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HEDGING IN COMPLETE MARKETS DRIVEN BY NORMAL MARTINGALES

Abstract. This paper aims at a unified treatment of hedging in market models driven by martingales with deterministic bracket $\langle M, M\rangle_t$, including Brownian motion and the Poisson process as particular cases. Replicating hedging strategies for European, Asian and Lookback options are explicitly computed using either the Clark–Ocone formula or an extension of the delta hedging method, depending on which is most appropriate.

1. Introduction. The Clark formula [Cla70] allows in principle the calculation of replicating hedging strategies in complete markets [KO91], but explicit computations are in general difficult to perform via this formula. For markets driven by Brownian motion a proof of the classical Black–Scholes formula via the Clark–Ocone formula can be found in [Øks96, Ch. 5, p. 13]. This method has recently been extended to markets driven by a Poisson process in [AOPU00]. Brownian motion and the compensated Poisson process share the important chaos representation property which is crucial for market completeness.

In this paper we consider a larger family $(M_t)_{t\in[0,T]}$ of martingales satisfying the following two conditions:

(a) the chaotic representation property (with respect to market completeness), i.e. every square-integrable functional, measurable with respect to the filtration generated by $(M_t)_{t\in[0,T]}$, can be expanded into a series of multiple stochastic integrals of deterministic functions with respect to $(M_t)_{t\in[0,T]}$,

(b) the condition $d\langle M, M\rangle_t = \alpha_t^2 dt$, where $(\alpha_t)_{t\in[0,T]}$ is a square integrable deterministic function.

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Hypothesis (b) implies that $([M,M]_t - \int_0^t \alpha_s^2\, ds)_{t\in[0,T]}$ is a martingale, hence from (a) there exists a process $(\phi_t)_{t\in[0,T]}$ such that $(M_t)_{t\in[0,T]}$ satisfies the structure equation

$$d[M,M]_t = \alpha_t^2 dt + \phi_t dM_t, \quad t \in [0,T] \tag{1.0.1}$$

(cf. [É90]). This equation can be viewed as a decomposition of $d[M,M]_t$ by projection on dt and dM_t, which yields a closed Itô type change of variable formula (see Prop. 2.3).

Brownian motion is obtained for $\alpha_t = 1$ and $\phi_t = 0$ for all $t \in [0,T]$, and the Poisson process corresponds to non-zero constant ϕ_t, $t\in[0,T]$. The choice $\phi_t = \beta M_t$, $-2 \le \beta < 0$, considered in [DP99], corresponds to the Azéma martingale and yields another complete market model with jumps. Choosing $(\phi_t)_{t\in[0,T]}$ to be a deterministic function allows the driving process to be alternatively Brownian or Poisson, depending on the vanishing of ϕ_t (see [JP02] for the corresponding market model).

The Clark–Ocone formula states the predictable representation of a random variable F as

$$F = E[F] + \int_0^T E[D_t F \mid \mathcal{F}_t]\, dM_t,$$

where $(\mathcal{F}_t)_{t\in[0,T]}$ is the filtration generated by $(M_t)_{t\in[0,T]}$ and D_t is the gradient operator that lowers the degree of multiple stochastic integrals with respect to $(M_t)_{t\in[0,T]}$. One of the goals of this paper is to compute the process $t \mapsto E[D_t F \mid \mathcal{F}_t]$ in several situations. We obtain explicit hedging formulas for European calls in the mixed Brownian–Poisson model of [JP02] and in the Azéma martingale model of [DP99] and for Asian and Lookback options.

More precisely, let $(S^x_{t,T})_{t\in[0,T]}$ denote the stock price process driven by $(M_t)_{t\in[0,T]}$, starting from x at time t, with volatility $(\sigma_t)_{t\in[0,T]}$, and let $i_t = 1_{\{\phi_t=0\}}$, $j_t = 1 - i_t$, $S_t = S^1_{0,t}$ for $t\in[0,T]$. In a model with deterministic structure equation, i.e. $d[M,M]_t = \alpha_t^2 dt + \phi_t dM_t$ with deterministic $(\phi_t)_{t\in[0,T]}$, the replicating hedging strategy of a European call with payoff $(S_T - K)^+$ is given by

$$t \mapsto \frac{e^{-\int_t^T r_s\, ds}}{\sigma_t S_t} E\Bigg[i_t \sigma_t S^x_{t,T} 1_{\{S^x_{t,T} \ge K\}} + \frac{j_t}{\phi_t}(\sigma_t \phi_t S^x_{t,T} - (K - S^x_{t,T})^+) 1_{\{S^x_{t,T} \ge K/(1+\sigma_t)\}} \Bigg]_{x=S_t}$$

(cf. (4.1.6) and Prop. 4.1). This formula extends both the classical Black–Scholes hedging formula in the Brownian case ($\phi_t = 0$ for all $t\in[0,T]$), and the hedging formula of [AOPU00, Th. 6.1] in the Poissonian jump case

($\phi_t = 1$ for all $t \in [0,T]$). It can be obtained both from the Clark formula and from martingale methods. The above conditional expectation can also be explicitly computed (see Prop. 4.2). The case of Asian options is treated in Proposition 4.3 in the deterministic structure equation model, and Lookback options are considered in Proposition 4.5 in a market driven by Brownian motion. In the Azéma martingale model of [DP99], i.e. $d[M,M]_t = dt + \beta M_{t^-} dM_t$, $-2 \le \beta < 0$, we obtain

$$t \mapsto \frac{e^{-\int_t^T r_s\,ds}}{\beta M_t \sigma_t S_t} E\Big[\sigma_t \beta (y + M_T - M_t) S_{t,T}^x 1_{[\frac{K}{1+\sigma_t\beta(y+M_T-M_t)},\infty[}(S_{t,T}^x) \\ + (S_{t,T}^x - K) 1_{[\frac{K}{1+\sigma_t\beta(y+M_T-M_t)},K]}(S_{t,T}^x)\Big]_{x=S_t}^{y=M_t}$$

(see Prop. 4.4).

We proceed as follows. In Section 2 we introduce the notation of chaotic calculus, the solutions of structure equations and the Clark–Ocone formula which gives the predictable representation of the random variable F. We also state the change of variable formula and Girsanov theorem, which hold in a particular form for solutions of structure equations. In Section 3 we describe different methods for the computation of predictable representations for the general class of normal martingales having the chaos representation property. The intrinsic expression of the gradient D is completely known if $(\phi_t)_{t\in[0,T]}$ is deterministic, i.e. for the Brownian, Poisson and deterministic structure equation models (Section 3.2). In the Markovian case (Section 3.3) it is possible to combine the Clark–Ocone and Itô formulas to obtain the explicit predictable representation of F. Section 4 is devoted to the computation of replicating portfolios. In Section 4.2 we hedge European calls using the Clark formula, extending the method applied in the Brownian case in [Øks96, Ch. 5, pp. 13–15]. In Sections 4.3–4.5 we deal with Asian, European and Lookback options, in particular we use the delta hedging approach to recover some results obtained in [Ber98] from the Clark formula.

2. Notation and preliminaries

2.1. *Chaotic calculus.* Let $(M_t)_{t\in[0,T]}$ be a martingale on the space Ω of càdlàg functions from $[0,T]$ to $\mathbb{R}$, having the chaos representation property. Let $(\alpha_t)_{t\in[0,T]} \in L^2([0,T])$ be a positive deterministic non-vanishing square-integrable function, and assume that $(M_t)_{t\in[0,T]}$ has deterministic angle bracket $d\langle M,M\rangle_t = \alpha_t^2 dt$. We denote by $(\mathcal{F}_t)_{t\in[0,T]}$ the filtration generated by $(M_t)_{t\in[0,T]}$, and by $L_\alpha^2([0,T])^{\circ n}$ the space $L^2([0,T],\alpha_t^2 dt)^{\circ n}$ of $\alpha_{t_1}^2 \dots \alpha_{t_n}^2 dt_1 \dots dt_n$ square-integrable symmetric functions. The multiple stochastic integral $I_n(f_n)$ is defined as

$$I_n(f_n) = n! \int_0^T \int_0^{t_n} \dots \int_0^{t_2} f_n(t_1, \dots, t_n)\, dM_{t_1} \dots dM_{t_n}, \qquad n \geq 1,$$

for $f_n \in L^2_\alpha([0,T])^{\circ n}$, with

$$E[I_n(f_n)I_m(g_m)] = n! 1_{\{n=m\}} \langle f_n, g_m \rangle_{L^2_\alpha([0,T])^{\circ n}}. \tag{2.1.1}$$

The *chaos representation property* for $(M_t)_{t\in[0,T]}$ states that every $F \in L^2(\Omega)$ has a decomposition as $F = \sum_{n=0}^\infty I_n(f_n)$. Let $D : \mathrm{Dom}(D) \to L^2(\Omega \times [0,T], dP \times \alpha_t^2 dt)$ denote the closable, unbounded gradient operator defined as

$$D_t F = \sum_{n=1}^\infty n I_{n-1}(f_n(*, t)) \qquad dP \times dt\text{-a.e.},$$

with $F = \sum_{n=0}^\infty I_n(f_n)$ in $\mathrm{Dom}(D)$, i.e.

$$\sum_{n=1}^\infty n n! \|f_n\|^2_{L^2(\mathbb{R}_+^n)} < \infty.$$

2.2. *Structure equations.* Let $L^\infty_{\mathrm{ad}}(\Omega \times [0,T])$ be the space of bounded, $(\mathcal{F}_t)_{t\in\mathbb{R}_+}$-adapted stochastic processes. We assume that $(M_t)_{t\in[0,T]}$ is a solution of the structure equation

$$d[M,M]_t = \alpha_t^2 dt + \phi_t dM_t, \quad t \in [0,T], \tag{2.2.1}$$

where $\phi_t = \varphi(t, M_{t^-})$ is a deterministic function of t and M_t. Existence and uniqueness of solutions are guaranteed when ϕ_t is a deterministic function [É90]. Existence is proved when $\phi_t = \varphi(M_{t^-})$ and φ is a continuous function [Mey89], and the solution is unique when $\varphi(x) = \beta x$ with $\beta \in [-2, 0)$ (cf. [É90]). See also [Pha00], [Tav99] for recent results on structure equations. Let $i_t = 1_{\{\phi_t = 0\}}$ and $j_t = 1_{\{\phi_t \neq 0\}} = 1 - i_t$ for $t \in [0,T]$. The continuous part of $(M_t)_{t\in[0,T]}$ is given by $dM_t^c = i_t dM_t$ and the possible jump of $(M_t)_{t\in[0,T]}$ at time $t \in [0,T]$ is $\Delta M_t = \phi_t$ (see [É90, p. 77]).

