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ON THE RATIO OF GAMMA AND RAYLEIGH RANDOM VARIABLES

Abstract. The gamma and Rayleigh distributions are two of the most applied distributions in engineering. Motivated by engineering issues, the exact distribution of the quotient X/Y is derived when X and Y are independent gamma and Rayleigh random variables. Tabulations of the associated percentage points and a computer program for generating them are also given.

1. Introduction. The gamma and Rayleigh distributions are two of the most applied distributions in engineering. There are many real situations where measurements could be modeled by these distributions. Some examples are:

1. in communication theory, X and Y could represent the random noise corresponding to two signals;
2. in ocean engineering, X and Y could represent distributions of navigation errors;
3. in image and speech recognition, X and Y could represent "input" distributions;
4. in chemical engineering, X and Y could represent the remission times of two chemicals when they are administered to two kinds of mechanical systems;
5. in civil engineering, X and Y could represent future observations on the strength of an engineering design (e.g. the strength of a bridge);
6. in hydrology, X and Y could represent the extreme rainfall at two stations.

In each of the examples above, it will be of interest to study the distribution of the quotient X/Y. For example, in communication theory, X/Y

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could represent the relative strength of the two different signals. In ocean engineering, X/Y could represent the relative safety of navigation. In mechanical engineering, X/Y could represent the relative effectiveness of the two chemicals. In civil engineering, X/Y could represent some measure of reliability of the engineering design. In hydrology, X/Y could represent the relative extremity of rainfall at the two stations.

The distribution of the quotient X/Y has been studied by several authors especially when X and Y are independent random variables and come from the same family. For instance, see Marsaglia (1965) and Korhonen and Narula (1989) for normal family, Press (1969) for Student's t family, Basu and Lochner (1971) for Weibull family, Shcolnick (1985) for stable family, Hawkins and Han (1986) for non-central chi-squared family, Provost (1989) for gamma family, and Pham-Gia (2000) for beta family.

However, there is relatively little work of the above kind when X and Y belong to different families. In this note, we study the exact distribution of X/Y when X and Y are independent random variables having the gamma and Rayleigh distributions specified by the probability density functions (pdfs)

$$f_X(x) = \frac{\mu^\alpha x^{\alpha-1} \exp(-\mu x)}{\Gamma(\alpha)} \tag{1}$$

and

$$f_Y(y) = 2\lambda^2 y \exp\left\{-(\lambda y)^2\right\}, \tag{2}$$

respectively, for $x > 0$, $y > 0$, $\alpha > 0$, $\lambda > 0$, and $\mu > 0$.

The results of this note are organized as follows: exact expressions for the pdf and the cumulative distribution function (cdf) of X/Y are given in Section 2; moment properties of X/Y including its characteristic function and moments are considered in Section 3; finally, tabulations of the percentile points of X/Y obtained by inverting the derived cdf are provided in Section 4.

The calculations of this note involve several special functions, including the error function defined by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2)\, dt,$$

the complementary error function defined by

$$\operatorname{erfc}(x) = 1 - \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2)\, dt,$$

the parabolic cylinder function defined by

$$D_p(x) = \frac{\exp(-x^2/4)}{\Gamma(-p)} \int_0^\infty \exp\{-(tx + t^2/2)\} t^{-(p+1)}\, dt,$$

the Kummer function defined by

$$\Psi(a, b; x) = \frac{1}{\Gamma(a)} \int_0^\infty \exp(-xt) t^{a-1} (1+t)^{b-a-1}\, dt,$$

the confluent hypergeometric function defined by

$${}_1F_1(a; b; x) = \sum_{k=0}^\infty \frac{(a)_k}{(b)_k} \frac{x^k}{k!},$$

the ${}_2F_2$ hypergeometric function defined by

$${}_2F_2(a, b; c, d; x) = \sum_{k=0}^\infty \frac{(a)_k (b)_k}{(c)_k (d)_k} \frac{x^k}{k!},$$

the incomplete gamma function defined by

$$\Gamma(a, x) = \int_x^\infty \exp(-t) t^{a-1}\, dt,$$

and the modified Bessel function of the first kind defined by

$$I_\nu(x) = \frac{x^\nu}{2^\nu \Gamma(\nu+1)} \sum_{k=0}^\infty \frac{1}{(\nu+1)_k k!} \left(\frac{x^2}{4}\right)^k,$$

where $(e)_k = e(e+1)\cdots(e+k-1)$ denotes the ascending factorial. The properties of the above special functions can be found in Prudnikov *et al.* (1986) and Gradshteyn and Ryzhik (2000).

