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AN EQUILIBRIUM MODEL FOR ELECTRICITY AUCTIONS

Abstract. This work discusses the process of price formation for electrical energy within an auction-like trading environment. Calculating optimal bid strategies of power producers by equilibrium arguments, we obtain the corresponding electricity price and estimate its tail behavior.

1. Introduction. The introduction of competitive wholesale electricity market is a key aspect of liberalization of energy production and trading, which has recently been effected around the world. Up to now, all experiences with deregulated electricity markets show that the electricity trading incorporates high risk resulting from volatile and "spiky" prices. This issue is intrinsic to electricity as a *flow* commodity, which cannot be economically stored. In our approach, we examine by equilibrium arguments an economical mechanism effecting price peaks. Let us mention some related work. In [2], questions of electricity pricing are presented and it is explained that the non-storability requires modeling the electricity production process. The economical mechanism of one-period-ahead price formation is discussed in [1], and in a different context in [5]. The concept of equilibrium asset pricing is widely used by economists; see, for example, [8], [6] and the references therein. A survey of the theory of auctions is found in [7].

Since electrical energy is not economically storable, a deregulated electricity market is different from the usual commodity markets. In general, it includes two parts: the *real-time market* for contracts on immediate production and the *electricity exchange* for those on future delivery of electricity. While the electricity exchange is similar to the usual forward market, the trading rules at the real-time market are designed to continuously match demand and supply to maintain network electrical equilibrium. This requirement is satisfied by auction-like trading subjected to several technical restrictions.

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Let us simplify the methods for price settlement applied at the real-time markets to explain two procedures: the *pay-as-bid* (PAB) and the *system-marginal price* (SMP) auctions. They work as follows: Each electricity producer submits for each hour of the next day his schedule consisting of a bid quantity and a bid price for power which he is willing to sell at least at this price. The *system operator* arranges the bids for each hour in the increasing price order. The *system price* set for the current hour equals the bid price of the last generator needed to meet the demand. Those producers who are *in merit* (i.e. whose bid price was below or equal to the system price) supply power and obtain a payment. Other producers suffer a loss since they have to pay fixed costs for their idle production units. For the PAB-auction, each producer who is in merit obtains his own bid price, while for the SMP-auction he obtains the system price. In this work, we restrict ourselves to the PAB-auction.

Denote by Q the electricity demand within one hour and agree that Q is a non-negative random variable on a probability space $(\Omega, \mathcal{F}, P)$ such that

$$(1) \quad F : [0, \infty[\to \mathbb{R}, \ q \mapsto P(Q \leq q), \text{ is continuous, strictly increasing.}$$

We suppose that the distribution F of Q is known to all producers.

Let $J \subseteq [0, \infty[$ be the set of all bid prices which are acceptable by the system operator. Note that J may be a proper subset of $[0, \infty[$ due to a possible price cap and/or discrete price levels. After all producers have submitted their schedules, the system operator determines the production capacity $\mathcal{I}(p)$ installed at the price $p \in J$ by summing up all amounts of bids with bid price at most p. The non-decreasing right-continuous installed capacity $\mathcal{I} : J \to [0, \infty[$ is *saturated* for an additional producer if it does not matter whether he does nothing or rents a small production unit and submits his order at an arbitrary price $p \in J$. The idea here is that in the real market, the installed capacity must always be saturated since otherwise some additional producers will install capacities until the saturation is reached. Let us make this concept more precise.

