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Dispersion of ratio block sequences and asymptotic density

by

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1. Introduction. We first recall some basic definitions. Denote by $\mathbb{N}$ and $\mathbb{R}^+$ the sets of all positive integers and positive real numbers, respectively. For $X \subset \mathbb{N}$ let $X(n) = \#\{x \in X : x \le n\}$. In the whole paper we will assume that X is infinite. Denote by $R(X) = \{x/y : x \in X,\, y \in X\}$ the *ratio set* of X and say that a set X is (R)-*dense* if $R(X)$ is (topologically) dense in $\mathbb{R}^+$. The concept of (R)-density was defined and first studied in [10] and [11]. The quantities

$$\underline{d}(X) = \liminf_{n\to\infty} \frac{X(n)}{n}, \qquad \overline{d}(X) = \limsup_{n\to\infty} \frac{X(n)}{n},$$
$$d(X) = \lim_{n\to\infty} \frac{X(n)}{n}$$

are the *lower asymptotic density*, *upper asymptotic density*, and *asymptotic density* (if defined), respectively. Relations between (R)-density, asymptotic density and logarithmic density have been studied, among others, in [5], [6], [14] and [15].

Now let $X = \{x_1, x_2, \dots\}$ where $x_n < x_{n+1}$ are positive integers. Then

$$\frac{x_1}{x_1}, \frac{x_1}{x_2}, \frac{x_2}{x_2}, \frac{x_1}{x_3}, \frac{x_2}{x_3}, \frac{x_3}{x_3}, \dots, \frac{x_1}{x_n}, \frac{x_2}{x_n}, \dots, \frac{x_n}{x_n}, \dots \tag{1}$$

is called the *ratio block sequence* of the set X. It is formed by the blocks $X_1, X_2, \dots$, where

$$X_n = \left(\frac{x_1}{x_n}, \frac{x_2}{x_n}, \dots, \frac{x_n}{x_n}\right), \qquad n = 1, 2, \dots.$$

This kind of block sequences have been studied in [16], [18] and [1]. Also other kinds of block sequences have been studied by several authors (see [2],

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[4], [7], [12] and [17]). Let $Y = (y_n)$ be an increasing sequence of positive integers. In [9], extending a result of [3], a sequence of blocks of the type

$$Y_n = \left(\frac{1}{y_n}, \frac{2}{y_n}, \dots, \frac{y_n}{y_n}\right)$$

was investigated. The authors obtained a complete theory of uniform distribution of the related block sequence (Y_n).

For every $n \in \mathbb{N}$ let

$$D(X_n) = \max\left\{\frac{x_1}{x_n}, \frac{x_2 - x_1}{x_n}, \dots, \frac{x_{i+1} - x_i}{x_n}, \dots, \frac{x_n - x_{n-1}}{x_n}\right\}$$

be the maximum distance between two consecutive terms in the nth block. We will consider the quantity (see [18])

$$\underline{D}(X) = \liminf_{n\to\infty} D(X_n),$$

called the *dispersion* of the block sequence (1) derived from X, and its relations to the previously mentioned asymptotic density of the original set X.

To end this section, let us mention the concept of dispersion of a general sequence of numbers in the interval $[0,1]$. Let $(x_n)_{n=0}^{\infty}$ be a sequence in $[0,1]$. For every $N \in \mathbb{N}$ let $x_{i_1} \le \dots \le x_{i_N}$ be a nondecreasing rearrangement of its first N terms and define

$$d_N = \frac{1}{2}\max\{\max\{x_{i_{j+1}} - x_{i_j} : j = 1, \dots, N-1\}, x_{i_1}, 1 - x_{i_N}\}$$

to be the dispersion of the finite sequence $x_0, x_1, \dots, x_N$. The properties of this concept can be found for example in [8] where it is also proved that

$$\limsup_{N\to\infty} N d_N \ge \frac{1}{\log 4}$$

for every one-to-one infinite sequence $x_n \in [0,1)$. Notice that the density of the whole sequence $(x_n)_{n=0}^{\infty}$ is equivalent to $\lim_{N\to\infty} d_N = 0$. The dispersion of block sequences defined in the present paper does not have the analogous property. Much more on these and related topics can be found in [13].