(a) If $(\phi_t)_{t\in[0,T]}$ is deterministic, let $\lambda_t = j_t \alpha_t^2 / \phi_t^2$ for $t \in [0,T]$. Then $(M_t)_{t\in[0,T]}$ can be represented as

$$dM_t = i_t \alpha_t dB_t + \phi_t (dN_t - \lambda_t dt), \qquad t \in [0,T], \qquad M_0 = 0, \tag{2.2.2}$$

where $(B_t)_{t\in[0,T]}$ is a standard Brownian motion, and $(N_t)_{t\in[0,T]}$ a Poisson process independent of $(B_t)_{t\in[0,T]}$, with intensity $\nu_t = \int_0^t \lambda_s\, ds$ for $t \in [0,T]$ (cf. [É90, Prop. 4] and [JP02]).

(b) If $\phi_t = \beta M_t$, $\beta \in \mathbb{R}$, then (2.2.1) has a unique solution called the *Azéma martingale* (cf. [É90]). If $-2 \leq \beta < 0$, this solution has the chaos representation property and it has been used to model a complete market with jumps in [DP99].

Figure 1 shows a sample path of $(S_t)_{t\in[0,T]}$ and the corresponding function $(i_t)_{t\in[0,T]}$ chosen to be a simple indicator function, with $S_0 = 4$, $\sigma_t = 1$, $\alpha_t^2 = 50$, $\phi_t = 1.6 i_t$, for $t \in [0,T]$.

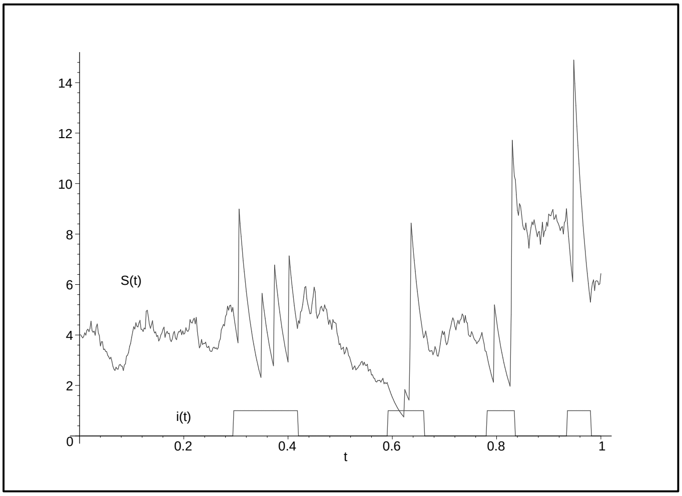

Fig. 1. Sample trajectory of $(S_t)_{t\in[0,T]}$ (vertical lines represent jumps)

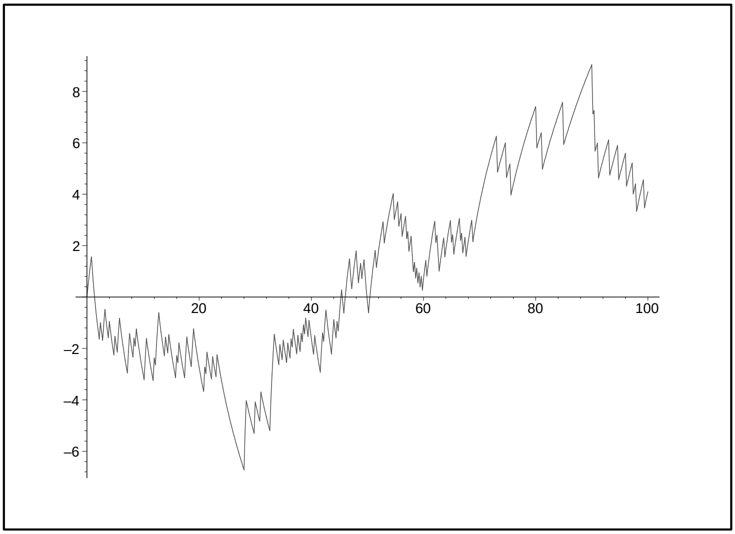

Fig. 2. Sample path of an Azéma martingale

Figure 2 is a simulation of an Azéma martingale with $\beta \neq -1$, from a discretization of the structure equation (2.2.1):

$$\Delta X_t = \frac{\beta X_{t^-}}{2} \pm \sqrt{(\beta X_{t^-}/2)^2 + \Delta t},$$

with probabilities

$$p = \frac{1}{2} \mp \frac{\beta X_{t^-}}{\sqrt{(\beta X_{t^-}/2)^2 + \Delta t}}.$$

In all cases of interest in this paper we have $(\phi_t)_{t\in[0,T]} \in L^\infty_{\mathrm{ad}}(\Omega \times [0,T])$, since if $(\phi_t)_{t\in[0,T]} = (\beta M_t)_{t\in[0,T]}$ then $\sup_{t\in[0,T]} |M_t| \le (-2/\beta)^{1/2}$ (see [É90, p. 83]). Given $(u_t)_{t\in[0,T]} \in L^\infty_{\mathrm{ad}}(\Omega \times [0,T])$, we denote by $(\xi_t(u))_{t\in[0,T]}$ the solution of the equation

$$Z_t = 1 + \int_0^t Z_{s^-} u_s \, dM_s, \qquad t \in [0,T], \tag{2.2.3}$$

which can be written as ([Pro90, Th. 36, p. 77])

$$\xi_t(u) = \exp\left(\int_0^t u_s \, dM_s - \frac{1}{2}\int_0^t u_s^2 \alpha_s^2 i_s \, ds\right) \prod_{s\in J_M^t} (1 + u_s \phi_s) e^{-u_s \phi_s}, \tag{2.2.4}$$

where J_M^t denotes the set of jump times of $(M_s)_{s\in[0,t]}$ for $t \in [0,T]$, and let $\xi(u) = \xi_T(u)$. If $u \in L^\infty([0,T])$ then $\xi_t(u)$ can be represented as

$$\xi_t(u) = \sum_{n=0}^\infty \frac{1}{n!} I_n((u 1_{[0,t]})^{\otimes n}),$$

and we have $D_s \xi_t(u) = 1_{[0,t]}(s) u_s \xi_t(u)$ for $s, t \in [0,T]$.

Definition 2.1. Let $\mathcal{S}$ denote the linear space generated by exponential vectors of the form $\xi(u)$, where $u \in L^\infty([0,T])$.

The space $\mathcal{S}$ is dense in $L^2(\Omega)$, and by the lemma below, $\mathcal{S}$ is an algebra for the pointwise multiplication of random variables if $(\phi_t)_{t\in[0,T]}$ is deterministic. The following is a version of Yor's formula [Yor76] or Theorem 37 of [Pro90, p. 79], for martingales with deterministic bracket $\langle M, M\rangle_t$.

Lemma 2.2. *For any $u, v \in L^\infty([0,T])$,*

$$\xi(u)\xi(v) = \exp(\langle u, v\rangle_{L^2_\alpha([0,T])}) \xi(u + v + \phi u v). \tag{2.2.5}$$

Proof. For $u, v \in L^\infty([0,T])$ we have

$$\begin{aligned} d(\xi_t(u)\xi_t(v)) &= u_t \xi_{t^-}(u)\xi_{t^-}(v) dM_t + v_t \xi_{t^-}(v)\xi_{t^-}(u) dM_t \\ &\quad + v_t u_t \xi_{t^-}(v)\xi_{t^-}(u) d[M,M]_t \\ &= u_t \xi_{t^-}(u)\xi_{t^-}(v) dM_t + v_t \xi_{t^-}(v)\xi_{t^-}(u) dM_t + v_t u_t \xi_t(v)\xi_t(u)\alpha_t^2 dt \\ &\quad + \phi_t u_t v_t \xi_{t^-}(v)\xi_{t^-}(u) dM_t \\ &= v_t u_t \xi_t(v)\xi_t(u)\alpha_t^2 dt + \xi_{t^-}(v)\xi_{t^-}(u)(u_t + v_t + \phi_t u_t v_t) dM_t. \end{aligned}$$

Hence

$$d(e^{-\int_0^t u_s v_s \alpha_s^2 ds}\xi_t(u)\xi_t(v)) = e^{-\int_0^t u_s v_s \alpha_s^2 ds}\xi_{t^-}(v)\xi_{t^-}(u)(u_t + v_t + \phi_t u_t v_t)dM_t,$$

which shows that $\exp(-\langle u,v\rangle_{L^2_\alpha([0,T])})\xi(u)\xi(v) = \xi(u+v+\phi uv)$. Relation (2.2.5) follows then by comparison with (2.2.3). ∎

2.3. *Change of variable formula.* We recall the following change of variable formula, which follows from Proposition 2 of [É90] after addition of an absolutely continuous drift term.

PROPOSITION 2.3. *Let $X = (X_t)_{t\in[0,T]}$ be an $\mathbb{R}^n$-valued process satisfying*

$$dX_t = R_t dt + K_t dM_t, \qquad X_0 > 0,$$

where $R = (R_t)_{t\in[0,T]}$ and $K = (K_t)_{t\in[0,T]}$ are two predictable square-integrable $\mathbb{R}^n$-valued processes. For any function $\mathbb{R}_+ \times \mathbb{R}^n \ni (t,x) \mapsto f_t(x)$ in $\mathcal{C}^2_{\mathrm{b}}(\mathbb{R}_+ \times \mathbb{R}^n;\mathbb{R})$ we have

$$f_t(X_t) = f_0(X_0) + \int_0^t L_s f_s(X_s)\,dM_s + \int_0^t U_s f_s(X_s)\,ds + \int_0^t \frac{\partial f_s}{\partial s}(X_s)\,ds, \tag{2.3.1}$$

where

$$L_s f_s(X_s) = i_s\langle K_s, \nabla f_s(X_s)\rangle + \frac{j_s}{\phi_s}\,(f_s(X_{s^-} + \phi_s K_{s^-}) - f_s(X_{s^-}))$$

and

$$\begin{aligned} U_s f_s(X_s) = R_s \nabla f_s(X_s) + \alpha_s^2\bigg(&\frac{1}{2}\, i_s\langle \operatorname{Hess} f_s(X_s), K_s \otimes K_s\rangle \\ &+ \frac{j_s}{\phi_s^2}\,(f_s(X_{s^-} + \phi_s K_{s^-}) - f_s(X_{s^-}) - \phi_s\langle K_s, \nabla f_s(X_s)\rangle)\bigg), \end{aligned}$$

with the convention $0/0 = 0$.