2. The equilibrium. Denote by $c > 0$ (MWh) the capacity amount of a small rentable production unit with full production costs $p^{fv} = p^f + p^v$, where $p^f > 0$ (USD/MWh) are fixed and $p^v > 0$ (USD/MWh) are variable costs. Then, for the installed capacity $\mathcal{I}$, the income of the additional producer depends on his strategy to rent the capacity and to submit a schedule at the price $p \in J$, which yields a random gain

$$G^{\mathcal{I}}(p) := c(p - p^v)1_{\{Q > \mathcal{I}(p)\}} - cp^f \quad \text{ for all } p \in J,$$

or to be idle, which gives a non-random zero gain: $G^{\mathcal{I}}(\text{idle}) = 0$. Suppose that the risk aversion of the additional producer is described by a strictly increasing concave utility function $U \in C(\mathbb{R})$, giving the utility functional $\mathcal{U}^{\mathcal{I}}$ which evaluates the producer's strategies by

$$\mathcal{U}^{\mathcal{I}}(p) = E(U(G^{\mathcal{I}}(p))) \quad \text{for all } p \in J$$

for the case when a schedule is submitted, and $\mathcal{U}^{\mathcal{I}}(\text{idle}) = E(U(G^{\mathcal{I}}(\text{idle}))) = U(0)$ otherwise.

DEFINITION 1. A non-decreasing right-continuous installed capacity $\mathcal{I} : J \to [0,\infty[$ is called *saturated* for the additional producer with c, p^f, p^v, U as above if

$$\mathcal{U}^{\mathcal{I}}(p) \le U(0) \quad \text{for all } p \in J \text{ with } \mathcal{I}(p) = 0, \tag{2}$$

$$\mathcal{U}^{\mathcal{I}}(p) = U(0) \quad \text{for all } p \in J \text{ with } \mathcal{I}(p) > 0. \tag{3}$$

Intuitively, this definition says that for a saturated installed capacity, the best strategy is to be idle, since either the price is too low to cover the production costs or there is already a sufficient amount of installed capacity.

Given the installed capacity $\mathcal{I}$, the system price p^S depends on demand and is set by the system operator as explained above to be

$$p^S(\omega) = \inf\{p \in J : \mathcal{I}(p) \ge Q(\omega)\}, \qquad \omega \in \Omega, \tag{4}$$

with the usual agreement $p^S(\omega) := \sup J$ if $\{p \in J : \mathcal{I}(p) \ge Q(\omega)\}$ is empty.

Let us use the following notations:

$$\widehat{\mathcal{I}}(p) = F^{-1}\bigg(\frac{U(cp - cp^{fv}) - U(0)}{U(cp - cp^{fv}) - U(-cp^f)} 1_{]p^{fv},\infty[}(p)\bigg) \qquad \text{for all } p \ge 0, \tag{5}$$

$$\varrho = p^{fv} + c^{-1}U^{-1}\bigg(\frac{U(0) - F(Q)U(-cp^f)}{1 - F(Q)}\bigg), \tag{6}$$

to calculate the system price for the saturated installed capacity.

PROPOSITION 1. *Suppose that the demand distribution satisfies* (1).

(i) *For the additional producer with* c, p^f, p^v, U *as above there exists a unique saturated installed capacity* $\mathcal{I} : J \to [0,\infty[$ *given by*

$$\mathcal{I}(p) = \widehat{\mathcal{I}}(p) \quad \textit{for all } p \in J. \tag{7}$$

(ii) *For the saturated installed capacity* $\mathcal{I}$ *from* (i), *the system price* p^S *from* (4) *satisfies*

$$p^S 1_{\{Q \ge \sup_{p\in J} \mathcal{I}(p)\}} = \sup J\, 1_{\{Q \ge \sup_{p\in J} \mathcal{I}(p)\}}, \tag{8}$$

$$p^S 1_{\{Q < \sup_{p\in J} \mathcal{I}(p)\}} \ge \varrho 1_{\{Q < \sup_{p\in J} \mathcal{I}(p)\}}, \tag{9}$$

almost surely.