2. Results. When calculating $\underline{D}(X)$, the following theorems are often useful (see [18, Theorem 1 and Corollary 1]).

(A1) Let

$$X = \{x_1, x_2, \dots\} = \bigcup_{n=1}^{\infty} (c_n, d_n] \cap \mathbb{N},$$

where $x_n < x_{n+1}$ and $c_n < d_n < c_{n+1}$, for $n \in \mathbb{N}$, are positive integers. Then

$$\underline{D}(X) = \liminf_{n\to\infty} \frac{\max\{c_{i+1} - d_i : i = 1, \ldots, n\}}{d_{n+1}}.$$

(A2) Let X be as in (A1). Suppose that there exists a positive integer n_0 such that

$$c_{n+1} - d_n \le c_{n+2} - d_{n+1} \quad \text{for all } n > n_0.$$

Then

$$\underline{D}(X) = \liminf_{n\to\infty} \frac{c_{n+1} - d_n}{d_{n+1}}.$$

The following theorem relates the asymptotic density of a set $X \subset \mathbb{N}$ and the dispersion of the block sequence (1).

THEOREM 1. *Let $X \subset \mathbb{N}$. Then*

(i) *If $d(X) > 0$ then $\underline{D}(X) = 0$.*
(ii) *If $a = \underline{d}(X) < \overline{d}(X) = b$ then*

$$\underline{D}(X) \le \frac{(1-b)(b-a)}{b(1-a)}.$$

(iii) *If $d(X) = 0$ then $\underline{D}(X)$ can be any number in the interval* $[0,1]$.

Proof. Write $X = \bigcup_{n=1}^{\infty}(c_n, d_n] \cap \mathbb{N}$ and define $h(n) = X(n)/n$ for $n \in \mathbb{N}$. For the purposes of this proof we introduce the following notation. For every $n \in \mathbb{N}$ let $\mu(n)$ be the smallest integer in $\{1, \ldots, n\}$ such that

$$c_{\mu(n)+1} - d_{\mu(n)} = \max\{c_{i+1} - d_i : i = 1, \ldots, n\}.$$

Notice that in this notation (A1) says that

$$\underline{D}(X) = \liminf_{n\to\infty} \frac{c_{\mu(n)+1} - d_{\mu(n)}}{d_{n+1}}. \tag{2}$$

Now there are two possibilities:

(a) $\mu = \lim_{n\to\infty} \mu(n) < \infty$,
(b) $\mu = \lim_{n\to\infty} \mu(n) = \infty$.

In case (a), (2) clearly yields $\underline{D}(X) = 0$ and the statement of the theorem holds. Thus in the following we assume that (b) holds.

By definition of the upper asymptotic density, for every $p \in \mathbb{N}$ there exists a $k = k(p) \in \mathbb{N}$ such that $k(p) \to \infty$ as $p \to \infty$ and

$$h(d_{k+1}) > \overline{d}(X) - 1/p. \tag{3}$$

To simplify the writing, set

$$c_{\mu(k)+1} = n = n(p); \quad c_{\mu(k)+1} - d_{\mu(k)} = t = t(p); \quad d_{k+1} - c_{\mu(k)+1} = u = u(p).$$

Notice that n, t, u are positive integers and

$$h(n+u) \geq h(n). \tag{4}$$

Also, the conditions $n \to \infty$ and $p \to \infty$ are equivalent.

