}} q^{(m+k+1/2)^2+|D|(k+1/2)^2}.\end{aligned}$$

It remains to use the transformation $(m+k,k)\mapsto(m,n)$. ∎

Lemma 2.4. *We have*

$$\sum_{m,n\in\mathbb{Z}} q^{2m^2+mn+2n^2} = \sum_{m,n\in\mathbb{Z}} q^{5m^2+3n^2} + \sum_{m,n\in\mathbb{Z}} q^{5(m+1/2)^2+3(n+1/2)^2}.$$

Proof. We sum the series on the left of the identity over even and odd values of both m and n to get

$$\begin{aligned}\sum_{j,k\in\mathbb{Z}} q^{8j^2+8k^2+4jk} + \sum_{j,k\in\mathbb{Z}} q^{8j^2+10j+8k^2+10k+4jk+5} + 2\sum_{j,k\in\mathbb{Z}} q^{8j^2+8j+8k^2+2k+4jk+2} \\ = \sum_{j,k\in\mathbb{Z}} q^{5(j+k)^2+3(j-k)^2} + \sum_{j,k\in\mathbb{Z}} q^{5(j+k+1)^2+3(j-k)^2} \\ + \sum_{j,k\in\mathbb{Z}} q^{5(j+k)^2+5(j+k)+3(j-k)^2+3(j-k)+2} \\ + \sum_{k,j\in\mathbb{Z}} q^{5(k+j-1)^2+5(k+j-1)+3(k-j)^2+3(k-j)+2}.\end{aligned}$$

We then apply

$$\sum_{m,n\in\mathbb{Z}} C_{m,n} = \sum_{j,k\in\mathbb{Z}} C_{j+k,j-k} + \sum_{j,k\in\mathbb{Z}} C_{j+k+1,j-k}$$

to establish the lemma. ∎

Lemma 2.5. *We have*

$$\sum_{n=1}^{\infty}\left(\frac{-3}{n}\right)\frac{q^n}{1-q^n} = \sum_{n=1}^{\infty}\left(\frac{-12}{n}\right)\frac{q^n}{1-q^{2n}} + \sum_{n=1}^{\infty}\left(\frac{-3}{n}\right)\frac{q^{4n}}{1-q^{4n}}.$$

Proof. We sum over n modulo 4 to get

$$\begin{aligned}
&\sum_{n=1}^{\infty}\left(\frac{-3}{n}\right)\frac{q^n}{1-q^n}\\
&\quad=\sum_{\substack{n=1\\ n\text{ odd}}}^{\infty}\left(\frac{-3}{n}\right)\frac{q^n}{1-q^n}+\sum_{\substack{n=1\\ n\text{ odd}}}^{\infty}\left(\frac{-3}{2n}\right)\frac{q^{2n}}{1-q^{2n}}+\sum_{n=1}^{\infty}\left(\frac{-3}{4n}\right)\frac{q^{4n}}{1-q^{4n}}\\
&\quad=\sum_{n=1}^{\infty}\left(\frac{-12}{n}\right)\frac{q^n}{1-q^n}-\sum_{\substack{n=1\\ n\text{ odd}}}^{\infty}\left(\frac{-3}{n}\right)\frac{q^{2n}}{1-q^{2n}}+\sum_{n=1}^{\infty}\left(\frac{-3}{n}\right)\frac{q^{4n}}{1-q^{4n}}\\
&\quad=\sum_{n=1}^{\infty}\left(\frac{-12}{n}\right)\frac{q^n+q^{2n}}{1-q^{2n}}-\sum_{n=1}^{\infty}\left(\frac{-12}{n}\right)\frac{q^{2n}}{1-q^{2n}}+\sum_{n=1}^{\infty}\left(\frac{-3}{n}\right)\frac{q^{4n}}{1-q^{4n}}. \blacksquare
\end{aligned}$$

Proof of Theorem 2.1. Lemmas 2.3 and 2.4 show that the theta series in each identity of (2.2) can be decomposed into the theta series in (2.1) plus another theta series of the form

$$\sum_{m,n\in\mathbb{Z}} q^{a(m+1/2)^2+b(n+1/2)^2}.$$

We shall proceed to show that these latter theta series have Lambert series representations via Lemma 2.2.

Summing over odd and even n, we obtain

$$\sum_{n\in\mathbb{Z}} q^{n(n+1)} = 2\sum_{n\in\mathbb{Z}} q^{4n^2+2n}.$$

So

$$\begin{aligned}
\sum_{m,n\in\mathbb{Z}} q^{(m+1/2)^2+7(n+1/2)^2} &= 4\sum_{m,n\in\mathbb{Z}} q^{4m^2+2m+28n^2+14n+2}\\
&= 4\sum_{j,k\in\mathbb{Z}} q^{4(j+k)^2+2(j+k)+28k^2+14k+2} = 4\sum_{j,k\in\mathbb{Z}} q^{4j^2+8jk+2j+32k^2+16k+2}\\
&= 4\sum_{j,k\in\mathbb{Z}} q^{2(2j^2+j(4k+1)+(4k+1)^2)} = 4\sum_{n=1}^{\infty}\left(\frac{-28}{n}\right)\frac{q^{2n}}{1-q^{2n}}.
\end{aligned}$$

This completes the proof of (2.1b). Similarly, we have

$$\begin{aligned}
\sum_{m,n\in\mathbb{Z}} q^{(m+1/2)^2+15(n+1/2)^2} &= 4\sum_{j,k\in\mathbb{Z}} q^{2(2j^2+j(4k+1)+2(4k+1)^2)},\\
\sum_{m,n\in\mathbb{Z}} q^{3(m+1/2)^2+5(n+1/2)^2} &= 4\sum_{j,k\in\mathbb{Z}} q^{2(4j^2+j(4k+1)+(4k+1)^2)}
\end{aligned}$$

which establishes (2.1c) and (2.1d).

Finally, we can also show that

$$\sum_{m,n\in\mathbb{Z}} q^{(m+1/2)^2+3(n+1/2)^2} = 2\sum_{j,k\in\mathbb{Z}} q^{j^2+j(4k+1)+(4k+1)^2} = 4\sum_{n=1}^{\infty}\left(\frac{-12}{n}\right)\frac{q^n}{1-q^{2n}}.$$

This means that

$$\sum_{m,n\in\mathbb{Z}} q^{m^2+3n^2} = 1+6\sum_{n=1}^{\infty}\left(\frac{-3}{n}\right)\frac{q^n}{1-q^n} - 4\sum_{n=1}^{\infty}\left(\frac{-12}{n}\right)\frac{q^n}{1-q^{2n}} = 1+2\sum_{n=1}^{\infty}\left(\frac{-3}{n}\right)\frac{q^n}{1-q^n} + 4\sum_{n=1}^{\infty}\left(\frac{-3}{n}\right)\frac{q^{4n}}{1-q^{4n}},$$

where we have used Lemma 2.5. ■

We end this section with some remarks. Identities (2.1a) to (2.1c) were previously studied by S. Ramanujan and proved by B. C. Berndt [2, Chapter 17, Entry 8(iv), p. 114, Chapter 19, Entry 17(ii), p. 302 and Chapter 20, Entry 10(vi), p. 379]. K. S. Williams [17] has also provided alternative proofs of (2.1a) to (2.1c). The pair of identities (2.1c) and (2.1d) were studied independently by A. Berkovich and H. Yesilyurt [1] and S. Cooper [4]. Our approach appears to be new.

