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NEW UNIFYING CONVERGENCE CRITERIA FOR NEWTON-LIKE METHODS

Abstract. We present a local and a semilocal analysis for Newton-like methods in a Banach space. Our hypotheses on the operators involved are very general. It turns out that by choosing special cases for the “majorizing” functions we obtain all previous results in the literature, but not vice versa. Since our results give a deeper insight into the structure of the functions involved, we can obtain semilocal convergence under weaker conditions and in the case of local convergence a larger convergence radius.

1. Introduction. In this study we are concerned with the problem of approximating a locally unique solution x^* of the equation

$$(1) \quad F(x) = 0,$$

where F is a Fréchet-differentiable operator defined on a closed convex subset D of a Banach space X with values in a Banach space Y .

Most problems in applied sciences (like systems of equations, differential equations, integral equations, integro-differentiable equations and others) can be brought in the form (1) for a suitable choice of the corresponding spaces.

Newton-like methods

$$(2) \quad x_{n+1} = x_n - A(x_n)^{-1}F(x_n) \quad (n \geq 0)$$

have been used extensively under various hypotheses to generate a sequence converging to x^* [3]–[7], [9], [10], [17], [18].

We present local and semilocal convergence theorems under very general conditions on the operators involved. This way we can handle a wider range

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of problems than was possible before. Moreover by choosing special cases for the “majorizing” functions, we obtain all previous results in the literature but not vice versa. Furthermore since our results give a deeper insight into the structure of the functions involved, we can in particular obtain semilocal convergence under weaker conditions and in the case of local convergence a larger convergence radius.

2. Semilocal convergence analysis. It is convenient for us to define the non-negative number a by

$$(3) \quad \|A(x_0)^{-1}F(x_0)\| \leq a$$

for some $x_0 \in D$ such that $A(x_0)^{-1} \in L(Y, X)$.

We suppose

$$(4) \quad \|A(x_0)^{-1}[F'(y) - A(x)]\| \leq w_1(\|y - x\|) + v_1(\|x - x_0\|) + v_0$$

for all $x, y \in \bar{U}(x_0, R) = \{x \in X : \|x - x_0\| \leq R\}$ and $R \geq 0$ such that

$$(5) \quad \bar{U}(x_0, R) \subseteq D,$$

where $v_1, w_1: [0, \infty) \rightarrow [0, \infty)$ are increasing, $v_0 \in [0, 1)$,

$$(6) \quad \|A(x_0)^{-1}[A(x) - A(x_0)]\| \leq w_0(\|x - x_0\|) \quad \text{for all } x \in \bar{U}(x_0, r) \quad (0 \leq r \leq R),$$

where again $w_0: [0, \infty) \rightarrow [0, \infty)$ is increasing and

$$(7) \quad \lim_{r \rightarrow 0} w_1(r) = \lim_{r \rightarrow 0} v_1(r) = \lim_{r \rightarrow 0} \bar{w}_0(r) = 0,$$

with $w_0(r) = \bar{w}_0(r) + \bar{v}_0$, $r \in [0, R]$, for some $\bar{v}_0 \in [0, 1)$. Define functions on $[0, R]$ by

$$(8) \quad w(r) = \sup\{w_1(t_1) + v_1(t_2) + v_0 + w_0(t_2) : t_1 + t_2 = r, t_1 \geq 0, t_2 \geq 0\},$$

$$(9) \quad \varphi(r) = \int_0^r w(t) dt - r + a$$

and a sequence

$$(10) \quad r_0 = 0, \quad r_{n+1} = r_n + \frac{\varphi(r_n)}{1 - w_0(r_n)} \quad (n \geq 0).$$

With the notation introduced above we can show the following semilocal convergence theorem for Newton-like methods:

THEOREM 1. *Suppose that a function φ has a unique zero $r^* \in [0, R]$ and $\varphi(R) \leq 0$. Then:*