2.4. *Girsanov theorem.* The Girsanov theorem holds in a particular form when $(M_t)_{t\in[0,T]}$ is the solution of a structure equation (1.0.1). Let $(\psi_t)_{t\in[0,T]}$ be a bounded predictable process such that $1 + \phi_t\psi_t > 0$ for all $t \in [0,T]$, let $(l_t)_{t\in[0,T]}$ denote the solution of the equation

$$dl_t = l_{t^-}\psi_t dM_t, \qquad t \in [0,T], \qquad l_0 = 1,$$

and let Q be the probability defined by

$$l_t = E\left[\frac{dQ}{dP}\,\middle|\, \mathcal{F}_t\right], \qquad t \in [0,T]. \tag{2.4.1}$$

PROPOSITION 2.4. *Under the probability Q, the process*

$$Z_t = M_t - \int_0^t \alpha_s^2 \psi_s\,ds, \qquad t \in [0,T], \tag{2.4.2}$$

is a local martingale which satisfies the structure equation

$$d[Z,Z]_t = \alpha_t^2(1+\phi_t\psi_t)dt + \phi_t dZ_t, \qquad t\in[0,T]. \tag{2.4.3}$$

Proof. From the Girsanov theorem,

$$dZ_t = dM_t - \frac{1}{l_{t^-}}\, d\langle L, M\rangle_t = dM_t - \alpha_t^2\psi_t dt$$

is a local martingale under Q, with $d\langle L,M\rangle_t = l_{t^-}\alpha_t^2\psi_t dt$, and

$$d[Z,Z]_t = d[M,M]_t = \alpha_t^2 dt + \phi_t dM_t = \alpha_t^2(1+\phi_t\psi_t)dt + \phi_t dZ_t. \ \blacksquare$$

If $(\psi_t)_{t\in[0,T]}$ and $(\phi_t)_{t\in[0,T]}$ are deterministic, then $(Z_t)_{t\in[0,T]}$ has the chaos representation property under Q, since (2.4.3) is a deterministic structure equation.

3. Computations of predictable representations

3.1. *Clark formula.* The Clark–Ocone formula (cf. [Cla70], [KO91]) is a consequence of the chaos representation property for $(M_t)_{t\in[0,T]}$, and states that any $F\in \mathrm{Dom}(D)\subset L^2(\Omega,\mathcal{F}_T,P)$ has a representation

$$F = E[F] + \int_0^T E[D_tF \mid \mathcal{F}_t]\, dM_t. \tag{3.1.1}$$

It can be proved as follows:

$$\begin{aligned}
F &= E[F] + \sum_{n=1}^{\infty} n! \int_0^T\int_0^{t_n}\cdots\int_0^{t_2} f_n(t_1,\dots,t_n)\, dM_{t_1}\dots dM_{t_n}\\
&= E[F] + \sum_{n=1}^{\infty} n\int_0^T I_{n-1}(f_n(*,t)1_{\{*\le t\}})\, dM_t\\
&= E[F] + \int_0^T E[D_tF\mid \mathcal{F}_t]\, dM_t.
\end{aligned}$$

Although $D: L^2(\Omega,\mathcal{F}_T,P)\to L^2(\Omega\times[0,T], dP\times\alpha_t^2dt)$ is unbounded, the representation formula (3.1.1) can be extended to $F\in L^2(\Omega,\mathcal{F}_T,P)$.

Proposition 3.1. *The operator $F\mapsto E[D_\cdot F\mid\mathcal{F}_\cdot]$ taking its values in the space of square-integrable adapted processes has a continuous extension from* $\mathrm{Dom}(D)$ *to* $L^2(\Omega,\mathcal{F}_T,P)$.

Proof. We use the bound

$$\begin{aligned}
\|E[D_\cdot F\mid\mathcal{F}_\cdot]\|^2_{L^2(\Omega\times[0,T])} &= \|E[F]-F\|^2_{L^2(\Omega)} = \mathrm{var}(F)\\
&\le \|F\|^2_{L^2(\Omega)}, \qquad F\in\mathrm{Dom}(D). \ \blacksquare
\end{aligned}$$

Instead of the adapted projection $(E[D_tF \mid \mathcal{F}_t])_{t\in[0,T]}$ one may also use the predictable projection $(E[D_tF \mid \mathcal{F}_{t-}])_{t\in[0,T]}$. Here this leads to the same representation since both processes coincide in $L^2(\Omega \times [0,T], dP \times \alpha_t^2 dt)$.

3.2. *Deterministic structure equation.* In this subsection we consider the case where $(\phi_t)_{t\in[0,T]} \in L^\infty([0,T])$ is a deterministic function. In this case, the probabilistic interpretation of D_t is known and D_tF is explicitly computable. We define the operator $D^B : \mathcal{S} \to L^2(\Omega \times \mathbb{R}_+, dP \times \alpha_t^2 dt)$ on $\mathcal{S}$ as

$$\langle D^B F, u\rangle_{L^2_\alpha([0,T])} = \frac{d}{d\varepsilon} F\Big(\omega(\cdot) + \varepsilon \int_0^{\cdot} i_s u_s\, ds\Big)\Big|_{\varepsilon=0}, \qquad F \in \mathcal{S}.$$

For $F = \xi(u)$ and $g \in L^2_\alpha([0,T])$ we have

$$\begin{aligned}\langle D^B F, g\rangle_{L^2_\alpha([0,T])} &= \frac{d}{d\varepsilon} \exp\Big(\varepsilon \int_0^T g_s u_s \alpha_s i_s\, ds\Big)\xi(u)\Big|_{\varepsilon=0} \\ &= \int_0^T g_s u_s \alpha_s i_s\, ds\, \xi(u),\end{aligned}$$

hence $D_t^B \xi(u) = i_t u_t \xi(u)$ for $t \in [0,T]$, where

$$\xi(u) = \exp\Big(\int_0^T u_s\, dM_s - \frac{1}{2}\int_0^T u_s^2 \alpha_s^2 i_s\, ds\Big) \prod_{s\in J_M^T} (1 + u_s\phi_s) e^{-u_s\phi_s}. \tag{3.2.1}$$

Note that the definition of $D^B F$ by duality in $L^2_\alpha([0,T])$ implies

$$D_t^B \int_0^T u_s \alpha_s\, dB_s = u_t, \qquad t \in [0,T].$$

We define a linear transformation T_t^ϕ of exponential vectors, and more generally of elements of $\mathcal{S}$, as

$$T_t^\phi \xi(u) = (1 + u_t\phi_t)\xi(u), \qquad u \in L^\infty([0,T]).$$

The transformation T_t^ϕ is well defined on $\mathcal{S}$ because $\xi(u_1), \dots, \xi(u_n)$ are linearly independent if $u_1, \dots, u_n$ are distinct elements of $L^2(\mathbb{R}_+)$. Since $\Delta M_t = 0$ $dt \times dP$-a.e., $T_t^\phi \xi(u)$ coincides $dt \times dP$-a.e. with the value at time T of the solution of the equation

$$Z_s^t = 1 + \int_0^s Z_{\tau-}^t u_\tau\, dM_\tau^t, \qquad s \in [0,T], \tag{3.2.2}$$

where $(M_s^t)_{s\in[0,T]}$ is defined as

$$M_s^t = M_s + \phi_t 1_{[t,T]}(s), \qquad s \in [0,T].$$

In order to see this we check that $Z^t_s = \xi_s(u)$ for $s < t$,

$$Z^t_t = (1 + \phi_t u_t) Z^t_{t^-} = (1 + \phi_t u_t)\xi_{t^-}(u) = (1 + \phi_t u_t)\xi_t(u)$$

(since $\xi_{t^-}(u) = \xi_t(u)$ a.s. for fixed t), and for $s > t$,

$$Z^t_s = Z^t_t + \int_t^s Z^t_{\tau^-} u_\tau \, dM_\tau = (1 + \phi_t u_t)\xi_t(u) + \int_t^s Z^t_{\tau^-} u_\tau \, dM_\tau,$$

hence

$$\frac{Z^t_s}{1 + \phi_t u_t} = \xi_t(u) + \int_t^s \frac{Z^t_{\tau^-}}{1 + \phi_t u_t} \, u_\tau \, dM_\tau, \qquad s > t,$$

which implies, from (2.2.3),

$$\frac{Z^t_s}{1 + \phi_t u_t} = \xi_s(u), \qquad s > t,$$

and $Z^t_T = (1 + \phi_t u_t)\xi(u) = T^\phi_t \xi(u)$ P-a.s. for $t \in [0, T]$.

In other terms, $T^\phi_t F$, $F \in \mathcal{S}$, can be interpreted as the evaluation of F on the trajectories of $(M_s)_{s\in[0,T]}$ perturbed by addition of a jump of height ϕ_t at time t.

Proposition 3.2. *The transformation T^ϕ_t is multiplicative, i.e.*

$$T^\phi_t(FG) = (T^\phi_t F)(T^\phi_t G), \qquad F, G \in \mathcal{S}.$$

Moreover,

$$D_t F = D^B_t F + \frac{j_t}{\phi_t} (T^\phi_t F - F), \qquad t \in [0, T], \; F \in \mathcal{S}, \tag{3.2.3}$$

and

$$D_t(FG) = F D_t G + G D_t F + \phi_t D_t F D_t G, \qquad t \in [0, T], \; F, G \in \mathcal{S}. \tag{3.2.4}$$

Proof. For the multiplicativity we note that

$$\begin{aligned} T^\phi_t(\xi(u)\xi(v)) &= \exp(\langle u, v\rangle_{L^2_\alpha([0,T])}) T^\phi_t \xi(u + v + \phi uv) \\ &= \exp(\langle u, v\rangle_{L^2_\alpha([0,T])})(1 + \phi_t(u_t + v_t + \phi_t u_t v_t))\xi(u + v + \phi uv) \\ &= (1 + \phi_t u_t)(1 + \phi_t v_t)\xi(u)\xi(v) \\ &= T^\phi_t \xi(u) T^\phi_t \xi(v). \end{aligned}$$

When $\phi_t = 0$ we have $D^B_t F = i_t u_t \xi(u) = i_t D_t F$, hence

$$\begin{aligned} D_t\xi(u) &= i_t D_t \xi(u) + j_t D_t \xi(u) = i_t u_t \xi(u) + j_t u_t \xi(u) \\ &= D^B_t \xi(u) + \frac{j_t}{\phi_t} (T^\phi_t \xi(u) - \xi(u)) \qquad t \in [0, T]. \end{aligned}$$

Concerning the product rule we have, from Lemma 2.2,

$$\begin{aligned} D_t(\xi(u)\xi(v)) &= \exp\Big(\int_0^T u_s v_s \alpha_s^2\, ds\Big) D_t \xi(u+v+\phi uv) \\ &= \exp\Big(\int_0^T u_s v_s \alpha_s^2\, ds\Big)(u_t + v_t + \phi_t u_t v_t)\xi(u+v+\phi uv) \\ &= (u_t + v_t + \phi_t u_t v_t)\xi(u)\xi(v) \\ &= \xi(u) D_t\xi(v) + \xi(v) D_t \xi(u) + \phi_t D_t\xi(u) D_t \xi(v) \end{aligned}$$

for $u, v \in L^\infty([0,T])$ (see also [Pri96, (6)]. ■

If $(\phi_t)_{t\in[0,T]}$ is random the probabilistic interpretation of D is unknown, but we have the product rule

$$\begin{aligned} E[D_t(FG)\,|\,\mathcal{F}_t] = E[F D_t G\,|\,\mathcal{F}_t] + E[G D_t F\,|\,\mathcal{F}_t] \\ + \phi_t E[D_t F D_t G\,|\,\mathcal{F}_t] \end{aligned} \tag{3.2.5}$$

for $F, G \in \mathcal{S}$ and $t \in [0,T]$ (cf. [PSV00, Prop. 5]).