3. Discriminants of the form $4D$ where D is even. There are 16 known idoneal discriminants (Table 5) of the form $d = 4D$ where D is an even idoneal fundamental discriminant. With the exception of $d = -16$ where $|C(-16)| = |C(-4)| = 1$, the remaining cases all satisfy $|C(4D)| = 2|C(D)|$. Thus, in addition to considering ideals relatively prime to $2\mathfrak{O}_K$, we require another operation, namely replacing q by $\sqrt{-1}\,q$, to obtain the full set of identities. We shall use the following theorem to illustrate this computation for discriminants $d = -32p$, $p = 3, 5, 11$ or 29.

Table 5. Non-fundamental idoneal discriminants, $d = 4D$, where D is even, fundamental and idoneal

$\lvert C(d)\rvert$	Theorem	d
1	3.2	−16
2	3.3	−32
4	3.1	−96, −160, −352, −928
8	3.4	−480, −672, −1248, −1632
8	3.5	−1120, −2080, −3040
16	3.6	−3360, −5280, −7392

Theorem 3.1. *Let $p = 3, 5, 11$ or 29, and set $\epsilon = \left(\frac{-1}{p}\right)$. Define*

$$S_1 = \sum_{m,n\in\mathbb{Z}} q^{m^2+8pn^2}, \qquad S_2 = \sum_{m,n\in\mathbb{Z}} q^{8m^2+pn^2},$$

$$S_3 = \sum_{m,n\in\mathbb{Z}} q^{(p+2)m^2+2(p-2)mn+(p+2)n^2},$$

$$S_4 = \sum_{m,n\in\mathbb{Z}} q^{(2p+1)m^2+2(2p-1)mn+(2p+1)n^2}.$$

Then

$$S_1 + S_2 + S_3 + S_4 = 4 + 2\sum_{k,n=1}^{\infty} \left(\frac{4}{k}\right)\left(\frac{-8p}{n}\right) q^{kn} + 4\sum_{k,n=1}^{\infty} \left(\frac{-8p}{n}\right) q^{4kn},$$

$$S_1 - S_2 - S_3 + S_4 = 2\sum_{k,n=1}^{\infty} \left(\frac{4p\epsilon}{k}\right)\left(\frac{-8\epsilon}{n}\right) q^{kn} + 4\sum_{k,n=1}^{\infty} \left(\frac{p\epsilon}{k}\right)\left(\frac{-8\epsilon}{n}\right) q^{4kn},$$

$$S_1 + \epsilon S_2 - \epsilon S_3 - S_4 = 2\sum_{k,n=1}^{\infty} \left(\frac{-4}{k}\right)\left(\frac{8p}{n}\right) q^{kn},$$

$$S_1 - \epsilon S_2 + \epsilon S_3 - S_4 = 2\sum_{k,n=1}^{\infty} \left(\frac{-4p\epsilon}{k}\right)\left(\frac{8\epsilon}{n}\right) q^{kn}.$$

We remark that it is possible to isolate each of S_i and write it as a sum of Lambert series.

Proof of Theorem 3.1. For $D = -8p$, $p = 3, 5, 11$ or 29, we have $|C(D)| = 2$ and it is known [16, p. 232] that

$$\sum_{m,n\in\mathbb{Z}} q^{m^2+2pn^2} + \sum_{m,n\in\mathbb{Z}} q^{pm^2+2n^2} = 2 + 2\sum_{k,n=1}^{\infty} \left(\frac{-8p}{n}\right) q^{kn}, \tag{3.1a}$$

$$\sum_{m,n\in\mathbb{Z}} q^{m^2+2pn^2} - \sum_{m,n\in\mathbb{Z}} q^{pm^2+2n^2} = 2\sum_{k,n=1}^{\infty} \left(\frac{p\epsilon}{k}\right)\left(\frac{-8\epsilon}{n}\right) q^{kn}. \tag{3.1b}$$

In these cases, the ideal $2\mathfrak{O}_K$ is ramified, and if we sum over all ideals relatively prime to $2\mathfrak{O}_K$, we obtain

$$\sum_{\substack{m,n\in\mathbb{Z}\\ m\equiv 1 \,(\mathrm{mod}\, 4)}} q^{m^2+2pn^2} + \sum_{\substack{m,n\in\mathbb{Z}\\ m\equiv 1 \,(\mathrm{mod}\, 4)}} q^{pm^2+2n^2} = \sum_{k,n=1}^{\infty} \left(\frac{4}{k}\right)\left(\frac{-8p}{n}\right) q^{kn}, \tag{3.2a}$$

$$\sum_{\substack{m,n\in\mathbb{Z}\\ m\equiv 1 \,(\mathrm{mod}\, 4)}} q^{m^2+2pn^2} - \sum_{\substack{m,n\in\mathbb{Z}\\ m\equiv 1 \,(\mathrm{mod}\, 4)}} q^{pm^2+2n^2} = \sum_{k,n=1}^{\infty} \left(\frac{4p\epsilon}{k}\right)\left(\frac{-8\epsilon}{n}\right) q^{kn}. \tag{3.2b}$$

The moduli of the characters in the above Lambert series both have a factor of 4 because the corresponding Dirichlet series are characterized by the absence of the prime 2.

Comparing (3.1a) and (3.2a), we can see that the first two identities in the theorem are equivalent to

$$2\sum_{\substack{m,n\in\mathbb{Z}\\ m\equiv 1\ (\mathrm{mod}\ 4)}} q^{m^2+2pn^2} + 2\sum_{m,n\in\mathbb{Z}} q^{4(m^2+2pn^2)} = S_1+S_4, \tag{3.3a}$$

$$2\sum_{\substack{m,n\in\mathbb{Z}\\ m\equiv 1\ (\mathrm{mod}\ 4)}} q^{pm^2+2n^2} + 2\sum_{m,n\in\mathbb{Z}} q^{4(pm^2+2n^2)} = S_2+S_3. \tag{3.3b}$$

Since the proofs are virtually identical, we will only establish (3.3a). The left hand side equals

$$\begin{aligned}
&\sum_{\substack{m,n\in\mathbb{Z}\\ m\text{ odd}}} q^{m^2+2pn^2} + 2\sum_{\substack{m,n\in\mathbb{Z}\\ m,n\text{ even}}} q^{m^2+2pn^2}\\
&\quad= \sum_{\substack{m,n\in\mathbb{Z}\\ m\text{ odd}\\ n\text{ even}}} q^{m^2+2pn^2} + \sum_{\substack{m,n\in\mathbb{Z}\\ m\text{ even}\\ n\text{ even}}} q^{m^2+2pn^2} + \sum_{\substack{m,n\in\mathbb{Z}\\ m\text{ odd}\\ n\text{ odd}}} q^{m^2+2pn^2} + \sum_{\substack{m,n\in\mathbb{Z}\\ m\text{ even}\\ n\text{ even}}} q^{m^2+2pn^2}\\
&\quad= \sum_{\substack{m,n\in\mathbb{Z}\\ n\text{ even}}} q^{m^2+2pn^2} + \sum_{a,b\in\mathbb{Z}} q^{(a-b)^2+2p(a+b)^2} = S_1+S_4.
\end{aligned}$$

Note that the identity

$$\sum_{\substack{m,n\in\mathbb{Z}\\ m\equiv n\ (\mathrm{mod}\ 2)}} C_{m,n} = \sum_{j,k\in\mathbb{Z}} C_{j-k,j+k}$$

was used in the penultimate equality.

