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GLOBAL REGULAR NONSTATIONARY FLOW FOR THE NAVIER–STOKES EQUATIONS IN A CYLINDRICAL PIPE

Abstract. Global existence of regular solutions to the Navier–Stokes equations describing the motion of an incompressible viscous fluid in a cylindrical pipe with large inflow and outflow is shown. Global existence is proved in two steps. First, by the Leray–Schauder fixed point theorem we prove local existence with large existence time. Next, the local solution is prolonged step by step.

The existence is proved without any restrictions on the magnitudes of the inflow, outflow, external force and initial velocity.

1. Introduction. We consider viscous incompressible fluid motions in a finite cylinder with large inflow and outflow and under boundary slip conditions. Therefore the following initial-boundary value problem is examined:

$$
\begin{aligned}
&v_{,t} + v\cdot\nabla v - \operatorname{div}\mathbb{T}(v,p) = f && \text{in } \Omega^T = \Omega\times(0,T),\\
&\operatorname{div} v = 0 && \text{in } \Omega^T,\\
&v\cdot\bar n = 0 && \text{on } S_1^T = S_1\times(0,T),\\
&\nu\bar n\cdot\mathbb{D}(v)\cdot\bar\tau_\alpha + \gamma v\cdot\bar\tau_\alpha = 0,\ \ \alpha=1,2, && \text{on } S_1^T,\\
&v\cdot\bar n = d && \text{on } S_2^T = S_2\times(0,T),\\
&\bar n\cdot\mathbb{D}(v)\cdot\bar\tau_\alpha = 0, \quad \alpha = 1,2, && \text{on } S_2^T,\\
&v|_{t=0} = v(0) && \text{in } \Omega,
\end{aligned}
\tag{1.1}
$$

where $\Omega\subset\mathbb{R}^3$, $S = S_1\cup S_2 = \partial\Omega$, $v = v(x,t) = (v_1(x,t), v_2(x,t), v_3(x,t))\in\mathbb{R}^3$ is the velocity vector of the fluid motion, $p = p(x,t)\in\mathbb{R}^1$ the pressure,

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$f = f(x,t) = (f_1(x,t), f_2(x,t), f_3(x,t)) \in \mathbb{R}^3$ the external force field, $\bar{n}$ the unit outward vector normal to the boundary S, and $\bar{\tau}_\alpha$, $\alpha = 1,2$, are tangent vectors to S. Moreover, $\mathbb{T}(v,p)$ is the stress tensor of the form

$$\mathbb{T}(v,p) = \nu\mathbb{D}(v) - pI,$$

where ν is the constant viscosity coefficient, I the unit matrix and $\mathbb{D}(v)$ the dilatation tensor

$$\mathbb{D}(v) = \{v_{i,x_j} + v_{j,x_i}\}_{i,j=1,2,3}.$$

Finally, $\gamma > 0$ is the slip coefficient.

By $\Omega \subset \mathbb{R}^3$ we denote a cylindrical type domain parallel to the x_3 axis with arbitrary cross section. We assume that S_1 is the part of the boundary which is parallel to the x_3 axis and S_2 is perpendicular to x_3. Hence

$$\begin{aligned} S_1 &= \{x \in \mathbb{R}^3 : \varphi(x_1,x_2) = c_0,\ -a < x_3 < a\},\\ S_2(-a) &= \{x \in \mathbb{R}^3 : \varphi(x_1,x_2) < c_0,\ x_3 = -a\},\\ S_2(a) &= \{x \in \mathbb{R}^3 : \varphi(x_1,x_2) < c_0,\ x_3 = a\}, \end{aligned}$$

where a, c_0 are given positive numbers and $\varphi(x_1,x_2) = c_0$ describes a sufficiently smooth closed curve in the plane $x_3 = \text{const}$.

To describe the inflow and outflow we define

$$d_1 = -v \cdot \bar{n}|_{S_2(-a)}, \qquad d_2 = v \cdot \bar{n}|_{S_2(a)}, \tag{1.2}$$

so $d_i \ge 0$, $i = 1,2$, and by $(1.1)_{2,3}$ and (1.2) we have the compatibility condition

$$\Phi \equiv \int\limits_{S_2(-a)} d_1\, dS_2 = \int\limits_{S_2(a)} d_2\, dS_2, \tag{1.3}$$

where Φ is the flux.

Let us introduce an extension $\alpha = \alpha(x,t) \in \mathbb{R}$ such that

$$\alpha|_{S_2(-a)} = d_1, \qquad \alpha|_{S_2(a)} = d_2. \tag{1.4}$$

Then equations $(1.1)_{2,3,6}$ and (1.3) imply the compatibility condition

$$\int\limits_\Omega \alpha_{,x_3}\, dx = -\int\limits_{S_2(-a)} \alpha|_{x_3=-a}\, dS_2 + \int\limits_{S_2(a)} \alpha|_{x_3=a}\, dS_2 = 0.$$

The aim of this paper is to prove the existence of global solutions to problem (1.1) without any restrictions on the magnitudes of the initial velocity, external force field, inflow and outflow. This will be done by increasing regularity of weak solutions. For this purpose we follow the ideas and methods from [9]. To show the existence of such solutions we need, however, some small parameters. By such parameters we have L_2-norms of derivatives of v and f with respect to x_3 and derivatives of d_1, d_2 with respect to x', $x' = (x_1,x_2)$.

Hence we introduce the quantities

$$
\begin{aligned}
&h = v_{,x_3}, \quad q = p_{,x_3}, \quad g = f_{,x_3},\\
&w = v_3, \quad \chi = v_{2,x_1} - v_{1,x_2}.
\end{aligned}
\tag{1.5}
$$

Finally, we recall the definitions of Besov spaces which are necessary to understand the main results:

$$
\begin{aligned}
W_p^{\sigma,\sigma/2}(\Omega^T) = \bigg\{ u = u(x,t) : \|u\|_{W_p^{\sigma,\sigma/2}(\Omega^T)} \equiv \|u\|_{L_p(\Omega^T)}\\
+ \bigg(\int_0^T\int_\Omega\int_\Omega \frac{|D_x^{[\sigma]}u(x,t) - D_{x'}^{[\sigma]}u(x',t)|^p}{|x-x'|^{3+p(\sigma-[\sigma])}}\,dx\,dx'\,dt\bigg)^{1/p}\\
+ \bigg(\int_\Omega\int_0^T\int_0^T \frac{|\partial_t^{[\sigma/2]}u(x,t) - \partial_{t'}^{[\sigma/2]}u(x,t')|^p}{|t-t'|^{1+p(\sigma/2-[\sigma/2])}}\,dx\,dt\,dt'\bigg)^{1/p} < \infty \bigg\},
\end{aligned}
$$

where $\Omega \subset \mathbb{R}^3$, $1 \le p \le \infty$, $\sigma \in \mathbb{R}_+$ noninteger and $[\sigma]$ is the integer part of σ, and

$$
\begin{aligned}
W_p^\sigma(\Omega) = \bigg\{ u = u(x) : \|u\|_{W_p^\sigma(\Omega)} \equiv \|u\|_{L_p(\Omega)}\\
+ \bigg(\int_\Omega\int_\Omega \frac{|D_x^{[\sigma]}u(x) - D_{x'}^{[\sigma]}u(x')|^p}{|x-x'|^{3+p(\sigma-[\sigma])}}dxdx'\bigg)^{1/p} < \infty\bigg\}.
\end{aligned}
$$

Now we formulate our main results.

THEOREM 1.1 (local existence). *Assume that*:

(a) $h(0) \in W_2^1(\Omega)$, $\chi(0) \in L_2(\Omega)$, $v(0) \in W_{5/2}^1(\Omega)$, $g \in L_2(\Omega^T)$, $d_{,t} \in L_2(0,T,W_{6/5}^1(S_2))$, $d_{,x'} \in W_2^{3/2,3/4}(S_2^T)$, $d \in W_{5/2}^{8/5,8/10}(S_2^T)$, $f \in L_{5/2}(\Omega^T)$, $F_3 \in L_{10/3}(\Omega^T)$, $d \in L_\infty(0,T,W_3^1(S_2))$.