Proof. (i) We show that (7) indeed defines a saturated installed capacity. First, we point out that $\mathcal{I}(p) > 0$ holds for $p \in J$ if and only if $p > p^{fv}$: The if-part holds since U is strictly increasing: $U(cp - cp^{fv}) > U(0) > U(-cp^f)$, ensuring that

$$\frac{U(cp - cp^{fv}) - U(0)}{U(cp - cp^{fv}) - U(-cp^f)} > 0,$$

which implies $\mathcal{I}(p) > 0$ since F^{-1} is strictly increasing on $[0, \infty[$ with $F^{-1}(0) = 0$. The only-if-part follows from the definition of $\mathcal{I}$: let $p \in J$; then $\mathcal{I}(p) > 0$ implies $p > p^{fv}$. Now we shall see that Definition 1 in fact applies to (7). If $\mathcal{I}(p) = 0$ then $p \leq p^{fv}$ as shown above, hence

$$G^{\mathcal{I}}(p) = c(p - p^v)1_{\{Q > \mathcal{I}(p)\}} - cp^f \leq 0,$$

which yields $\mathcal{U}^{\mathcal{I}}(p) = E(U(G^{\mathcal{I}}(p))) \leq U(0)$. If $\mathcal{I}(p) > 0$, then $p > p^{fv}$ as shown above, and

$$\mathcal{U}^{\mathcal{I}}(p) = U(cp - cp^{fv})(1 - F(\mathcal{I}(p))) + U(-cp^f)F(\mathcal{I}(p))$$

equals $U(0)$ by the definition of $\mathcal{I}$.

Let us show the uniqueness. Suppose that $\widetilde{\mathcal{I}}$ is some saturated installed capacity and $\widetilde{p} \in [0, p^{fv}] \cap J$. Then $\widetilde{\mathcal{I}}(\widetilde{p}) = 0$ since $\widetilde{\mathcal{I}}(\widetilde{p}) > 0$ would yield

$$\mathcal{U}^{\widetilde{\mathcal{I}}}(\widetilde{p}) = U(c\widetilde{p} - cp^{fv})P(Q > \widetilde{\mathcal{I}}(\widetilde{p})) + U(-cp^f)P(Q \leq \widetilde{\mathcal{I}}(\widetilde{p})) < U(0),$$

which contradicts (3). Now suppose that $\widetilde{p} \in]p^{fv}, \infty[\cap J$. Then $\widetilde{\mathcal{I}}(\widetilde{p}) > 0$ since otherwise $\widetilde{\mathcal{I}}(\widetilde{p}) = 0$ and we would obtain

$$\mathcal{U}^{\widetilde{\mathcal{I}}}(\widetilde{p}) = U(c\widetilde{p} - cp^{fv})P(Q > \widetilde{\mathcal{I}}(\widetilde{p})) + U(-cp^f)P(Q \leq \widetilde{\mathcal{I}}(\widetilde{p})) > U(0),$$

contrary to (2). That is, for $\widetilde{p} \in]p^{fv}, \infty[\cap J$ we have

$$\mathcal{U}^{\widetilde{\mathcal{I}}}(\widetilde{p}) = U(c\widetilde{p} - cp^{fv})P(Q > \widetilde{\mathcal{I}}(\widetilde{p})) + U(-cp^f)P(Q \leq \widetilde{\mathcal{I}}(\widetilde{p})) = U(0),$$

which is equivalently rewritten as

$$\widetilde{\mathcal{I}}(\widetilde{p}) = F^{-1}\left(\frac{U(c\widetilde{p} - cp^{fv}) - U(0)}{U(c\widetilde{p} - cp^{fv}) - U(-cp^f)}\right) \qquad \text{for all } \widetilde{p} \in]p^{fv}, \infty[\cap J.$$

Hence, $\widetilde{\mathcal{I}}$ coincides with the $\mathcal{I}$ from (7), giving the uniqueness.

(ii) In the case $Q(\omega) \geq \sup_{p \in J} \mathcal{I}(p)$ we obtain $p^S(\omega) = \sup J$ by definition. Since (ii) should hold for almost all $\omega \in \{Q < \sup_{p \in J} \mathcal{I}(p)\}$, it suffices to prove it for all ω with $0 < Q(\omega) < \sup_{p \in J} \mathcal{I}(p)$ due to $P(Q = 0) = 0$. By definition (4) and from the right-continuity of $\mathcal{I}$, we deduce that

$$\mathcal{I}(p^S(\omega)) \geq Q(\omega) > 0. \tag{10}$$

The positivity of $\mathcal{I}(p^S(\omega))$ yields

$$\mathcal{I}(p^S(\omega)) = F^{-1}\left(\frac{U(cp^S(\omega) - cp^{fv}) - U(0)}{U(cp^S(\omega) - cp^{fv}) - U(-cp^f)}\right). \tag{11}$$