3.3. *Markovian case.* This section presents a representation method which is based on the Itô formula and the Markov property (see also [Pro01] in the continuous case). Let $(X_t)_{t\in[0,T]}$ be an $\mathbb{R}^n$-valued Markov (not necessarily time homogeneous) process defined on Ω, satisfying a change of variable formula of the form

$$f(X_t) = f(X_0) + \int_0^t L_s f(X_s)\, dM_s + \int_0^t U_s f(X_s)\, ds, \qquad t \in [0,T], \tag{3.3.1}$$

where L_s, U_s are operators defined on $\mathcal{C}^2$ functions. We assume that the semigroup $(P_{s,t})_{0\le s\le t\le T}$ associated to $(X_t)_{t\in[0,T]}$, i.e.

$$P_{s,t} f(X_s) = E[f(X_t)\,|\,\mathcal{F}_s] = E[f(X_t)\,|\,X_s], \qquad 0 \le s \le t \le T,$$

acts on $\mathcal{C}_\mathrm{b}^2(\mathbb{R}^n)$ functions, with $P_{s,t} \circ P_{t,u} = P_{s,u}$ for $0 \le s \le t \le u \le T$. Although the probabilistic interpretation of D is not known when $(\phi_t)_{t\in[0,T]}$ is random, it is still possible to compute the explicit predictable representation of $f(X_T)$ using the Itô formula and the Markov property.

LEMMA 3.3. *Let $f \in \mathcal{C}_\mathrm{b}^2(\mathbb{R}^n)$. Then*

$$E[D_t f(X_T)\,|\,\mathcal{F}_t] = (L_t(P_{t,T} f))(X_t), \quad t \in [0,T]. \tag{3.3.2}$$

Proof. We apply the change of variable formula (3.3.1) to $t \mapsto P_{t,T} f(X_t) = E[f(X_T)\,|\,\mathcal{F}_t]$, since $P_{t,T} f$ is $\mathcal{C}^2$. Using the fact that the finite variation term vanishes since $t \mapsto P_{t,T} f(X_t)$ is a martingale (see e.g. [Pro90, Cor. 1,

p. 64]), we obtain

$$P_{t,T}f(X_t) = P_{0,T}f(X_0) + \int_0^t (L_s(P_{s,T}f))(X_s)\, dM_s, \qquad t \in [0,T],$$

with $P_{0,T}f(X_0) = E[f(X_T)]$. Letting $t = T$, we obtain (3.3.2) by uniqueness of the representation (3.1.1) applied to $F = f(X_T)$. ■

In practice, we will use Proposition 3.1 to extend $(E[D_t f(X_T) \mid \mathcal{F}_t])_{t\in[0,T]}$ to a less regular function $f : \mathbb{R}^n \to \mathbb{R}$. As an example, if ϕ_t is written as $\phi_t = \varphi(t, M_t)$, and $dS_t = \sigma(t, S_t)dM_t + \mu(t, S_t)dt$, we can apply Proposition 2.3 with $(X_t)_{t\in[0,T]} = ((S_t, M_t))_{t\in[0,T]}$ and

$$\begin{aligned} L_t f(S_t, M_t) &= i_t\sigma(t, S_t)\partial_1 f(S_t, M_t) + i_t\partial_2 f(S_t, M_t) \\ &\quad + \frac{j_t}{\varphi(t, M_t)} \left(f(S_t + \varphi(t, M_t)\sigma(t, S_t), M_t + \varphi(t, M_t)) - f(S_t, M_t)\right), \end{aligned}$$

since the possible jump of $(M_t)_{t\in[0,T]}$ at time t is $\varphi(t, M_t)$. Here ∂_1, resp. ∂_2, denotes the partial derivative with respect to the first, resp. second, variable. Hence

$$\begin{aligned} &E[D_t f(S_T, M_T) \mid \mathcal{F}_t] \\ &\quad = i_t\sigma(t, S_t)(\partial_1 P_{t,T}f)((S_t, M_t)) + i_t(\partial_2 P_{t,T}f)((S_t, M_t)) \\ &\qquad + \frac{j_t}{\varphi(t, M_t)} (P_{t,T}f)((S_t + \varphi(t, M_t)\sigma(t, S_t), M_t + \varphi(t, M_t))) \\ &\qquad - \frac{j_t}{\varphi(t, M_t)} (P_{t,T}f)((S_t, M_t)). \end{aligned}$$

If $(M_t)_{t\in[0,T]}$ is an Azéma martingale ($\phi_t = \beta M_t$ for $t \in [0,T]$), then $i_t = 0$ $dP \times dt$ a.s.

4. Computations of hedging strategies

4.1. *Market model.* In this subsection we introduce the price process which will be considered in what follows. Let $\mu : [0,T] \to \mathbb{R}$ and $\sigma : [0,T] \to]0,\infty[$ be deterministic bounded functions. Let $(r_t)_{t\in[0,T]}$ be a deterministic non-negative function which models a riskless asset, and let $(\psi_t)_{t\in[0,T]}$ be defined as

$$\psi_t = \frac{r_t - \mu_t}{\sigma_t \alpha_t^2}, \quad t \in [0,T].$$

We assume that $1 + \phi_t\psi_t > 0$ for $t \in [0,T]$. If $(\phi_t)_{t\in[0,T]}$ is not deterministic this choice is still possible due to the boundedness of $(\phi_t)_{t\in[0,T]}$ or from [DP99, Th. 2.1(iii)]. Let Q denote the probability defined by $E\left[\frac{dQ}{dP} \,\middle|\, \mathcal{F}_t\right] = l_t$ for $t \in [0,T]$, where $dl_t = l_{t^-}\psi_t dM_t$ for $t \in [0,T]$, with $l_0 = 1$. From Proposition 2.4,

$$Z_t = M_t - \int_0^t \alpha_s^2 \psi_s \, ds, \qquad t \in [0, T],$$

is a local martingale under Q with angle bracket $d\langle Z, Z\rangle_t = \alpha_t^2(1 + \phi_t\psi_t)dt$. If $(\phi_t)_{t\in[0,T]}$ is deterministic then $(Z_t)_{t\in[0,T]}$ is a true martingale under Q. This also holds if

$$(\phi_t)_{t\in[0,T]} = (\beta M_t)_{t\in[0,T]},$$

from the boundedness of $(M_t)_{t\in[0,T]}$. If $(\phi_t)_{t\in[0,T]}$ is deterministic, we may start from the data of $(\widetilde{\alpha}_t)_{t\in[0,T]}$, take $d[Z, Z]_t = \widetilde{\alpha}_t^2 dt + \phi_t dZ_t$ and define $(\alpha_t)_{t\in[0,T]}$ by $\alpha_t^2 = \widetilde{\alpha}_t^2/(1 + \phi_t\psi_t)$ for $t \in [0, T]$. Let the price $(S_t)_{t\in[0,T]}$ of a risky asset satisfy the equation

$$dS_t = \mu_t S_t dt + \sigma_t S_{t^-} dZ_t, \qquad t \in [0, T], \qquad S_0 = 1. \tag{4.1.1}$$

We have

$$dS_t = r_t S_t dt + \sigma_t S_{t^-} dM_t, \qquad t \in [0, T], \tag{4.1.2}$$

and under P, $(S_t e^{-\int_0^t r_s \, ds})_{t\in[0,T]}$ is a martingale, i.e. the market is arbitrage free. Let η_t and ζ_t denote the number of units invested at time t in the risky and riskless assets respectively. Thus the value V_t of the portfolio at time t is given by

$$V_t = \zeta_t A_t + \eta_t S_t, \qquad t \in [0, T], \tag{4.1.3}$$

where

$$dA_t = r_t A_t dt, \qquad A_0 = 1, \qquad t \in [0, T]. \tag{4.1.4}$$

We assume that the portfolio is self-financing, i.e.

$$dV_t = \zeta_t dA_t + \eta_t dS_t, \qquad t \in [0, T],$$

therefore

$$dV_t = r_t V_t dt + \sigma_t \eta_t S_{t^-} dM_t, \qquad t \in [0, T],$$

and

$$V_T e^{-\int_0^T r_t \, dt} = V_0 + \int_0^T \sigma_t \eta_t S_{t^-} e^{-\int_0^t r_s \, ds} \, dM_t, \tag{4.1.5}$$

Suppose that we are required to find a portfolio $(\zeta_t, \eta_t)_{t\in[0,T]}$ which leads to a given value $V_T = F$. By the Clark–Ocone formula,

$$F = E[F] + \int_0^T E[D_t F \mid \mathcal{F}_t] \, dM_t,$$

and comparing with (4.1.5) we obtain

$$V_0 = e^{-\int_0^T r_s \, ds} E[F],$$

$$\eta_t = \sigma_t^{-1} S_t^{-1} E[D_t F \mid \mathcal{F}_t] e^{-\int_t^T r_s \, ds}, \qquad t \in [0, T]. \tag{4.1.6}$$

Next we consider different models with explicit computations of hedging strategies.

4.2. *European options and deterministic structure.* In this section we hedge European calls using the Clark formula. Let $(M_t)_{t\in[0,T]}$ be the martingale described in Section 2.2(a), with deterministic $(\phi_t)_{t\in[0,T]}$, i.e. $(M_t)_{t\in[0,T]}$ is alternatively Brownian or Poisson depending on the vanishing of $(\phi_t)_{t\in[0,T]}$. We assume that $1+\sigma_t\phi_t>0$ for all $t\in[0,T]$. We have

$$S_t = \exp\left(\int_0^t \sigma_s\alpha_s i_s\, dB_s + \int_0^t (r_s - \phi_s\lambda_s\sigma_s - \frac{1}{2}\, i_s\sigma_s^2\alpha_s^2)\, ds\right)\prod_{k=1}^{N_t}(1+\sigma_{T_k}\phi_{T_k})$$

for $0\le t\le T$, where $(T_k)_{k\ge 1}$ denotes the jump times of $(N_t)_{t\in\mathbb{R}_+}$. We will denote by $(S^x_{t,u})_{u\in[t,T]}$ the process defined as

$$dS^x_{t,u} = r_t S^x_{t,u} du + \sigma_u S^x_{t,u^-} dM_u, \qquad u\in[t,T], \qquad S^x_{t,t}=x.$$

We have

$$S^x_{t,T} = x\exp\left(\int_t^T \sigma_u\alpha_u i_u\, dB_u + \int_t^T \left(r_u - \phi_u\lambda_u\sigma_u - \frac{1}{2}\, i_u\sigma_u^2\alpha_u^2\right) du\right) \times \prod_{k=1+N_t}^{N_T}(1+\sigma_{T_k}\phi_{T_k})$$

for $0\le t\le T$, with $S_t = S^1_{0,t}$ for $t\in[0,T]$.