(b) *There exists a constant A such that $\varphi(A, G(T))\eta_1^2(T) + G'(T) \le A$, where*

$$
\begin{aligned}
l_1(T) &= \psi(\|d\|_{L_\infty(0,t,W_3^1(S_2))}, T)(\|d\|_{L_2(0,t,H^1(S_2))}^2 + |\!|\!| d_{,t} |\!|\!|_{6/5,2,S_2^T}^2\\
&\quad + |\!|\!| f |\!|\!|_{6/5,2,\Omega^t}^2 + |v(0)|_{2,\Omega}^2),\\
G(T) &= l_1(T) + \|d\|_{8/5,5/2,S_2^t} + \|v(0)\|_{1,5/2,\Omega} + |f|_{5/2,\Omega^T}\\
&\quad + |F_3|_{10/3,\Omega^T} + |\chi(0)|_{2,\Omega} + |\!|\!| d |\!|\!|_{3,6,S_2^T} + |\!|\!| d |\!|\!|_{3,\infty,S_2^T},\\
G'(T) &= |g|_{2,\Omega^T} + \|h(0)\|_{1,2,\Omega} + l_1(T) + \|d_{,x'}\|_{3/2,2,S_2^T},\\
\eta_1(T) &= \sup_{t'<T} \|d_{,x'}(t')\|_{1,S_2} + \|d_{,x'}\|_{L_2(0,t,H^1(S_2))}\\
&\quad + \|d_{,t}\|_{L_2(0,T,W_{6/5}^1(S_2))} + |\!|\!| f_3 |\!|\!|_{4/3,2,S_2^T} + |\!|\!| g |\!|\!|_{6/5,2,\Omega^T} + |h(0)|_{2,\Omega}
\end{aligned}
$$

and φ, ψ are increasing positive functions.

Then there exists a solution to problem (1.1) *such that*

$$\|h\|_{W_2^{2,1}(\Omega^T)} \le A, \quad \|v\|_{W_{5/2}^{2,1}(\Omega^T)} \le \varphi_0(A, G(T), G'(T), \eta_1(T), l_1(T)),$$

where φ_0 *is an increasing positive function.*

THEOREM 1.2 (global existence). *Let the assumptions of Theorem* 1.1 *be satisfied. Then there exists a sequence* $\{T_n\}_{n=1}^\infty$, *increasing to infinity, such that in each interval* $[T_n, T_{n+1}]$ *with* $T_{n+1} - T_n \le T$ *there exists a local solution to problem* (1.1) *satisfying the estimates*

$$\|h\|_{W_2^{2,1}(\Omega\times(T_n,T_{n+1}))} \le A,$$

$$\|v\|_{W_{5/2}^{2,1}(\Omega\times(T_n,T_{n+1}))} \le \varphi_0(A, G(T_n, T_{n+1}), G'(T_n, T_{n+1}), \eta_1(T_n, T_{n+1}), l_1(T_n, T_{n+1}))$$

for all $n \in \mathbb{N}$.

2. Notation and auxiliary results. To simplify the writing we introduce the following notation:

$$\begin{aligned}
|u|_{p,Q} &= \|u\|_{L_p(Q)}, && Q \in \{\Omega^T, S^T, \Omega, S\},\ p \in [1,\infty],\\
\|u\|_{s,Q} &= \|u\|_{H^s(Q)}, && Q \in \{\Omega, S\},\ s \in \mathbb{R}_+ \cup \{0\},\\
\|u\|_{s,Q^T} &= \|u\|_{W_2^{s,s/2}(Q^T)}, && Q \in \{\Omega, S\},\ s \in \mathbb{R}_+ \cup \{0\},\\
|u|_{p,q,Q^T} &= \|u\|_{L_q(0,T;L_p(Q))}, && Q \in \{\Omega, S\},\ p,q \in [1,\infty],\\
\|u\|_{s,q,Q^T} &= \|u\|_{W_q^{s,s/2}(Q^T)}, && Q \in \{\Omega, S\},\ s \in \mathbb{R}_+ \cup \{0\},\ q \in [1,\infty],\\
\|u\|_{s,q,Q} &= \|u\|_{W_q^s(Q)}, && Q \in \{\Omega, S\},\ s \in \mathbb{R}_+ \cup \{0\},\ q \in [1,\infty].
\end{aligned}$$

By c we denote a generic constant which changes its magnitude from formula to formula. By $\bar{c}(\sigma)$ and $\varphi(\sigma)$ we understand generic functions which are always positive and increasing. Finally, we do not distinguish scalar and vector-valued functions in notation.

We introduce the space

$$V_2^k(\Omega^T) = \Big\{u : \|u\|_{V_2^k(\Omega^T)} = \operatorname*{ess\,sup}_{t\in(0,T)} \|u\|_{H^k(\Omega)} + \Big(\int_0^T \|\nabla u(t)\|^2_{H^k(\Omega)}\,dt\Big)^{1/2} < \infty\Big\}, \quad k \in \mathbb{N}.$$

Now we recall a certain imbedding for anisotropic Sobolev spaces. Let $\Omega \subset \mathbb{R}^3$. Then we define

$$\|u\|_{W_2^{1,k}(\Omega)} = \Big[\int_\Omega (|u|^2 + |\nabla' u|^2 + |\nabla^k_{x_3} u|^2)\,dx\Big]^{1/2}, \quad k \in \mathbb{N},$$

where $\nabla' = (\partial_{x_1}, \partial_{x_2})$. From [6] we have

$$\|u\|_{q,r,\Omega^T} \le c\|u\|^{2/r}_{L_2(0,T,W_2^{1,k}(\Omega))} \operatorname*{ess\,sup}_t |u|^{1-2/r}_{2,\Omega}, \tag{2.1}$$

where

$$\frac{2}{r} + \frac{2k+1}{qk} = \frac{2k+1}{2k}. \tag{2.2}$$

For $r = q$ we have

$$q = \frac{2(4k+1)}{2k+1} \equiv q(k) \tag{2.3}$$

and the inequality

$$|u|_{q,\Omega^T} \le c\|u\|^{2/q}_{L_2(0,T,W_2^{1,k}(\Omega))} \operatorname*{ess\,sup}_t |u|^{1-2/q}_{2,\Omega}, \tag{2.4}$$

where $2/q < 1$. By the Young inequality, (2.4) gives

$$|u|_{q,\Omega^T} \le \varepsilon^{q/2}\|u\|_{L_2(0,T,W_2^{1,k}(\Omega))} + c\varepsilon^{-q/(q-2)} \operatorname*{ess\,sup}_t |u|_{2,\Omega}, \tag{2.5}$$

Finally, from [9] we get, for a weak solution to problem (1.1):

LEMMA 2.1. *Assume that* $d \in L_\infty(0,T;W_3^1(S_2)) \cap L_2(0,T;H^1(S_2))$, $d_{,t} \in L_2(0,T;L_{6/5}(S_2))$, $f \in L_2(0,T;L_{6/5}(\Omega))$, $v(0) \in L_2(\Omega)$. *Then*

$$\begin{aligned} \|v\|_{V_2^0(\Omega^t)} \le \varphi(\|d\|_{L_\infty(0,t,W_3^1(S_2))}, t)[\|d\|^2_{L_2(0,t,H^1(S_2))} \\ + \|d_{,t}\|^2_{6/5,2,S_2^t} + \|f\|^2_{6/5,2,\Omega^t} + |v(0)|^2_{2,\Omega}] \equiv l_1^2(t). \end{aligned} \tag{2.6}$$

where φ is an increasing positive function.

3. Basic formulations. To prove the existence of global solutions to problem (1.1) we follow [9]. Therefore we need problems satisfied by quantities (1.5). First we have, from [9]:

LEMMA 3.1. *The quantities h, q are solutions to the problem*

$$\begin{aligned} & h_{,t} - \operatorname{div}\mathbb{T}(h,q) = -v\cdot\nabla h - h\cdot\nabla v + g && \text{in } \Omega^T, \\ & \operatorname{div} h = 0 && \text{in } \Omega^T, \\ & \bar n\cdot h = 0 && \text{on } S_1^T, \\ & \nu\bar n\cdot\mathbb{D}(h)\cdot\bar\tau_\alpha + \gamma h\cdot\bar\tau_\alpha = 0,\ \alpha = 1,2, && \text{on } S_1^T, \\ & h_i = -d_{,x_i}, \quad i = 1,2, && \text{on } S_2^T, \\ & h_{3,x_3} = \Delta' d && \text{on } S_2^T, \\ & h|_{t=0} = h(0) && \text{in } \Omega, \end{aligned} \tag{3.1}$$

where $\Delta' = \partial^2_{x_1} + \partial^2_{x_2}$, d stands for d_1 and d_2, because $d|_{S_2(-a)} = d_1$, $d|_{S_2(+a)} = d_2$.

Proof. Equations $(3.1)_{1,2,3,4,7}$ follow directly from the corresponding equations in (1.1) by differentiation with respect to x_3, because S_1 is parallel to the x_3 axis.

To show $(3.1)_{5,6}$ we recall that

$$v_3|_{S_2} = d, \quad (v_{i,x_3} + v_{3,x_i})|_{S_2} = 0, \quad i = 1,2. \tag{3.2}$$

Hence $v_{i,x_3}|_{S_2} = -d_{,x_i}$, $i = 1,2$, and $(3.1)_5$ holds.

