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# Euler constants for a Fuchsian group of the first kind

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**1. Introduction.** Let $\Gamma \subseteq \mathrm{PSL}(2,\mathbb{R})$ be a Fuchsian group of the first kind, $\chi$ an $r$-dimensional unitary representation of the group $\Gamma$, and $Z_{\Gamma,\chi}$ the corresponding zeta function.

In this paper we will evaluate the Euler–Selberg constant $\gamma_0^{(\Gamma,\chi)}$ and higher Euler–Selberg constants $\gamma_n^{(\Gamma,\chi)}$ ($n \in \mathbb{N}$) appearing in the Laurent series expansion of $Z'_{\Gamma,\chi}/Z_{\Gamma,\chi}$ around $s = 1$:

$$(1)\qquad \frac{Z'_{\Gamma,\chi}}{Z_{\Gamma,\chi}}(s) = \frac{1}{s-1} + \gamma_0^{(\Gamma,\chi)} + \gamma_1^{(\Gamma,\chi)}(s-1) + \gamma_2^{(\Gamma,\chi)}(s-1)^2 + \cdots .$$

More precisely, in Theorem 3.2 we prove that

$$\gamma_j^{(\Gamma,\chi)} = \frac{(-1)^j}{j!} \lim_{x\to\infty} \Bigg( \sum_{N(P)<x} \frac{\mathrm{Tr}(\chi(P))\Lambda(P)}{N(P)} \log^j N(P) - \frac{\log^{j+1} x}{j+1} \Bigg)$$

for $j = 0, 1, \ldots$, where the sum on the right is taken over all hyperbolic conjugacy classes of $\Gamma$, $N(P)$ denotes the norm of the class $P$, and $\Lambda(P) = \log N(P_0)/(1 - N(P)^{-1})$ for the primitive element $P_0$ such that $P = P_0^n$ for some $n$.

This extends the result proved by Y. Hashimoto, Y. Iijima, N. Kurokawa and M. Wakayama [6] from the case of compact Riemann surfaces to the case of non-compact Riemann surfaces of a finite volume. The proof of Theorem 3.2 relies on the representation of the logarithmic derivative of the Selberg zeta function $Z_{\Gamma,\chi}$ obtained in [4], and introduces an approach that differs from the one used in [6].

In Theorem 4.1 and Proposition 4.4, we obtain upper bounds (in terms of topological and spectral theoretical invariants of $\Gamma$) for the Euler–Selberg constant $\gamma_0^{(\Gamma)}$ in the case when $r = 1$ and $\chi = \mathrm{id}$, which improve the bound

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obtained by J. Jorgenson and J. Kramer in [10]. Lower bounds for the constant $\gamma_0^{(\Gamma)}$ are given in Propositions 5.1 and 5.2. The role of $\gamma_0^{(\Gamma)}$ in measuring the difference between the Arakelov and the Petersson metrics is discussed in [10, p. 2].

## 2. Preliminaries

**2.1.** *The Selberg zeta function.* Let $\mathcal{H}$ be the upper half-plane and $\Gamma$ a Fuchsian group of the first kind containing $n_1 \geq 1$ inequivalent parabolic classes. Then $\Gamma\backslash\mathcal{H}$ can be identified with a non-compact, hyperbolic Riemann surface of a finite volume with $n_1$ cusps. We will denote by $\mathfrak{F}$ the fundamental domain of that surface, and by $|\mathfrak{F}|$ its volume.

The group $\Gamma$ contains inequivalent hyperbolic, elliptic and parabolic classes. We denote the set of inequivalent hyperbolic resp. elliptic classes by $\{P\}$ resp. $\{R\}$, whereas the set of inequivalent, primitive hyperbolic classes is denoted by $\{P_0\}$. All elements of an elliptic class are conjugate in $\mathrm{SL}(2,\mathbb{R})$ to a rotation $\left(\begin{smallmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{smallmatrix}\right)$, for some $\theta \in (0,\pi)$. The order of the primitive element $R_0$ associated to $R$ is denoted by $M_R/2$.

The *Selberg zeta function* (see [7], [8] and [12]) associated to the pair $(\Gamma,\chi)$, where $\chi$ is an $r$-dimensional unitary representation of $\Gamma$ (without loss of generality, we may assume that $\chi$ is irreducible, see [8, p. 267]), is defined as an Euler product

$$(2)\qquad Z_{\Gamma,\chi}(s) = \prod_{\{P_0\}_\Gamma}\prod_{k=0}^{\infty}\det(I_r - \chi(P_0)N(P_0)^{-s-k}) = \prod_{\{P_0\}_\Gamma} Z_{\Gamma,\chi,P_0}(s),$$

converging absolutely for $\operatorname{Re} s > 1$. Here, $N(P_0)$ denotes the norm of the class $P_0$.

Investigation of $Z_{\Gamma,\chi}$ is closely related to the $L^2$ spectral theory of the operator $\Delta = y^2(\partial/\partial x^2 + \partial/\partial y^2)$ on $X = \Gamma\backslash\mathcal{H}$ (see, e.g., [9]). The operator $-\Delta$ is essentially self-adjoint on the space $\mathcal{D}$ of all twice continuously differentiable functions $f : \mathcal{H} \to V$ ($V$ is an $r$-dimensional vector space over $\mathbb{C}$) such that $f$ and $\Delta f$ are square integrable on $\mathfrak{F}$ and satisfy the equality $f(Sz) = \chi(S)f(z)$ for all $z \in \mathcal{H}$ and $S \in \Gamma$. It has the unique (self-adjoint) extension $-\widetilde{\Delta}$ to the space $\widetilde{\mathcal{D}}$.

Let $T_j$, $j = 1,\ldots,n_1$, denote all parabolic classes of the group $\Gamma$. Then $\chi(T_j)$ does not depend on the choice of the representative of the class $T_j$ and can be considered as a matrix from $\mathbb{C}^{r\times r}$. We will denote by $m_j$ the multiplicity of 1 as an eigenvalue of the matrix $\chi(T_j)$, and $n_1^* = \sum_{j=1}^{n_1} m_j$ will be the degree of singularity of $\chi$.

If $n_1^* \geq 1$, the operator $-\widetilde{\Delta}$ has both discrete and continuous spectrum; if $n_1^* = 0$, it has only a discrete spectrum. Let $\{\lambda_n\}_{n\geq 0}$ $(0 = \lambda_0 < \lambda_1 < \cdots,$

$\lambda_n \to \infty$) be the discrete spectrum of $-\widetilde{\Delta}$. The non-trivial zeros $s_n = 1/2 \pm ir_n$ of $Z_{\Gamma,\chi}(s)$, lying on the critical line, are related to the discrete spectrum, the numbers $r_n$ being solutions of the equations $1/4 + r_n^2 = \lambda_n$.

Let $0 < \varepsilon < \varepsilon_0 < 1/4$. We will denote by $N_{\varepsilon,\Gamma,\chi,\varepsilon_0}$ the number of eigenvalues $\lambda_n$ such that $\varepsilon \le \lambda_n < \varepsilon_0$.

The continuous spectrum of $-\widetilde{\Delta}$ is expressed through zeros (or equivalently, poles) of the hyperbolic scattering determinant

$$\phi(s) = \left(\frac{\Gamma(s-1/2)}{\Gamma(s)}\right)^{n_1^*} \sum_{n=1}^{\infty} \frac{a_n}{\mathfrak{g}_n^{2s}},$$

where the coefficients $a_n$ and $\mathfrak{g}_n$ depend on the group $\Gamma$ (see [8] or [5]).