For the third and fourth identities of Theorem 3.1, we observe that since we are only summing over odd values of k and n on the right of (3.2a), if we replace q by iq (where $i=\sqrt{-1}$), we get

$$\begin{aligned}
&i\sum_{\substack{m,n\in\mathbb{Z}\\ m\equiv 1\ (\mathrm{mod}\ 4)}} (-1)^n q^{m^2+2pn^2} + i^p \sum_{\substack{m,n\in\mathbb{Z}\\ m\equiv 1\ (\mathrm{mod}\ 4)}} (-1)^n q^{pm^2+2n^2}\\
&\qquad= \sum_{k,n=0}^{\infty} \left(\frac{4}{2k+1}\right)\left(\frac{-8p}{2n+1}\right)(iq)^{4kn+2k+2n+1}\\
&\qquad= i\sum_{k,n=0}^{\infty} (-1)^{k+n}\left(\frac{4}{2k+1}\right)\left(\frac{-8p}{2n+1}\right)q^{4kn+2k+2n+1}
\end{aligned}$$

$$= i \sum_{k,n=0}^{\infty} \left(\frac{-4}{2k+1}\right)\left(\frac{8p}{2n+1}\right) q^{4kn+2k+2n+1}.$$

We summarize the above calculation as

$$\text{(3.4a)} \qquad \sum_{k,n=1}^{\infty} \left(\frac{-4}{k}\right)\left(\frac{8p}{n}\right) q^{kn} = \sum_{\substack{m,n\in\mathbb{Z}\\ m\equiv 1\,(\mathrm{mod}\,4)}} (-1)^n q^{m^2+2pn^2} + \epsilon \sum_{\substack{m,n\in\mathbb{Z}\\ m\equiv 1\,(\mathrm{mod}\,4)}} (-1)^n q^{pm^2+2n^2},$$

$$\text{(3.4b)} \qquad \sum_{k,n=1}^{\infty} \left(\frac{-4p\epsilon}{k}\right)\left(\frac{8\epsilon}{n}\right) q^{kn} = \sum_{\substack{m,n\in\mathbb{Z}\\ m\equiv 1\,(\mathrm{mod}\,4)}} (-1)^n q^{m^2+2pn^2} - \epsilon \sum_{\substack{m,n\in\mathbb{Z}\\ m\equiv 1\,(\mathrm{mod}\,4)}} (-1)^n q^{pm^2+2n^2}.$$

It remains to use similar manipulations to show that

$$2 \sum_{\substack{m,n\in\mathbb{Z}\\ m\equiv 1\,(\mathrm{mod}\,4)}} (-1)^n q^{m^2+2pn^2} = S_1 - S_4,$$

$$2 \sum_{\substack{m,n\in\mathbb{Z}\\ m\equiv 1\,(\mathrm{mod}\,4)}} (-1)^n q^{pm^2+2n^2} = S_2 - S_3. \quad \blacksquare$$

There is a more succinct way to represent Theorem 3.1. We first define

$$\chi_A(x) = \begin{cases} 1 & \text{if } x = A, \\ 0 & \text{otherwise.} \end{cases}$$

Now let p take on one of the values $3, 5, 11$ or 29. Set $D = -8p$ and $\epsilon = \left(\frac{-1}{p}\right)$. We further define $m_1 = 1$ and let m_j be a prime that is represented by the quadratic form associated with S_j such that $m_j \nmid D$. For example, when $p = 3$, we can pick $m_2 = 11$, $m_3 = 5$ and $m_4 = 7$. Then for $d_p = D$ or $d_p = -8\epsilon$ we have

$$\text{(3.5a)} \qquad \sum_{j=1}^{4} \left(\frac{d_p}{m_j}\right) S_j = 4\chi_D(d_p) + 2 \sum_{k,n=1}^{\infty} \left(\frac{4D/d_p}{k}\right)\left(\frac{d_p}{n}\right) q^{kn} + 4 \sum_{k,n=1}^{\infty} \left(\frac{D/d_p}{k}\right)\left(\frac{d_p}{n}\right) q^{4kn},$$

$$\text{(3.5b)} \qquad \sum_{j=1}^{4} \left(\frac{-d_p}{m_j}\right) S_j = 2 \sum_{k,n=1}^{\infty} \left(\frac{-4D/d_p}{k}\right)\left(\frac{-d_p}{n}\right) q^{kn}.$$

The rest of the results in this section can all be proved in a similar manner, so we just record them without proofs.

THEOREM 3.2. *Let* $D = -4$. *Then*

$$\sum_{m,n\in\mathbb{Z}} q^{m^2+4n^2} = 1 + 2\sum_{k,n=1}^{\infty}\left(\frac{4}{k}\right)\left(\frac{D}{n}\right)q^{kn} + 4\sum_{k,n=1}^{\infty}\left(\frac{D}{n}\right)q^{4kn}.$$

THEOREM 3.3. *Let* $D = -8$ *and define*

$$S_1 = \sum_{m,n\in\mathbb{Z}} q^{m^2+8n^2} \quad \text{and} \quad S_2 = \sum_{m,n\in\mathbb{Z}} q^{3m^2+2mn+3n^2}.$$

Then

$$S_1 + S_2 = 2 + 2\sum_{k,n=1}^{\infty}\left(\frac{4}{k}\right)\left(\frac{D}{n}\right)q^{kn} + 4\sum_{k,n=1}^{\infty}\left(\frac{D}{n}\right)q^{4kn},$$

$$S_1 - S_2 = 2\sum_{k,n=1}^{\infty}\left(\frac{-4}{k}\right)\left(\frac{-D}{n}\right)q^{kn}.$$

THEOREM 3.4. *Let* $p = 5, 7, 13$ *or* 17, $D = -24p$ *and set* $\epsilon = \left(\frac{-1}{p}\right)$. *Define*

$$S_1 = \sum_{m,n\in\mathbb{Z}} q^{m^2+24pn^2}, \quad S_2 = \sum_{m,n\in\mathbb{Z}} q^{3m^2+8pn^2}, \quad S_3 = \sum_{m,n\in\mathbb{Z}} q^{pm^2+24n^2},$$

$$S_4 = \sum_{m,n\in\mathbb{Z}} q^{8m^2+3pn^2}, \quad S_5 = \sum_{m,n\in\mathbb{Z}} q^{(6p+1)m^2+2(6p-1)mn+(6p+1)n^2},$$

$$S_6 = \sum_{m,n\in\mathbb{Z}} q^{(3p+2)m^2+2(3p-2)mn+(3p+2)n^2},$$

$$S_7 = \sum_{m,n\in\mathbb{Z}} q^{(2p+3)m^2+2(2p-3)mn+(2p+3)n^2},$$

$$S_8 = \sum_{m,n\in\mathbb{Z}} q^{(p+6)m^2+2(p-6)mn+(p+6)n^2}.$$

For each p, *define* $m_1 = 1$ *and let* m_j *be a prime that is represented by the quadratic form associated with* S_j *such that* $m_j \nmid D$. *Then for* $d_p = D, 8\epsilon, 8p, -24\epsilon$, *we have*

$$\sum_{j=1}^{8}\left(\frac{d_p}{m_j}\right)S_j = 8\chi_D(d_p) + 2\sum_{k,n=1}^{\infty}\left(\frac{4D/d_p}{k}\right)\left(\frac{d_p}{n}\right)q^{kn} + 4\sum_{k,n=1}^{\infty}\left(\frac{D/d_p}{k}\right)\left(\frac{d_p}{n}\right)q^{4kn}, \tag{3.6a}$$