From $(1.1)_2$ we have $v_{3,x_3x_3}|_{S_2} = -(v_{1,x_3x_1} + v_{2,x_3x_2})|_{S_2} = d_{,x_1x_1} + d_{,x_2x_2} = \Delta' d$. Hence $(3.1)_6$ follows. This ends the proof.

LEMMA 3.2. *The function $\chi = v_{2,x_1} - v_{1,x_2}$ is a solution to the problem*

$$\begin{aligned}
&\chi_{,t} + v\cdot\nabla\chi - h_3\chi + h_2 w_{,x_1} - h_1 w_{,x_2} - \nu\Delta\chi = F_3 && \text{in } \Omega^T,\\
&\chi|_{S_1} = -v_i(n_{i,x_j}\tau_{1j} + \tau_{1i,x_j}n_j) + \frac{\gamma}{\nu}v_j\tau_{1j} &&\\
&\qquad + v\cdot\bar\tau_1(\tau_{12,x_1} - \tau_{11,x_2}) \equiv \chi_* && \text{on } S_1^T,\\
&\chi_{,x_3} = 0 && \text{on } S_2^T,\\
&\chi|_{t=0} = \chi(0) && \text{in } \Omega,
\end{aligned} \tag{3.3}$$

where

$$\bar n|_{S_1} = \frac{(\varphi_{,x_1}, \varphi_{,x_2}0)}{\sqrt{\varphi_{,x_1}^2 + \varphi_{,x_2}^2}}, \quad \bar\tau_1|_{S_1} = \frac{(-\varphi_{,x_2}, \varphi_{,x_1}, 0)}{\sqrt{\varphi_{,x_1}^2 + \varphi_{,x_2}^2}}, \quad \bar\tau_2|_{S_1} = (0,0,1) \equiv \bar e_3,$$

$$\bar n|_{S_2} = \bar e_3, \quad \bar\tau_1|_{S_2} = \bar e_1, \quad \bar\tau_2|_{S_2} = \bar e_2,$$

where $\bar e_1 = (1,0,0)$, $\bar e_2 = (0,1,0)$ and $F_3 = f_{2,x_1} - f_{1,x_2}$.

Proof. Differentiating the first equation of $(1.1)_1$ with respect to x_2, the second equation of $(1.1)_1$ with respect to x_1, and subtracting the results yields $(3.3)_1$.

To show $(3.3)_2$ we extend the vectors $\bar\tau_1$, $\bar n$ into a neighbourhood of S_1. In this neighourhood $v' = (v_1, v_2)$ can be expressed in the form

$$v' = v\cdot\bar\tau_1\bar\tau_1 + v\cdot\bar n\bar n.$$

Then

$$\begin{aligned}
\chi|_{S_1} &= [(v\cdot\bar\tau_1\tau_{12} + v\cdot\bar n n_2)_{,x_1} - (v\cdot\bar\tau_1\tau_{11} + v\cdot\bar n n_1)_{,x_2}]|_{S_1}\\
&= [-\bar n\cdot\nabla(v\cdot\bar\tau_1) + v\cdot\bar\tau_1(\tau_{12,x_1} - \tau_{11,x_2})]|_{S_1},
\end{aligned} \tag{3.4}$$

where $(1.1)_3$ was employed and τ_{1i}, n_i are the ith Cartesian coordinates.

Utilizing $(1.1)_3$ in $(1.1)_4$ for $\alpha = 1$ yields

$$\nu\bar n\cdot\nabla(v\cdot\bar\tau_1) - \nu v_i(n_{i,x_j}\tau_{1j} + \tau_{1i,x_j}n_j) + \gamma v\cdot\bar\tau_1 = 0. \tag{3.5}$$

Exploiting (3.5) in (3.4) yields $(3.3)_2$. By the definition of χ and $(3.1)_5$ we have

$$\chi_{,x_3}|_{S_2} + (v_{2,x_1x_3} - v_{1,x_2x_3})|_{S_2} = -(d_{,x_1x_2} - d_{,x_2x_1})|_{S_2} = 0.$$

This ends the proof.

For solutions to problem (3.1) we get (see [9])

LEMMA 3.3. *Assume that v is a weak solution to problem* (1.1). *Then*

$$|h(t)|^2_{2,\Omega}+\nu\int_0^t\|h(t')\|^2_{1,\Omega}dt'+\gamma|h\cdot\bar\tau_\alpha|^2_{2,S_1^t}\le\varphi(|d_1|_{3,6,S_2^t},|\nabla v|_{3,2,\Omega^t},l_1,|d|_{3,\infty,S_2^t})\eta_1^2(t),\tag{3.6}$$

where φ is an increasing positive function, l_1 is defined by (2.6) *and*

$$\eta_1(t)=\sup_{t'\le t}\|d_{,x'}(t')\|_{1,S_2}+\|d_{,x'}\|_{L_2(0,t,H^1(S_2))}+\|d_{,t}\|_{L_2(0,t,W^1_{6/5}(S_2))}+|f_3|_{4/3,2,S_2^t}+|g|_{6/5,2,\Omega^t}+|h(0)|_{2,\Omega}.\tag{3.7}$$

4. Estimates. First we examine problem (3.3). Let $\tilde\chi$ be a solution of the problem

$$\begin{aligned}&\tilde\chi_{,t}-\nu\Delta\tilde\chi=0 &&\text{in } \Omega^T,\\ &\tilde\chi=\chi_* &&\text{on } S_1^T,\\ &\tilde\chi_{,x_3}=0 &&\text{on } S_2^T,\\ &\tilde\chi|_{t=0}=0 &&\text{in } \Omega.\end{aligned}\tag{4.1}$$

Then $\chi'=\chi-\tilde\chi$ is a solution of

$$\begin{aligned}&\chi'_{,t}+v\cdot\nabla\chi'-h_3(v_{,2x_1}-v_{1,x_2})+h_2w_{,x_1}-h_1w_{,x_2}\\ &\qquad-\nu\Delta\chi'=F_3-v\cdot\nabla\tilde\chi &&\text{in } \Omega^T,\\ &\chi'=0 &&\text{on } S_1^T,\\ &\chi'_{,x_3}=0 &&\text{on } S_2^T,\\ &\chi'|_{t=0}=\chi(0) &&\text{in } \Omega.\end{aligned}\tag{4.2}$$

LEMMA 4.1. *Assume that $h\in L_5(\Omega^T)$, $F_3\in L_{10/7}(\Omega^T)$, $v'\in L_\infty(0,T,W^1_{9/5}(\Omega))$, $v\in W_r^{s,s/2}(\Omega^T)$ with $5/r-3/2\le s$, $\chi(0)\in L_2(\Omega)$. Assume also that v is a weak solution satisfying* (2.6). *Then a solution of problem* (4.2) *satisfies the inequality*

$$|\chi(t)|^2_{2,\Omega}+\int_0^t\|\chi(t')\|^2_{1,\Omega}dt'\le c(l_1^2(t)(\|v'\|^2_{L_\infty(0,t,W^1_{9/5}(\Omega))}+|h|^2_{5,\Omega^t})+\|v'\|^2_{s,r,\Omega^t}+|F_3|^2_{10/7,\Omega^t}+|\chi(0)|^2_{2,\Omega}).\tag{4.3}$$

Proof. Multiplying $(4.2)_1$ by χ' and integrating the result over Ω we get

$$\frac12\frac{d}{dt}|\chi'(t)|^2_{2,\Omega}+|\nabla\chi'|^2_{2,\Omega}=\int_\Omega(v_{2,x_1}-v_{1,x_2})h_3\chi'dx-\int_\Omega(h_2w_{,x_1}-h_1w_{,x_2})\chi'dx+\int_\Omega v\cdot\nabla\tilde\chi\chi'dx+\int_\Omega F_3\chi'dx.$$

Utilizing the Poincaré inequality and integrating with respect to time yields

$$
(4.4)\qquad |\chi'(t)|^2_{2,\Omega} + \int_0^t \|\chi'(t')\|^2_{1,\Omega}\,dt' \le c\Big(\int_{\Omega^t} |h_3|\,|\nabla v'|\,|\chi'|\,dx\,dt' + \int_{\Omega^t} |h'|\,|\nabla' w|\,|\chi'|\,dx\,dt + \Big|\int_0^t\int_\Omega v(t')\cdot\nabla\tilde\chi(t')\chi'(t')\,dx\,dt'\Big| + \int_{\Omega^t} |F_3|\,|\chi'|\,dx\,dt + |\chi(0)|^2_{2,\Omega}\Big).
$$