- (i) *The scalar sequence $\{r_n\}$ ($n \geq 0$) converges monotonically to r^* .*

(ii) A Newton-like method $\{x_n\}$ ($n \geq 0$) is well defined, remains in $\bar{U}(x_0, r^*)$ for all $n \geq 0$ and converges to a solution x^* of the equation $F(x) = 0$. Moreover the following error bounds hold for all $n \geq 0$:

$$(11) \quad \|x_{n+1} - x_n\| \leq r_{n+1} - r_n$$

and

$$(12) \quad \|x^* - x_n\| \leq r^* - r_n.$$

(iii) If $r^* = R$ then the solution x^* is unique in $\bar{U}(x_0, r^*)$ provided that

$$(13) \quad w_1(r^*) < 1.$$

If $r^* < R$ then the solution x^* is unique in $U(x_0, R_0)$ provided that

$$(14) \quad R_0 \leq R \quad \text{and} \quad \int_0^1 w_1[tR_0 + (1-t)r^*] dt < 1.$$

Proof. (i) This follows exactly as in the proof of Lemma 1 in [12, p. 214] (see also [10], [19]).

In particular note that since r^* is the only zero of φ on $[0, R]$, this function is strictly positive on $[0, r^*)$. The same is true for the function $r \rightarrow 0 < -\varphi'(r) \leq 1 - w_0(r)$ (by (8)). Moreover the map

$$r \mapsto r + \frac{\varphi(r)}{1 - w_0(r)}$$

is increasing on $[0, r^*)$ (as can easily be seen by showing the non-negativity of the derivative). The rest of the proof is standard.

(ii) We show (11) and (12) by induction on $n \geq 0$. For $n = 0$ inequality (11) holds as equality. Indeed,

$$\|x_1 - x_0\| = \|A(x_0)^{-1}F(x_0)\| \leq a \leq r_1 = r_1 - r_0.$$

Suppose (11) holds for all $n < k$. Then we have

$$\|x_k - x_0\| \leq \sum_{j=1}^k \|x_j - x_{j-1}\| \leq \sum_{j=1}^k (r_j - r_{j-1}) = r_k \leq r^*.$$

That is, $x_k \in \bar{U}(x_0, r_k) \subseteq \bar{U}(x_0, r^*)$.

Using (6) and (9) we obtain

$$(15) \quad \|A(x_0)^{-1}[A(x_k) - A(x_0)]\| \leq w_0(\|x_k - x_0\|) \leq w_0(r_k) < 1$$

(see part (i)). It follows by (15) and the Banach lemma on invertible operators [14] that $A(x_k)^{-1}$ exists and

$$(16) \quad \|A(x_k)^{-1}A(x_0)\| \leq [1 - w_0(\|x_k - x_0\|)]^{-1} \leq [1 - w_0(r_k)]^{-1}.$$

Moreover, by (2), (4)–(10) and (16) we obtain in turn

$$\begin{aligned}
 (17) \quad & \|x_{k+1} - x_k\| \\
 &= \|A(x_k)^{-1}F(x_k)\| \leq \|A(x_k)^{-1}A(x_0)\| \cdot \|A(x_0)^{-1}F(x_k)\| \\
 &\leq \|A(x_k)^{-1}A(x_0)\| \\
 &\quad \times \|A(x_0)^{-1}[F(x_k) - F(x_{k-1}) - A(x_{k-1})(x_k - x_{k-1})]\| \\
 &\leq \|A(x_k)^{-1}A(x_0)\| \\
 &\quad \times \int_0^1 \|F'[x_k + t(x_k - x_{k-1})] - A(x_{k-1})\| \cdot \|x_k - x_{k-1}\| dt \\
 &\leq \frac{1}{1 - w_0(r_k)} \\
 &\quad \times \int_0^1 \{w_1[r_{k-1} + t(r_k - r_{k-1}) - r_{k-1}] + v_1(r_{k-1}) + v_0\}(r_k - r_{k-1}) dt \\
 &\leq \frac{1}{1 - w_0(r_k)} \int_0^1 \{w(r_{k-1} + t(r_k - r_{k-1})) - w_0(r_{k-1})\}(r_k - r_{k-1}) dt \\
 &\leq \frac{\varphi(r_k) - \varphi(r_{k-1}) - (w_0(r_{k-1}) - 1)(r_k - r_{k-1})}{1 - w_0(r_k)} = r_{k+1} - r_k,
 \end{aligned}$$