PROPOSITION 4.1. *Assume that $\phi_t\ge 0$ for all $t\in[0,T]$. Then for $0\le t\le T$ we have*

$$E[D_t(S_T-K)^+\mid \mathcal{F}_t] = E\Big[i_t\sigma_t S^x_{t,T}1_{[K,\infty[}(S^x_{t,T}) + \frac{j_t}{\phi_t}\,(\sigma_t\phi_t S^x_{t,T} - (K-S^x_{t,T})^+)1_{[\frac{K}{1+\sigma_t},\infty[}(S^x_{t,T})\Big]_{x=S_t}.$$

Proof. By Proposition 3.2, for any $F\in\mathcal{S}$ we have

$$D_tF = D^B_tF + \frac{j_t}{\phi_t}\,(T^\phi_tF - F), \qquad t\in[0,T]. \tag{4.2.1}$$

We have $T^\phi_t S_T = (1+\sigma_t\phi_t)S_T$ for $t\in[0,T]$, and the chain rule $D^Bf(F) = f'(F)D^BF$ holds for $F\in\mathcal{S}$ and $f\in\mathcal{C}^2_{\mathrm{b}}(\mathbb{R})$. Since $\mathcal{S}$ is an algebra for deterministic $(\phi_t)_{t\in[0,T]}$, we may approach $x\mapsto(x-K)^+$ by polynomials on compact intervals and proceed e.g. as in [Øks96, pp. 5–13]. By dominated convergence, $(S_T-K)^+\in\mathrm{Dom}(D)$ and (4.2.1) becomes

$$D_t(S_T-K)^+ = i_t\sigma_t S_T1_{[K,\infty[}(S_T) + \frac{j_t}{\phi_t}\,((1+\sigma_t\phi_t)S_T-K)^+ - (S_T-K)^+)$$

for $0 \le t \le T$. The Markov property of $(S_t)_{t\in[0,T]}$ implies

$$E[D_t^B (S_T - K)^+ \mid \mathcal{F}_t] = i_t \sigma_t E[S^x_{t,T} 1_{[K,\infty[}(S^x_{t,T})]_{x=S_t},$$

and

$$\begin{aligned}
&\frac{j_t}{\phi_t} E[(T_t^\phi S_T - K)^+ - (S_T - K)^+ \mid \mathcal{F}_t] \\
&\quad = \frac{j_t}{\phi_t} E[((1+\sigma_t\phi_t) S^x_{t,T} - K)^+ - (S^x_{t,T} - K)^+]_{x=S_t} \\
&\quad = \frac{j_t}{\phi_t} E[((1+\sigma_t\phi_t) S^x_{t,T} - K) 1_{[\frac{K}{1+\sigma_t\phi_t},\infty[}(S^x_{t,T})]_{x=S_t} \\
&\qquad - \frac{j_t}{\phi_t} E[(S^x_{t,T} - K)^+ 1_{[K,\infty[}(S^x_{t,T})]_{x=S_t} \\
&\quad = \frac{j_t}{\phi_t} E[\sigma_t\phi_t S^x_{t,T} 1_{[\frac{K}{1+\sigma_t\phi_t},\infty[}(S^x_{t,T}) + (S^x_{t,T} - K) 1_{[\frac{K}{1+\sigma_t\phi_t},K]}(S^x_{t,T})]_{x=S_t} \\
&\quad = \frac{j_t}{\phi_t} E[\sigma_t\phi_t S^x_{t,T} 1_{[\frac{K}{1+\sigma_t\phi_t},\infty[}(S^x_{t,T}) - (K - S^x_{t,T})^+ 1_{[\frac{K}{1+\sigma_t\phi_t},\infty]}(S^x_{t,T})]_{x=S_t} \\
&\quad = \frac{j_t}{\phi_t} E[(\sigma_t\phi_t S^x_{t,T} - (K - S^x_{t,T})^+) 1_{[\frac{K}{1+\sigma_t\phi_t},\infty[}(S^x_{t,T})]_{x=S_t}. \quad \blacksquare
\end{aligned}$$

If $(\phi_t)_{t\in[0,T]}$ is not constrained to be positive then

$$\begin{aligned}
E[D_t(S_T - K)^+ \mid \mathcal{F}_t] &= i_t\sigma_t E[S^x_{t,T} 1_{[K,\infty[}(S^x_{t,T})]_{x=S_t} \\
&\quad + \frac{j_t}{\phi_t} E[\sigma_t\phi_t S^x_{t,T} 1_{[\frac{K}{1+\sigma_t\phi_t},\infty[}(S^x_{t,T}) + (S^x_{t,T} - K) 1_{[\frac{K}{1+\sigma_t\phi_t},K]}(S^x_{t,T})]_{x=S_t},
\end{aligned}$$

with the convention $1_{[b,a]} = -1_{[a,b]}$ for $0 \le a < b \le T$. Proposition 4.1 can also be proved using Lemma 3.3 and the Itô formula (2.3.1). In the deterministic case, the semigroup $P_{t,T}$ can be explicitly computed. Let $\Gamma^\sigma_{t,T} = \int_t^T i_s \alpha_s^2 \sigma_s^2 \, ds$ denote the variance of $\int_t^T i_s \alpha_s \sigma_s \, dB_s$ for $t \in [0,T]$, and let $\Gamma_{t,T} = \int_t^T \gamma_s ds$, $t \in [0,T]$, denote the intensity of $N_T - N_t$ under Q, where $\gamma_t = \lambda_t(1+\phi_t\psi_t)$ for $t \in [0,T]$.

Proposition 4.2. *For $f \in \mathcal{C}_{\mathrm{b}}(\mathbb{R})$ we have*

$$\begin{aligned}
P_{t,T} f(x) &= \sum_{k=0}^\infty \frac{e^{-\Gamma_{t,T}}}{k!} \int_{-\infty}^{\infty} \nu_{t_0} \int_{[t,T]^k} \gamma_{t_1} \dots \gamma_{t_k} \\
&\qquad \times f\Big(x e^{-\Gamma^\sigma_{t,T}/2 + (\Gamma^\sigma_{t,T})^{1/2} t_0 - \int_t^T \phi_s\gamma_s\sigma_s \, ds} \prod_{i=1}^k (1+\sigma_{t_i}\phi_{t_i}) \Big) \, dt_1 \dots dt_k \, dt_0,
\end{aligned}$$

where ν denotes the standard Gaussian density.

Proof. We have $P_{t,T}f(x) = E[f(S_T) \mid S_t = x] = E[f(S^x_{t,T})]$, and

$$P_{t,T}f(x) = \sum_{k=0}^{\infty} E[f(S^x_{t,T}) \mid N_T - N_t = k] \exp(-\Gamma_{t,T}) \frac{\Gamma^k_{t,T}}{k!}.$$

For $k \in \mathbb{N}$, since $(N_{\Gamma^{-1}_{t,s}} - N_t)_{s\in[t,T]}$ is a standard Poisson process, conditionally on $\{N_T - N_t = k\}$, the first n jump times $(T_1, \ldots, T_n)$ of $(N_s)_{s\in[t,T]}$ have the law

$$\frac{k!}{(\Gamma_{t,T})^k} 1_{\{t<t_1<\ldots<t_k<T\}} \gamma_{t_1} \cdots \gamma_{t_k} dt_1 \ldots dt_k,$$

and conditionally on $\{N_T - N_t = k\}$, the jump times $(\Gamma_{t,T_1}, \ldots, \Gamma_{t,T_k})$ have a uniform law on $[0, \Gamma_{t,T}]^k$. We then use the identity in law between $S^x_{t,T}$ and

$$xX_{t,T} \exp\Big(- \int_t^T \phi_s \lambda_s (1 + \phi_s \psi_s) \sigma_s \, ds\Big) \prod_{k=1+N_t}^{N_T} (1 + \sigma_{T_k} \phi_{T_k}),$$

where

$$X_{t,T} = \exp(-\Gamma^\sigma_{t,T}/2 + (\Gamma^\sigma_{t,T})^{1/2} W),$$

and W a standard Gaussian random variable, independent of $(N_t)_{t\in[0,T]}$. This identity holds because $(B_t)_{t\in[0,T]}$ is a standard Brownian motion, independent of $(N_t)_{t\in[0,T]}$. ∎

See Proposition 8 of [JP02] for a computation of

$$E\Big[\exp\Big(-\int_0^T r_s \, ds\Big)(S_T - K)^+\Big]$$

in terms of the classical Black–Scholes function

$$\mathrm{BS}(x, T; r, \sigma^2; K) = E[e^{-rT}(xe^{rT-\sigma^2 T/2+\sigma W_t} - K)^+],$$

where W_t is a centered Gaussian random variable with variance t.

4.3. *Asian options and deterministic structure.* The price at time t of such an option is

$$E\bigg[e^{-\int_t^T r_s \, ds}\bigg(\frac{1}{T}\int_0^T S_u \, du - K\bigg)^+ \,\bigg|\, \mathcal{F}_t\bigg].$$

The next proposition gives us a replicating hedging strategy for Asian options in the case of a deterministic structure equation model. Following [LL96, p. 91], we define the auxiliary process

$$Y_t = \frac{1}{S_t}\bigg(\frac{1}{T}\int_0^t S_u \, du - K\bigg), \qquad t \in [0, T].$$

PROPOSITION 4.3. *There exists a measurable function $\widetilde{C}$ on $\mathbb{R}_+ \times \mathbb{R}$ such that $\widetilde{C}(t,\cdot)$ is $\mathcal{C}^1$ for all $t \in \mathbb{R}_+$, and*

$$S_t \widetilde{C}(t, Y_t) = E\left[\left(\frac{1}{T}\int_0^T S_u\,du - K\right)^+ \,\Big|\, \mathcal{F}_t\right].$$

Moreover, the replicating portfolio for an Asian option with payoff

$$\left(\frac{1}{T}\int_0^T S_u\,du - K\right)^+$$

is given by (4.1.3) *and*

$$\eta_t = \frac{1}{\sigma_t} e^{-\int_t^T r_s\,ds}\Big[\widetilde{C}(t, Y_t)\sigma_t + (1+\sigma_t\phi_t)\Big(\frac{j_t}{\phi_t}\Big(\widetilde{C}\Big(t, \frac{Y_t}{1+\sigma_t\phi_t}\Big) - \widetilde{C}(t, Y_t)\Big) - i_t\sigma_t Y_t \partial_2 \widetilde{C}(t, Y_t)\Big)\Big]. \tag{4.3.1}$$