We estimate the first term on the r.h.s. of (4.4) by $|h_3|_{5,\Omega^t}\cdot|\nabla v'|_{2,\Omega^t}|\chi'|_{10/3,\Omega^t}$ and the second by $|\nabla' w|_{2,\Omega^t}|h'|_{5,\Omega^t}|\chi'|_{10/3,\Omega^t}$. The third term on the r.h.s. of (4.4) can be expressed in the form

$$
\Big|\int_0^t\int_\Omega v(t')\cdot\nabla\chi'(t')\tilde\chi(t')\,dx\,dt'\Big|
$$

and estimated by

$$
\varepsilon|\nabla\chi'|^2_{2,\Omega^t} + |\nabla v|^2_{2,\Omega^t}|\!|\!|\tilde\chi|\!|\!|^2_{3,\infty,\Omega^t}.
$$

We bound the fourth integral on the r.h.s. of (4.4) by

$$
|\chi'|_{10/3,\Omega^t}|F_3|_{10/7,\Omega^t}.
$$

Utilizing the above estimates in (4.4) we obtain

$$
|\chi'|^2_{2,\Omega} + \int_0^t \|\chi'(t')\|^2_{1,\Omega}dt' \le c(\varepsilon(|\chi'|^2_{10/3,\Omega^t} + |\nabla\chi'|^2_{2,\Omega^t}) + |\nabla v|^2_{2,\Omega^t}|\!|\!|\tilde\chi|\!|\!|^2_{3,\infty,\Omega^t} + |h|^2_{5,\Omega^t}|\nabla v|^2_{2,\Omega^t} + |F_3|^2_{10/7,\Omega^t} + |\chi(0)|^2_{2,\Omega}).
$$

Applying the transformation $\chi' = \chi - \tilde\chi$ and taking ε sufficiently small we have

$$
|\chi(t)|^2_{2,\Omega} + \int_0^t \|\chi(t')\|^2_{1,\Omega}dt' \le c\Big(|\nabla v|^2_{2,\Omega^t}|\!|\!|\tilde\chi|\!|\!|^2_{3,\infty,\Omega^t} + |h|^2_{5,\Omega^t}|\nabla v|^2_{2,\Omega^t} + |\tilde\chi(t)|^2_{2,\infty,\Omega^t} + \int_0^t \|\tilde\chi(t')\|^2_{1,\Omega}\,dt' + |F_3|^2_{10/7,\Omega^t} + |\chi(0)|^2_{2,\Omega}\Big).
$$

Now using the inequalities

$$
|u|_{10/3,\Omega^t} \le c(|u|_{2,\infty,\Omega^t} + \|u\|_{L_2(0,T;W^1_2(\Omega))}) \le c\|u\|_{s,r,\Omega^t},
$$

where $5/r - 3/2 \le s$, we obtain

$$
|\chi(t)|^2_{2,\Omega} + \int_0^t \|\chi(t')\|^2_{1,\Omega}\,dt' \le c(l_1^2(t)|\!|\!|\tilde\chi|\!|\!|^2_{3,\infty,\Omega^t} + l_1^2(t)|h|^2_{5,\Omega^t} + \|\tilde\chi\|^2_{s,r,\Omega^t} + |F_3|^2_{10/7,\Omega^t} + |\chi(0)|^2_{2,\Omega}),
$$

and using the inequalities $\|\tilde\chi\|_{s,r,\Omega^t} \le c\|\chi_*\|_{s-1/r,r,S_1^t} \le c\|v\|_{s,r,\Omega^t}$ and $|\tilde\chi|_{3,\infty,\Omega^t} \le |v'|_{3,\infty,S_1^t} \le \|v'\|_{L_\infty(0,t,W_{9/5}(\Omega))}$, we obtain (4.3). This concludes the proof.

Next we consider the problem

$$
\begin{aligned}
&v_{1,x_2} - v_{2,x_1} = \chi && \text{in } \Omega',\\
&v_{1,x_1} + v_{2,x_2} = -h_3 && \text{in } \Omega',\\
&v'\cdot n' = 0 && \text{on } S_1',
\end{aligned}
\tag{4.5}
$$

where $\Omega' = \Omega \cap \{x_3 = \text{const} \in (-a,a)\}$, $S_1' = S_1 \cap \{x_3 = \text{const} \in (-a,a)\}$, and x_3, t are treated as parameters.

LEMMA 4.2. *Let the assumptions of Lemma* 4.1 *be satisfied. Then any solution of problem* (4.5) *satisfies*

$$
\begin{aligned}
\sup_{t'\le t}\|v'(t')\|_{1,\Omega}^2 + \|\nabla v'\|^2_{L_2(0,t,W_2^1(\Omega))} &\le c\Big(l_1^2(t)(|h|^2_{5,\Omega^t}+1) + \|v'\|^2_{s,r,\Omega^t}\\
&+ |F_3|^2_{10/7,\Omega^t} + |\chi(0)|^2_{2,\Omega} + \sup_{t'<t}|h'(t')|^2_{2,\Omega^t} + \int_0^t \|h(t')\|^2_{1,\Omega}dt'\Big) \equiv A^2(t).
\end{aligned}
\tag{4.6}
$$

Proof. For solutions of problem (4.5) we get the estimates

$$\|v'\|^2_{1,\Omega'} \le c(|\chi|^2_{2,\Omega'} + |h_3|^2_{2,\Omega'}), \qquad \|v'\|^2_{2,\Omega'} \le c(\|\chi\|^2_{1,\Omega'} + \|h_3\|^2_{1,\Omega'}).$$

where $v' = (v_1, v_2)$. Integrating the above estimates with respect to x_3 and the second one also with respect to time, and adding the results, we obtain

$$
\begin{aligned}
\sup_{t'<t}\int_{-a}^{a}\|v'(x_3,t')\|^2_{1,\Omega'}\,dx_3 + \int_0^t\int_{-a}^{a}\|v'(x_3,t')\|^2_{2,\Omega'}\,dx_3\,dt'&\\
\le c\Big(\int_0^t\|\chi(t')\|^2_{1,\Omega}\,dt' + \int_0^t\|h_3(t')\|^2_{1,\Omega}\,dt'\Big)&.
\end{aligned}
$$

Adding to the last inequality $\sup_{t'<t}|h'|^2_{2,\Omega} + \int_0^t\|h'(t')\|^2_{1,\Omega}\,dt'$, we obtain

$$
\begin{aligned}
\sup_{t'<t}\|v'\|^2_{1,\Omega} + \int_0^t\|v'\|^2_{2,\Omega}\,dt' \le c\Big(\int_0^t\|\chi(t')\|^2_{1,\Omega}\,dt' + \sup_{t'<t}|h'|^2_{2,\Omega}&\\
+ \int_0^t\|h'(t')\|^2_{1,\Omega}\,dt' + \int_0^t\|h_3(t')\|^2_{1,\Omega}\,dt'\Big)&.
\end{aligned}
$$

Utilizing (4.3) to estimate the first norm in the last inequality and the inequality $\|v'\|_{L_\infty(0,t,W^1_{9/5}(\Omega))} \le \varepsilon\|v'\|_{L_\infty(0,t,H^1(\Omega))} + c(1/\varepsilon)|v|_{2,\infty,\Omega^t}$, we obtain (4.6). This concludes the proof.

Now we increase the regularity of v. For problem (1.1) we obtain

$$
(4.7)\qquad \|v\|_{2,q,\Omega^t}+|\nabla p|_{q,\Omega^t}\le c(|v'\cdot\nabla v|_{q,\Omega^t}+|w\cdot h|_{q,\Omega^t} + \|d\|_{2-1/q,S_2^t}+\|\gamma v\cdot\tau_\alpha\|_{1-1/q,qS_1^t}+\|v(0)\|_{1,q,\Omega}+|f|_{q,\Omega^t}).
$$

We estimate the first term of the r.h.s. by

$$
|v'\cdot\nabla v|_{q,\Omega^t}\le|v'|_{\lambda_1 q,\Omega^t}|\nabla v|_{\lambda_2 q,\Omega^t},
$$

where $\lambda_1 q=10$, $\lambda_2 q=2$ and $1/\lambda_1+1/\lambda_2=1$, so $q=5/3$. The second term is estimated by

$$
|w\cdot h|_{5/3,\Omega^t}\le|w|_{10/3,\Omega^t}|h|_{10/3,\Omega^t}.
$$

Using the above estimates and the inequality

$$
\|v\|_{2/5,5/3,S_1^t}\le\varepsilon\|v\|_{2,5/3,\Omega^t}+c(1/\varepsilon)|v|_{2,\Omega^t},
$$

we obtain

$$
(4.8)\qquad \|v\|_{2,5/3,\Omega^t}+|\nabla p|_{5/3,\Omega^t}\le c(A(t)l_1(t)+\|d\|_{7/5,5/3,S_2^t} + \|v(0)\|_{1,5/3,\Omega^t}+l_1(t)+|f|_{5/3,\Omega^t}).
$$

Now in (4.8) we use (4.6) and the inequality $\|v\|_{s,r,\Omega^t}\le\varepsilon\|v\|_{2,5/3,\Omega^t}+c(\frac{1}{\varepsilon})|v|_{2,\Omega^t}$, with $1+s<5/r$, to obtain

$$
(4.9)\qquad \|v\|_{2,5/3,\Omega^t}\le c\Big(l_1(t)(1+|h|_{5,\Omega^t})+|F_3|_{10/3,\Omega^t}+|\chi(0)|_{2,\Omega} + |f|_{5/3,\Omega^t}+\sup_{t'<t}|h'(t)|_{2,\Omega}+\Big(\int_0^t\|h(t')\|_{1,\Omega}^2dt'\Big)^{1/2} + \|d\|_{7/5,5/3,S_2^t}+\|v(0)\|_{1,5/3,\Omega^t}\Big)\equiv A'(t),
$$

so $v\in W^{2,1}_{5/3}(\Omega^t)$.