which shows (11) for $n = k$. Furthermore, for $m > n$ we get

$$(18) \quad \|x_m - x_n\| \leq r_m - r_n.$$

It follows from (18) that $\{x_n\}$ ($n \geq 0$) is a Cauchy sequence in the Banach space X and hence it converges to some $x^* \in \overline{U}(x_0, r^*)$ (since $\overline{U}(x_0, r^*)$ is a closed set). By letting $n \rightarrow \infty$ in (2) we obtain $F(x^*) = 0$ and by letting $m \rightarrow \infty$ in (18) we obtain (12).

Finally to show uniqueness, let $y^* \in U(x_0, R_0)$ be a solution of equation (1). We have, by (4),

$$\begin{aligned}
 & \int_0^1 \|A(x_0)^{-1}[F'(x + t(y - x)) - A(x_0)]\| dt \\
 & \leq \int_0^1 w_1[\|x + t(y - x) - x_0\|] dt \\
 & \leq \int_0^1 w_1[t\|y - x_0\| + (1 - t)\|x - x_0\|] dt \\
 & \leq \int_0^1 w_1[tR_0 + (1 - t)r^*] dt < 1.
 \end{aligned}$$

Hence, the inverse of the linear operator

$$L = \int_0^1 F'(x^* + t(y^* - x^*)) dt$$

exists. But we can write

$$0 = F(y^*) - F(x^*) = L(y^* - x^*),$$

which shows that under (13) or (14),

$$x^* = y^*. \blacksquare$$

REMARK 1. (a) Let $A(x) = F'(x)$ ($x \in D$) and set

$$\bar{v}_0 = v_0 = 0, \quad v_1(r) = 0 \quad (0 \leq r \leq R).$$

Then Theorem 1 reduces to Theorem 2 in [12, p. 215]. Moreover, if we choose $w_1(r) = w_0(r)$, $r \in [0, R]$, we obtain the main theorem in [1]. Note however that uniqueness results are not given in either [1] or [12]. Furthermore, choose

$$w_1(r) = \ell r, \quad r \in [0, R], \quad \ell > 0.$$

It follows by (8) that

$$w(r) = \ell r, \quad r \in [0, R],$$

and (9) becomes

$$\varphi_1(r) = \frac{1}{2}\ell r^2 - r + a.$$

In this case the function φ has a zero if and only if

$$(19) \quad 2\ell a \leq 1.$$

We set

$$R = \frac{1}{\ell} \quad \text{and} \quad r^* = \frac{1 - \sqrt{1 - 2\ell a}}{\ell}.$$

Condition (19) is the Newton–Kantorovich hypothesis for the convergence of Newton’s method given in [14].

(b) Assume there exist increasing functions $w_2, w_3: [0, \infty) \rightarrow [0, \infty)$ such that

$$(20) \quad \|A(x_0)^{-1}[F'(y) - F'(x)]\| \leq w_2(\|y - x\|),$$

$$(21) \quad \|A(x_0)^{-1}[F'(x) - A(x)]\| \leq w_3(\|x - x_0\|) + v_0$$

for all $x, y \in \bar{U}(x_0, r)$, and

$$\lim_{r \rightarrow 0} w_2(r) = \lim_{r \rightarrow 0} w_3(r) = 0.$$

Define functions on $[0, R]$ by

$$w_4(r) = \sup\{w_2(t_1) + w_0(t_2) : t_1 + t_2 = r, t_1 \geq 0, t_2 \geq 0\},$$

$$\varphi_2(r) = \int_0^r w_4(t) dt + r w_3(r) + v_0 r - r + a,$$

and an iteration

$$s_0 = 0, \quad s_{n+1} = s_n + \frac{\varphi_2(s_n)}{1 - w_0(s_n)} \quad (n \geq 0).$$