Using (A1) and observing that $\{d_{k(p)}\}_{p=1}^{\infty}$ is a subsequence of $\{d_n\}_{n=1}^{\infty}$ we find that

$$\begin{aligned} \underline{D}(X) &= \liminf_{n\to\infty} \frac{c_{\mu(n)+1} - d_{\mu(n)}}{d_{n+1}} \\ &\leq \liminf_{p\to\infty} \frac{c_{\mu(k(p))+1} - d_{\mu(k(p))}}{d_{k(p)+1}} = \liminf_{p\to\infty} \frac{t(p)}{n(p)+u(p)}. \end{aligned} \tag{5}$$

To upper bound the last fraction (notice that $X(n-t) = X(n)$)

$$\frac{t}{n+u} = \frac{n-(n-t)}{n+u} = \frac{\frac{X(n)}{h(n)} - \frac{X(n)}{h(n-t)}}{\frac{X(n)}{h(n)} + u},$$

we estimate u from below as follows. We have

$$h(n+u) = \frac{X(n+u)}{n+u} \leq \frac{X(n)+u}{n+u},$$

and consequently

$$u \geq \frac{nh(n+u) - X(n)}{1 - h(n+u)}.$$

Thus

$$\begin{aligned} \frac{t}{n+u} &\leq \frac{\frac{X(n)}{h(n)} - \frac{X(n)}{h(n-t)}}{\frac{X(n)}{h(n)} + \frac{nh(n+u)-X(n)}{1-h(n+u)}} = \frac{\frac{1}{h(n)} - \frac{1}{h(n-t)}}{\frac{1}{h(n)} + \frac{\frac{nh(n+u)}{X(n)}-1}{1-h(n+u)}} \\ &= \frac{\frac{1}{h(n)} - \frac{1}{h(n-t)}}{\frac{1}{h(n)} + \frac{\frac{h(n+u)}{h(n)}-1}{1-h(n+u)}} = \frac{\frac{h(n-t)-h(n)}{h(n)h(n-t)}}{\frac{1-h(n)}{h(n)(1-h(n+u))}} \\ &= (1-h(n+u)) \frac{h(n-t)-h(n)}{h(n-t)(1-h(n))}. \end{aligned} \tag{6}$$

(i) Suppose that $d(X) > 0$. Then $\lim_{p\to\infty}(n(p) - t(p)) = \infty$. Using the previously derived relations and taking into account (4), we have

$$\begin{aligned} \underline{D}(X) &\leq \liminf_{n\to\infty} \frac{1-h(n+u)}{1-h(n)} \frac{h(n-t)-h(n)}{h(n-t)} \\ &\leq \lim_{n\to\infty} \frac{h(n-t)-h(n)}{h(n-t)} = 0. \end{aligned}$$

The last equality follows from the fact that $n - t \to \infty$ as $p \to \infty$ and $d(X) = \lim_{m\to\infty} h(m) > 0$.

(ii) Let $a = \underline{d}(X) < \overline{d}(X) = b$. Then

$$\lim_{p\to\infty} h(n(p)+u(p)) = b, \quad \liminf_{p\to\infty} h(n(p)) \ge a, \quad \limsup_{p\to\infty} h(n(p)-t(p)) \le b.$$

A simple analysis of the last term in (6) shows that it is increasing in $h(n-t)$ and decreasing in $h(n)$. Thus, by (5), (6) and taking into account (3), we have, for $n = n(p), u = u(p)$ and $t = t(p)$,

$$\begin{aligned}\underline{D}(X) &\le (1 - \lim_{p\to\infty} h(n+u))\frac{\limsup_{p\to\infty} h(n-t) - \liminf_{p\to\infty} h(n)}{\limsup_{p\to\infty} h(n-t)(1-\liminf_{p\to\infty} h(n))}\\ &\le \frac{(1-b)(b-a)}{b(1-a)}.\end{aligned}$$

(iii) Let $\alpha \in (0,1)$. Then put $a = 1/(1-\alpha) > 1$ and consider the set $X = \{[a^n] : n \in \mathbb{N}\}$. Then

$$d(X) = \lim_{n\to\infty} \frac{n}{[a^n]} = 0.$$

Moreover, X satisfies the conditions of (A2) and so

$$\underline{D}(X) = \liminf_{n\to\infty} \frac{[a^{n+1}] - 1 - [a^n]}{[a^{n+1}]} = \frac{a-1}{a} = \alpha.$$

Now let $\alpha = 0$. Then set $X = \{n^2 : n \in \mathbb{N}\}$. One can easily show that $d(X) = 0$ and, again using (A2), also $\underline{D}(X) = 0 = \alpha$.