Next for problem (1.1) we obtain the inequality

$$
(4.10)\qquad \|v\|_{2,2,\Omega^t}+|\nabla p|_{2,\Omega^t}\le c(|v'\cdot\nabla v|_{2,\Omega^t}+|w\cdot h|_{2,\Omega^t}+\|d\|_{3/2,2,S_2^t} + \|\gamma v\cdot\tau_\alpha\|_{1/2,2,S_1^t}+\|v(0)\|_{1,2,\Omega}+|f|_{2,\Omega^t}).
$$

We estimate the first term of the r.h.s. by

$$
|v'\cdot\nabla v|_{2,\Omega^t}\le|v'|_{10,\Omega^t}|\nabla v|_{5/2,\Omega^t},
$$

and the second term by

$$
|w\cdot h|_{2,\Omega^t}\le|w|_{5,\Omega^t}|h|_{10/3,\Omega^t}.
$$

Using the inequalities

$$
|\nabla v|_{5/2,\Omega^t}+|v|_{5,\Omega^t}\le c\|v\|_{2,5/3,\Omega^t},
$$
$$
\|v\|_{1/2,2,S_1^t}\le\varepsilon\|v\|_{2,2,\Omega^t}+c(1/\varepsilon)|v|_{2,\Omega^t},
$$

we obtain

$$(4.11)\quad \|v\|_{2,2,\Omega^t}+|\nabla p|_{2,\Omega^t}\le c(\|v\|_{5/3,2,\Omega^t}(|v'|_{10,\Omega^t}+|h|_{10/3,\Omega^t}) + \|v(0)\|_{1,2,\Omega}+|f|_{2,\Omega^t}+l_1(t)+\|d\|_{3/2,2,S_2^t}).$$

We need more regularity for v. Hence we prove

LEMMA 4.4. *Assume that* $v\in W_2^{2,1}(\Omega^t)$. *Then*

$$(4.12)\quad \|v\|_{2,5/2,\Omega^t}+|\nabla p|_{5/2,\Omega^t}\le c((A'(t)+l_1(t))(\|v\|_{2,2,\Omega^t}+A'(t)+l_1(t)) + \|v(0)\|_{1,5/2,\Omega}+|f|_{5/2,\Omega^t}+l_1(t)+\|d\|_{8/5,5/2,S_2^t}).$$

Proof. From (1.1) we obtain

$$(4.13)\quad \|v\|_{2,5/2,\Omega^t}+|\nabla p|_{5/2,\Omega^t}\le c(|v'\cdot\nabla v|_{5/2,\Omega^t}+|w\cdot h|_{5/2,\Omega^t} + \|d\|_{8/5,5/2,S_2^t}+\|\gamma v\cdot\tau_\alpha\|_{3/5,5/2,S_1^t}+\|v(0)\|_{1,5/2,\Omega^t}+|f|_{5/2,\Omega^t}).$$

Then from (4.13), using the inequality

$$\|v\|_{3/5,5/2,S_1^t}\le\varepsilon\|v\|_{5/2,2,\Omega^t}+c(1/\varepsilon)|v|_{2,\Omega^t},$$

we obtain

$$(4.14)\quad \|v\|_{2,5/2,\Omega^t}+|\nabla p|_{5/2,\Omega^t}\le c(|v'|_{10,\Omega^t}|\nabla v|_{10/3,\Omega^t}+|v|_{10,\Omega^t}|h|_{10/3,\Omega^t} + \|v(0)\|_{1,5/2,\Omega}+|f|_{5/2,\Omega^t}+l_1(t)).$$

Now using in (4.14) the inequalities $|v|_{10,\Omega^t}\le c\|v\|_{2,2,\Omega^t}$ and

$$|\nabla v|_{10/3,\Omega^t}\le\varepsilon\|v\|_{2,5/2,\Omega^t}+c(1/\varepsilon)|v|_{2,\Omega^t},$$

we obtain (4.12). This concludes the proof.

To prove the existence of local solutions to problem (1.1) we apply the Leray–Schauder fixed point theorem. We show existence of a fixed point of a transformation generated by problem (3.1).

LEMMA 4.5. *Let* $v\in W_{5/2}^{2,1}(\Omega^t)$, $g\in L_2(\Omega^t)$, $h(0)\in L_2(\Omega)$. *Then a solution of* (3.1) *satisfies*

$$(4.15)\quad \|h\|_{2,2,\Omega^t}+|\nabla q|_{2,\Omega^t}\le c(\varphi(\|v\|_{5/2,2,\Omega^t})|h|_{2,\Omega^t}+\|d_{,x'}\|_{3/2,2,\Omega^t} + \|v(0)\|_{1,2,\Omega}+|g|_{2,\Omega^t}),$$

where φ *an increasing positive function.*

Proof. From (3.1) we get

$$(4.16)\quad \|h\|_{2,2,\Omega^t}+|\nabla q|_{2,\Omega^t}\le c(|v\cdot\nabla h|_{2,\Omega^t}+|h\cdot\nabla v|_{2,\Omega^t} + \|\gamma h\cdot\tau_\alpha\|_{1/2,2,S_1^t}+\|d_{,x'}\|_{3/2,2,S_2^t}+\|v(0)\|_{1,2,\Omega}+|g|_{2,\Omega^t}).$$

Using the Hölder inequality in (4.16) we obtain

$$(4.17)\quad \|h\|_{2,2,\Omega^t}+|\nabla q|_{2,\Omega^t}\le c(|v|_{10,\Omega^t}|\nabla h|_{5/2,\Omega^t}+|h|_{5,\Omega^t}|\nabla v|_{10/3,\Omega^t} + \|h\|_{1/2,2,S_1^t}+\|d_{,x'}\|_{3/2,2,S_2^t}+\|v(0)\|_{1,2,\Omega}+|g|_{2,\Omega^t}).$$

Now using the inequalities

$$\|h\|_{1/2,2,S_1^t}\le\varepsilon\|h\|_{2,2,\Omega^t}+C(1/\varepsilon)|h|_{2,\Omega^t},$$

$$|v|_{10,\Omega^t}|\nabla h|_{5/2,\Omega^t} \le \varepsilon\|h\|_{2,2,\Omega^t} + \varphi_1(|v|_{10,\Omega^t})|h|_{2,\Omega^t}$$

and

$$|\nabla v|_{10/3,\Omega^t}|h|_{5,\Omega^t} \le \varepsilon\|h\|_{2,2,\Omega^t} + \varphi_2(|\nabla v|_{10/3,\Omega^t})|h|_{2,\Omega^t},$$

where φ_1, φ_2 are increasing positive functions, we obtain (4.15).

From (4.9) and (4.11) we get

$$\begin{aligned}\|v\|_{2,2,\Omega^t} \le c(&A'(t)(A'(t)+l_1(t)) + \|v(0)\|_{2,2,\Omega^t}\\ &+ |f|_{2,\Omega^t} + l_1(t) + \|d\|_{3/2,2,S_2^t}) \equiv B(t).\end{aligned} \tag{4.18}$$

Next from (4.18) and (4.12) we get

$$\begin{aligned}\|v\|_{2,5/2,\Omega^t} \le c(&(A'(t)+l_1(t))(B(t)+A'(t)+l_1(t))\\ &+ \|v(0)\|_{1,5/2,\Omega} + |f|_{5/2,\Omega^t} + l_1(t) + \|d\|_{8/5,5/2,S_2^t}).\end{aligned} \tag{4.19}$$

Finally, from (4.15) and (4.19) we obtain

$$\|h\|_{2,2,\Omega^t} \le \varphi(\|h\|_{2,2,\Omega^t}, G(t))|h|_{2,\Omega^t} + G'(t), \tag{4.20}$$

where

$$\begin{aligned}G(t) &= l_1(t) + \|d\|_{8/5,5/2,S_2^t} + \|v(0)\|_{1,5/2,\Omega} + |f|_{5/2,\Omega^t}\\ &\quad + |F_3|_{10/3,\Omega^t} + |\chi(0)|_{2,\Omega},\\ G'(t) &= |g|_{2,\Omega^t} + \|h(0)\|_{1,2,\Omega} + l_1(t) + \|d_{,x'}\|_{3/2,2,S_2^t} + |h|_{2,\Omega^t}\end{aligned} \tag{4.21}$$

and φ is an increasing positive function.