(3.6b) $$\sum_{j=1}^{8}\left(\frac{-d_p}{m_j}\right)S_j = 2\sum_{k,n=1}^{\infty}\left(\frac{-4D/d_p}{k}\right)\left(\frac{-d_p}{n}\right)q^{kn}.$$

THEOREM 3.5. *Let $p = 7, 13$ or 19, $D = -40p$, and set $\epsilon = \left(\frac{-1}{p}\right)$. Define*

$$S_1 = \sum_{m,n\in\mathbb{Z}} q^{m^2+40pn^2}, \quad S_2 = \sum_{m,n\in\mathbb{Z}} q^{5m^2+8pn^2}, \quad S_3 = \sum_{m,n\in\mathbb{Z}} q^{pm^2+40n^2},$$
$$S_4 = \sum_{m,n\in\mathbb{Z}} q^{8m^2+5pn^2}, \quad S_5 = \sum_{m,n\in\mathbb{Z}} q^{(10p+1)m^2+2(10p-1)mn+(10p+1)n^2},$$
$$S_6 = \sum_{m,n\in\mathbb{Z}} q^{(5p+2)m^2+2(5p-2)mn+(5p+2)n^2},$$
$$S_7 = \sum_{m,n\in\mathbb{Z}} q^{(2p+5)m^2+2(2p-5)mn+(2p+5)n^2},$$
$$S_8 = \sum_{m,n\in\mathbb{Z}} q^{(p+10)m^2+2(p-10)mn+(p+10)n^2}.$$

For each p, define $m_1 = 1$ and let m_j be a prime that is represented by the quadratic form associated with S_j such that $m_j \nmid D$. Then for $d_p = D, -8\epsilon, -8p, -40\epsilon$, we have

(3.7a) $$\sum_{j=1}^{8}\left(\frac{d_p}{m_j}\right)S_j = 8\chi_D(d_p) + 2\sum_{k,n=1}^{\infty}\left(\frac{4D/d_p}{k}\right)\left(\frac{d_p}{n}\right)q^{kn} + 4\sum_{k,n=1}^{\infty}\left(\frac{D/d_p}{k}\right)\left(\frac{d_p}{n}\right)q^{4kn},$$

(3.7b) $$\sum_{j=1}^{8}\left(\frac{-d_p}{m_j}\right)S_j = 2\sum_{k,n=1}^{\infty}\left(\frac{-4D/d_p}{k}\right)\left(\frac{-d_p}{n}\right)q^{kn}.$$

THEOREM 3.6. *Let $(p,\ell) = (5,7), (5,11)$ or $(7,11)$, $D = -24p\ell$, and set $\epsilon = \left(\frac{-1}{p}\right)$. Define*

$$S_1 = \sum_{m,n\in\mathbb{Z}} q^{m^2+24p\ell n^2}, \quad S_2 = \sum_{m,n\in\mathbb{Z}} q^{3m^2+8p\ell n^2}, \quad S_3 = \sum_{m,n\in\mathbb{Z}} q^{pm^2+24\ell n^2},$$
$$S_4 = \sum_{m,n\in\mathbb{Z}} q^{\ell m^2+24pn^2}, \quad S_5 = \sum_{m,n\in\mathbb{Z}} q^{8m^2+3p\ell n^2}, \quad S_6 = \sum_{m,n\in\mathbb{Z}} q^{3pm^2+8\ell n^2},$$
$$S_7 = \sum_{m,n\in\mathbb{Z}} q^{3\ell m^2+8pn^2}, \quad S_8 = \sum_{m,n\in\mathbb{Z}} q^{24m^2+p\ell n^2},$$
$$S_9 = \sum_{m,n\in\mathbb{Z}} q^{4m^2+4mn+(6p\ell+1)n^2}, \quad S_{10} = \sum_{m,n\in\mathbb{Z}} q^{8m^2+8mn+(3p\ell+2)n^2},$$

$$S_{11} = \sum_{m,n\in\mathbb{Z}} q^{12m^2+12mn+(2p\ell+3)n^2}, \quad S_{12} = \sum_{m,n\in\mathbb{Z}} q^{4pm^2+4pmn+(6\ell+p)n^2},$$
$$S_{13} = \sum_{m,n\in\mathbb{Z}} q^{4\ell m^2+4\ell mn+(6p+\ell)n^2}, \quad S_{14} = \sum_{m,n\in\mathbb{Z}} q^{24m^2+24mn+(p\ell+6)n^2},$$
$$S_{15} = \sum_{m,n\in\mathbb{Z}} q^{8pm^2+8pmn+(2p+3\ell)n^2}, \quad S_{16} = \sum_{m,n\in\mathbb{Z}} q^{8\ell m^2+8\ell mn+(2\ell+3p)n^2}.$$

For each (p,ℓ), *define* $m_1 = 1$ *and let* m_j *be a prime that is represented by the quadratic form associated with* S_j *such that* $m_j \nmid D$. *Then for* $d_p = D, -8\epsilon, -8p, -24\ell\epsilon, 8\ell\epsilon, 24p, 24\epsilon, 8p\ell$, *we have*

$$\text{(3.8a)} \quad \sum_{j=1}^{16} \left(\frac{d_p}{m_j}\right) S_j = 16\chi_D(d_p) + 2\sum_{k,n=1}^{\infty} \left(\frac{4D/d_p}{k}\right)\left(\frac{d_p}{n}\right) q^{kn} + 4\sum_{k,n=1}^{\infty} \left(\frac{D/d_p}{k}\right)\left(\frac{d_p}{n}\right) q^{4kn},$$

$$\text{(3.8b)} \quad \sum_{j=1}^{16} \left(\frac{-d_p}{m_j}\right) S_j = 2\sum_{k,n=1}^{\infty} \left(\frac{-4D/d_p}{k}\right)\left(\frac{-d_p}{n}\right) q^{kn}.$$

4. Discriminants of the form $d = p^2D$ where p is an odd prime. There are nine idoneal discriminants of the form $d = p^2D$, where p is an odd prime and D is also idoneal. They are $d = -27, -36, -72, -75, -99, -100, -147, -180$ and -315. As the proofs are similar, we will provide the details only for the case $d = -99$ after stating the identities.