Finally, let $\alpha = 1$. Then set $X = \{2^{2^n} : n \in \mathbb{N}\}$. It can be easily shown that $d(X) = 0$ and, once again by (A2), also $\underline{D}(X) = 1 = \alpha$. ∎

Item (ii) of the above theorem implies that if $\overline{d}(X) = b = 1$ then we immediately have $\underline{D}(X) = 0$. In [18, Theorem 2] it is proved that $\underline{D}(X) = 0$ implies density of the block sequence (1). This gives the result of [10] that $\overline{d}(X) = 1$ implies (R)-density of X.

The following theorem says that the bound in (ii) is the best possible, as the set of values of the dispersion of sets with $\underline{d}(X) = a$ and $\overline{d}(X) = b$ is the whole interval $[0, \overline{\alpha}]$ where

$$\overline{\alpha} := \frac{(1-b)(b-a)}{b(1-a)}.$$

THEOREM 2. *Let $0 \le a < b < 1$ and $\alpha \in [0, \overline{\alpha}]$ with $\overline{\alpha}$ defined above. Then there exists a set $X \subseteq \mathbb{N}$ such that $\underline{d}(X) = a$, $\overline{d}(X) = b$ and $\underline{D}(X) = \alpha$.*

Proof. First we consider the case $\alpha = \overline{\alpha}$. We have two possibilities: $a > 0$ and $a = 0$.

The case $a > 0$. Let $0 < a < b < 1$. Choose a real number d_0 such that

$$\min\left\{d_0\,\frac{b}{a}; d_0\,\frac{1-a}{1-b}\right\} > d_0 + 1 \tag{7}$$

and define the sequences $\{c_n\}_{n=1}^{\infty}$ and $\{d_n\}_{n=1}^{\infty}$ as follows:

$$(8) \qquad c_n = \frac{b}{a}\, d_{n-1} \quad \text{and} \quad d_n = \frac{b(1-a)}{a(1-b)}\, d_{n-1} \quad \text{for all } n \in \mathbb{N}.$$

Then (7) and (8) imply

$$(9) \qquad [c_n] < [d_n] < [c_{n+1}] \quad \text{for all } n \in \mathbb{N},$$

$$(10) \qquad n^2 = o(c_n) \quad \text{and} \quad n^2 = o(d_n).$$

By (8) and (9), we can apply both statements (A1) and (A2) to the set $X \subset \mathbb{N}$ defined by

$$(11) \qquad X = \bigcup_{n=1}^{\infty} ([c_n], [d_n]] \cap \mathbb{N}.$$

We are going to show that $\underline{d}(X) = a$, $\overline{d}(X) = b$ and $\underline{D}(X) = \overline{\alpha}$. From the definition of X, clearly

$$(12) \qquad \underline{d}(X) = \liminf_{n\to\infty} \frac{X([c_n])}{[c_n]},$$

$$(13) \qquad \overline{d}(X) = \limsup_{n\to\infty} \frac{X([d_n])}{[d_n]}.$$

From (8), (10) and (11) one can derive

$$\begin{aligned} \frac{X([d_n])}{[d_n]} &= \frac{\sum_{i=1}^{n}([d_i]-[c_i])}{[d_n]} = \frac{\sum_{i=1}^{n}(d_i - c_i)}{d_n} + o(1) \\ &= \frac{\sum_{i=1}^{n}\left(\left(\frac{b(1-a)}{a(1-b)} - \frac{b}{a}\right)d_{i-1}\right)}{d_n} + o(1) = \frac{b(b-a)}{a(1-b)} \sum_{i=1}^{n} \frac{d_{i-1}}{d_n} + o(1) \\ &= \frac{b(b-a)}{a(1-b)} \sum_{i=1}^{n} \left(\frac{a(1-b)}{b(1-a)}\right)^{i} + o(1). \end{aligned}$$