5. Local existence and uniqueness. To prove the existence of local solutions to problem (1.1) we look for a fixed point of the transformation

$$h = \phi(\tilde h, \lambda), \qquad \lambda \in [0,1], \tag{5.1}$$

defined by the following system:

$$\begin{aligned}&h_{,t} - \operatorname{div}\mathbb{T}(h,q) = -\lambda(v(\tilde h)\cdot\nabla\tilde h + \tilde h\cdot\nabla v(\tilde h)) + g && \text{in } \Omega^T,\\ &\operatorname{div} h = 0 && \text{in } \Omega^T,\\ &n\cdot h = 0 && \text{on } S_1^T,\\ &\nu n\cdot\mathbb{D}(h)\cdot\tau_\alpha + \gamma h\cdot\tau_\alpha = 0, \quad \alpha = 1,2, && \text{on } S_1^T,\\ &h_i = -d_{,x_i}, \quad i = 1,2, && \text{on } S_2^T,\\ &h_{3,x_3} = \Delta' d && \text{on } S_2^T,\\ &h|_{t=0} = h(0) && \text{in } \Omega.\end{aligned} \tag{5.2}$$

Moreover, the dependence $v = v(\tilde h)$ is determined by Lemma 4.4. The main problem of this section is to show the existence of a fixed point of the transformation (5.1) for $\lambda = 1$.

The above formulation suggests that the Leray–Schauder fixed point theorem should be applied.

To define the domain of ϕ we first examine the mapping $v = v(h)$ defined by Lemma 4.4. In view of the assumptions of (4.18) we define

(5.3) $\mathcal{M}_0(\Omega^T) = \{h : h \in L_{10/3}(\Omega^t) \cap L_\infty(0,t,L_2(\Omega)) \cap L_2(0,t,H^1(\Omega))\}.$

Then (4.18) shows that

$$v : \mathcal{M}_0 \to W^{2,1}_{5/2}(\Omega^t). \tag{5.4}$$

Let ϕ be the transformation defined by problem (5.2). Then from (5.4) and Lemma 4.5 we get

$$\mathcal{M} = \mathcal{M}_0(\Omega^T) \cap W^{1,1/2}_2(\Omega^T) \cap L_5(\Omega^T) \cap \{h : \nabla h \in L_{5/2}(\Omega^T)\}$$

and $\phi : \mathcal{M} \to W^{2,1}_2(\Omega^T)$.

LEMMA 5.1. *Assume that*

$$g \in L_2(\Omega^T), \quad d \in W^{5/2,5/4}_{8/5}(S^T_2), \quad f \in L_{5/2}(\Omega^T), \quad F_3 \in L_{10/3}(\Omega^T),$$
$$\chi(0) \in L_2(\Omega), \quad v(0) \in W^1_{5/2}(\Omega), \quad h(0) \in W^1_2(\Omega), \quad d_{,x'} \in W^{2,1}_{3/2}(S^T_2),$$

Then the imbedding $W^{2,1}_2(\Omega^T) \subset \mathcal{M}(\Omega^T)$ is compact.

Proof. In view of interpolation inequalities the following imbeddings are compact:

$$W^{2,1}_2(\Omega^T) \subset \mathcal{M}_0(\Omega^T), \quad W^{2,1}_2(\Omega^T) \subset W^{1,1/2}_2(\Omega^T), \quad W^{2,1}_2(\Omega^T) \subset L_5(\Omega^T),$$

and $W^{2,1}_2(\Omega^T) \subset \{h : \nabla h \in L_{5/2}(\Omega^T)\}$. Hence $W^{2,1}_2(\Omega^T) \subset \mathcal{M}(\Omega^T)$ is compact.

LEMMA 5.2. *With the assumptions of Lemma* 4.5, *there exists a constant $A > 0$ such that a fixed point of ϕ satisfies*

$$\|h\|_{2,2,\Omega^t} + |\nabla q|_{2,\Omega} \le A. \tag{5.5}$$

Proof. From (3.6) and (4.20) we get

$$\|h\|_{2,2,\Omega^t} \le \varphi(\|h\|_{2,2,\Omega^t}, G(t))\eta_1(t) + G'(t),$$

where

$$G(t) = l_1(t) + \|d\|_{8/5,5/2,S^t_2} + \|v(0)\|_{1,5/2,\Omega^t} + \|f\|_{5/2,\Omega^t} + |F_3|_{10/3,\Omega^t} + |\chi(0)|_{2,\Omega} + |d|_{3,6,S^t_2} + |d|_{3,\infty,S^t_2},$$
$$G'(t) = |g|_{2,\Omega^t} + \|h(0)\|_{1,2,\Omega} + l_1(t) + \|d_{,x'}\|_{3/2,2,2,S^t_2}$$

and φ is an increasing positive function. For $\eta_1(t)$ sufficiently small there exists a constant A such that $\varphi(A, G(t))\eta_1(t) + G'(t) \le A$ and $G'(t) \le A$. Hence the estimate (5.5) holds. This concludes the proof.

Finally, we show the uniform continuity of ϕ.

LEMMA 5.3. *Let the assumptions of Lemma* 5.2 *hold. Then the mapping ϕ is uniformly continuous in $\mathcal{M}(\Omega^T) \times [0,1]$.*

Proof. The uniform continuity with respect to $\lambda \in [0,1]$ is evident. Therefore we examine the uniform continuity with respect to elements of $\mathcal{M}(\Omega^t)$ for any $\lambda \in [0,1]$.

Since the dependence on λ is elementary we omit λ in the considerations below because this does not change the proof.

Let $\tilde{h}_s \in \mathcal{M}(\Omega^t)$, $s = 1,2$. We consider the problem

$$
\begin{aligned}
&h_{s,t} - \operatorname{div} \mathbb{T}(h_s, q_s) = -v_s \cdot \nabla \tilde{h}_s - \tilde{h}_s \cdot \nabla v_s + g && \text{in } \Omega^T,\\
&\operatorname{div} h_s = 0 && \text{in } \Omega^T,\\
&h_s \cdot n = 0 && \text{on } S_1^T,\\
&\nu n \cdot \mathbb{D}(h) \cdot \tau_\alpha + \gamma h \cdot \tau_\alpha = 0, \quad \alpha = 1,2, && \text{on } S_1^T,\\
&h_{si} = -d_{,x_i}, \quad i = 1,2, && \text{on } S_2^T,\\
&h_{sx_3} = \Delta' d && \text{on } S_2^T,\\
&h_s|_{t=0} = h(0) && \text{in } \Omega,
\end{aligned}
\tag{5.6}
$$

where $s = 1,2$ and $v_s = v(h_s)$. Moreover, we have

$$
\begin{aligned}
&\chi_{s,t} + v_s \cdot \nabla \chi_s + \tilde{h}_{s2} w_{s,x_1} - \tilde{h}_{s1} w_{s,x_2} - \tilde{h}_{s3} \chi_s \\
&\qquad - \nu \Delta \chi_s = F_3 && \text{in } \Omega^T,\\
&\chi_s = \sum_{i=1}^{2} v_{si} a_i \equiv \chi_{s*} && \text{on } S_1^T,\\
&\chi_{s,x_3} = 0 && \text{on } S_2^T,\\
&\chi_s|_{t=0} = \chi(0) && \text{in } \Omega,
\end{aligned}
\tag{5.7}
$$

where $s = 1,2$, and a_i, $i = 1,2$, are defined by $(3.3)_2$.

Next we have the elliptic problem

$$
\begin{aligned}
&v_{s,2x_1} - v_{s1,x_2} = \chi_s && \text{in } \Omega',\\
&v_{s1,x_1} + v_{s2,x_2} = -h_{s3} && \text{in } \Omega',\\
&v_s' \cdot n' = 0 && \text{on } S_1',
\end{aligned}
\tag{5.8}
$$

where $s = 1,2$, Ω' and S_1' are cross-sections of Ω and S_1 with a plane perpendicular to the x_3 axis, and $n' = (n_1, n_2, 0)$.