One of the properties of the continuous spectrum is that it is possible to choose column vector $f_{h_j}$ (for a fixed $j \in \{1, \ldots, n_1\}$ and $1 \le h \le r$) so that $\chi(T_j^{-1}) f_{h_j} = e^{2\pi i \alpha_{h_j}} f_{h_j}$, where $0 \le \alpha_{h_j} < 1$ and $\alpha_{h_j} = 0$ iff $1 \le h \le m_j$ (see [8, pp. 268–269]).

As proved in [8, pp. 496–501], the Selberg zeta function $Z_{\Gamma,\chi}(s)$ is a meromorphic function of a finite order that satisfies the functional equation

$$Z_{\Gamma,\chi}(s)\Psi_{\Gamma,\chi}(s) = Z_{\Gamma,\chi}(1-s),$$

with the factor $\Psi_{\Gamma,\chi}$ given by

$$\Psi_{\Gamma,\chi}(s) = \phi(s) \cdot \eta\left(\frac{1}{2}\right) \exp\left(\int_{1/2}^{s} \frac{\eta'}{\eta}(u)\, du\right),$$

where

$$\begin{aligned} \frac{\eta'}{\eta}(s) = &-r|\mathfrak{F}| \frac{(s-1/2)\sin 2\pi(s-1/2)}{\cos 2\pi(s-1/2)+1} - 2\sum_{\alpha_{h_j}\neq 0} \log|1-e^{2\pi i\alpha_{h_j}}| \\ &+ \pi \sum_{\substack{\{R\}\\ 0<\theta(R)<\pi}} \frac{\mathrm{Tr}(\chi(R))}{M_R \sin\theta} \frac{\cos 2(\pi-\theta)(s-1/2) + \cos 2\theta(s-1/2)}{\cos 2\pi(s-1/2)+1} \\ &- 2n_1^*\left[\log 2 - \frac{\Gamma'}{\Gamma}(s) + \frac{\Gamma'}{\Gamma}(1-s)\right] - n_1^*\left[\frac{\Gamma'}{\Gamma}\left(s+\frac{1}{2}\right) - \frac{\Gamma'}{\Gamma}\left(\frac{3}{2}-s\right)\right]. \end{aligned}$$

**2.2.** *The logarithmic derivative of the Selberg zeta function.* Properties of the Selberg zeta function, such as finiteness of its order, Euler product representation and the functional equation, make it a representative member of the fundamental class of functions introduced by Jorgenson and Lang ([11, pp. 45–46]). The Jorgenson–Lang explicit formula has been generalized to a wider class of test functions [1] and given a new form applicable in the case when the factor of functional equation has infinitely many zeros or poles in the critical strip [3]. In [2] and [4], we have proved that the Selberg trace formula, when interpreted as an explicit formula, holds for a class of test functions that need not satisfy Selberg's boundedness condition.

As an application of the trace formula obtained, we deduced the following theorem, which gave a new integral representation of the logarithmic derivative of the Selberg zeta function.

THEOREM 2.A ([4]). (a) *For* $\operatorname{Re}\alpha > 0$ *and* $x > 1$,

$$\frac{Z'_{\Gamma,\chi}}{Z_{\Gamma,\chi}}\left(\frac{1}{2}+\alpha\right) = \frac{1}{1+x^{2\alpha}} \sum_{N(P)<x} \frac{\operatorname{Tr}(\chi(P))\Lambda(P)}{N(P)^{\alpha+1/2}}(x^{2\alpha} - N(P)^{2\alpha}) \tag{3}$$
$$+ \frac{4\alpha x^{\alpha}}{1+x^{2\alpha}}\Bigg(\sum_{n=0}^{M} \frac{\cos yr_n}{\alpha^2+r_n^2} + \int_0^\infty \frac{\cos yt\, dR(t)}{\alpha^2+t^2} - \frac{|\mathfrak{F}|}{2\pi}\int_0^\infty \frac{t\cdot r(t)\cos yt}{\alpha^2+t^2}\,dt$$
$$+ \frac{n_1^*}{\pi}\int_0^\infty \frac{H(t)\cos yt\, dt}{\alpha^2+t^2}$$
$$- \sum_{\substack{\{R\}_\Gamma \\ 0<\theta(R)<\pi}} \frac{\operatorname{Tr}(\chi(R))}{2M_R\sin\theta}\int_0^\infty \frac{\cos yt}{\alpha^2+t^2}\,\frac{\cosh 2(\pi-\theta)t + \cosh 2\theta t}{\cosh 2\pi t + 1}\,dt\Bigg).$$

(b)

$$\frac{Z'_{\Gamma,\chi}}{Z_{\Gamma,\chi}}\left(\frac{1}{2}+\alpha\right) = O\left(\min\left\{\frac{T}{\sigma\log|T|}, \frac{T^{1-2\sigma}}{\sigma}\right\}\right) \quad \textit{as } |T|\to\infty$$

*for* $\alpha = \sigma + iT,\ 1/2 > \sigma > 0$.

Here, we put $r_n = -i\sqrt{1/4-\lambda_n} = -i\mu_n$ for $\lambda_n \le 1/4$, $n = 0,\dots,M$,

$$R(t) = N[0 \le r_n \le t] - \frac{1}{4\pi}\int_{-t}^{t} \frac{\phi'}{\phi}\left(\frac{1}{2}+iu\right)du - r\,\frac{|\mathfrak{F}|}{4\pi}\,t^2 + \frac{n_1^*}{\pi}\,t\log t$$
$$-\frac{t}{\pi}\Big(n_1^* - n_1^*\log 2 - \sum_{\alpha_{h_j}\neq 0} \log|1-e^{2\pi i\alpha_{h_j}}|\Big),$$

$r(t) = \tanh \pi t - 1$ and $H(t) = \frac{\Gamma'}{\Gamma}(1+it) - \log t$.

In [8, Th. 2.29 on p. 468 with $m = 0$], it is shown that $R(t) = O(|t|/\log|t|)$ as $|t|\to\infty$ and that

$$R_1(t) = \int_1^t R(u)\,du = O\left(\frac{|t|}{\log^2|t|}\right) \quad \text{as } |t|\to\infty.$$

Obviously, $r(t) = O(e^{-t})$ and $H(t) = O(1/t)$ as $t\to\infty$.

**2.3.** *The function $\psi_{\Gamma,\chi}(x)$ and the prime geodesic theorem.* The prime geodesics counting function $\pi_{\Gamma,\chi}(x)$ is defined as

$$\pi_{\Gamma,\chi}(x) = \sum_{N(P_0)\le x} \operatorname{Tr}(\chi(P_0)),$$

where the sum is taken over all primitive hyperbolic classes $P_0$ of $\Gamma$ the norm of which does not exceed $x$. When $r = 1$ and $\chi = \mathrm{id}$, $\pi_\Gamma(x)$ is simply the number of primitive hyperbolic classes $P_0$ of $\Gamma$ with norms not exceeding $x$, or, equivalently, the number of primitive geodesics of lengths not larger than $\log x$.

The function $\psi_{\Gamma,\chi}(x)$ is given by

$$\psi_{\Gamma,\chi}(x) = \sum_{N(P)\le x} \Lambda(P) \operatorname{Tr}(\chi(P)),$$

where the sum is taken over all hyperbolic classes $P$ of $\Gamma$ whose norm does not exceed $x$.

In [4, Th. 6.1] we have proved that

$$\psi_{\Gamma,\chi}(x) = \sum_{n=0}^{M} \frac{x^{s_n}}{s_n} + g_{\Gamma,\chi}(x) \quad \text{for } x \ge 2,$$

where $s_n = 1/2 + ir_n$ for $n = 0, \dots, M$ and $-C_{\Gamma,\chi} x^{3/4} \le g_{\Gamma,\chi}(x) \le C_{\Gamma,\chi} x^{3/4}$ for $x \ge 2$. The constant $C_{\Gamma,\chi}$ depends only upon $\Gamma$ and $\chi$.