Proof. With the above notation, the price of the Asian option at time t becomes

$$E[e^{-\int_t^T r_s\,ds} S_T (Y_T)^+ \,|\, \mathcal{F}_t].$$

For $0 \le s \le t \le T$, we have

$$d(S_t Y_t) = \frac{1}{T}\,d\Big(\int_0^t S_u\,du - K\Big) = \frac{S_t}{T}\,dt,$$

hence

$$\frac{S_t Y_t}{S_s} = Y_s + \frac{1}{T}\int_s^t \frac{S_u}{S_s}\,du.$$

Let $H \in \mathcal{C}_{\mathrm{b}}^2(\mathbb{R})$. We have

$$\begin{aligned} E[H(S_T Y_T) \,|\, \mathcal{F}_t] &= E\left[H\Big(S_t Y_t + \frac{1}{T}\int_t^T S_u\,du\Big) \,\Big|\, \mathcal{F}_t\right] \\ &= E\left[H\Big(xy + \frac{x}{T}\int_t^T \frac{S_u}{S_t}\,du\Big)\right]_{y=Y_t,\, x=S_t}. \end{aligned}$$

Let $C \in \mathcal{C}_{\mathrm{b}}^2(\mathbb{R}_+ \times \mathbb{R}^2)$ be defined as

$$C(t,x,y) = E\left[H\Big(xy + \frac{x}{T}\int_t^T \frac{S_u}{S_t}\,du\Big)\right],$$

i.e.

$$C(t, S_t, Y_t) = E[H(S_T Y_T) \,|\, \mathcal{F}_t].$$

When $H(x) = x^+$, since for any $t \in [0,T]$, S_t is positive and $\mathcal{F}_t$-measurable, and S_u/S_t is independent of $\mathcal{F}_t$ for $u \geq t$, we have

$$\begin{aligned}
E[H(S_T Y_T) \,|\, \mathcal{F}_t] = E[S_T (Y_T)^+ \,|\, \mathcal{F}_t] &= S_t E\left[\left(\frac{S_T}{S_t} Y_T\right)^+ \,\Big|\, \mathcal{F}_t\right] \\
&= S_t E\left[\left(Y_t + \frac{1}{T}\int_t^T \frac{S_u}{S_t}\,du\right)^+ \,\Big|\, \mathcal{F}_t\right] \\
&= S_t E\left[\left(y + \frac{1}{T}\int_t^T \frac{S_u}{S_t}\,du\right)^+ \,\Big|\,\right]_{y=Y_t} = S_t \widetilde{C}(t, Y_t)
\end{aligned}$$

with

$$\widetilde{C}(t,y) = E\left[\left(y + \frac{1}{T}\int_t^T \frac{S_u}{S_t}\,du\right)^+\right].$$

We now proceed as in [Bel99], which deals with the sum of a Brownian motion and a Poisson process. From the expression for $1/S_t$ we have

$$d\left(\frac{1}{S_t}\right) = \frac{1}{S_{t^-}}\left[\left(-r_t + \frac{\alpha_t^2\sigma_t^2}{1+\sigma_t\phi_t}\right)dt - \frac{\sigma_t}{1+\sigma_t\phi_t}\,dM_t\right],$$

hence by (2.3.1),

$$dY_t = Y_t\left(-r_t + \frac{\alpha_t^2\sigma_t^2}{1+\sigma_t\phi_t}\right)dt + \frac{1}{T}\,dt - \frac{Y_{t^-}\sigma_t}{1+\sigma_t\phi_t}\,dM_t.$$

Applying Lemma 3.3 we get

$$\begin{aligned}
\text{(4.3.2)}\quad E[D_t H(S_T Y_T) \,|\, \mathcal{F}_t] &= L_t C(t, S_t, Y_t) \\
&= i_t\left(\sigma_t S_{t^-}\partial_2 C(t,S_t,Y_t) - \frac{Y_t\sigma_t}{1+\sigma_t\phi_t}\,\partial_3 C(t,S_t,Y_t)\right) \\
&\quad + \frac{j_t}{\phi_t}\left(C\left(t, S_{t^-} + \sigma_t S_{t^-}, Y_{t^-} - \frac{Y_t\sigma_t}{1+\sigma_t\phi_t}\right) - C(t, S_{t^-}, Y_{t^-})\right).
\end{aligned}$$

Given a family $(H_n)_{n\in\mathbb{N}}$ of $\mathcal{C}_{\mathrm{b}}^2$ functions such that $|H_n(x)| \leq x^+$ and $|H_n'(x)| \leq 2$ for $x \in \mathbb{R}$ and $n \in \mathbb{N}$, and converging pointwise to $x \mapsto x^+$, by dominated convergence (4.3.2) holds for $C(t,x,y) = x\widetilde{C}(t,y)$ and we obtain

$$\begin{aligned}
&E\left[D_t\left(\frac{1}{T}\int_0^T S_u\,du - K\right)^+ \,\Big|\, \mathcal{F}_t\right] \\
&\quad = i_t \widetilde{C}(t,Y_t)\sigma_t S_t \\
&\qquad + S_t\left(\frac{j_t}{\phi_t}\left(\widetilde{C}\left(t, \frac{Y_t}{1+\sigma_t\phi_t}\right) - \widetilde{C}(t,Y_t)\right) - i_t\sigma_t Y_t \partial_2\widetilde{C}(t,Y_t)\right) \\
&\qquad + S_t\sigma_t\phi_t\left(\frac{j_t}{\phi_t}\left(\widetilde{C}\left(t, \frac{Y_t}{1+\sigma_t\phi_t}\right) - \widetilde{C}(t,Y_t)\right) - i_t\sigma_t Y_t \partial_2\widetilde{C}(t,Y_t)\right). \quad \blacksquare
\end{aligned}$$

As a particular case we consider the Brownian motion model, i.e. $\phi_t = 0$ for all $t \in [0,T]$, so $i_t = 1$, $j_t = 0$ for all $t \in [0,T]$. In this case we have

$$\begin{aligned}\eta_t &= e^{-\int_t^T r_s\,ds}(-Y_t\partial_2\widetilde{C}(t,Y_t) + \widetilde{C}(t,Y_t))\\ &= e^{-\int_t^T r_s\,ds}\left(S_t\frac{\partial}{\partial x}\widetilde{C}\left(t,\frac{1}{x}\left(\frac{1}{T}\int_0^t S_u\,du - K\right)\right)\Big|_{x=S_t} + \widetilde{C}(t,Y_t)\right)\\ &= \frac{\partial}{\partial x}\left(xe^{-\int_t^T r_s ds}\widetilde{C}\left(t,\frac{1}{x}\left(\frac{1}{T}\int_0^t S_u\,du - K\right)\right)\right)\Big|_{x=S_t}, \qquad t \in [0,T],\end{aligned}$$

which can be denoted informally as a partial derivative with respect to S_t.

4.4. *European options and Azéma martingales.* Let $-2 \le \beta < 0$, and let $(M_t)_{t\in[0,T]}$ be the unique solution of the structure equation

$$d[M,M]_t = dt + \beta M_{t^-}dM_t, \qquad t \in [0,T]. \tag{4.4.1}$$

This process has the chaos representation property, hence the results of Section 3 apply. This allows us to obtain an explicit hedging formula for the model of [DP99]. We use the convention $1_{[b,a]} = -1_{[a,b]}$ for $0 \le a < b \le T$.

PROPOSITION 4.4. *We have*

$$\begin{aligned}&E[D_t(S_T-K)^+ \mid \mathcal{F}_t]\\ &\quad= \frac{1}{\beta M_t}E\Big[\sigma_t\beta(y+M_T-M_t)S^x_{t,T}1_{[\frac{K}{1+\sigma_t\beta(y+M_T-M_t)},\infty[}(S^x_{t,T})\\ &\qquad + (S^x_{t,T}-K)1_{[\frac{K}{1+\sigma_t\beta(y+M_T-M_t)},K]}(S^x_{t,T})\Big]^{y=M_t}_{x=S_t}.\end{aligned}$$

Proof. Let $(X_t)_{t\in[0,T]} = ((S_t,M_t))_{t\in[0,T]}$, $(R_t)_{t\in[0,T]} = ((r_tS_t,0))_{t\in[0,T]}$, $(K_t)_{t\in[0,T]} = ((\sigma_tS_t,1))_{t\in[0,T]}$, and $X_0 = (1,0)$. By Lemma 3.3, for $f \in \mathcal{C}^2_{\mathrm{b}}(\mathbb{R}^2)$ we have

$$\begin{aligned}E[D_tf(X_T)\mid\mathcal{F}_t] &= (L_t(P_{t,T}f))(X_t)\\ &= \frac{1}{\beta M_t}((P_{t,T}f)(X_t+\beta M_tK_t) - (P_{t,T}f)(X_t))\\ &= \frac{1}{\beta M_t}((P_{t,T}f)((1+\beta M_t\sigma_t)S_t,(1+\beta)M_t) - (P_{t,T}f)(S_t,M_t))\\ &= \frac{1}{\beta M_t}E[f((1+\sigma_t\beta(y+M_T-M_t))S^x_{t,T},\\ &\qquad\qquad (1+\beta(y+M_T-M_t)))]^{y=M_t}_{x=S_t}\\ &\quad - \frac{1}{\beta M_t}E[f(S^x_{t,T},y+M_T-M_t)]^{y=M_t}_{x=S_t}.\end{aligned}$$

In particular if $f(x,y)=f(x)$ depends only on the first variable we have

$$E[D_t f(S_T)\,|\,\mathcal{F}_t] = \frac{1}{\beta M_t} E[f((1+\sigma_t\beta(y+M_T-M_t))S^x_{t,T}) - f(S^x_{t,T})]^{y=M_t}_{x=S_t}.$$

Approaching the function $x\mapsto (x-K)^+$ with a sequence $(f_n)_{n\in\mathbb{N}}$ of $\mathcal{C}^2_{\mathrm{b}}$ functions converging pointwise with $|f_n(x)|\le (x-K)^+$ and $|f_n'(x)|\le 2$ for $x\in\mathbb{R}$ and $n\in\mathbb{N}$, we obtain

$$\begin{aligned}
&E[D_t(S_T-K)^+\,|\,\mathcal{F}_t]\\
&= \frac{1}{\beta M_t} E[((1+\sigma_t\beta(y+M_T-M_t))S^x_{t,T}-K)^+ - (S^x_{t,T}-K)^+]^{y=M_t}_{x=S_t}\\
&= \frac{1}{\beta M_t} E\Big[((1+\sigma_t\beta(y+M_T-M_t))S^x_{t,T}-K)1_{[\frac{K}{1+\sigma_t\beta(y+M_T-M_t)},\infty[}(S^x_{t,T})\\
&\quad - (S^x_{t,T}-K)1_{[K,\infty[}(S^x_{t,T})\Big]^{y=M_t}_{x=S_t}\\
&= \frac{1}{\beta M_t} E\Big[\sigma_t\beta(y+M_T-M_t)S^x_{t,T}1_{[\frac{K}{1+\sigma_t\beta(y+M_T-M_t)},\infty[}(S^x_{t,T})\\
&\quad + (S^x_{t,T}-K)1_{[\frac{K}{1+\sigma_t\beta(y+M_T-M_t)},K]}(S^x_{t,T})\Big]^{y=M_t}_{x=S_t}. \ \blacksquare
\end{aligned}$$