First we examine problem (5.7). Let $\tilde{\chi}_s$ solve the problem

$$
\begin{aligned}
&\tilde{\chi}_{s,t} - \nu \Delta \tilde{\chi}_s = 0 && \text{in } \Omega^T,\\
&\tilde{\chi}_s = \tilde{\chi}_{s*} && \text{on } S_1^T,\\
&\tilde{\chi}_{s,x_3} = 0 && \text{on } S_2^T,\\
&\tilde{\chi}_s|_{t=0} = 0 && \text{on } S_2^T,
\end{aligned}
\tag{5.9}
$$

where $s=1,2$. Then $\chi_s'=\chi_s-\tilde\chi_s$, $s=1,2$, is a solution to the problem

$$
(5.10)\qquad
\begin{aligned}
&\chi'_{s,t}+v_s\cdot\nabla\chi_s'-\tilde h_{s3}\chi_s'+h_{s2}w_{s,x_1}-\tilde h_{s1}w_{s,x_2}-\nu\Delta\chi_s' \\
&\qquad=F_3-v_s\cdot\nabla\chi_s'+\tilde h_{s3}\tilde\chi_s && \text{in } \Omega^T,\\
&\chi_s'=0 && \text{on } S_1^T,\\
&\chi'_{s,x_3}=0 && \text{on } S_1^T,\\
&\chi_s'|_{t=0}=\chi'(0) && \text{in } \Omega.
\end{aligned}
$$

Since we are looking for a solution which is a regularization of a weak solution we use the energy type estimate for the weak solution

$$
(5.11)\qquad |v|_{2,\infty,\Omega^t}+|\nabla v|_{2,\Omega^t}\le l_1(t),\quad t\le T.
$$

Repeating the considerations leading to (4.19) we obtain

$$
(5.12)\qquad
\begin{aligned}
\|v_s\|_{2,5/2,\Omega^t}\le c(&(A'(t)+d_2(t))(B(t)+A'(t)+d_2(t))\\
&+\|v(0)\|_{1,5/2,\Omega}+\|f\|_{5/2,\Omega^t}+l_1(t)+\|d\|_{8/5,5/2,S_2^t}),
\end{aligned}
$$

where h is replaced by $\tilde h_s$. In view of (5.12) we have

$$
(5.13)\qquad \|v_s\|_{2,5/2,\Omega^t}\le\varphi(\|\tilde h\|_{\mathcal M(\Omega^t)},\gamma_1(t),\gamma_2(0)),
$$

where

$$
\begin{aligned}
\gamma_1(t)&=l_1(t)+\|d\|_{8/5,5/2,S_2^t}+|f|_{5/2,\Omega^t}+|F_3|_{10/3,\Omega^t},\\
\gamma_2(0)&=\|v(0)\|_{1,5/2,\Omega^t}+|\chi(0)|_{2,\Omega}
\end{aligned}
$$

and φ is an increasing positive function. By problem (5.2), from Lemma 4.5 and (4.20) we get

$$
(5.14)\qquad \|h_s\|_{\mathcal M(\Omega^t)}\le\varphi(\|\tilde h_s\|_{\mathcal M(\Omega^t)},G(t))l_1(t)+G'(t),
$$

where

$$
\begin{aligned}
G&=l_1(t)+\|d\|_{8/5,5/2,S_2^t}+\|v(0)\|_{1,5/2,\Omega}+|f|_{5/2,\Omega^t}+|F_3|_{10/3,\Omega^t}\\
&\quad+|\chi(0)|_{2,\Omega},\\
G'&=|g|_{2,\Omega^t}+\|h(0)\|_{1,2,\Omega}+l_1(t)+\|d_{x'}\|_{3/2,2,S_2^t}
\end{aligned}
$$

and φ is an increasing positive function.

Hence the mapping ϕ transforms bounded sets in $\mathcal M(\Omega^T)$ into bounded sets in $\mathcal M(\Omega^T)$.

Now we show the uniform continuity of ϕ. For this purpose we introduce

$$
\begin{aligned}
H&=h_1-h_2, &\quad V&=v_1-v_2,\\
Q&=q_1-q_2, &\quad K&=\chi_1-\chi_2.
\end{aligned}
$$

Then H satisfies

$$
(5.15)\quad
\begin{aligned}
&H_{,t} - \operatorname{div} \mathbb{T}(H,Q) = -V\cdot\nabla\tilde{h}_1 - v_2\cdot\nabla\tilde{H} \\
&\qquad\qquad - \tilde{H}\cdot\nabla v_1 - \tilde{h}_2\cdot\nabla V && \text{in } \Omega^T,\\
&\operatorname{div} H = 0 && \text{in } \Omega^T,\\
&H\cdot n = 0 && \text{on } S_1^T,\\
&\nu n\cdot\mathbb{D}(H)\cdot\tau_\alpha + \gamma H\cdot\tau_\alpha = 0, \quad \alpha=1,2, && \text{on } S_1^T,\\
&H_i = 0, \quad i=1,2, && \text{on } S_2^T,\\
&H_{3,x_3} = 0 && \text{on } S_2^T,\\
&H|_{t=0} = 0 && \text{in } \Omega.
\end{aligned}
$$

For solutions to problem (5.15) we obtain

$$
(5.16)\quad
\begin{aligned}
\|H\|_{2,2,\Omega^t} + |\nabla Q|_{2,\Omega^t} &\le c(|V\cdot\tilde{h}_1|_{2,\Omega^t} + |v_2\cdot\nabla\tilde{H}|_{2,\Omega^t}\\
&+ |\tilde{H}\cdot\nabla v_1|_{2,\Omega^t} + |\tilde{h}_2\cdot\nabla V|_{2,\Omega^t} + \|\gamma\tilde{H}\cdot\tau_\alpha\|_{1/2,2,S_1^t}).
\end{aligned}
$$

Assume that h_s, $s=1,2$, belong to a bounded set in $\mathcal{M}(\Omega^t)$. Hence there exists a constant A such that

$$
(5.17)\qquad \|\tilde{h}_s\|_{2,2,\Omega^t} \le A, \quad s=1,2.
$$

Then from (5.16) we obtain

$$
(5.18)\qquad \|H\|_{2,2,\Omega^t} + |\nabla Q|_{2,\Omega^t} \le \varphi(A)(\|V\|_{2,5/2,\Omega^t} + \|\tilde{H}\|_{\mathcal{M}(\Omega^t)}),
$$

where $t\le T$ and φ is an increasing positive function.

To show the continuity of ϕ we have to find an estimate for $\|V\|_{2,5/2,\Omega^t}$. For this purpose we consider the problem

$$
(5.19)\quad
\begin{aligned}
&V_t - \operatorname{div}\mathbb{T}(V,Q) = -V'\cdot\nabla v_1 - v_2'\cdot\nabla V - Wh_1 - w_2 H && \text{in } \Omega^T,\\
&\operatorname{div} V = 0 && \text{in } \Omega^T,\\
&V\cdot n = 0 && \text{on } S^T,\\
&n\cdot\mathbb{T}(v,Q)\cdot\tau_\alpha + \gamma V\cdot\tau_\alpha = 0, \quad \alpha=1,2, && \text{on } S^T,\\
&V|_{t=0} = 0 && \text{in } \Omega,
\end{aligned}
$$

where $V'=(V_1,V_2)$, $W=V_3$, $v_s'=(v_{s1},v_{s2})$, $w_s=v_{s3}$, $s=1,2$. For solutions of (5.19) we have

$$
(5.20)\quad
\begin{aligned}
\|V\|_{2,5/2,\Omega^t} + |\nabla Q|_{2,\Omega^t} &\le c(|V'\cdot\nabla v_1|_{5/2,\Omega^t} + |v_2'\cdot\nabla V|_{5/2,\Omega^t}\\
&+ |Wh_1|_{5/2,\Omega^t} + |w_2 H|_{5/2,\Omega^t}).
\end{aligned}
$$

We bound the first term on the r.h.s. by

$$
\begin{aligned}
|V'\cdot\nabla v_1|_{5/2,\Omega^t} &\le |V|_{10,\Omega^t}|\nabla v_1|_{10/3,\Omega^t}\\
&\le \varepsilon\|V\|_{2,5/2,\Omega^t} + \varphi(\|v\|_{2,5/2,\Omega^t})|V|_{2,\Omega^t}.
\end{aligned}
$$

The second term on the r.h.s. of (5.20) is estimated by

$$|v_2' \cdot \nabla V|_{5/2,\Omega^t} \le |v_2'|_{8,\Omega^t} |\nabla V|_{40/11,\Omega^t} \le \varepsilon \|V\|_{2,5/2,\Omega^t} + \varphi(\|v\|_{2,5/2,\Omega^t}) |V|_{2,\Omega^t}.$$

The third term on the r.h.s. of (5.20) is estimated by

$$|Wh_1|_{5/2,\Omega^t} \le |W|_{5,\Omega^t} |h_1|_{5,\Omega^t} \le \varepsilon \|V\|_{2,5/2,\Omega^t} + \varphi(\|h\|_{2,2,\Omega^t}) |V|_{2,\Omega^t}.$$