The prime geodesic theorem proved in [4] states that

$$\pi_{\Gamma,\chi}(x) = \sum_{n=0}^{M} \operatorname{li}(x^{s_n}) + h_{\Gamma,\chi}(x) \quad \text{for } x \ge 2,$$

where $-c_{\Gamma,\chi} x^{3/4} \log^{-1} x \le h_{\Gamma,\chi}(x) \le c_{\Gamma,\chi} x^{3/4} \log^{-1} x$ for some constant $c_{\Gamma,\chi}$ depending only upon $\Gamma$ and $\chi$.

**3. Euler–Selberg constants.** In the proof of our next theorem, we shall make use of the following lemma that can be easily verified by induction.

Lemma 3.1. *Let $f$ be a meromorphic function with a pole of order $m$ at a point $s = s_0$, and the corresponding Laurent series expansion*

$$f(s) = \sum_{n=-m}^{\infty} a_n (s - s_0)^n.$$

*Assume that for a fixed $\delta > 0$ and $|s - s_0| < \delta$ the function $f$ can also be represented as*

$$f(s) = \sum_{n=-m}^{\infty} f_n(x, s)(s - s_0)^n$$

*for all* $x > x_0$. *Suppose* $\lim_{x\to\infty} f_n(x,s) = b_n \in \mathbb{C}$ *for* $n \geq -m$, *independently of* $s$ *in the above disc. Then* $b_n = a_n$ *for all* $n \geq -m$.

THEOREM 3.2. *Let*

$$\frac{Z'_{\Gamma,\chi}}{Z_{\Gamma,\chi}}(s) = \frac{1}{s-1} + \gamma_0^{(\Gamma,\chi)} + \gamma_1^{(\Gamma,\chi)}(s-1) + \gamma_2^{(\Gamma,\chi)}(s-1)^2 + \cdots$$

*be the Laurent series expansion of the logarithmic derivative of the Selberg zeta function around* $s = 1$. *Then*

$$\gamma_j^{(\Gamma,\chi)} = \frac{(-1)^j}{j!} \lim_{x\to\infty} \bigg( \sum_{N(P)<x} \frac{\operatorname{Tr}(\chi(P))\Lambda(P)}{N(P)} \log^j N(P) - \frac{\log^{j+1} x}{j+1} \bigg)$$

*for all* $j = 0, 1, \ldots$.

*Proof.* Putting $s = 1/2 + \alpha$ into (3), taking into account that

$$\frac{\cos yr_0}{(s-1/2)^2 + r_0^2} = \frac{x^{1/2} + x^{-1/2}}{2s(s-1)},$$

and conveniently regrouping the terms, we get

$$\begin{aligned}
(4)\quad \frac{Z'_{\Gamma,\chi}}{Z_{\Gamma,\chi}}(s) &= \frac{x^{2s-1}}{1+x^{2s-1}} \bigg( \sum_{N(P)<x} \frac{\operatorname{Tr}(\chi(P))\Lambda(P)}{N(P)} N(P)^{1-s} + \frac{x^{1-s}}{s-1} \bigg) \\
&\quad + \frac{(4s-2)x^{s-1/2}}{1+x^{2s-1}} \bigg( \sum_{n=1}^{M} \frac{\cos yr_n}{(s-1/2)^2 + r_n^2} + \int_0^\infty \frac{\cos yt\, dR(t)}{(s-1/2)^2+t^2} \\
&\quad - \frac{|\mathfrak{F}|}{2\pi} \int_0^\infty \frac{t\cdot r(t)\cos yt}{(s-1/2)^2+t^2}\, dt + \frac{n_1^*}{\pi} \int_0^\infty \frac{H(t)\cos yt\, dt}{(s-1/2)^2+t^2} \\
&- \sum_{\substack{\{R\}_\Gamma \\ 0<\theta(R)<\pi}} \frac{\operatorname{Tr}(\chi(R))}{2M_R \sin\theta} \int_0^\infty \frac{\cos yt}{(s-1/2)^2+t^2}\, \frac{\cosh 2(\pi-\theta)t + \cosh 2\theta t}{\cosh 2\pi t + 1}\, dt \bigg) \\
&- \frac{1}{1+x^{2s-1}} \bigg( \sum_{N(P)<x} \operatorname{Tr}(\chi(P))\Lambda(P)N(P)^{s-1} - \frac{x^s}{s} \bigg) + \frac{x^{s-1}}{1+x^{2s-1}} \bigg( \frac{1}{s} + \frac{1}{s-1} \bigg)
\end{aligned}$$

for $x > 1$. Now,

$$\frac{x^{1-s}}{s-1} = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(n+1)!} (s-1)^n \log^{n+1} x$$

and

$$\begin{aligned}
&\sum_{N(P)<x} \frac{\operatorname{Tr}(\chi(P))\Lambda(P)}{N(P)} N(P)^{1-s} \\
&\qquad = \sum_{N(P)<x} \frac{\operatorname{Tr}(\chi(P))\Lambda(P)}{N(P)} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} (s-1)^n \log^n N(P).
\end{aligned}$$

If we put

$$f_{-1}(x,s)=\frac{x^{2s-1}}{1+x^{2s-1}}+\frac{x^{s-1}}{1+x^{2s-1}},$$

$$\begin{aligned}f_0(x,s)=\frac{(4s-2)x^{s-1/2}}{1+x^{2s-1}}&\bigg(\sum_{n=1}^{M}\frac{\cos yr_n}{(s-1/2)^2+r_n^2}+\int_0^\infty\frac{\cos yt\,dR(t)}{(s-1/2)^2+t^2}\\&-\frac{|\mathfrak{F}|}{2\pi}\int_0^\infty\frac{t\cdot r(t)\cos yt}{(s-1/2)^2+t^2}\,dt+\frac{n_1^*}{\pi}\int_0^\infty\frac{H(t)\cos yt\,dt}{(s-1/2)^2+t^2}\\&-\sum_{\substack{\{R\}_\Gamma\\0<\theta(R)<\pi}}\frac{\mathrm{Tr}(\chi(R))}{2M_R\sin\theta}\int_0^\infty\frac{\cos yt}{(s-1/2)^2+t^2}\,\frac{\cosh 2(\pi-\theta)t+\cosh 2\theta t}{\cosh 2\pi t+1}\,dt\bigg)\\&-\frac{1}{1+x^{2s-1}}\bigg(\sum_{N(P)<x}\mathrm{Tr}(\chi(P))\Lambda(P)N(P)^{s-1}-\frac{x^s}{s}-\frac{x^{s-1}}{s}\bigg)\\&+\frac{x^{2s-1}}{1+x^{2s-1}}\bigg(\sum_{N(P)<x}\frac{\mathrm{Tr}(\chi(P))\Lambda(P)}{N(P)}-\log x\bigg)\\&=A(x,s)-B(x,s)+C(x,s)\end{aligned}$$

and

$$f_n(x,s)=\frac{x^{2s-1}}{1+x^{2s-1}}\,\frac{(-1)^n}{n!}\bigg(\sum_{N(P)<x}\frac{\mathrm{Tr}(\chi(P))\Lambda(P)}{N(P)}\log^n N(P)-\frac{\log^{n+1}x}{n+1}\bigg)$$

for $n\ge 1$, then (4) becomes

$$\frac{Z'_{\Gamma,\chi}}{Z_{\Gamma,\chi}}(s)=\sum_{n=-1}^{\infty}f_n(x,s)(s-1)^n.$$

We will first evaluate $\lim_{x\to\infty}A(x,s)$ for $s$ in the disc $|s-1|<\delta$, where $\delta=\min\{1/4,1/2-\mu_1\}$ if $0<\lambda_1<1/4$, and $\delta=1/4$ if $\lambda_1\ge 1/4$. (Recall that, in the first case, $\mu_1=\sqrt{1/4-\lambda_1}<1/2$.) If $M\ge 1$, we have

$$\sum_{n=1}^{M}\frac{\cos yr_n}{(s-1/2)^2+r_n^2}\le C_1x^{\mu_1}\tag{5}$$

for some constant $C_1$, uniformly in the disc $|s-1|<\delta$.