Based on the identity

$$(22) \quad F(x_k) = \int_0^1 [F'(x_{k-1} + t(x_k - x_{k-1})) - F'(x_{k-1})](x_k - x_{k-1}) dt \\ + [F'(x_{k-1}) - A(x_{k-1})](x_k - x_{k-1})$$

instead of (17), the conclusions of Theorem 1 hold if we replace $\varphi, \{r_n\}$ by $\varphi_2, \{s_n\}$ ($n \geq 0$) respectively. Moreover, if we set, for all $x, y \in \bar{U}(x_0, r)$, $\ell_i \geq 0, i = 1, \dots, 5$,

$$(23) \quad w_2(\|y - x\|) = \ell_1 \|y - x\|,$$

$$(24) \quad w_3(\|x - x_0\|) = \ell_2 \|x - x_0\|, \quad \ell_3 = v_0,$$

$$(25) \quad w_0(\|x - x_0\|) = \ell_4 \|x - x_0\| + \ell_5, \quad \ell_5 = \bar{v}_0,$$

we obtain the theorem due to Yamamoto [17]. For $A(x) = F'(x)$ ($x \in D$) and $w_3 = 0$ we again obtain Theorem 2 in [12].

(c) Consider the iteration

$$(26) \quad x_{n+1} = x_n - A(x_n)^{-1}(F(x_n) + G(x_n)) \quad (n \geq 0)$$

for approximating a solution of the equation

$$(27) \quad F(x) + G(x) = 0,$$

where F is as in Theorem 1 and $G: D \rightarrow Y$ is a continuous operator.

Assume there exists a continuous, non-decreasing function $w_5: [0, \infty) \rightarrow [0, \infty)$ such that

$$(28) \quad \|A(x_0)^{-1}[G(x) - G(y)]\| \leq w_5(r)\|x - y\|$$

for all $x, y \in \bar{U}(x_0, r)$, and set

$$w_6(r) = \int_0^r w_5(t) dt.$$

Define a function φ_3 by

$$\varphi_3(r) = w_6(r) + \int_0^r w(t) dt - r + a$$

and an iteration $\{b_n\}$ ($n \geq 0$) by

$$b_0 = 0, \quad b_{n+1} = b_n + \frac{\varphi_3(b_n)}{1 - w_0(b_n)} \quad (n \geq 0).$$

Then the conclusions of Theorem 1 concerning (26) and (27) hold with φ , $\{r_n\}$ replaced by φ_3 , $\{b_n\}$ ($n \geq 0$) respectively. If we let

$$w_0(\|x - x_0\|) = w_7(\|x - x_0\|) + c_0, \\ w_1(r + t) = w_8(r + t) - w_7(r) + c_1, \quad t \geq 0, r \in [0, R],$$

for some $c_0, c_1 \geq 0$ and continuous non-decreasing functions $w_7, w_8(r + t) - w_7(r)$ ($t \geq 0$), then we obtain the main theorem in [10].