Applying the non-decreasing function F to (10) and to (11) we obtain

$$F(\mathcal{I}(p^S))(\omega) \geq F(Q)(\omega), \tag{12}$$

$$F(\mathcal{I}(p^S))(\omega) = \frac{U(cp^S(\omega) - cp^{fv}) - U(0)}{U(cp^S(\omega) - cp^{fv}) - U(-cp^f)}, \tag{13}$$

which gives

$$\frac{U(cp^S(\omega) - cp^{fv}) - U(0)}{U(cp^S(\omega) - cp^{fv}) - U(-cp^f)} \geq F(Q)(\omega).$$

This inequality implies

$$p^S(\omega) \geq p^{fv} + c^{-1}U^{-1}\left(\frac{U(0) - F(Q(\omega))U(-cp^f)}{1 - F(Q(\omega))}\right)$$

finishing the proof. ∎

3. System price distribution. The assertion (ii) of the previous proposition allows an interpretation of the system price distribution. An interesting feature here is that the tail $p \mapsto P(p^S > p)$ admits an estimation from below which does not involve the distribution of Q. This shows that the PAB-auction procedure will produce system prices with a "fat tail" distribution. To give a precise statement, we need

PROPOSITION 2. *With the above notations we have*:

(i) $\{\varrho < \sup J\} = \{Q < \sup_{p\in J} \mathcal{I}(p)\}$ *almost surely.*

(ii) *If* $p \leq \sup J$ *then* $\varrho \geq p$ *implies that* $p^S \geq p$ *almost surely.*

(iii) *For each* $p \leq \sup J$, *the probability that the system price* p^S *reaches or exceeds the price* p *is estimated from below by*

$$P(p^S \geq p) \geq \frac{U(0) - U(-cp^f)}{U(cp - cp^{fv}) - U(-cp^f)} \quad \text{for all } p \in [p^{fv}, \sup J]. \tag{14}$$

Proof. (i) Again, since the statement is to hold almost surely, we suppose $Q(\omega) > 0$. From the definition of ϱ, it follows that $\varrho(\omega) > p^{fv}$, hence

$$\widehat{\mathcal{I}}(\varrho(\omega)) = F^{-1}\left(\frac{U(c\varrho(\omega) - cp^{fv}) - U(0)}{U(c\varrho(\omega) - cp^{fv}) - U(-cp^f)}\right).$$

Using the definition of ϱ, we verify

$$U(c\varrho(\omega) - cp^{fv}) = \frac{U(0) - F(Q(\omega))U(-cp^{fv})}{1 - F(Q(\omega))},$$

which gives, together with the previous equation,

$$\widehat{\mathcal{I}}(\varrho(\omega)) = F(F^{-1}(Q(\omega))) = Q(\omega). \tag{15}$$

If $\varrho(\omega) < \sup J$, then there exists $\widetilde{p} \in J$ with $\varrho(\omega) < \widetilde{p}$. On the other hand $\widehat{\mathcal{I}}$ is strictly increasing on $[p^{fv}, \infty[$, giving, with (15), $Q(\omega) = \widehat{\mathcal{I}}(\varrho(\omega)) <$

$\widehat{\mathcal{I}}(\widetilde{p}) = \mathcal{I}(\widetilde{p}) \le \sup_{p\in J} \mathcal{I}(p)$, which implies that

$$\{\varrho < \sup J\} \subseteq \{Q < \sup_{p\in J} \mathcal{I}(p)\}. \tag{16}$$

Vice versa, if $\varrho(\omega) \ge \sup J$, then by the same arguments $Q(\omega) = \widehat{\mathcal{I}}(\varrho(\omega)) \ge \widehat{\mathcal{I}}(\sup J) \ge \mathcal{I}(\widetilde{p})$ for all $\widetilde{p} \in J$. Consequently,

$$\{\varrho \ge \sup J\} \subseteq \{Q \ge \sup_{p\in J} \mathcal{I}(p)\}, \tag{17}$$

and the assertion follows by (16) and (17).