Furthermore,

$$\bigg|\int_0^\infty\frac{\cos yt\,dR(t)}{(s-1/2)^2+t^2}\bigg|\le y\bigg|\int_3^\infty\frac{R(t)\sin yt\,dt}{(s-1/2)^2+t^2}\bigg|+\bigg|\int_3^\infty\frac{2tR(t)\cos yt\,dt}{((s-1/2)^2+t^2)^2}\bigg|+C_1y+C_2$$

for some constants $C_1$ and $C_2$, and

$$\left|\int_3^\infty \frac{R(t)\sin yt\,dt}{(s-1/2)^2+t^2}\right| \le y\left|\int_3^\infty \frac{R_1(t)\cos yt\,dt}{(s-1/2)^2+t^2}\right| + \left|\int_3^\infty \frac{2tR_1(t)\sin yt\,dt}{((s-1/2)^2+t^2)^2}\right|.$$

Since $R_1(t) = O(t/\log^2 t)$ as $t \to \infty$, the two integrals on the right-hand side of the last inequality are bounded by some constants $K_1$ and $K_2$, uniformly in $s$ for $|s-1| < \delta$. Therefore,

$$\left|\int_0^\infty \frac{\cos yt\,dR(t)}{(s-1/2)^2+t^2}\right| \le K_1 \log^2 x + (C_1+K_2)\log x + C_2. \tag{6}$$

Using (5), (6), the facts that

$$r(t) = O(e^{-t}), \qquad H(t) = O(1/|t|), \qquad \frac{\cosh 2(\pi-\theta)t + \cosh 2\theta t}{\cosh 2\pi t + 1} = O(1),$$

as $|t| \to \infty$, and having in mind that the sum over elliptic classes is finite, we obtain

$$|A(x,s)| \le C\left|\frac{(4s-2)x^{s-1/2}}{1+x^{2s-1}}\right| \max\{x^{\mu_1}, \log^2 x\}$$

for large $x$, and $s$ in the disc $|s-1| < \delta$. Therefore, $\lim_{x\to\infty} A(x,s) = 0$ uniformly for $s$ in the disc $|s-1| < \delta$.

The next step is to prove that $\lim_{x\to\infty} B(x,s) = 0$ uniformly in that disc. To do so, we will use the integral representation of the sum appearing in $B(x,s)$:

$$\sum_{N(P)<x} \operatorname{Tr}(\chi(P))\Lambda(P)N(P)^{s-1} = \int_\tau^x t^{s-1}\,d\psi_{\Gamma,\chi}(t),$$

where $\tau > 1$ is less than the smallest $N(P)$.

Now, for $x > 2$, one gets

$$\begin{aligned}
&|B(x,s)| \\
&= \left|\frac{1}{1+x^{2s-1}}\right| \bigg| \int_\tau^2 t^{s-1}\,d\psi_{\Gamma,\chi}(t) - 2^{s-1}\psi_{\Gamma,\chi}(2) + \sum_{n=0}^M \frac{x^{s+s_n-1}}{s_n} + x^{s-1}g_{\Gamma,\chi}(x) \\
&\quad - (s-1)\sum_{n=0}^M \int_2^x \frac{t^{s+s_n-2}}{s_n}\,dt - (s-1)\int_2^x t^{s-2}g_{\Gamma,\chi}(t)\,dt - \frac{x^s}{s} - \frac{x^{s-1}}{s}\bigg| \\
&\le \left|\frac{1}{1+x^{2s-1}}\right| \bigg( C + \sum_{n=1}^M \left|\frac{x^{s+s_n-1}}{s+s_n-1}\right| + |x^{s-1}|\,|g_{\Gamma,\chi}(x)| \\
&\quad + \delta\left|\int_2^x t^{s-2}g_{\Gamma,\chi}(t)\,dt\right| + \left|\frac{x^{s-1}}{s}\right|\bigg)
\end{aligned}$$

uniformly in $|s-1|<\delta$, where the constant $C$ depends on $\Gamma$ and $\chi$ only. Since $g_{\Gamma,\chi}(x)=O(x^{3/4})$ as $x\to\infty$, passing to the limit we conclude that $\lim_{x\to\infty} B(x,s)=0$ uniformly in the disc $|s-1|<\delta$.

Finally, we will prove that $\lim_{x\to\infty} C(x,s)$ is a finite number, not depending on $s$ in the disc $|s-1|<\delta$. First, we represent the sum appearing in the expression for $C(x,s)$ as the Stieltjes integral

$$\sum_{N(P)<x} \frac{\mathrm{Tr}(\chi(P))\Lambda(P)}{N(P)} = \int_{\tau}^{x} \frac{d\psi_{\Gamma,\chi}(t)}{t}.$$

Integration by parts yields

$$\int_{\tau}^{x} \frac{d\psi_{\Gamma,\chi}(t)}{t} - \log x = 1 - \sum_{n=1}^{M} \frac{x^{s_n-1}}{\lambda_n} + \frac{g_{\Gamma,\chi}(x)}{x} + \int_{2}^{x} t^{-2} g_{\Gamma,\chi}(t)\,dt + K,$$

where the constant $K$ depends on $\Gamma$ and $\chi$ only. Boundedness of $g_{\Gamma,\chi}(x)$ by $x^{3/4}$ and the fact that $s_n<1$ for $n=1,\dots,M$ imply that the limit of the right-hand side of the above equality is finite as $x\to\infty$. Since $x^{2s-1}/(1+x^{2s-1})\to 1$ as $x\to\infty$, uniformly in $|s-1|<1/4$, we obtain

$$\lim_{x\to\infty} f_0(x,s) = \lim_{x\to\infty}\biggl(\sum_{N(P)<x} \frac{\mathrm{Tr}(\chi(P))\Lambda(P)}{N(P)} - \log x\biggr) = b_0 \in \mathbb{C}$$

for all $s$ in the disc $|s-1|<\delta$.

It is left to prove that $\lim_{x\to\infty} f_n(x,s)=b_n\in\mathbb{C}$ for all $n\in\mathbb{N}$. Arguing as above, we represent the sum appearing in the expression for $f_n(x,s)$ as the Stieltjes integral and integrate by parts to obtain

$$\begin{aligned}\int_{\tau}^{x} \frac{\log^n t}{t}\,d\psi_{\Gamma,\chi}(t) - \frac{\log^{n+1} x}{n+1} &= \frac{\log^n x}{x}\,\psi_{\Gamma,\chi}(x) - \log^n x \\ &+ \sum_{n=1}^{M} \frac{1}{s_n}\int_{2}^{x} \frac{\log^{n-1} t\,(\log t - n)}{t^{2-s_n}}\,dt + \int_{2}^{x} g_{\Gamma,\chi}(t)\,\frac{\log^{n-1} t\,(\log t - n)}{t^2}\,dt + K_1,\end{aligned}$$

where the constant $K_1$ depends on $\Gamma$ and $\chi$ only.