The above relations yield

$$\begin{aligned} (14) \qquad \overline{d}(X) &= \limsup_{n\to\infty} \frac{X([d_n])}{[d_n]} = \lim_{n\to\infty} \frac{X([d_n])}{[d_n]} \\ &= \frac{b(b-a)}{a(1-b)} \lim_{n\to\infty} \sum_{i=1}^{n} \left(\frac{a(1-b)}{b(1-a)}\right)^{i} = \frac{b(b-a)}{a(1-b)} \frac{\frac{a(1-b)}{b(1-a)}}{1 - \frac{a(1-b)}{b(1-a)}} = b. \end{aligned}$$

Similarly, from (8), (14) and the definition of X we obtain

$$\begin{aligned} \underline{d}(X) &= \liminf_{n\to\infty} \frac{X([c_n])}{[c_n]} = \liminf_{n\to\infty} \frac{[d_{n-1}]}{[c_n]} \frac{X([c_n])}{[d_{n-1}]} \\ &= \liminf_{n\to\infty} \frac{[d_{n-1}]}{[c_n]} \frac{X([d_{n-1}])}{[d_{n-1}]} = \frac{a}{b}\, b = a. \end{aligned}$$

By the definitions of $\{c_n\}_{n=1}^{\infty}$ and $\{d_n\}_{n=1}^{\infty}$,

$$[c_{n+1}] - [d_n] \le [c_{n+2}] - [d_{n+1}] \tag{15}$$

for all sufficiently large $n \in \mathbb{N}$. By use of (A2) we obtain

$$\underline{D}(X) = \liminf_{n\to\infty} \frac{[c_{n+1}] - [d_n]}{[d_{n+1}]} = \liminf_{n\to\infty} \frac{\frac{c_{n+1}}{d_n} - 1}{\frac{d_{n+1}}{d_n}} = \frac{(1-b)(b-a)}{b(1-a)} = \overline{\alpha}.$$

The case $a = 0$. Let $0 = a < b < 1$. Choose $d_0 \in \mathbb{R}^+$ such that

$$d_0 > \frac{1-b}{b}. \tag{16}$$

Now define

$$c_n = n d_{n-1} \quad \text{and} \quad d_n = \frac{n}{1-b}\, d_{n-1} \quad \text{for all } n \in \mathbb{N}. \tag{17}$$

By (16) and (17), we see that (9) and (10) hold again, so we can apply (A1) and (A2) to

$$X = \bigcup_{n=1}^{\infty} ([c_n], [d_n]] \cap \mathbb{N}.$$

Again, we are going to show that $\underline{d}(X) = a$, $\overline{d}(X) = b$ and $\underline{D}(X) = \overline{\alpha}$. One can easily check the following bounds:

$$\frac{[d_n] - [c_n]}{[d_n]} \le \frac{X([d_n])}{[d_n]} = \frac{\sum_{i=1}^{n}([d_i] - [c_i])}{[d_n]} \le \frac{[d_n] - [c_n]}{[d_n]} + \frac{[d_{n-1}]}{[d_n]}.$$

By (17) we have

$$\lim_{n\to\infty} \frac{[d_n] - [c_n]}{[d_n]} = b \quad \text{and} \quad \lim_{n\to\infty} \frac{[d_{n-1}]}{[d_n]} = 0.$$

Again, the definition of X allows us to use (12) and (13) giving

$$\overline{d}(X) = \limsup_{n\to\infty} \frac{X([d_n])}{[d_n]} = \lim_{n\to\infty} \frac{[d_n] - [c_n]}{[d_n]} = b,$$

$$\underline{d}(X) = \liminf_{n\to\infty} \frac{X([c_n])}{[c_n]} = \liminf_{n\to\infty} \frac{[d_{n-1}]}{[c_n]} \frac{X([d_{n-1}])}{[d_{n-1}]} = \lim_{n\to\infty} \frac{1}{n} b = 0.$$