2. Exact distribution of X/Y. The following theorem expresses the cdf of X/Y in terms of the confluent hypergeometric function.

THEOREM. *Suppose X and Y are independent random variables distributed according to* (1) *and* (2), *respectively. The cdf of $Z = X/Y$ can be expressed as*

$$(3) \qquad F(z) = \frac{\mu^\alpha z^\alpha}{\alpha \lambda^\alpha \Gamma(\alpha)} \Gamma\left(\frac{\alpha+2}{2}\right) {}_1F_1\left(\frac{\alpha}{2}; \frac{1}{2}; \frac{\mu^2 z^2}{4\lambda^2}\right) - \frac{\mu^{\alpha+1} z^{\alpha+1} \Gamma((\alpha+3)/2)}{(\alpha+1)\lambda^{\alpha+1}\Gamma(\alpha)} {}_1F_1\left(\frac{\alpha+1}{2}; \frac{3}{2}; \frac{\mu^2 z^2}{4\lambda^2}\right)$$

for $z > 0$. *The corresponding pdf is*

$$f(z) = \alpha(\alpha+1) 2^{-\alpha/2} \lambda^{-\alpha} \mu^{\alpha} z^{\alpha-1} \exp\left(\frac{\mu^2 z^2}{8\lambda^2}\right) D_{-2-\alpha}\left(\frac{\mu z}{\sqrt{2}\lambda}\right) \tag{4}$$

for $z > 0$. *If* α *is an integer then*

$$f(z) = -\frac{\sqrt{\pi}\lambda(-\mu)^{\alpha} z^{\alpha-1}}{\Gamma(\alpha)} \frac{\partial^{\alpha+1}}{\partial q^{\alpha+1}} \left[\exp\left(\frac{q^2}{4\lambda^2}\right) \operatorname{erfc}\left(\frac{q}{2\lambda}\right)\right]\bigg|_{q=\mu z} \tag{5}$$

for $z > 0$.

Proof. The cdf corresponding to (1) is $1 - \Gamma(\alpha, \mu x)/\Gamma(\alpha)$. Thus, one can write the cdf of X/Y as

$$\begin{aligned} \Pr(X/Y \leq z) &= \int_0^{\infty} F_X(zy) f_Y(y)\, dy \\ &= 1 - \frac{2\lambda^2}{\Gamma(\alpha)} \int_0^{\infty} \Gamma(\alpha, \mu y z) y \exp(-\lambda^2 y^2)\, dy \\ &= 1 - \frac{2\lambda^2}{\Gamma(\alpha)} I. \end{aligned} \tag{6}$$

Application of equation (2.10.3.9) in Prudnikov *et al.* (1986, Vol. 2) shows that the integral I can be calculated as

$$\begin{aligned} I = {} & \frac{\Gamma(\alpha)}{2\lambda^2} - \frac{(\mu z)^{\alpha}}{2\alpha\lambda^{\alpha+2}} \Gamma\left(\frac{\alpha+2}{2}\right) {}_2F_2\left(\frac{\alpha}{2}, \frac{\alpha+2}{2}; \frac{1}{2}, \frac{\alpha}{2}+1; \frac{\mu^2 z^2}{4\lambda^2}\right) \\ & + \frac{(\mu z)^{\alpha+1}}{2(\alpha+1)\lambda^{\alpha+3}} \Gamma\left(\frac{\alpha+3}{2}\right) {}_2F_2\left(\frac{\alpha+1}{2}, \frac{\alpha+3}{2}; \frac{3}{2}, \frac{\alpha+3}{2}; \frac{\mu^2 z^2}{4\lambda^2}\right). \end{aligned} \tag{7}$$