Since $s_n<1$ (for $n\ge 1$) and $g_{\Gamma,\chi}(x)=O(x^{3/4})$ as $x\to\infty$, the limit as $x\to\infty$ of the two integrals on the right-hand side of the above equality is finite. This proves that $\lim_{x\to\infty} f_n(x,s)=b_n\in\mathbb{C}$ for all $n\in\mathbb{N}$, uniformly in the disc $|s-1|<\delta$, and completes the proof of the theorem.

The case $r=1$ and $\chi=\mathrm{id}$ deserves our special attention. In this case, the Euler–Selberg constant $\gamma_0^{(\Gamma)}$ is a real number that plays an important role in applications. We have just proved that

$$\gamma_0^{(\Gamma)} = \lim_{x\to\infty}\biggl(\sum_{N(P)<x} \frac{\Lambda(P)}{N(P)} - \log x\biggr).$$

However, the constant $\gamma_0^{(\Gamma)}$ can be expressed in terms of primitive geodesics only. A benefit from such an expression will be clear in the next two sections, where we consider upper and lower bounds for $\gamma_0^{(\Gamma)}$. In the following, we shall assume that $r=1$ and $\chi=\mathrm{id}$, and omit the index $\chi$ in further notation.

PROPOSITION 3.3. *For $x>T\geq 2$ we have*

$$\gamma_0^{(\Gamma)} = \sum_{N(P_0)\leq T}\frac{Z'_{P_0}}{Z_{P_0}}(1) + \lim_{x\to\infty}\Big(\sum_{T<N(P_0)<x}\frac{\log N(P_0)}{N(P_0)-1} - \log x\Big) + \sum_{k=2}^{\infty}\int_T^{\infty}\frac{\log t\, d\pi_\Gamma(t)}{t^k-1}. \tag{7}$$

*Proof.* For $x>T\geq 2$ one has

$$\begin{aligned}\gamma_0^{(\Gamma)} &= \lim_{x\to\infty}\Big(\sum_{N(P)<x}\frac{\Lambda(P)}{N(P)} - \log x\Big)\\ &= \lim_{x\to\infty}\Big[\sum_{N(P_0)\leq T}\ \sum_{k:\,N(P_0)^k<x}\frac{\log N(P_0)}{N(P_0)^k-1}\\ &\quad + \Big(\sum_{T<N(P_0)<x}\frac{\log N(P_0)}{N(P_0)-1} - \log x\Big) + \sum_{k=2}^{\infty}\ \sum_{T<N(P_0)<\sqrt[k]{x}}\frac{\log N(P_0)}{N(P_0)^k-1}\Big].\end{aligned} \tag{8}$$

The second sum on the right-hand side of (8) is taken over all natural numbers $k$ such that $N(P_0)^k<x$. Since the first sum is finite, we get

$$\lim_{x\to\infty}\sum_{N(P_0)\leq T}\ \sum_{k:\,N(P_0)^k<x}\frac{\log N(P_0)}{N(P_0)^k-1} = \sum_{N(P_0)\leq T}\sum_{k=1}^{\infty}\frac{\log N(P_0)}{N(P_0)^k-1} = \sum_{N(P_0)\leq T}\frac{Z'_{P_0}}{Z_{P_0}}(1). \tag{9}$$

Furthermore,

$$\begin{aligned}\sum_{T<N(P_0)<x}\frac{\log N(P_0)}{N(P_0)-1} - \log x &= \int_T^x\frac{\log t\, d\pi_\Gamma(t)}{t-1} - \log x\\ &= \sum_{n=0}^{M}\int_T^x\frac{\log t\, d(\mathrm{li}(t^{s_n}))}{t-1} + \int_T^x\frac{\log t\, dh_\Gamma(t)}{t-1} - \log x\\ &= \sum_{n=1}^{M}\int_T^x\frac{t^{s_n-1}\,dt}{t-1} + \log(x-1) - \log x + \frac{\log x\cdot h_\Gamma(x)}{x-1}\\ &\quad - \log(T-1) - \frac{\log T\cdot h_\Gamma(T)}{T-1} + \int_T^x\frac{\log t\cdot h_\Gamma(t)\,dt}{(t-1)^2} - \int_T^x\frac{h_\Gamma(t)\,dt}{t(t-1)}.\end{aligned} \tag{10}$$

Therefore, letting $x \to \infty$ and having in mind that $|h_\Gamma(t)| \le c_\Gamma t^{3/4} \log^{-1} t$ for all $t \ge 2$, we obtain

$$(11) \qquad \lim_{x\to\infty}\Bigg(\sum_{T<N(P_0)<x} \frac{\log N(P_0)}{N(P_0)-1} - \log x\Bigg) = B(T) < \infty.$$

It is left to consider the limit of the last sum on the right-hand side of (8). We have

$$\sum_{T<N(P_0)<\sqrt[k]{x}} \frac{\log N(P_0)}{N(P_0)^k-1} = \int_T^{\sqrt[k]{x}} \frac{\log t\, d\pi_\Gamma(t)}{t^k-1}.$$

Hence,

$$\lim_{x\to\infty} \sum_{T<N(P_0)<\sqrt[k]{x}} \frac{\log N(P_0)}{N(P_0)^k-1} = \int_T^{\infty} \frac{\log t\, d\pi_\Gamma(t)}{t^k-1}.$$

For $k \ge 2$ one has

$$\int_T^{\infty} \frac{\log t\, d\pi_\Gamma(t)}{t^k-1} = \sum_{n=0}^{M} \int_T^{\infty} \frac{t^{s_n-1}dt}{t^k-1} + \int_T^{\infty} \frac{\log t\, dh_\Gamma(t)}{t^k-1}.$$

It is easy to see that

$$(12) \qquad \int_T^{\infty} \frac{t^{s_n-1}\,dt}{t^k-1} \le \int_T^{\infty} \frac{dt}{(t-1)^{k+1-s_n}} = \frac{(T-1)^{s_n-k}}{k-s_n}.$$

Integration by parts and simple estimations yield

$$(13) \qquad \begin{aligned} \left|\int_T^{\infty} \frac{\log t\, dh_\Gamma(t)}{t^k-1}\right| &\le \left|\frac{h_\Gamma(T)\log T}{T^k-1}\right| + \left|\int_T h_\Gamma(T)\, d\Big(\frac{\log t}{t^k-1}\Big)\right| \\ &\le \frac{c_\Gamma T^{3/4}}{T^k-1} + c_\Gamma \int_T^{\infty} t^{3/4}\, \frac{kt^{k-1}\,dt}{(t^k-1)^2} \\ &= c_\Gamma\bigg(\frac{T^{3/4}}{T^k-1} + \frac{T^{3/4}}{T^k-1} + \frac{3}{4}\int_T^{\infty} \frac{dt}{t^{1/4}(t^k-1)}\bigg) \\ &\le c_\Gamma\bigg(\frac{2T^{3/4}}{T^k-1} + \frac{3}{4}\,\frac{c_\Gamma}{k-3/4}\,\frac{(T^k-1)^{3/4k}}{T^k-1}\bigg) \\ &< \frac{13c_\Gamma T^{3/4}}{5(T^k-1)}. \end{aligned}$$

Therefore,

$$\int_T^{\infty} \frac{\log t\, d\pi_\Gamma(t)}{t^k-1} \le \sum_{n=0}^{M} \frac{(T-1)^{s_n-k}}{k-s_n} + \frac{13c_\Gamma T^{3/4}}{5(T-1)^k}.$$