THEOREM 4.1. *Let* $D = -3$. *Then*

$$\text{(4.1)} \quad \sum_{m,n\in\mathbb{Z}} q^{m^2+mn+7n^2} = 1 + 2\sum_{k,n=1}^{\infty} \left(\frac{9}{k}\right)\left(\frac{D}{n}\right) q^{kn} + 6\sum_{k,n=1}^{\infty} \left(\frac{D}{n}\right) q^{9kn}.$$

THEOREM 4.2. *Let* $D = -8$ *and define*

$$S_1 = \sum_{m,n\in\mathbb{Z}} q^{m^2+18n^2} \quad \text{and} \quad S_2 = \sum_{m,n\in\mathbb{Z}} q^{2m^2+9n^2}.$$

Then

$$\text{(4.2a)} \quad S_1 + S_2 = 2 + 2\sum_{k,n=1}^{\infty} \left(\frac{9}{k}\right)\left(\frac{9D}{n}\right) q^{kn} + 4\sum_{k,n=1}^{\infty} \left(\frac{D}{n}\right) q^{9kn},$$

$$\text{(4.2b)} \quad S_1 - S_2 = 2\sum_{k,n=1}^{\infty} \left(\frac{-3}{k}\right)\left(\frac{-3D}{n}\right) q^{kn}.$$

THEOREM 4.3. *Let $D = -11$ and define*

$$S_1 = \sum_{m,n\in\mathbb{Z}} q^{m^2+mn+25n^2} \quad \text{and} \quad S_2 = \sum_{m,n\in\mathbb{Z}} q^{5m^2+11mn+11n^2}.$$

Then

$$S_1 + S_2 = 2 + 2\sum_{k,n=1}^{\infty}\left(\frac{9}{k}\right)\left(\frac{9D}{n}\right)q^{kn} + 4\sum_{k,n=1}^{\infty}\left(\frac{D}{n}\right)q^{9kn}, \tag{4.3a}$$

$$S_1 - S_2 = 2\sum_{k,n=1}^{\infty}\left(\frac{-3}{k}\right)\left(\frac{-3D}{n}\right)q^{kn}. \tag{4.3b}$$

THEOREM 4.4. *Let $p = 3$ or 5, $D = -4$, and set $\epsilon = \left(\frac{-1}{p}\right)$. Define*

$$S_1 = \sum_{m,n\in\mathbb{Z}} q^{m^2+p^2n^2} \quad \text{and} \quad S_2 = \sum_{m,n\in\mathbb{Z}} q^{2m^2+2mn+(\frac{p^2+1}{2})n^2}.$$

Then

$$S_1 + S_2 = 2 + 2\sum_{k,n=1}^{\infty}\left(\frac{p^2}{k}\right)\left(\frac{p^2D}{n}\right)q^{kn} + 8\sum_{k,n=1}^{\infty}\left(\frac{D}{n}\right)q^{p^2kn}, \tag{4.4a}$$

$$S_1 - S_2 = 2\sum_{k,n=1}^{\infty}\left(\frac{\epsilon p}{k}\right)\left(\frac{\epsilon pD}{n}\right)q^{kn}. \tag{4.4b}$$

THEOREM 4.5. *Let $p = 5$ or 7, $D = -3$, and set $\epsilon = \left(\frac{-1}{p}\right)$. Define*

$$S_1 = \sum_{m,n\in\mathbb{Z}} q^{m^2+mn+(\frac{3p^2+1}{4})n^2} \quad \text{and} \quad S_2 = \sum_{m,n\in\mathbb{Z}} q^{3m^2+3mn+(\frac{p^2+3}{4})n^2}.$$

Then

$$S_1 + S_2 = 2 + 2\sum_{k,n=1}^{\infty}\left(\frac{p^2}{k}\right)\left(\frac{p^2D}{n}\right)q^{kn} + 12\sum_{k,n=1}^{\infty}\left(\frac{D}{n}\right)q^{p^2kn}, \tag{4.5a}$$

$$S_1 - S_2 = 2\sum_{k,n=1}^{\infty}\left(\frac{\epsilon p}{k}\right)\left(\frac{\epsilon pD}{n}\right)q^{kn}. \tag{4.5b}$$

THEOREM 4.6. *Let $D = -20$ and define*

$$S_1 = \sum_{m,n\in\mathbb{Z}} q^{m^2+45n^2}, \qquad S_2 = \sum_{m,n\in\mathbb{Z}} q^{5m^2+9n^2},$$

$$S_3 = \sum_{m,n\in\mathbb{Z}} q^{2m^2+2mn+23n^2}, \qquad S_4 = \sum_{m,n\in\mathbb{Z}} q^{7m^2+4mn+7n^2}.$$

Further define $m_1 = 1$ and let m_j be a prime that is represented by the quadratic form associated with S_j such that $m_j \nmid 3D$. Then for $d_p = D, -4,$

we have

$$\sum_{j=1}^{4}\left(\frac{d_p}{m_j}\right)S_j = 4\chi_D(d_p) + 2\sum_{k,n=1}^{\infty}\left(\frac{9D/d_p}{k}\right)\left(\frac{9d_p}{n}\right)q^{kn} + 4\sum_{k,n=1}^{\infty}\left(\frac{D/d_p}{k}\right)\left(\frac{d_p}{n}\right)q^{9kn}, \tag{4.6a}$$

$$\sum_{j=1}^{4}\left(\frac{-3d_p}{m_j}\right)S_j = 2\sum_{k,n=1}^{\infty}\left(\frac{-3D/d_p}{k}\right)\left(\frac{-3d_p}{n}\right)q^{kn}. \tag{4.6b}$$

THEOREM 4.7. *Let $D=-35$ and define*

$$S_1 = \sum_{m,n\in\mathbb{Z}} q^{m^2+mn+79n^2}, \qquad S_2 = \sum_{m,n\in\mathbb{Z}} q^{5m^2+5mn+17n^2},$$
$$S_3 = \sum_{m,n\in\mathbb{Z}} q^{7m^2+7mn+13n^2}, \qquad S_4 = \sum_{m,n\in\mathbb{Z}} q^{9m^2+9mn+11n^2}.$$

Further define $m_1 = 1$ and let m_j be a prime that is represented by the quadratic form associated with S_j such that $m_j \nmid 3D$. Then for $d_p = D, -7$, we have

$$\sum_{j=1}^{4}\left(\frac{d_p}{m_j}\right)S_j = 4\chi_D(d_p) + 2\sum_{k,n=1}^{\infty}\left(\frac{9D/d_p}{k}\right)\left(\frac{9d_p}{n}\right)q^{kn} + 4\sum_{k,n=1}^{\infty}\left(\frac{D/d_p}{k}\right)\left(\frac{d_p}{n}\right)q^{9kn}, \tag{4.7a}$$

$$\sum_{j=1}^{4}\left(\frac{-3d_p}{m_j}\right)S_j = 2\sum_{k,n=1}^{\infty}\left(\frac{-3D/d_p}{k}\right)\left(\frac{-3d_p}{n}\right)q^{kn}. \tag{4.7b}$$

We end this section with the proof of the case $d=-99$.

Proof of Theorem 4.3. As in all the previous cases, we begin with the identity for the fundamental discriminant $D=-11$, namely,

$$\sum_{m,n\in\mathbb{Z}} q^{m^2+mn+3n^2} = 1 + 2\sum_{k,n=1}^{\infty}\left(\frac{-11}{n}\right)q^{kn}. \tag{4.8}$$

In $\mathfrak{O}_{11} = \mathbb{Z}[(1+\sqrt{-11})/2]$, we know that $3\mathfrak{O}_{11}$ splits into $\mathfrak{s}\mathfrak{s}'$. We now sum $q^{N(\mathfrak{a})}$ over all non-zero ideals $\mathfrak{a}$ that are relatively prime to both $\mathfrak{s}$ and $\mathfrak{s}'$. This gives us the sum

$$\sum_{\substack{m,n\in\mathbb{Z}\\ m\equiv 1\,(\mathrm{mod}\,3)\\ n\equiv 0\,(\mathrm{mod}\,3)}} q^{m^2+mn+3n^2} + \sum_{\substack{m,n\in\mathbb{Z}\\ m\equiv 1\,(\mathrm{mod}\,3)\\ n\equiv 1\,(\mathrm{mod}\,3)}} q^{m^2+mn+3n^2},$$

whose corresponding Dirichlet series is characterized by the absence of the prime factor 3, i.e.