Note that the two hypergeometric terms in (7) simplify as

$${}_2F_2\left(\frac{\alpha}{2}, \frac{\alpha+2}{2}; \frac{1}{2}, \frac{\alpha}{2}+1; \frac{\mu^2 z^2}{4\lambda^2}\right) = {}_1F_1\left(\frac{\alpha}{2}; \frac{1}{2}; \frac{\mu^2 z^2}{4\lambda^2}\right)$$

and

$${}_2F_2\left(\frac{\alpha+1}{2}, \frac{\alpha+3}{2}; \frac{3}{2}, \frac{\alpha+3}{2}; \frac{\mu^2 z^2}{4\lambda^2}\right) = {}_1F_1\left(\frac{\alpha+1}{2}; \frac{3}{2}; \frac{\mu^2 z^2}{4\lambda^2}\right).$$

The result in (3) follows by substituting (7) into (6). The pdf of X/Y in (4) can be obtained by writing

$$\begin{aligned} f(z) &= \int_0^{\infty} y f_X(zy) f_Y(y)\, dy \\ &= \frac{2\lambda^2 \mu^{\alpha} z^{\alpha-1}}{\Gamma(\alpha)} \int_0^{\infty} y^{\alpha+1} \exp(-\lambda^2 y^2 - \mu z y)\, dy \end{aligned} \tag{8}$$

and applying equation (2.3.15.3) in Prudnikov *et al.* (1986, Vol. 1) to the integral in (8). The result in (5) follows by special properties of the parabolic cylinder function. ∎

Using special properties of the hypergeometric functions, one can derive simpler forms for (3). This is illustrated in the corollaries below for integer and half-integer values of α. Note that the forms of (3) involve the error function for integer α and the Bessel function for half-integer α.

COROLLARY 1. *If* $\alpha = 1, 2, 3, 4, 5$ *then* (3) *can be reduced to the simpler forms*

$$
\begin{aligned}
F(z) &= \sqrt{\pi x}\exp(x)\{1 - \operatorname{erf}(\sqrt{x})\},\\
F(z) &= (x/\lambda)\{2\lambda + \mu z\sqrt{\pi}\exp(x)\operatorname{erf}(\sqrt{x}) + \mu z\sqrt{\pi}\exp(x)\},\\
F(z) &= (x^{3/2}/(2\lambda^2))\{2\sqrt{\pi}\exp(x)\lambda^2 + \sqrt{\pi}\exp(x)\mu^2 z^2 + 2\mu z\lambda\\
&\qquad + 2\sqrt{\pi}\exp(x)\operatorname{erf}(\sqrt{x})\lambda^2 + \sqrt{\pi}\exp(x)\operatorname{erf}(\sqrt{x})\mu^2 z^2\},\\
F(z) &= (x^2/(6\lambda^3))\{8\lambda^3 + 2\mu^2 z^2\lambda + 6\mu z\sqrt{\pi}\exp(x)\operatorname{erf}(\sqrt{x})\lambda^2\\
&\qquad + \mu^3 z^3\sqrt{\pi}\exp(x)\operatorname{erf}(\sqrt{x}) + 6\mu z\sqrt{\pi}\exp(x)\lambda^2 + \mu^3 z^3\sqrt{\pi}\exp(x)\},\\
F(z) &= (x^{5/2}/(24\lambda^4))\{12\sqrt{\pi}\exp(x)\mu^2 z^2\lambda^2 + \sqrt{\pi}\exp(x)\mu^4 z^4\\
&\qquad + 12\sqrt{\pi}\exp(x)\lambda^4 + 20\mu z\lambda^3 + 2\mu^3 z^3\lambda + 12\sqrt{\pi}\exp(x)\operatorname{erf}(\sqrt{x})\mu^2 z^2\lambda^2\\
&\qquad + \sqrt{\pi}\exp(x)\operatorname{erf}(\sqrt{x})\mu^4 z^4 + 12\sqrt{\pi}\exp(x)\operatorname{erf}(\sqrt{x})\lambda^4\},
\end{aligned}
$$

respectively, where $x = \mu^2 z^2/(4\lambda^2)$.