The definitions of $\{c_n\}_{n=1}^{\infty}$ and $\{d_n\}_{n=1}^{\infty}$ imply (15) once more, so using (A2) we obtain

$$\underline{D}(X) = \liminf_{n\to\infty} \frac{[c_{n+1}] - [d_n]}{[d_{n+1}]} = \liminf_{n\to\infty} \frac{n}{\frac{n+1}{1-b}} = 1 - b = \overline{\alpha},$$

proving our statement in the case when $\alpha = \overline{\alpha}$.

The above sequences $\{c_n\}_{n\in\mathbb{N}}$ and $\{d_n\}_{n\in\mathbb{N}}$ and the set X will also be used to prove the statement of our theorem for any $0 \le \alpha < \overline{\alpha}$.

We will distinguish two cases: $\alpha > 0$ and $\alpha = 0$.

Let $0 < \alpha < \overline{\alpha}$. Define

$$Z = \bigcup_{n=1}^{\infty} \left\{ [d_n] + k\left[\frac{\alpha}{\overline{\alpha}}([c_{n+1}] - [d_n])\right] : k = 1, \dots, m_n \right\},$$

where

$$m_n\left[\frac{\alpha}{\overline{\alpha}}([c_{n+1}]-[d_n])\right] \le [c_{n+1}]-[d_n] < (m_n+1)\left[\frac{\alpha}{\overline{\alpha}}([c_{n+1}]-[d_n])\right]. \tag{18}$$

First we show that $d(Z) = 0$. To see this, let $x \in \mathbb{N}$ and $[c_n] < x \le [c_{n+1}]$. Then, using (10) and (18), we obtain

$$0 \le \frac{Z(x)}{x} \le \frac{\sum_{i=1}^{n} m_i}{[c_n]} = O\left(\frac{\frac{\overline{\alpha}}{\alpha} n}{[c_n]}\right) = o\left(\frac{1}{n}\right),$$

which yields $d(Z) = 0$.

Set $Y = X \cup Z$. As $d(Z) = 0$, we have

$$\underline{d}(Y) = \underline{d}(X) = a \quad \text{and} \quad \overline{d}(Y) = \overline{d}(X) = b.$$

It remains to show that $\underline{D}(Y) = \alpha$.

By definition of Y, we can apply (A1) to obtain

$$\underline{D}(Y) = \liminf_{n\to\infty} \frac{\left[\frac{\alpha}{\overline{\alpha}}([c_{n+1}] - [d_n])\right]}{[d_{n+1}]} = \frac{\alpha}{\overline{\alpha}}\,\overline{\alpha} = \alpha.$$

Now let $\alpha = 0$. Define

$$Z = \bigcup_{n=1}^{\infty} \left\{ [d_n] + k\left[\frac{[c_{n+1}] - [d_n]}{n}\right] : k = 1, \dots, n \right\}.$$

Again $d(Z) = 0$, as for every $x \in \mathbb{N}$ with $[c_n] < x \le [c_{n+1}]$ we have

$$\frac{Z(x)}{x} \le \frac{\sum_{i=1}^{n} i}{[c_n]} = O\left(\frac{n^2}{[c_n]}\right) = o(1).$$

Put $Y = X \cup Z$. We have immediately

$$\underline{d}(Y) = \underline{d}(X) = a \quad \text{and} \quad \overline{d}(Y) = \overline{d}(X) = b.$$

By definition of Y, we can apply (A1) to obtain

$$\underline{D}(Y) = \liminf_{n\to\infty} \frac{\left[\frac{1}{n}([c_{n+1}] - [d_n])\right]}{[d_{n+1}]} = \lim_{n\to\infty} \frac{1}{n}\,\overline{\alpha} = 0,$$

finishing the proof. ∎

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