$$\left(1-\frac{1}{11^s}\right)^{-1}\prod_{p\text{ is inert}}\left(1-\frac{1}{p^{2s}}\right)^{-1}\prod_{\substack{p\text{ splits}\\ p\neq 3}}\left(1-\frac{1}{p^s}\right)^{-2} = L\left(\left(\frac{9}{\cdot}\right),s\right)L\left(\left(\frac{-99}{\cdot}\right),s\right).$$

We thus have the identity

$$\sum_{\substack{m,n\in\mathbb{Z}\\ m\equiv 1\,(\mathrm{mod}\,3)\\ n\equiv 0\,(\mathrm{mod}\,3)}} q^{m^2+mn+3n^2}+\sum_{\substack{m,n\in\mathbb{Z}\\ m\equiv 1\,(\mathrm{mod}\,3)\\ n\equiv 1\,(\mathrm{mod}\,3)}} q^{m^2+mn+3n^2}=\sum_{k,n=1}^{\infty}\left(\frac{9}{k}\right)\left(\frac{-99}{n}\right)q^{kn}. \tag{4.9}$$

We can extract more information from (4.9) by observing that the exponents of q in the first and second sums are congruent respectively to 1 and 2 modulo 3. This means that

$$\begin{aligned}\sum_{\substack{m,n\in\mathbb{Z}\\ m\equiv 1\,(\mathrm{mod}\,3)\\ n\equiv 0\,(\mathrm{mod}\,3)}} q^{m^2+mn+3n^2} &= \sum_{i,j=0}^{\infty}\left(\frac{9}{3i+1}\right)\left(\frac{-99}{3j+1}\right)q^{(3i+1)(3j+1)}\\ &\quad+\sum_{i,j=0}^{\infty}\left(\frac{9}{3i+2}\right)\left(\frac{-99}{3j+2}\right)q^{(3i+2)(3j+2)}\\ &=\sum_{i,j=0}^{\infty}\left(\frac{-3}{3i+1}\right)\left(\frac{33}{3j+1}\right)q^{(3i+1)(3j+1)}\\ &\quad+\sum_{i,j=0}^{\infty}\left(\frac{-3}{3i+2}\right)\left(\frac{33}{3j+2}\right)q^{(3i+2)(3j+2)}.\end{aligned} \tag{4.10}$$

Similarly,

$$\begin{aligned}\sum_{\substack{m,n\in\mathbb{Z}\\ m\equiv 1\,(\mathrm{mod}\,3)\\ n\equiv 1\,(\mathrm{mod}\,3)}} q^{m^2+mn+3n^2} &= \sum_{i,j=0}^{\infty}\left(\frac{9}{3i+1}\right)\left(\frac{-99}{3j+2}\right)q^{(3i+1)(3j+2)}\\ &\quad+\sum_{i,j=0}^{\infty}\left(\frac{9}{3i+2}\right)\left(\frac{-99}{3j+1}\right)q^{(3i+2)(3j+1)}\\ &=-\sum_{i,j=0}^{\infty}\left(\frac{-3}{3i+1}\right)\left(\frac{33}{3j+2}\right)q^{(3i+1)(3j+2)}\\ &\quad-\sum_{i,j=0}^{\infty}\left(\frac{-3}{3i+2}\right)\left(\frac{33}{3j+1}\right)q^{(3i+2)(3j+1)}.\end{aligned} \tag{4.11}$$

Combining (4.10) and (4.11), we arrive at the companion identity of (4.9),

(4.12)
$$\sum_{\substack{m,n\in\mathbb{Z}\\ m\equiv 1\ (\mathrm{mod}\ 3)\\ n\equiv 0\ (\mathrm{mod}\ 3)}} q^{m^2+mn+3n^2} - \sum_{\substack{m,n\in\mathbb{Z}\\ m\equiv 1\ (\mathrm{mod}\ 3)\\ n\equiv 1\ (\mathrm{mod}\ 3)}} q^{m^2+mn+3n^2} = \sum_{k,n=1}^{\infty} \left(\frac{-3}{k}\right)\left(\frac{33}{n}\right) q^{kn}.$$

It remains to use elementary series manipulations to show that

(4.13a)
$$\sum_{m,n\in\mathbb{Z}} q^{m^2+mn+25n^2} = \sum_{\substack{m,n\in\mathbb{Z}\\ m\equiv 0\ (\mathrm{mod}\ 3)\\ n\equiv 0\ (\mathrm{mod}\ 3)}} q^{m^2+mn+3n^2} + 2 \sum_{\substack{m,n\in\mathbb{Z}\\ m\equiv 1\ (\mathrm{mod}\ 3)\\ n\equiv 0\ (\mathrm{mod}\ 3)}} q^{m^2+mn+3n^2},$$

(4.13b)
$$\sum_{m,n\in\mathbb{Z}} q^{5m^2+11mn+11n^2} = \sum_{\substack{m,n\in\mathbb{Z}\\ m\equiv 0\ (\mathrm{mod}\ 3)\\ n\equiv 0\ (\mathrm{mod}\ 3)}} q^{m^2+mn+3n^2} + 2 \sum_{\substack{m,n\in\mathbb{Z}\\ m\equiv 1\ (\mathrm{mod}\ 3)\\ n\equiv 1\ (\mathrm{mod}\ 3)}} q^{m^2+mn+3n^2},$$

thereafter to replace the theta series on the right with the appropriate Lambert series using (4.9) and (4.12). ∎

5. The eight remaining non-fundamental discriminants. In this final section, we discuss the four discriminants of the form $d = 16D$, namely $d = -48, -64, -112, -240$, the three discriminants of the form $d = 64D$, namely $d = -192, -448, -960$, and the discriminant $d = -288$. For these discriminants, our methods of proof given in the previous sections would not work. However, we are able to "guess" the identities associated with these discriminants using similar generalized Lambert series we discovered in the previous sections. Once we derive these "identities", we can provide a proof (or more accurately a verification) using the theory of modular forms. More precisely, we note from [6, Theorem 10.9] that the theta series

$$\sum_{m,n\in\mathbb{Z}} q^{2am^2+2bmn+2cn^2}$$

is a modular form on $\Gamma_0(|d|)$ of weight 1 and multiplier $\left(\frac{d}{\cdot}\right)$, where $d = b^2 - 4ac$ is the discriminant. Consequently, it is also a modular form of level $m|d|$, some multiple of the discriminant. On the other hand, Theorem 4.7.1 of [10] can be used to show that each of the generalized Lambert series given below is also a modular form of the same weight and level with the same multiplier as the corresponding theta series. Thus these "identities" can be verified by checking that the coefficients of q of the modular forms on both sides agree beyond the Sturm bound (see [13, Corollary 9.20], [8, Theorem 3.13] or [14]). For example, in the proof of Theorem 5.3, which is associated with the discriminant $d = -240$, each of S_1 to S_4 and the Lambert series given in (5.3) are of level $N = 4|d| = 960$. The required Sturm bound is given by

$$\frac{1}{12}\left(N\prod_{p|N}\left(1+\frac{1}{p}\right)\right) = 192.$$

The respective levels and Sturm bounds for all the identities in this section are given in Table 6.