Summation over $k$ yields

$$\sum_{k=2}^{\infty}\int_{T}^{\infty}\frac{\log t\,d\pi_{\Gamma}(t)}{t^k-1}\le\sum_{n=0}^{M}\frac{(T-1)^{s_n-1}}{T-2}+\frac{13c_{\Gamma}T^{3/4}}{5(T-1)(T-2)}. \tag{14}$$

Thus, the series

$$\sum_{k=2}^{\infty}\sum_{T<N(P_0)<\sqrt[k]{x}}\frac{\log N(P_0)}{N(P_0)^k-1}$$

is of the form $\sum_{k=2}^{\infty} f_k(x)$, where

$$f_k(x)\nearrow A_k=\int_{T}^{\infty}\frac{\log t\,d\pi_{\Gamma}(t)}{t^k-1}\qquad (x\to\infty)$$

and $\sum_{k=2}^{\infty} A_k<\infty$. Hence,

$$\lim_{x\to\infty}\sum_{k=2}^{\infty}\sum_{T<N(P_0)<\sqrt[k]{x}}\frac{\log N(P_0)}{N(P_0)^k-1}=\sum_{k=2}^{\infty}\lim_{x\to\infty}\Big(\sum_{T<N(P_0)<\sqrt[k]{x}}\frac{\log N(P_0)}{N(P_0)^k-1}\Big)$$
$$=\sum_{k=2}^{\infty}\int_{T}^{\infty}\frac{\log t\,d\pi_{\Gamma}(t)}{t^k-1}.$$

Now, taking into account (8), (9) and (11) we get (7).

**4. Upper bounds for the Euler–Selberg constant.** In this section we shall make use of Theorem 3.2 and Proposition 3.3 to obtain upper bounds for $\gamma_0^{(\Gamma)}$.

THEOREM 4.1. (a)

$$\gamma_0^{(\Gamma)}\le\min_{T\ge 2}\Big\{\sum_{n=1}^{M}\frac{T^{s_n-1}}{1-s_n}+\sum_{N(P)\le T}\frac{\Lambda(P)}{N(P)}+5C_{\Gamma}T^{-1/4}-\log T\Big\}.$$

(b) *Let* $0<\varepsilon<1/4$ *and* $s_{\varepsilon}=1/2+\sqrt{1/4-\varepsilon}$. *Then*

$$\gamma_0^{(\Gamma)}\le\sum_{0<\lambda_n<\varepsilon}\frac{T^{s_n-1}}{\lambda_n}+\frac{1}{1-s_{\varepsilon}}T^{s_{\varepsilon}-1}N_{\varepsilon,\Gamma,1/4}+\sum_{N(P)\le T}\frac{\Lambda(P)}{N(P)}+5C_{\Gamma}T^{-1/4}-\log T \tag{15}$$

*for* $T\ge 2$.

*Proof.* (a) For $x > T \geq 2$ we have

$$\begin{aligned}\sum_{N(P)<x} \frac{\Lambda(P)}{N(P)} - \log x &= \sum_{N(P)\leq T} \frac{\Lambda(P)}{N(P)} + \int_T^x \frac{d\psi_\Gamma(t)}{t} - \log x \\ &= \sum_{N(P)\leq T} \frac{\Lambda(P)}{N(P)} + \frac{\psi_\Gamma(x)}{x} - \frac{\psi_\Gamma(T)}{T} - \log T \\ &\quad + \sum_{n=1}^{M} \frac{T^{s_n-1}}{\lambda_n} - \sum_{n=1}^{M} \frac{x^{s_n-1}}{\lambda_n} + \int_T^x \frac{g_\Gamma(t)}{t^2}\, dt.\end{aligned}$$

Letting $x \to \infty$, and taking into account that $s_n - 1 < 0$ for $n \geq 1$ and $\psi_\Gamma(x)/x \to 1$ as $x \to \infty$, we get

$$\gamma_0^{(\Gamma)} = 1 + \sum_{n=1}^{M} \frac{T^{s_n-1}}{\lambda_n} + \sum_{N(P)\leq T} \frac{\Lambda(P)}{N(P)} + \int_T^\infty \frac{g_\Gamma(t)}{t^2}\, dt - \frac{\psi_\Gamma(T)}{T} - \log T.$$

Since $|g_\Gamma(t)| \leq C_\Gamma t^{3/4}$ and $\psi_\Gamma(T)/T \geq 1 + \sum_{n=1}^{M} T^{s_n-1}/s_n - C_\Gamma T^{-1/4}$, we have

$$\gamma_0^{(\Gamma)} \leq \sum_{n=1}^{M} \frac{T^{s_n-1}}{1-s_n} + \sum_{N(P)\leq T} \frac{\Lambda(P)}{N(P)} + 5C_\Gamma T^{-1/4} - \log T. \tag{16}$$

(b) Obviously,

$$\frac{T^{s_n-1}}{1-s_n} \leq \frac{1}{1-s_\varepsilon} T^{s_\varepsilon-1} \qquad \text{for all } \varepsilon \leq \lambda_n < 1/4.$$

On the other hand, $\lambda_n < 1 - s_n$ for all $0 < \lambda_n < 1/4$. These two facts combined with (16) give (15).

Corollary 4.2. (a) *If* $T \geq \max\{e, (5C_\Gamma)^4\}$, *then*

$$\gamma_0^{(\Gamma)} \leq \sum_{n=1}^{M} \frac{T^{s_n-1}}{1-s_n} + \sum_{N(P)\leq T} \frac{\Lambda(P)}{N(P)}. \tag{17}$$

(b) *Let* $T_\varepsilon$ *be such that*

$$\frac{1}{1-s_\varepsilon} T_\varepsilon^{s_\varepsilon-1} N_{\varepsilon,\Gamma,1/4} + 5C_\Gamma T_\varepsilon^{-1/4} = \log T_\varepsilon.$$

*If* $T \geq \max\{2, T_\varepsilon\}$, *then*

$$\gamma_0^{(\Gamma)} \leq \sum_{0<\lambda_n<\varepsilon} \frac{T^{s_n-1}}{\lambda_n} + \sum_{N(P)\leq T} \frac{\Lambda(P)}{N(P)}.$$

*Proof.* Straightforward.

Notice that the left-hand side of the equation in (b), as a function of $T$, is decreasing from $+\infty$ to 0 and the right-hand side strictly increases to $+\infty$. Therefore, $T_\varepsilon$ is unique.

REMARK 4.3. By representing the function $Z'_\Gamma/Z_\Gamma$ as a Gauss transform of the hyperbolic heat trace, J. Jorgenson and J. Kramer [10, Theorem 4.7] have given an upper bound for the Euler–Selberg constant involving the constant $c'_\Gamma$ that appears in the error term of the prime number theorem. They have used a weaker version of the prime geodesic theorem, proved in [8, p. 475], asserting that

$$\pi_{\Gamma,\chi}(x) = \sum_{n=0} \mathrm{li}(x^{s_n}) + h_{\Gamma,\chi}(x) \quad \text{for } x \ge 2,$$

where $-c'_{\Gamma,\chi} x^{3/4}\log^{-1/2} x \le h_{\Gamma,\chi}(x) \le c'_{\Gamma,\chi} x^{3/4}\log^{-1/2} x$ for some constant $c'_{\Gamma,\chi}$ depending only upon $\Gamma$ and $\chi$. The constant $c'_{\Gamma,\chi}$ differs from our constant $c_{\Gamma,\chi}$. In our notation, the upper bound for $\gamma_0^{(\Gamma)}$ obtained by Jorgenson and Kramer can be stated as

$$\gamma_0^{(\Gamma)} \le 3 + \sum_{0<\lambda_n<\varepsilon} \frac{1}{\lambda_n} + \sum_{N(P_0)<e^{\delta_{\Gamma,\varepsilon}}} \frac{Z'_{P_0}}{Z_{P_0}}(1), \tag{18}$$

where $0 < \varepsilon \le 7/64$, the second sum on the right-hand side is taken over primitive hyperbolic conjugacy classes $P_0$, and

$$\delta_{\Gamma,\varepsilon} = \max\left\{ \frac{s_\varepsilon}{\varepsilon} \log\left( \frac{4(4-3s_\varepsilon)}{\varepsilon} \left(c'_\Gamma + N_{\varepsilon,\Gamma,1/4}\right)\right), 5 + 2\log N_{0,\Gamma,\varepsilon} \right\}.$$

Proposition 3.3 enables us to obtain also an upper bound for $\gamma_0^{(\Gamma)}$ with the constant $c_\Gamma$ instead of $C_\Gamma$. We state this as follows.