COROLLARY 2. *If* $\alpha = 1/2, 3/2, 5/2, 7/2, 9/2$ *then* (3) *can be reduced to the simpler forms*

$$
\begin{aligned}
F(z) &= \sqrt{\pi x}\exp(x)\{I_{-1/4}(x) + I_{1/4}(x)\},\\
F(z) &= 4\sqrt{\pi}x^{3/2}\exp(x)\{I_{1/4}(x) + I_{-3/4}(x) + I_{-1/4}(x) + I_{3/4}(x)\},\\
F(z) &= (4\sqrt{\pi}/(3\lambda^2))x^{3/2}\exp(x)\{\mu^2 z^2 I_{-1/4}(x) + 2\lambda^2 I_{-1/4}(x) + \mu^2 z^2 I_{3/4}(x)\\
&\qquad + \mu^2 z^2 I_{1/4}(x) + 2\lambda^2 I_{1/4}(x) + \mu^2 z^2 I_{-3/4}(x)\},\\
F(z) &= (32\sqrt{\pi}/(15\lambda^2))x^{5/2}\exp(x)\{\mu^2 z^2 I_{1/4}(x) + 5\lambda^2 I_{1/4}(x) + \mu^2 z^2 I_{-3/4}(x)\\
&\qquad + 3\lambda^2 I_{-3/4}(x) + \mu^2 z^2 I_{-1/4}(x) + 5\lambda^2 I_{-1/4}(x) + \mu^2 z^2 I_{3/4}(x)\\
&\qquad + 3\lambda^2 I_{3/4}(x)\},\\
F(z) &= (32\sqrt{\pi}/(105\lambda^4))x^{5/2}\exp(x)\{10\mu^2 z^2\lambda^2 I_{-1/4}(x) + \mu^4 z^4 I_{-1/4}(x)\\
&\qquad + 10\lambda^4 I_{-1/4}(x) + 8\mu^2 z^2\lambda^2 I_{3/4}(x) + \mu^4 z^4 I_{3/4}(x) + 10\mu^2 z^2 I_{1/4}(x)\\
&\qquad + \mu^4 z^4 I_{1/4}(x) + 10\lambda^4 I_{1/4}(x) + 8\mu^2 z^2\lambda^2 I_{-3/4}(x) + \mu^4 z^4 I_{-3/4}(x)\},
\end{aligned}
$$

respectively, where $x = \mu^2 z^2/(8\lambda^2)$.

Figure 1 illustrates possible shapes of the pdf of X/Y for selected values of α. As expected, the densities are unimodal and the effect of the parameter is evident.

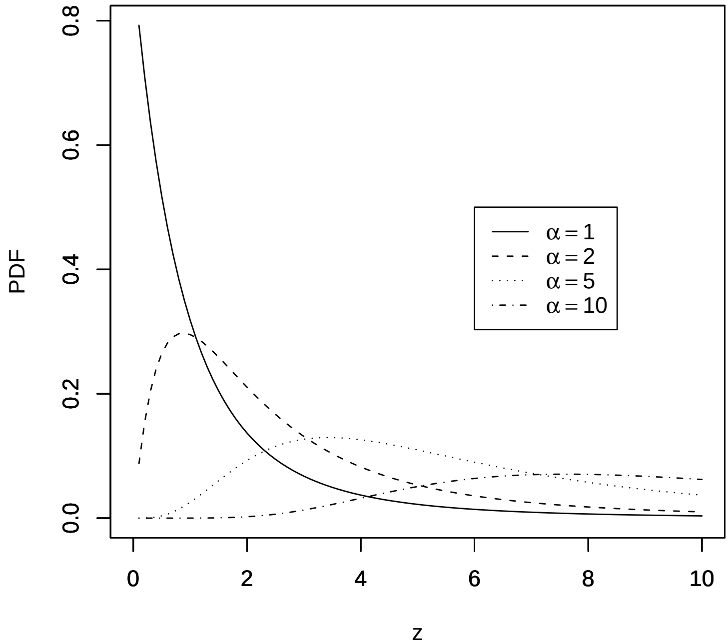

Fig. 1. Plots of the pdf of (3) for $\lambda = 1$, $\mu = 1$ and $\alpha = 1, 2, 5, 10$

3. Moment properties of X/Y. The moment properties of X/Y can be derived by knowing the same for X and Y. It is well known (see, for example, Johnson *et al.* (1994, 1995)) that

$$E(X^n) = \frac{\Gamma(n+\alpha)}{\mu^n \Gamma(\alpha)} \quad \text{and} \quad E(Y^n) = \frac{\Gamma(1+n/2)}{\lambda^n}.$$

Thus, the nth moment of $Z = X/Y$ is

$$E(Z^n) = \frac{\lambda^n \Gamma(n+\alpha)\Gamma(1-n/2)}{\mu^n \Gamma(\alpha)}.$$