Table 6. Sturm bounds for identities proved via modular forms

d	Theorem	level	Sturm bound
-48	5.1	192	32
-112	5.1	448	64
-64	5.2	256	32
-240	5.3	960	192
-192	5.4	768	128
-448	5.4	1792	256
-960	5.5	3840	768
-288	5.6	5184	864

Theorem 5.1. *Let* $p = 3$ *or* 7, $D = -p$, *and set* $N_3 = 6$, $N_7 = 2$. *Define*

$$S_1 = \sum_{m,n\in\mathbb{Z}} q^{m^2+4pn^2} \quad \text{and} \quad S_2 = \sum_{m,n\in\mathbb{Z}} q^{4m^2+pn^2}.$$

Then

$$\text{(5.1a)} \qquad S_1 + S_2 = 2 + 2\sum_{k,n=1}^{\infty}\left(\frac{4}{k}\right)\left(\frac{4D}{n}\right)q^{kn} + 4\sum_{k,n=1}^{\infty}\left(\frac{4}{k}\right)\left(\frac{4D}{n}\right)q^{4kn} + 2N_p\sum_{k,n=1}^{\infty}\left(\frac{D}{n}\right)q^{16kn},$$

$$\text{(5.1b)} \qquad S_1 - S_2 = 2\sum_{k,n=1}^{\infty}\left(\frac{-4}{k}\right)\left(\frac{-4D}{n}\right)q^{kn}.$$

Theorem 5.2. *Let* $D = -4$ *and define*

$$S_1 = \sum_{m,n\in\mathbb{Z}} q^{m^2+16n^2} \quad \text{and} \quad S_2 = \sum_{m,n\in\mathbb{Z}} q^{4m^2+4mn+5n^2}.$$

Then

$$S_1+S_2=2+2\sum_{k,n=1}^{\infty}\left(\frac{4}{k}\right)\left(\frac{4D}{n}\right)q^{kn}+4\sum_{k,n=1}^{\infty}\left(\frac{4}{k}\right)\left(\frac{4D}{n}\right)q^{4kn}+8\sum_{k,n=1}^{\infty}\left(\frac{D}{n}\right)q^{16kn}, \tag{5.2a}$$

$$S_1-S_2=2\sum_{k,n=1}^{\infty}\left(\frac{-2}{k}\right)\left(\frac{-2D}{n}\right)q^{kn}. \tag{5.2b}$$

THEOREM 5.3. *Let $D=-15$ and define*

$$S_1=\sum_{m,n\in\mathbb{Z}}q^{m^2+60n^2},\qquad S_2=\sum_{m,n\in\mathbb{Z}}q^{3m^2+20n^2},$$
$$S_3=\sum_{m,n\in\mathbb{Z}}q^{4m^2+15n^2},\qquad S_4=\sum_{m,n\in\mathbb{Z}}q^{5m^2+12n^2}.$$

Further define $m_1=1$ and let m_j be a prime that is represented by the quadratic form associated with S_j such that $m_j\nmid D$. Then for $d_p=D,5$, we have

$$\sum_{j=1}^{4}\left(\frac{d_p}{m_j}\right)S_j=4\chi_D(d_p)+2\sum_{k,n=1}^{\infty}\left(\frac{4D/d_p}{k}\right)\left(\frac{4d_p}{n}\right)q^{kn}+4\sum_{k,n=1}^{\infty}\left(\frac{4D/d_p}{k}\right)\left(\frac{4d_p}{n}\right)q^{4kn}+4\sum_{k,n=1}^{\infty}\left(\frac{D/d_p}{k}\right)\left(\frac{d_p}{n}\right)q^{16kn}, \tag{5.3a}$$

$$\sum_{j=1}^{4}\left(\frac{-d_p}{m_j}\right)S_j=2\sum_{k,n=1}^{\infty}\left(\frac{-4D/d_p}{k}\right)\left(\frac{-4d_p}{n}\right)q^{kn}. \tag{5.3b}$$

THEOREM 5.4. *Let $p=3$ or 7, $D=-p$, and set $N_3=6$ and $N_7=2$. Define*

$$S_1=\sum_{m,n\in\mathbb{Z}}q^{m^2+16pn^2},\qquad S_2=\sum_{m,n\in\mathbb{Z}}q^{pm^2+16n^2},$$
$$S_3=\sum_{m,n\in\mathbb{Z}}q^{4m^2+4mn+(4p+1)n^2},\qquad S_4=\sum_{m,n\in\mathbb{Z}}q^{(p+4)m^2+2(p-4)mn+(p+4)n^2}.$$

Further define $m_1=1$ and let m_j be a prime that is represented by the quadratic form associated with S_j such that $m_j\nmid D$. Then

$$\sum_{j=1}^{4}\left(\frac{D}{m_j}\right)S_j = 4+2\sum_{k,n=1}^{\infty}\left(\frac{4}{k}\right)\left(\frac{4D}{n}\right)q^{kn}+4\sum_{k,n=1}^{\infty}\left(\frac{4}{k}\right)\left(\frac{4D}{n}\right)q^{4kn} + 8\sum_{k,n=1}^{\infty}\left(\frac{4}{k}\right)\left(\frac{4D}{n}\right)q^{16kn}+4N_p\sum_{k,n=1}^{\infty}\left(\frac{D}{n}\right)q^{64kn}, \tag{5.4a}$$

$$\sum_{j=1}^{4}\left(\frac{-4D}{m_j}\right)S_j = 2\sum_{k,n=1}^{\infty}\left(\frac{-4}{k}\right)\left(\frac{-4D}{n}\right)q^{kn}+4\sum_{k,n=1}^{\infty}\left(\frac{-4}{k}\right)\left(\frac{-4D}{n}\right)q^{4kn}, \tag{5.4b}$$

$$\sum_{j=1}^{4}\left(\frac{2}{m_j}\right)S_j = 2\sum_{k,n=1}^{\infty}\left(\frac{2}{k}\right)\left(\frac{2D}{n}\right)q^{kn}, \tag{5.4c}$$

$$\sum_{j=1}^{4}\left(\frac{-2}{m_j}\right)S_j = 2\sum_{k,n=1}^{\infty}\left(\frac{-2}{k}\right)\left(\frac{-2D}{n}\right)q^{kn}. \tag{5.4d}$$

THEOREM 5.5. *Let $D=-15$ and define*

$$S_1=\sum_{m,n\in\mathbb{Z}}q^{m^2+240n^2},\quad S_2=\sum_{m,n\in\mathbb{Z}}q^{3m^2+80n^2},\quad S_3=\sum_{m,n\in\mathbb{Z}}q^{4m^2+4mn+61n^2},$$
$$S_4=\sum_{m,n\in\mathbb{Z}}q^{5m^2+48n^2},\quad S_5=\sum_{m,n\in\mathbb{Z}}q^{12m^2+12mn+23n^2},\quad S_6=\sum_{m,n\in\mathbb{Z}}q^{15m^2+16n^2},$$
$$S_7=\sum_{m,n\in\mathbb{Z}}q^{16m^2+16mn+19n^2},\quad S_8=\sum_{m,n\in\mathbb{Z}}q^{17m^2+14mn+17n^2}.$$

Further define $m_1=1$ and let m_j be a prime that is represented by the quadratic form associated with S_j such that $m_j\nmid D$. Then for $d_p=D,5$, we have

$$\sum_{j=1}^{8}\left(\frac{d_p}{m_j}\right)S_j = 8\chi_D(d_p)+2\sum_{k,n=1}^{\infty}\left(\frac{4d_p}{k}\right)\left(\frac{4D/d_p}{n}\right)q^{kn} + 4\sum_{k,n=1}^{\infty}\left(\frac{4d_p}{k}\right)\left(\frac{4D/d_p}{n}\right)q^{4kn} + 8\sum_{k,n=1}^{\infty}\left(\frac{4d_p}{k}\right)\left(\frac{4D/d_p}{n}\right)q^{16kn} + 8\sum_{k,n=1}^{\infty}\left(\frac{d_p}{k}\right)\left(\frac{D/d_p}{n}\right)q^{64kn}, \tag{5.5a}$$