PROPOSITION 4.4. (a) *Let $T_\Gamma$ be the constant such that*

$$\log(T_\Gamma - 1) = \frac{6c_\Gamma T_\Gamma^{3/4}}{T_\Gamma - 1} + \sum_{n=0}^{M} \frac{(T_\Gamma - 1)^{s_n - 1}}{T_\Gamma - 2}.$$

*Then, for $T \ge \max\{e^2, T_\Gamma\}$,*

$$\gamma_0^{(\Gamma)} \le \sum_{N(P_0)\le T} \frac{Z'_{P_0}}{Z_{P_0}}(1) + \sum_{n=1}^{M} \frac{(T-1)^{s_n - 1}}{1 - s_n}. \tag{19}$$

(b) *Let $0 < \varepsilon < 1/4$ and denote by $T_{\Gamma,\varepsilon}$ the solution of the equation*

$$(T-1)^{s_\varepsilon - 1} N_{\varepsilon,\Gamma,1/4}\left( \frac{1}{1-s_\varepsilon} + \frac{1}{T-2}\right) + \frac{6c_\Gamma T^{3/4}}{T-1} + \frac{N_{0,\Gamma,\varepsilon}}{T-2} = \log(T-1).$$

*Then, for* $T \geq \max\{e^2, T_{\Gamma,\varepsilon}\}$,

$$\gamma_0^{(\Gamma)} \leq \sum_{0<\lambda_n<\varepsilon} \frac{(T-1)^{s_n-1}}{\lambda_n} + \sum_{N(P_0)\leq T} \frac{Z'_{P_0}}{Z_{P_0}}(1). \tag{20}$$

*Proof.* Note that the uniqueness of numbers $T_\Gamma$ and $T_{\Gamma,\varepsilon}$ follows by the same argument applied to $T_\varepsilon$ in Corollary 4.2(b).

(a) For any $T > 2$, combining (7) and (10) we obtain

$$\begin{aligned}\gamma_0^{(\Gamma)} \leq \sum_{N(P_0)\leq T} \frac{Z'_{P_0}}{Z_{P_0}}(1) + \sum_{n=1}^{M} \int_T^\infty \frac{t^{s_n-1}\,dt}{t-1} - \log(T-1) - \frac{\log T\cdot h_\Gamma(T)}{T-1} \\ + \int_T^\infty \frac{\left(\log t - \frac{t-1}{t}\right)h_\Gamma(t)\,dt}{(t-1)^2} + \sum_{k=2}^\infty \int_T^\infty \frac{\log t\,d\pi_\Gamma(t)}{t^k-1}.\end{aligned} \tag{21}$$

Let us estimate the right-hand side of (21). First,

$$\begin{aligned}&\left|\int_T^\infty \frac{\left(\log t - \frac{t-1}{t}\right)h_\Gamma(t)\,dt}{(t-1)^2}\right| \\ &\quad\leq c_\Gamma \int_T^\infty \frac{t^{3/4}\,dt}{(t-1)^2} \leq c_\Gamma\left(\frac{T^{3/4}}{T-1} + \frac{3}{4}\int_T^\infty \frac{dt}{t^{1/4}(t-1)}\right) \\ &\quad\leq c_\Gamma\left(\frac{T^{3/4}}{T-1} + \frac{3}{4}\int_T^\infty \frac{dt}{(t-1)^{5/4}}\right) = c_\Gamma\left(\frac{T^{3/4}}{T-1} + 3(T-1)^{-1/4}\right).\end{aligned} \tag{22}$$

The other two terms containing integrals in (21) are taken care of by (12) and (14). Thus, having in mind that $\left|\frac{\log T\cdot h_\Gamma(T)}{T-1}\right| \leq \frac{c_\Gamma T^{3/4}}{T-1}$, we get

$$\begin{aligned}\gamma_0^{(\Gamma)} &\leq \sum_{N(P_0)\leq T} \frac{Z'_{P_0}}{Z_{P_0}}(1) + \sum_{n=1}^{M} \frac{(T-1)^{s_n-1}}{1-s_n} \\ &+ c_\Gamma\left(\frac{T^{3/4}}{T-1}\left(2+\frac{13}{5(T-2)}\right) + 3(T-1)^{-1/4}\right) + \sum_{n=0}^{M} \frac{(T-1)^{s_n-1}}{T-2} - \log(T-1).\end{aligned}$$

Finally, since $3(T-1)^{-1/4} < \frac{3T^{3/4}}{T-1}$ and $\frac{13}{5(T-2)} < 1$, for $T > 5$ we have

$$\begin{aligned}\gamma_0^{(\Gamma)} \leq \sum_{N(P_0)\leq T} \frac{Z'_{P_0}}{Z_{P_0}}(1) + \sum_{n=1}^{M} \frac{(T-1)^{s_n-1}}{1-s_n} + \frac{6c_\Gamma T^{3/4}}{T-1} \\ + \sum_{n=0}^{M} \frac{(T-1)^{s_n-1}}{T-2} - \log(T-1),\end{aligned} \tag{23}$$

which proves (a) in the given range of $T$.

(b) Let $0<\varepsilon<1/4$. Then, since $\lambda_n=s_n(1-s_n)$ and $s_n<1$ for $n\ge 1$, we have

$$\sum_{n=1}^{M}\frac{(T-1)^{s_n-1}}{1-s_n}\le\sum_{0<\lambda_n<\varepsilon}\frac{(T-1)^{s_n-1}}{\lambda_n}+\sum_{s_n\le s_\varepsilon}\frac{(T-1)^{s_n-1}}{1-s_n}\tag{24}$$
$$\le\sum_{0<\lambda_n<\varepsilon}\frac{(T-1)^{s_n-1}}{\lambda_n}+N_{\varepsilon,\Gamma,1/4}\,\frac{(T-1)^{s_\varepsilon-1}}{1-s_\varepsilon}.$$

Similarly,

$$\sum_{n=0}^{M}\frac{(T-1)^{s_n-1}}{T-2}\le\frac{N_{0,\Gamma,\varepsilon}}{T-2}+\sum_{s_n\le s_\varepsilon}\frac{(T-1)^{s_n-1}}{T-2}\tag{25}$$
$$\le\frac{N_{0,\Gamma,\varepsilon}}{T-2}+N_{\varepsilon,\Gamma,1/4}\,\frac{(T-1)^{s_\varepsilon-1}}{T-2}.$$

This, together with (23) and (24), yields the statement.