In particular,

$$E(Z) = \frac{\sqrt{\pi}\,\lambda\alpha}{\mu}.$$

Note that moments of even order do not exist. Using the fact that the characteristic function (chf) of X is

$$E[\exp(itX)] = \left(\frac{\mu}{\mu - it}\right)^{\alpha},$$

where $i = \sqrt{-1}$ denotes the complex unit, the chf of X/Y can be expressed as

$$
\begin{aligned}
(9)\qquad E[\exp(itX/Y)] &= 2\lambda^2 \int_0^\infty \left(\frac{\mu}{\mu - it/y}\right)^\alpha y \exp\{-(\lambda y)^2\}\, dy \\
&= 2\lambda^2 \int_0^\infty \frac{y^{\alpha+1} \exp\{-(\lambda y)^2\}}{(y - it/\mu)^\alpha}\, dy \\
&= 2\lambda^2 \int_0^\infty \frac{y^{\alpha+1} (y + it/\mu)^\alpha \exp\{-(\lambda y)^2\}}{(y^2 + t^2/\mu^2)^\alpha}\, dy.
\end{aligned}
$$

If α is an integer then (9) can be simplified as

$$
\begin{aligned}
E[\exp(itX/Y)] &= 2\lambda^2 \sum_{k=0}^{\alpha} \binom{\alpha}{k} \left(\frac{it}{\mu}\right)^{\alpha-k} \int_0^\infty \frac{y^{\alpha+k+1} \exp\{-(\lambda y)^2\}}{(y^2 + t^2/\mu^2)^\alpha}\, dy \\
&= \lambda^2 \sum_{k=0}^{\alpha} \binom{\alpha}{k} \left(\frac{it}{\mu}\right)^{\alpha-k} \int_0^\infty \frac{x^{(\alpha+k)/2} \exp\{-\lambda^2 x\}}{(x + t^2/\mu^2)^\alpha}\, dx \\
&= \frac{\lambda^2 t^2}{\mu^2} \sum_{k=0}^{\alpha} \binom{\alpha}{k} i^{\alpha-k} \Gamma\left(\frac{\alpha+k}{2} + 1\right) \Psi\left(\frac{\alpha+k}{2} + 1, \frac{k-\alpha}{2} + 2; \frac{\lambda^2 t^2}{\mu^2}\right),
\end{aligned}
$$

where the last step follows by equation (2.3.6.9) in Prudnikov *et al.* (1986, Vol. 1).

4. Percentiles of X/Y. In this section, we provide tabulations of percentage points z_p associated with the cdf of $Z = X/Y$. These values are obtained by numerically solving the equation

$$
\frac{\mu^\alpha z_p^\alpha}{\alpha \lambda^\alpha \Gamma(\alpha)} \Gamma\left(\frac{\alpha+2}{2}\right) {}_1F_1\left(\frac{\alpha}{2}; \frac{1}{2}; \frac{\mu^2 z_p^2}{4\lambda^2}\right) - \frac{\mu^{\alpha+1} z_p^{\alpha+1} \Gamma((\alpha+3)/2)}{(\alpha+1)\lambda^{\alpha+1} \Gamma(\alpha)} {}_1F_1\left(\frac{\alpha+1}{2}; \frac{3}{2}; \frac{\mu^2 z_p^2}{4\lambda^2}\right) = p.
$$

Table 1. Percentage points of $Z = X/Y$

α	$p = 0.01$	$p = 0.05$	$p = 0.1$	$p = 0.9$	$p = 0.95$	$p = 0.99$
0.1	0.000	0.000	0.000	0.365	0.847	2.908
0.2	0.000	0.000	0.000	0.867	1.612	4.573
0.3	0.000	0.000	0.000	1.310	2.253	5.960
0.4	0.000	0.001	0.003	1.717	2.834	7.223
0.5	0.000	0.002	0.010	2.102	3.380	8.416
0.6	0.000	0.007	0.022	2.471	3.903	9.565
0.7	0.001	0.014	0.039	2.829	4.410	10.685

Table 1 (cont.)