4.5. *Lookback options.* Hedging strategies for Lookback options have been computed in [Ber98] using the Clark–Ocone formula. In this section we show that classical martingale methods also apply in this case. We assume that $(M_t)_{t\in[0,T]}=(B_t)_{t\in[0,T]}$ is a standard Brownian motion, i.e. $\alpha_t=1$ and $\phi_t=0$ for every $t\in[0,T]$, and take $r_t=r\ge 0$ and $\sigma_t=\sigma\ge 0$ for every $t\in[0,T]$. Under the risk-free probability P the asset price $(S_t)_{t\in[0,T]}$ has the dynamics

$$dS_t = rS_t dt + \sigma S_t dB_t, \qquad t\in[0,T],$$

so (4.1.5) becomes

$$V_T e^{-rT} = V_0 + \int_0^T \sigma\eta_t S_t e^{-rt}\,dB_t, \qquad t\in[0,T].$$

Let $m^t_s=\inf_{u\in[s,t]} S_u$ and $M^t_s=\sup_{u\in[s,t]} S_u$ for $0\le s\le t\le T$, and let $\mathcal{M}^t_s$ be either m^t_s or M^t_s. In the Lookback option case the payoff $H(S_T,\mathcal{M}^T_0)$ depends not only on the price of the underlying asset at maturity but also on all prices of the asset from the initial time to maturity. Let Look_t be the price of the Lookback option given by

$$\mathrm{Look}_t = e^{-r(T-t)}E[H(S_T,\mathcal{M}^T_0)\,|\,\mathcal{F}_t], \qquad H\in\mathcal{C}^2_{\mathrm{b}}(\mathbb{R}^2),\ t\in[0,T].$$

Proposition 4.5. *There exists a $\mathcal{C}^1$ function f such that*

$$f(S_t,\mathcal{M}^t_0,t) = E[H(S_T,\mathcal{M}^T_0)\,|\,\mathcal{F}_t], \qquad 0\le t\le T.$$

The replicating portfolio of a Lookback option with payoff $H(S_T, \mathcal{M}_0^T)$ *and price* $\mathrm{Look}_t = f(S_t, \mathcal{M}_0^t, t)$ *at time* t *is given by* (4.1.3), *and*

$$\eta_t = e^{-r(T-t)} \partial_1 f(S_t, \mathcal{M}_0^t, t), \qquad t \in [0, T]. \tag{4.5.1}$$

Proof. It suffices to deal with the case $\mathcal{M}_s^t = m_s^t$. The existence of f follows from the Markov property, more precisely

$$f(x, y, t) = E[H(S_{t,T}^x, y \wedge \mathcal{M}_t^T)].$$

Applying the change of variable formula, for $t \in [0, T]$ we have

$$\begin{aligned} df(S_t, \mathcal{M}_0^t, t) &= \left[\partial_3 f + rS_t \partial_1 f + \frac{1}{2} \sigma^2 S_t^2 \partial_1^2 f \right] (S_t, \mathcal{M}_0^t, t) dt \\ &\quad + \partial_2 f(S_t, \mathcal{M}_0^t, t) d\mathcal{M}_0^t + \sigma S_t \partial_1 f(S_t, \mathcal{M}_0^t, t) dB_t. \end{aligned}$$

Since $(E[H(S_T, \mathcal{M}_0^T) \mid \mathcal{F}_t])_{t \in [0,T]}$ is a P-martingale, we have

$$df(S_t, \mathcal{M}_0^t, t) = \sigma S_t \partial_1 f(S_t, \mathcal{M}_0^t, t) dB_t, \qquad t \in [0, T].$$

Then

$$e^{-rT} F = e^{-rT} E[F] + \int_0^T e^{-rT} \sigma S_t \frac{\partial}{\partial x} f(x, \mathcal{M}_0^t, t) \Big|_{x = S_t} dB_t, \qquad t \in [0, T],$$

which shows (4.5.1). ∎

It is stated in Bermin [Ber98] that we should have

$$\int_0^t \partial_2 f(S_s, \mathcal{M}_0^s, s) \, d\mathcal{M}_0^s = 0 \tag{4.5.2}$$

for the delta hedging method to work. We showed in Proposition 4.5 that the delta hedging approach can be applied without having to verify (4.5.2), since $(\mathcal{M}_0^t)_{t \in [0,T]}$ is a monotone process with finite variation.

Relation (4.5.1) can be written informally as

$$\eta_t = \frac{\partial}{\partial S_t} \mathrm{Look}_t, \qquad t \in [0, T].$$

Let

$$d_t^T(y) = \frac{\log \frac{S_t}{y} + \left(r + \frac{1}{2}\sigma^2\right)(T - t)}{\sigma \sqrt{T - t}}, \qquad \mathcal{N}(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^y e^{-\frac{1}{2}u^2} \, du.$$

A standard Lookback call option is the right to buy the underlying asset at the historically lowest price. In this case the strike is m_0^T and the payoff is

$$G = S_T - m_0^T.$$

From [DJ98, Prop. 4, p. 271], the price Look_t at time t is given by

$$\begin{aligned} \mathrm{Look}_t &= E_Q[e^{-r(T-t)}(S_T - m_0^T) \mid \mathcal{F}_t] \\ &= S_t \mathcal{N}(d_t^T(m_0^t)) - e^{-r(T-t)} m_0^t \mathcal{N}(d_t^T(m_0^t) - \sigma \sqrt{T - t}) \end{aligned} \tag{4.5.3}$$

$$+ e^{-r(T-t)} \frac{S_t \sigma^2}{2r} \left[\left(\frac{S_t}{m_0^t} \right)^{-2r/\sigma^2} \mathcal{N}\left(-d_t^T(m_0^t) + \frac{2r\sqrt{T-t}}{\sigma} \right) \right.$$
$$\left. - e^{r(T-t)} \mathcal{N}(-d_t^T(m_0^t)) \right].$$

In the following proposition we recover the result of [Ber98, §2.6.1, p. 29], using the delta hedging approach instead of the Clark formula, as an application of Proposition 4.5.

PROPOSITION 4.6. *The hedging strategy for a standard Lookback call option is given by*

$$\eta_t = \mathcal{N}(d_t^T(m_0^t)) - \frac{\sigma^2}{2r} \mathcal{N}(-d_t^T(m_0^t)) + e^{-r(T-t)} \left(\frac{S_t}{m_0^t} \right)^{-2r/\sigma^2} \left(\frac{\sigma^2}{2r} - 1 \right) \mathcal{N}\left(-d_t^T(m_0^t) + \frac{2r\sqrt{T-t}}{\sigma} \right). \tag{4.5.4}$$

Proof. We need to compute the following derivatives:

$$\begin{aligned}
&\frac{\partial}{\partial S_t} (\mathcal{N}(d_t^T(m_0^t))) \\
&\quad = \frac{\partial}{\partial S_t} \left(\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_t^T(m_0^t)} e^{-\frac{1}{2}u^2} du \right) \\
&\quad = \frac{1}{\sqrt{2\pi}} \left[\frac{\partial}{\partial S_t} (d_t^T(m_0^t)) \right] \exp\left(-\frac{1}{2} (d_t^T(m_0^t))^2 \right) \\
&\quad = \frac{1}{\sqrt{2\pi}} \left[\frac{\partial}{\partial S_t} \left(\frac{\log \frac{S_t}{m_0^t} + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \right) \right] \exp\left(-\frac{1}{2} (d_t^T(m_0^t))^2 \right) \\
&\quad = \frac{1}{S_t \sigma \sqrt{2\pi(T-t)}} \exp\left(-\frac{1}{2} (d_t^T(m_0^t))^2 \right),
\end{aligned}$$

and

$$\begin{aligned}
&\frac{\partial}{\partial S_t} (\mathcal{N}(d_t^T(m_0^t) - \sigma\sqrt{T-t})) \\
&\qquad = \frac{1}{S_t \sigma \sqrt{2\pi(T-t)}} \exp\left(-\frac{1}{2} (d_t^T(m_0^t) - \sigma\sqrt{T-t})^2 \right).
\end{aligned}$$

Similarly we have

$$\begin{aligned}
&\frac{\partial}{\partial S_t} \left(\mathcal{N}\left(-d_t^T(m_0^t) + \frac{2r\sqrt{T-t}}{\sigma} \right) \right) \\
&\qquad = -\frac{1}{S_t \sigma \sqrt{2\pi(T-t)}} \exp\left(-\frac{1}{2} \left(-d_t^T(m_0^t) + \frac{2r\sqrt{T-t}}{\sigma} \right)^2 \right),
\end{aligned}$$

and

$$\frac{\partial}{\partial S_t}(\mathcal{N}(-d_t^T(m_0^t))) = -\frac{1}{S_t\sigma\sqrt{2\pi(T-t)}}\exp\left(-\frac{1}{2}\,(d_t^T(m_0^t))^2\right).$$

Finally,

$$\frac{\partial}{\partial S_t}\left(\frac{S_t}{m_0^t}\right)^{-2r/\sigma^2} = \frac{-2r}{m_0^t\sigma^2}\left(\frac{S_t}{m_0^t}\right)^{-2r/\sigma^2-1}.$$