Remark 4.5. The bounds (20) and (18) cannot be compared since the constants $c_\Gamma$ and $c'_\Gamma$ are different. However, repeating the steps of the proof of Proposition 4.4, now using the weaker bound on $h_\Gamma$ that involves the constant $c'_\Gamma$, it is easy to see that

$$\gamma_0^{(\Gamma)}\le\sum_{N(P_0)\le T}\frac{Z'_{P_0}}{Z_{P_0}}(1)+\sum_{n=1}^{M}\frac{(T-1)^{s_n-1}}{1-s_n}+\frac{6c'_\Gamma T^{3/4}}{T-1}\log^{1/2}T$$
$$+\sum_{n=0}^{M}\frac{(T-1)^{s_n-1}}{T-2}-\log(T-1).$$

This implies that, for any $0<\varepsilon<1/4$ and $T=T'_{\Gamma,\varepsilon}$ satisfying

$$(T-1)^{s_\varepsilon-1}N_{\varepsilon,\Gamma,1/4}\biggl(\frac{1}{1-s_\varepsilon}+\frac{1}{T-2}\biggr)+\frac{6c'_\Gamma T^{3/4}}{T-1}\log^{1/2}T+\frac{N_{0,\Gamma,\varepsilon}}{T-2}=\log(T-1),$$

the inequality (20) remains valid.

Simple but somewhat lengthy calculations yield $e^{\delta_{\Gamma,\varepsilon}}>T'_{\Gamma,\varepsilon}$. Thus, our bound (20), with $T_{\Gamma,\varepsilon}$ replaced by $T'_{\Gamma,\varepsilon}$, is better than the bound (18).

**5. Lower bounds for the Euler–Selberg constant.** In this section we shall use Theorem 3.2 to prove lower bounds for the Euler–Selberg constant $\gamma_0^{(\Gamma)}$ involving the constant $C_\Gamma$. Proposition 3.3 will provide lower bounds involving primitive hyperbolic classes and the constant $c_\Gamma$.

Proposition 5.1.

$$\gamma_0^{(\Gamma)}\ge\max_{T\ge 2}\biggl\{\sum_{n=1}^{M}\frac{T^{s_n-1}}{1-s_n}+\sum_{N(P)\le T}\frac{\Lambda(P)}{N(P)}-5C_\Gamma T^{-1/4}-\log T\biggr\}.$$

*Proof.* Let us recall that for all $x > T \geq 2$,

$$\gamma_0^{(\Gamma)} = 1 + \sum_{n=1}^{M} \frac{T^{s_n-1}}{\lambda_n} + \sum_{N(P)\leq T} \frac{\Lambda(P)}{N(P)} + \int_T^\infty \frac{g_\Gamma(t)}{t^2}\,dt - \frac{\psi_\Gamma(T)}{T} - \log T.$$

The facts that $g_\Gamma(t) \geq -C_\Gamma t^{3/4}$ and $\psi_\Gamma(T)/T \leq 1 + \sum_{n=1}^{M} T^{s_n-1}/s_n + C_\Gamma T^{3/4}$ for $T \geq 2$ complete the proof.

The following proposition gives us lower bounds for $\gamma_0^{(\Gamma)}$ involving the spectra of the Laplace–Beltrami operator and the constant $c_\Gamma$.

PROPOSITION 5.2.

$$\gamma_0^{(\Gamma)} \geq \max_{T\geq 5}\Big\{ \sum_{n=1}^{M} \frac{s_n T^{s_n-1}}{1-s_n} + \sum_{N(P_0)\leq T} \frac{Z'_{P_0}}{Z_{P_0}}(1) + \log\frac{T}{T-1}\sum_{n=0}^{M} T^{s_n} - \frac{6c_\Gamma T^{3/4}}{T-1} - \log(T-1) - 1\Big\}. \tag{26}$$

*Proof.* We use Proposition 3.3 to estimate the last two summands in (7). For $k \geq 2$, the relation (13) implies

$$\begin{aligned}\int_T^\infty \frac{\log t\, d\pi_\Gamma(t)}{t^k - 1} &= \sum_{n=0}^{M} \int_T^\infty \frac{t^{s_n-1}}{t^k-1}\,dt + \int_T^\infty \frac{\log t\, dh_\Gamma(t)}{t^k-1}\\ &\geq \sum_{n=0}^{M} \frac{T^{s_n-k}}{k-s_n} - \frac{13c_\Gamma T^{3/4}}{5(T-1)^k}.\end{aligned}$$

Since $T^{s_n-k}/(k-s_n) > T^{s_n-k}/k$, we have

$$\begin{aligned}\sum_{k=2}^{\infty}\int_T^\infty \frac{\log t\, d\pi_\Gamma(t)}{t^k-1} &\geq \sum_{n=0}^{M} T^{s_n} \sum_{k=2}^{\infty} \frac{1}{kT^k} - \frac{13c_\Gamma T^{3/4}}{5(T-1)(T-2)}\\ &\geq \left[-\log\left(1-\frac{1}{T}\right) - \frac{1}{T}\right]\sum_{n=0}^{M} T^{s_n} - \frac{13c_\Gamma T^{3/4}}{5(T-1)(T-2)}\\ &\geq \log\frac{T}{T-1}\sum_{n=0}^{M} T^{s_n} - 1 - \sum_{n=1}^{M} T^{s_n-1} - \frac{c_\Gamma T^{3/4}}{T-1}.\end{aligned} \tag{27}$$

Now, consider the limit on the right-hand side of (7). Letting $x \to \infty$ in (10), having in mind that $c_\Gamma t^{3/4}\log^{-1} t \geq h_\Gamma(t) \geq -c_\Gamma t^{3/4}\log^{-1} t$ and using (22), we obtain

$$\lim_{x\to\infty}\Bigg(\sum_{T<N(P_0)<x}\frac{\log N(P_0)}{N(P_0)-1}-\log x\Bigg)$$
$$\geq \sum_{n=1}^{M}\int_{T}^{\infty} t^{s_n-2}\,dt-\log(T-1)-\frac{c_\Gamma T^{3/4}}{T-1}+\int_{T}^{\infty}\frac{(t\log t-t+1)\cdot h_\Gamma(t)\,dt}{t(t-1)^2}$$
$$\geq \sum_{n=1}^{M}\frac{T^{s_n-1}}{1-s_n}-\log(T-1)-\frac{c_\Gamma T^{3/4}}{T-1}-c_\Gamma\Bigg(\frac{T^{3/4}}{T-1}+3(T-1)^{-1/4}\Bigg)$$
$$\geq \sum_{n=1}^{M}\frac{T^{s_n-1}}{1-s_n}-\log(T-1)-\frac{5c_\Gamma T^{3/4}}{T-1}.$$

This, together with (7) and (27), implies that

$$\gamma_0^{(\Gamma)}\geq \sum_{N(P_0)\leq T}\frac{Z'_{P_0}}{Z_{P_0}}(1)+\sum_{n=1}^{M}\frac{T^{s_n-1}}{1-s_n}-\log(T-1)-\frac{6c_\Gamma T^{3/4}}{T-1}-\sum_{n=1}^{M}T^{s_n-1}-1$$
$$+\log\frac{T}{T-1}\sum_{n=0}^{M}T^{s_n}$$
$$=\sum_{N(P_0)\leq T}\frac{Z'_{P_0}}{Z_{P_0}}(1)+\sum_{n=1}^{M}\frac{s_nT^{s_n-1}}{1-s_n}+\log\frac{T}{T-1}\sum_{n=0}^{M}T^{s_n}-\frac{6c_\Gamma T^{3/4}}{T-1}-1$$
$$-\log(T-1).$$

The proof is complete.

Corollary 5.3.

$$\gamma_0^{(\Gamma)}=\lim_{x\to\infty}\Bigg(\sum_{N(P_0)\leq x}\frac{Z'_{P_0}}{Z_{P_0}}(1)-\log(x-1)\Bigg).$$

*Proof.* Immediate consequence of (23) and (26) as $T\to\infty$.

## References


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