(ii) Suppose that $\varrho(\omega) \ge p$ with $p \le \sup J$. If $\varrho(\omega) \ge \sup J$, then (i) implies that $Q(\omega) \ge \sup_{p\in J} \mathcal{I}(p)$ and from (8) we obtain $p^S(\omega) = \sup J \ge p$. If $\varrho(\omega) < \sup J$, then (ii) implies that $Q(\omega) < \sup_{p\in J} \mathcal{I}(p)$, and (9) yields again $p^S(\omega) \ge \varrho(\omega) \ge p$.

(iii) Let $p \le \sup J$. Then (ii) is interpreted as an almost sure inclusion

$$\{p^S \ge p\} \supseteq \{\varrho \ge p\}$$

and for $p \in [p^{fv}, \sup J]$ we have to calculate the probability of the event

$$\begin{aligned}\{\varrho \ge p\} &= \left\{p^{fv} + c^{-1}U^{-1}\left(U(0) + \frac{F(Q)}{1-F(Q)}\,(U(0) - U(-cp^f))\right) \ge p\right\}\\ &= \left\{\frac{F(Q)}{1-F(Q)} \ge \frac{U(cp - cp^{fv}) - U(0)}{U(0) - U(-cp^f)}\right\},\end{aligned}$$

which does not depend on the distribution of Q since $F(Q)$ is uniformly distributed on $]0,1[$. In other words, we can use the formula

$$P\left(\frac{F(Q)}{1-F(Q)} \ge y\right) = P\left(F(Q) \ge \frac{y}{1+y}\right) = (1+y)^{-1} \quad \text{for all } y \in [0,\infty[,$$

and the proof is completed by concluding for each $p \in [p^{fv}, \sup J]$ that

$$\begin{aligned}P(p^S \ge p) \ge P(\varrho \ge p) &= \left(1 + \frac{U(cp - cp^{fv}) - U(0)}{U(0) - U(-cp^f)}\right)^{-1}\\ &\ge \frac{U(0) - U(-cp^f)}{U(cp - cp^{fv}) - U(-cp^f)}. \ \blacksquare\end{aligned}$$

The estimate (14) shows that the tail $p \mapsto P(p^S \ge p)$ of the system price distribution decreases rather slowly. Assigning the system price distribution to a class of known "fat tail" distributions seems impossible at this level of generality, since the distribution depends heavily on the set J and on the utility function U; furthermore, we have to take production costs into account. The impact of the risk aversion is hard to evaluate exactly since in a real market the utility function is not observed explicitly. We give a rough quantitative estimate of the tail decrease which applies to each utility function and involves merely the production costs.

PROPOSITION 3. *Under the assumptions of Proposition* 2, *for* $p \in [p^{fv}, \sup J]$,

$$P(p^S \geq p) \geq \frac{1}{1 + (p - p^{fv})/p^f}.$$

Proof. For each $h > 0$ the concavity of U yields

$$U(h) \leq U(0) + h(U(0) - U(-cp^f))(cp^f)^{-1}.$$

Subtracting $U(-cp^f)$ on both sides, we obtain

$$U(h) - U(-cp^f) \leq U(0) - U(-cp^f) + h(U(0) - U(-cp^f))(cp^f)^{-1}.$$

This inequality yields the estimate

$$\frac{U(0) - U(-cp^f)}{U(h) - U(-cp^f)} \geq \frac{U(0) - U(-cp^f)}{U(0) - U(-cp^f) + h(U(0) - U(-cp^f))(cp^f)^{-1}} \geq \frac{1}{1 + h(cp^f)^{-1}}.$$

The assertion follows by putting $h = cp - cp^{fv}$ and using Proposition 2(iii). ∎

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