α	$p = 0.01$	$p = 0.05$	$p = 0.1$	$p = 0.9$	$p = 0.95$	$p = 0.99$
0.8	0.003	0.026	0.062	3.179	4.906	11.782
0.9	0.007	0.040	0.089	3.522	5.394	12.864
1	0.011	0.058	0.121	3.861	5.875	13.933
1.1	0.018	0.079	0.155	4.196	6.350	14.992
1.2	0.026	0.104	0.193	4.528	6.821	16.043
1.3	0.036	0.130	0.233	4.857	7.289	17.088
1.4	0.048	0.159	0.276	5.183	7.754	18.128
1.5	0.061	0.190	0.321	5.508	8.217	19.163
1.6	0.076	0.223	0.367	5.832	8.677	20.194
1.7	0.093	0.258	0.415	6.154	9.136	21.222
1.8	0.111	0.294	0.464	6.475	9.593	22.248
1.9	0.130	0.331	0.514	6.794	10.048	23.270
2	0.151	0.370	0.565	7.113	10.503	24.291
2.1	0.173	0.410	0.618	7.431	10.957	25.310
2.2	0.196	0.451	0.671	7.748	11.409	26.328
2.3	0.220	0.492	0.725	8.065	11.861	27.343
2.4	0.245	0.535	0.779	8.381	12.312	28.358
2.5	0.271	0.578	0.835	8.697	12.763	29.371
2.6	0.298	0.623	0.890	9.012	13.213	30.384
2.7	0.326	0.667	0.947	9.326	13.662	31.395
2.8	0.354	0.713	1.004	9.641	14.111	32.406
2.9	0.383	0.759	1.061	9.954	14.559	33.415
3	0.413	0.806	1.119	10.268	15.007	34.424
3.1	0.444	0.853	1.177	10.581	15.455	35.433
3.2	0.475	0.900	1.236	10.894	15.902	36.441
3.3	0.507	0.948	1.294	11.207	16.349	37.448
3.4	0.539	0.996	1.354	11.519	16.796	38.455
3.5	0.572	1.045	1.413	11.832	17.243	39.461
3.6	0.605	1.094	1.473	12.144	17.689	40.467
3.7	0.638	1.144	1.533	12.456	18.135	41.472
3.8	0.672	1.193	1.593	12.767	18.581	42.477
3.9	0.707	1.243	1.653	13.079	19.026	43.482
4	0.742	1.294	1.714	13.390	19.472	44.486
4.1	0.777	1.344	1.775	13.702	19.917	45.490
4.2	0.812	1.395	1.836	14.013	20.362	46.494
4.3	0.848	1.446	1.897	14.324	20.807	47.498
4.4	0.885	1.497	1.959	14.635	21.252	48.501
4.5	0.921	1.549	2.020	14.945	21.697	49.504
4.6	0.958	1.601	2.082	15.256	22.141	50.507
4.7	0.995	1.653	2.144	15.567	22.586	51.509
4.8	1.032	1.705	2.206	15.877	23.030	52.512
4.9	1.070	1.757	2.268	16.187	23.474	53.514
5	1.108	1.809	2.330	16.498	23.918	54.516

Evidently, this involves computation of the confluent hypergeometric function and routines for this are widely available. We used the function hypergeom ([·],[·],·) in the algebraic manipulation package MAPLE. Table 1 provides the numerical values of z_p for $\lambda = 1$, $\mu = 1$ and $\alpha = 0.1, 0.2, \ldots, 5$.

Tables of this kind will be of use to the practitioners mentioned in Section 1. Similar tabulations could be easily derived for other values of p, λ, μ and α by using the hypergeom (·) function in MAPLE. A sample program is shown in the Appendix below.

Appendix. The following procedure in MAPLE can be used to generate tables similar to that presented in Section 4.

```
percent:=proc(p,lambda,mu,alpha)
local c1,c2,f1,f2,z;
c1:=((mu*z)**alpha)*GAMMA((alpha+2)/2);
c1:=c1/(alpha*(lambda**alpha)*GAMMA(alpha));
c2:=((mu*z)**(alpha+1))*GAMMA((alpha+3)/2);
c2:=c2/((alpha+1)*(lambda**(alpha+1))*GAMMA(alpha));
f1:=hypergeom([alpha/2],[1/2],(mu*z)**2/(4*lambda*lambda));
f2:=hypergeom([(alpha+1)/2],[3/2],(mu*z)**2/(4*lambda*lambda));
fsolve(c1*f1-c2*f2=p,z=0..10000);
end proc;
```

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