The last term on the r.h.s. of (5.20) is estimated by

$$|w_2 H|_{5/2,\Omega^t} \le |w_2|_{10,\Omega^t} |H|_{10/3,\Omega^t} \le \|v\|_{5/2,2,\Omega^t} |H|_{10/3,\Omega^t}.$$

Utilizing the above estimates in (5.20) we obtain

$$\|V\|_{2,5/2,\Omega^t} + |\nabla Q|_{5/2,\Omega^t} \le \varphi(A)(|V|_{2,\Omega^t} + |H|_{10/3,\Omega^t}). \tag{5.21}$$

Now we have to estimate the r.h.s. of (5.21) in terms of $\tilde{H}$. Multiplying $(5.19)_1$ by V and integrating over Ω yields

$$\frac{1}{2}\frac{d}{dt}|V|^2_{2,\Omega} + \nu \|V\|^2_{1,\Omega} \le c(|\nabla v_1|^2_{3,\Omega} + |h_1|^2_{3,\Omega})|V|^2_{2,\Omega} + |w_2|^2_{3,\Omega}|H|^2_{2,\Omega}). \tag{5.22}$$

Multiplying $(5.15)_1$ by H and integrating over Ω yields

$$\frac{1}{2}\frac{d}{dt}|H|^2_{2,\Omega} + \nu\|H\|^2_{1,\Omega} \le c(|\nabla \tilde{h}_1|^2_{3,\Omega}|V|^2_{2,\Omega} + |v_2|^2_{3,\Omega}|\nabla \tilde{H}|^2_{2,\Omega} + |\nabla v_1|^2_{3,\Omega}|\tilde{H}|^2_{2,\Omega} + \sup_t |\tilde{h}_2|^2_{3,\Omega} |\nabla V|^2_{2,\Omega}). \tag{5.23}$$

Adding (5.22) and (5.23) gives

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}(c_1|V|^2_{2,\Omega} + |H|^2_{2,\Omega}) &+ \nu\bigg(\frac{c_1}{2}\|V\|^2_{1,\Omega} + \|H\|^2_{1,\Omega}\bigg) \\ &\le c((|\nabla v_1|^2_{3,\Omega} + |\tilde{h}_1|^2_{3,\Omega} + |\nabla\tilde{h}_1|^2_{3,\Omega})|V|^2_{2,\Omega} + |w_2|^2_{3,\Omega}|H|^2_{2,\Omega} \\ &\quad + |v_2|^2_{3,\Omega}|\nabla\tilde{H}|^2_{2,\Omega} + |\nabla v_1|^2_{3,\Omega}|\tilde{H}|^2_{2,\Omega}), \end{aligned} \tag{5.24}$$

where $c_1/2 \ge \sup_t |\tilde{h}_2|^2_{3,\Omega}$.

Integrating (5.24) with respect to time yields

$$\begin{aligned} |V(t)|^2_{2,\Omega} + |H(t)|^2_{2,\Omega} &+ \nu\int_0^t (\|V(t')\|^2_{1,\Omega} + \|H(t')\|^2_{1,\Omega})\,dt' \\ &\le c\exp c(|\nabla v_1|^2_{3,2,\Omega^t} + |h_1|^2_{3,2,\Omega^t} + |\nabla h_1|^2_{3,2,\Omega^t} + |v_2|^2_{3,2,\Omega^t}) \\ &\quad \cdot (|v_2|^2_{3,\infty,\Omega^t}|\nabla\tilde{H}|^2_{2,\Omega^t} + |v_1|^2_{3,2,\Omega^t}|\tilde{H}|^2_{2,\infty,\Omega^t}) \equiv J. \end{aligned} \tag{5.25}$$

By imbedding theorems we have

$$\begin{aligned} J \le c\exp c(\|v_1\|^2_{2,5/2,\Omega^t} + \|v_2\|^2_{2,5/2,\Omega^t} + \|h_1\|^2_{2,2,\Omega^t}) \\ \cdot (\|v_2\|^2_{2,5/2,\Omega^t}|\nabla\tilde{H}|^2_{2,\Omega^t} + \|v_1\|^2_{2,5/2,\Omega^t}|\tilde{H}|^2_{2,\infty,\Omega^t}) \equiv J_1. \end{aligned}$$

By (5.17) we obtain

$$J_1 \le \varphi(A)(|\nabla\tilde{H}|^2_{2,\Omega^t} + |\tilde{H}|^2_{2,\infty,\Omega^t}).$$

Hence (5.25) takes the form

$$\|V\|_{V_2^0(\Omega^t)} + \|H\|_{V_2^0(\Omega^t)} \le \varphi(A)(|\nabla\tilde{H}|_{2,\Omega^t} + |\tilde{H}|_{2,\infty,\Omega^t}). \tag{5.26}$$

Finally, from (5.18), (5.21) and (5.26) we obtain

$$\|H\|_{\mathcal{M}(\Omega^T)} \le \varphi(A)\|\tilde{H}\|_{\mathcal{M}(\Omega^T)}.$$

This implies the continuity of ϕ and ends the proof.

Finally, by the Leray–Schauder fixed point theorem we deduce Theorem 1.1 (local existence).

6. Global existence

THEOREM 6.1. *Let the assumptions of Theorem* 1.1 *be satisfied. Then there exists a sequence* $\{t_i\}_{i=0}^{\infty}$ *increasing to infinity such that the local solution determined by Theorem* 1.1 *exists in each interval* $[t_i, t_{i+1}]$, $i = 0, 1, \ldots,$ *where* $t_0 = 0$.

Proof. Assume that we have proved the existence of a local solution with sufficiently large existence time T. Then

$$\int_0^T |v(t)|_{2,\Omega}^2\,dt \le c, \quad \int_0^T \|v(t)\|_{2,5/2,\Omega}^{5/2}\,dt \le c,$$
$$\int_0^T |h(t)|_{2,\Omega}^2\,dt \le c, \quad \int_0^T \|h(t)\|_{2,2,\Omega}^2\,dt \le c.$$

Then there exists $T_* < T$ sufficiently large and there exists $t_* \in [T_*, T]$ such that

$$|v(t_*)|_{2,\Omega}, \quad |h(t_*)|_{2,\Omega}, \quad \|v(t_*)\|_{2,5/2,\Omega}, \quad \|h(t_*)\|_{2,2,\Omega}$$

are so small that

$$|v(t_*)|_{2,\Omega} \le |v(0)|_{2,\Omega},$$
$$|h(t_*)|_{2,\Omega} \le |h(0)|_{2,\Omega},$$
$$\|v(t_*)\|_{6/5,5/2,\Omega} \le \|v(0)\|_{6/5,5/2,\Omega},$$
$$\|h(t_*)\|_{1,2,\Omega} \le \|h(0)\|_{1,2,\Omega}.$$

Then we can prove the existence of local solutions in $[T, t_* + T]$. Hence the global existence follows.

References

[1] W. Alame, *On the existence of solutions for the nonstationary Stokes system with slip boundary conditions in general Sobolev–Slobodetskii and Besov spaces*, in: Banach Center Publ. 70, Inst. Math., Polish Acad. Sci., 2005, 21–49.

[2] O. V. Besov, V. P. Il'in and S. M. Nikol'skiĭ, *Integral Representations of Functions and Imbedding Theorems*, Nauka, Moscow, 1975 (in Russian).
[3] T. Kobayashi and W. M. Zajączkowski, *On global motion of a compressible barotropic visous fluid with boundary slip conditions*, Appl. Math. (Warsaw) 26 (1999), 159–194.
[4] O. A. Ladyzhenskaya and V. A. Solonnikov, *On some problems of vector analysis and generalized boundary value problems for the Navier–Stokes equations*, Zap. Nauchn. Sem. LOMI 59 (1976), 81–116 (in Russian).
[5] O. A. Ladyzhenskaya, V. A. Solonnikov and N. N. Uraltseva, *Linear and Quasilinear Equations of Parabolic Type*, Nauka, Moscow, 1967 (in Russian).
[6] W. M. Zajączkowski, *Long time existence of regular solutions to Navier–Stokes equations in cylindrical domains under boundary slip conditions*, Studia Math. 169 (2005), 243–285.
[7] —, *Global existence of axially symmetric solutions to Navier–Stokes equations with large angular component of velocity*, Colloq. Math. 100 (2004), 243–263.
[8] —, *Global special regular solutions to the Navier–Stokes equations in a cylindrical domain without the axis of symmetry*, Topol. Methods Nonlinear Anal. 24 (2004), 60–105.
[9] —, *Global regular nonstationary flow for the Navier–Stokes equations in a cylindrical pipe*, ibid. 26 (2005), 221–286.

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