The above expressions can be combined to compute the derivative of Look_t in (4.5.3), and to obtain

$$\begin{aligned}
\eta_t &= \mathcal{N}(d_t^T(m_0^t)) + \frac{1}{\sigma\sqrt{2\pi(T-t)}}\exp\left(-\frac{1}{2}\,(d_t^T(m_0^t))^2\right)\\
&\quad - e^{-r(T-t)}m_0^t\,\frac{1}{S_t\sigma\sqrt{2\pi(T-t)}}\exp\left(-\frac{1}{2}\,(d_t^T(m_0^t)-\sigma\sqrt{T-t})^2\right)\\
&\quad + e^{-r(T-t)}\frac{\sigma^2}{2r}\left[\left(\frac{S_t}{m_0^t}\right)^{-2r/\sigma^2}\mathcal{N}\left(-d_t^T(m_0^t)+\frac{2r\sqrt{T-t}}{\sigma}\right)\right.\\
&\quad \left. - e^{r(T-t)}\mathcal{N}(-d_t^T(m_0^t))\right]\\
&\quad + e^{-r(T-t)}\,\frac{S_t\sigma^2}{2r}\left[\frac{-2r}{m_0^t\sigma^2}\left(\frac{S_t}{m_0^t}\right)^{-2r/\sigma^2-1}\mathcal{N}\left(-d_t^T(m_0^t)+\frac{2r\sqrt{T-t}}{\sigma}\right)\right.\\
&\quad - \left(\frac{S_t}{m_0^t}\right)^{-2r/\sigma^2}\frac{1}{S_t\sigma\sqrt{2\pi(T-t)}}\exp\left(-\frac{1}{2}\left(-d_t^T(m_0^t)+\frac{2r\sqrt{T-t}}{\sigma}\right)^2\right)\\
&\quad \left. + e^{r(T-t)}\frac{1}{S_t\sigma\sqrt{2\pi(T-t)}}\exp\left(-\frac{1}{2}\,(d_t^T(m_0^t))^2\right)\right]\\
&= \mathcal{N}\left(-d_t^T(m_0^t)+\frac{2r\sqrt{T-t}}{\sigma}\right)\left[e^{-r(T-t)}\,\frac{\sigma^2}{2r}\left(\frac{S_t}{m_0^t}\right)^{-2r/\sigma^2} + e^{-r(T-t)}\,\frac{S_t\sigma^2}{2r}\right.\\
&\quad \left. \times\frac{-2r}{m_0^t\sigma^2}\left(\frac{S_t}{m_0^t}\right)^{-2r/\sigma^2-1}\right] + \mathcal{N}(d_t^T(m_0^t))\\
&\quad - \mathcal{N}(-d_t^T(m_0^t))\left[-e^{-r(T-t)}\,\frac{\sigma^2}{2r}\cdot e^{r(T-t)}\right]\\
&\quad + \frac{1}{\sigma\sqrt{2\pi(T-t)}}\left\{\exp\left(-\frac{1}{2}\,(d_t^T(m_0^t))^2\right)\left[1+e^{-r(T-t)}\,\frac{S_t\sigma^2}{2r}\cdot\frac{e^{r(T-t)}}{S_t}\right]\right.\\
&\quad - e^{-r(T-t)}\left[\frac{m_0^t}{S_t}\exp\left(-\frac{1}{2}\,(d_t^T(m_0^t)-\sigma\sqrt{T-t})^2\right)\right.\\
&\quad \left.\left. + \frac{\sigma^2}{2r}\left(\frac{S_t}{m_0^t}\right)^{-2r/\sigma^2}\exp\left(-\frac{1}{2}\left(-d_t^T(m_0^t)+\frac{2r\sqrt{T-t}}{\sigma}\right)^2\right)\right]\right\}
\end{aligned}$$

$$
\begin{aligned}
&= e^{-r(T-t)} \left(\frac{S_t}{m_0^t} \right)^{-2r/\sigma^2} \left(\frac{\sigma^2}{2r} - 1 \right) \mathcal{N}\left(-d_t^T(m_0^t) + \frac{2r\sqrt{T-t}}{\sigma} \right) \\
&\quad + \mathcal{N}(d_t^T(m_0^t)) - \frac{\sigma^2}{2r} \mathcal{N}(-d_t^T(m_0^t)) \\
&\quad + \frac{1}{\sigma\sqrt{2\pi(T-t)}} \left\{ \left(1 + \frac{\sigma^2}{2r} \right) \exp\left(-\frac{1}{2} d_t^T(m_0^t)^2 \right) \right. \\
&\quad - e^{-r(T-t)} \left[\frac{m_0^t}{S_t} \exp\left(-\frac{1}{2} (d_t^T(m_0^t) - \sigma\sqrt{T-t})^2 \right) \right. \\
&\quad \left. \left. + \frac{\sigma^2}{2r} \left(\frac{S_t}{m_0^t} \right)^{-2r/\sigma^2} \exp\left(-\frac{1}{2} \left(-d_t^T(m_0^t) + \frac{2r\sqrt{T-t}}{\sigma} \right)^2 \right) \right] \right\}.
\end{aligned}
$$

To obtain (4.5.4), it is sufficient to show that

$$
\begin{aligned}
0 &= \left(1 + \frac{\sigma^2}{2r} \right) \exp\left(-\frac{1}{2} d_t^T(m_0^t)^2 \right) \\
&\quad - e^{-r(T-t)} \left[\frac{m_0^t}{S_t} \exp\left(-\frac{1}{2} (d_t^T(m_0^t) - \sigma\sqrt{T-t})^2 \right) \right. \\
&\quad \left. + \frac{\sigma^2}{2r} \left(\frac{S_t}{m_0^t} \right)^{-2r/\sigma^2} \exp\left(-\frac{1}{2} \left(-d_t^T(m_0^t) + \frac{2r\sqrt{T-t}}{\sigma} \right)^2 \right) \right].
\end{aligned}
$$

To see this, one can observe that

$$
\begin{aligned}
&\exp\left(-\frac{1}{2} (d_t^T(y) - \sigma\sqrt{T-t})^2 \right) \\
&\qquad = \exp\left(-\frac{1}{2} [(d_t^T(y))^2 + \sigma^2(T-t) - 2d_t^T(y)\sigma\sqrt{T-t}] \right) \\
&\qquad = \exp\left(-\frac{1}{2} (d_t^T(y))^2 \right) \\
&\qquad\quad \times \exp\left(-\frac{1}{2} \left[\sigma^2(T-t) - 2\left(\log\frac{S_t}{y} + \left(r + \frac{1}{2}\sigma^2 \right)(T-t) \right) \right] \right) \\
&\qquad = \exp\left(-\frac{1}{2} \left[-2\log\frac{S_t}{y} - 2r(T-t) \right] - \frac{1}{2} (d_t^T(y))^2 \right) \\
&\qquad = e^{r(T-t)} \frac{S_t}{y} \exp\left(-\frac{1}{2} (d_t^T(y))^2 \right),
\end{aligned}
$$

and

$$
\begin{aligned}
&\exp\left(-\frac{1}{2} (-d_t^T(y) + 2r\sqrt{T-t}/\sigma)^2 \right) \\
&\qquad\qquad = \exp\left(-\frac{1}{2} (d_t^T(y))^2 - \frac{1}{2} \left[\frac{4r^2}{\sigma^2}(T-t) - \frac{4r}{\sigma} d_t^T(y)\sqrt{T-t} \right] \right)
\end{aligned}
$$

$$
\begin{aligned}
&= \exp\left(-\frac{1}{2}\,(d_t^T(y))^2\right)\\
&\quad \times \exp\left(-\frac{1}{2}\left[\frac{4r^2}{\sigma^2}\,(T-t) - \frac{4r}{\sigma^2}\left(\log\frac{S_t}{y} + \left(r + \frac{1}{2}\,\sigma^2\right)(T-t)\right)\right]\right)\\
&= \exp\left(-\frac{1}{2}\,(d_t^T(y))^2\right)\\
&\quad \times \exp\left(\frac{-2r^2}{\sigma^2}\,(T-t) + \frac{2r}{\sigma^2}\log\frac{S_t}{y} + \frac{2r^2}{\sigma^2}\,(T-t) + r(T-t)\right)\\
&= e^{r(T-t)}\left(\frac{S_t}{y}\right)^{2r/\sigma^2} \exp\left(-\frac{1}{2}\,(d_t^T(y))^2\right). \ \blacksquare
\end{aligned}
$$

Similar calculations using (4.5.1) are possible for other Lookback options, such as options on extrema and partial Lookback options (cf. [Kha02]).

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References

[AOPU00] K. Aase, B. Øksendal, N. Privault, and J. Ubøe, *White noise generalizations of the Clark–Haussmann–Ocone theorem with application to mathematical finance*, Finance Stoch. 4 (2000), 465–496.

[Bel99] N. Bellamy, *Evaluation et couverture dans un marché dirigé par des processus discontinus*, thèse, Univ. d'Evry, 1999.

[Ber98] H. Bermin, *Essays on Lookback options*: *a Malliavin calculus approach*, PhD thesis, Lund Univ., 1998.

[Cla70] J. M. C. Clark, *The representation of functionals of Brownian motion by stochastic integrals*, Ann. Math. Statist. 41 (1970), 1281–1295.

[DJ98] R.-A. Dana et M. Jeanblanc, *Marchés financiers en temps continu*, Economica, Paris, 1998.

[DP99] M. Dritschel and P. Protter, *Complete markets with discontinuous security price*, Finance Stoch. 3 (1999), 203–214.

[É90] M. Émery, *On the Azéma martingales*, in: *Séminaire de Probabilités XXIII*, Lecture Notes in Math. 1372, Springer, 1990, 66–87.

[JP02] M. Jeanblanc and N. Privault, *A complete market model with Poisson and Brownian components*, in: R. Dalang *et al.* (eds.), *Seminar on Stochastic Analysis*, *Random Fields and Applications* (Ascona, 1999), Progr. Probab., Birkhäuser, Basel, 2002, 189–204.

[Kha02] Y. El-Khatib, *Contributions to the study of discontinuous markets via the Malliavin calculus*, thèse, Univ. de La Rochelle, 2002.

[KO91] I. Karatzas and D. L. Ocone, *A generalized Clark representation formula with application to optimal portfolios*, Stochastics Stochastics Rep. 34 (1991), 187–230.

[LL96] D. Lamberton and B. Lapeyre, *Introduction to Stochastic Calculus Applied to Finance*, Chapman & Hall, London, 1996.

[Mey89] P.-A. Meyer, *Construction de solutions d'"équations de structure"*, in: *Séminaire de Probabilités XXIII*, Lecture Notes in Math. 1372, Springer, Berlin, 1989, 142–145.

[Øks96] B. Øksendal, *An introduction to Malliavin calculus with applications to economics*, working paper no. 3, Inst. Finance and Management Sci., Norwegian School of Economics and Business Administration, 1996.

[Pha00] A. Phan, *Contributions à l'étude des équations de structure markoviennes*, thèse, Univ. de Strasbourg, 2000.

[Pri96] N. Privault, *On the independence of multiple stochastic integrals with respect to a class of martingales*, C. R. Acad. Sci. Paris Sér. I Math. 323 (1996), 515–520.

[Pro90] P. Protter, *Stochastic Integration and Differential Equations. A New Approach*, Springer, Berlin, 1990.

[Pro01] —, *A partial introduction to financial asset pricing theory*, Stochastic Process. Appl. 91 (2001), 169–203.

[PSV00] N. Privault, J. L. Solé, and J. Vives, *Chaotic Kabanov formula for the Azéma martingales*, Bernoulli 6 (2000), 633–651.

[Tav99] G. Taviot, *Martingales et équations de structure : étude géométrique*, thèse, Univ. de Strasbourg, 1999.

[Yor76] M. Yor, *Sur les intégrales stochastiques optionnelles et une suite remarquable de formules exponentielles*, in: *Séminaire de Probabilités X* (*Seconde partie*: *Théorie des intégrales stochastiques*, *Strasbourg*, *1974/1975*), Lecture Notes in Math. 511, Springer, Berlin, 1976, 481–500.

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