(5.5b)
$$\sum_{j=1}^{8}\left(\frac{-4d_p}{m_j}\right)S_j = 2\sum_{k,n=1}^{\infty}\left(\frac{-4d_p}{k}\right)\left(\frac{-4D/d_p}{n}\right)q^{kn} + 4\sum_{k,n=1}^{\infty}\left(\frac{-4d_p}{k}\right)\left(\frac{-4D/d_p}{n}\right)q^{4kn},$$

(5.5c)
$$\sum_{j=1}^{8}\left(\frac{2d_p}{m_j}\right)S_j = 2\sum_{k,n=1}^{\infty}\left(\frac{2d_p}{k}\right)\left(\frac{2D/d_p}{n}\right)q^{kn},$$

(5.5d)
$$\sum_{j=1}^{8}\left(\frac{-2d_p}{m_j}\right)S_j = 2\sum_{k,n=1}^{\infty}\left(\frac{-2d_p}{k}\right)\left(\frac{-2D/d_p}{n}\right)q^{kn}.$$

THEOREM 5.6. *Let $H = -32$ and define*

$$S_1 = \sum_{m,n\in\mathbb{Z}} q^{m^2+72n^2}, \quad S_2 = \sum_{m,n\in\mathbb{Z}} q^{4m^2+4mn+19n^2},$$
$$S_3 = \sum_{m,n\in\mathbb{Z}} q^{8m^2+9n^2}, \quad S_4 = \sum_{m,n\in\mathbb{Z}} q^{8m^2+8mn+11n^2}.$$

Further define $m_1 = 1$ and let m_j be a prime that is represented by the quadratic form associated with S_j such that $m_j \nmid 3H$. Then for $d_p = 4$, we have

(5.6a)
$$\sum_{j=1}^{4}\left(\frac{d_p}{m_j}\right)S_j = 4 + 2\sum_{k,n=1}^{\infty}\left(\frac{9H/d_p}{k}\right)\left(\frac{9d_p}{n}\right)q^{kn} + 4\sum_{k,n=1}^{\infty}\left(\frac{H/d_p}{k}\right)\left(\frac{d_p}{n}\right)q^{9kn} + 4\sum_{k,n=1}^{\infty}\left(\frac{9H/d_p}{k}\right)\left(\frac{9d_p/4}{n}\right)q^{4kn} + 8\sum_{k,n=1}^{\infty}\left(\frac{H/d_p}{k}\right)\left(\frac{d_p/4}{n}\right)q^{36kn},$$

(5.6b)
$$\sum_{j=1}^{4}\left(\frac{-3d_p}{m_j}\right)S_j = 2\sum_{k,n=1}^{\infty}\left(\frac{-3H/d_p}{k}\right)\left(\frac{-3d_p}{n}\right)q^{kn} + 4\sum_{k,n=1}^{\infty}\left(\frac{-3H/d_p}{k}\right)\left(\frac{-3d_p/4}{n}\right)q^{4kn}.$$

For $d_p = -4$, *we have*

$$\sum_{j=1}^{4}\left(\frac{d_p}{m_j}\right)S_j = 2\sum_{k,n=1}^{\infty}\left(\frac{9H/d_p}{k}\right)\left(\frac{9d_p}{n}\right)q^{kn} + 4\sum_{k,n=1}^{\infty}\left(\frac{H/d_p}{k}\right)\left(\frac{d_p}{n}\right)q^{9kn}, \tag{5.6c}$$

$$\sum_{j=1}^{4}\left(\frac{-3d_p}{m_j}\right)S_j = 2\sum_{k,n=1}^{\infty}\left(\frac{-3H/d_p}{k}\right)\left(\frac{-3d_p}{n}\right)q^{kn}. \tag{5.6d}$$

Acknowledgements. It is our pleasure to thank Professor Y. Tanigawa for bringing [10, Theorem 4.7.1] to our attention. We would also like to thank the anonymous referee for pointing out several errors in our original manuscript and informing us of the article [11] by Patane. Part of the present paper was written during the first author's stay at the University of Hong Kong. He would like to thank K. M. Tsang and Y. K. Lau for their warm hospitality. The second author was partially supported by the NIE Academic Research Fund RI 3/12 TPC.

References

[1] A. Berkovich and H. Yesilyurt, *Ramanujan's identities and representation of integers by certain binary and quaternary quadratic forms*, Ramanujan J. 20 (2009), 375–408.

[2] B. C. Berndt, *Ramanujan's Notebooks, Part III*, Springer, New York, 1991.

[3] D. A. Buell, *Binary Quadratic Forms*, Springer, New York, 1989.

[4] S. Cooper, *Theta function identities for level 15*, in: Ramanujan Rediscovered, Ramanujan Math. Soc. Lect. Notes Ser. 14, Ramanujan Math. Soc., Mysore, 2010, 79–86.

[5] J. G. Huard, P. Kaplan and K. S. Williams, *The Chowla–Selberg formula for genera*, Acta Arith. 73 (1995), 271–301.

[6] H. Iwaniec, *Topics in Classical Automorphic forms*, Grad. Stud. Math. 17, Amer. Math. Soc., Providence, RI, 1997.

[7] E. Kani, *Idoneal numbers and some generalizations*, Ann. Sci. Math. Québec 35 (2011), 197–227.

[8] L. J. P. Kilford, *Modular Forms. A Classical and Computational Introduction*, Imperial College Press, London, 2008.

[9] E. Landau, *Elementary Number Theory*, Chelsea, New York, 1958.

[10] T. Miyake, *Modular Forms*, Springer, Berlin, 1989.

[11] F. Patane, *An identity connecting theta series associated with binary quadratic forms of discriminant* Δ *and* $\Delta(\text{prime})^2$, arXiv:1409.6280 (2014).

[12] L. C. Shen, *On a class of q-series related to quadratic forms*, Bull. Inst. Math. Acad. Sinica 26 (1998), 111–126.

[13] W. A. Stein, *Modular Forms. A Computational Approach*, Grad. Stud. Math. 79, Amer. Math. Soc., Providence, RI, 2007.

[14] J. Sturm, *On the congruence of modular forms*, in: Number Theory, Lecture Notes in Math. 1240, Springer, Berlin, 1987, 275–280.
[15] Z. H. Sun and K. S. Williams, *On the number of representations of n by $ax^2+bxy+cy^2$*, Acta Arith. 122 (2006), 101–171.
[16] P. C. Toh, *Representations of certain binary quadratic forms as Lambert series*, Acta Arith. 143 (2010), 227–237.
[17] K. S. Williams, *Some Lambert series expansions of products of theta functions*, Ramanujan J. 3 (1999), 367–384.

Heng Huat Chan
Department of Mathematics
National University of Singapore
10 Lower Kent Ridge Road
Singapore 119076
E-mail: matchh@nus.edu.sg

Pee Choon Toh
Mathematics & Mathematics Education
National Institute of Education
Nanyang Technological University
1 Nanyang Walk
Singapore 637616
E-mail: peechoon.toh@nie.edu.sg

Received on 5.9.2014
and in revised form on 11.3.2015 (7919)