(d) Theorem 1 can be further generalized (similarly for cases (a)–(c)) if instead of (4) we assume that

$$\|A(x_0)^{-1}[F'(y) - A(x)]\| \\ \leq v_2(\|x - y\|, \|x - x_0\|, \|y - x_0\|) + v_3(\|x - x_0\|) + v_4(\|y - x_0\|) + v_5$$

for all $x, y \in \bar{U}(x_0, r) \subseteq \bar{U}(x_0, R)$, where $v_2: [0, \infty)^3 \rightarrow [0, \infty)$, $v_3, v_4: [0, \infty) \rightarrow [0, \infty)$ are increasing, $v_5 \in [0, 1)$ and

$$\lim_{r \rightarrow 0} v_2(r, r, r) = \lim_{r \rightarrow 0} v_3(r) = \lim_{r \rightarrow 0} v_4(r) = 0.$$

Define functions on $[0, R]$ by

$$v(r) = \sup\{v_2(t_1, t_2, t_1) + v_3(t_2) + v_4(t_1) + v_5 + w_0(t_2) : \\ t_1 + t_2 = r, t_1 \geq 0, t_2 \geq 0\}, \\ \varphi_3(r) = \int_0^r v(r) - r + a$$

and a sequence

$$p_0 = 0, \quad p_{n+1} = p_n + \frac{\varphi_3(p_n)}{1 - w_0(p_n)} \quad (n \geq 0).$$

Then the conclusions of Theorem 1 hold with φ , $\{s_n\}$ replaced by φ_3 , $\{p_n\}$ ($n \geq 0$). If we let

$$v_2(r, r, r) + v_3(r) + v_4(r) + v_5 = w_0(r), \quad r \in [0, R],$$

we essentially obtain the results in [5].

(e) Consider a Newton-like method (2) in the form

$$(29) \quad x_{n+1} = x_n - A(x_n)^\# F(x_n) \quad (n \geq 0)$$

for solving the equation

$$(30) \quad A^\# F(x) = 0,$$

where a $A^\#$ is a bounded outer inverse of $A(x_0)$, and $A(x_n)^\#$ denotes an outer inverse of $A(x_n)$, i.e., $A(x_n)^\# A(x_n) A(x_n)^\# = A(x_n)^\#$ ($n \geq 0$). The conclusions of Theorem 1 hold for (29) and (30) if we simply replace $A(x_0)$ by $A^\#$ in the hypotheses of the theorem, and set

$$A(x_n)^\# = [I + A^\#(A(x_n) - A(x_0))]^{-1} A^\#(x_0),$$

with uniqueness holding in $\bar{U}(x_0, r^*) \cap \{R(A^\#(x_0)) + x_0\}$, where

$$R(A^\#(x_0)) + x_0 = \{x + x_0 : x \in R(A^\#(x_0))\}.$$

In the special case when the functions w_2, w_3, w_0 are given by (23), (24), (25) respectively, we obtain Theorem 3.1 in [9, p. 241] concerning the convergence of Newton-like methods using outer inverses.

(f) With the exception of the uniqueness part (iii) in Theorem 1 the rest of the conclusions hold if we replace condition (4) by the Mysovskii-type condition

$$\int_0^1 \|A(x_0)^{-1}[F'(x + t(y - x)) - A(x)](y - x)\| dt \leq [\bar{w}_1(\|y - x\|) + \bar{v}_1(\|x - x_0\|) + \bar{v}_0]\|y - x\|$$

for all $x, y \in \bar{U}(x_0, R)$, where $\bar{w}_1, \bar{v}_1, \bar{v}_0$ are like w_1, v_1, v_0 respectively. Since $\|L(x)\| \leq \|L\| \cdot \|x\|$ for any linear operator L , it is expected that $\bar{w}_1 \leq w_1$, which leads to smaller error bounds on the distances $\|x_{n+1} - x_n\|$ ($n \geq 0$).

3. Local convergence analysis. We complete this study by showing the following local convergence theorem for Newton-like methods.

THEOREM 2. *Suppose there exist: a simple zero $x^* \in D$ of the equation $F(x) = 0$ with $A(x^*)^{-1} \in L(Y, X)$; increasing functions $\alpha, \alpha_1, \beta: [0, \infty) \rightarrow [0, \infty)$ with*

$$\lim_{r \rightarrow 0} \alpha(r) = \lim_{r \rightarrow 0} \alpha_1(r) = \lim_{r \rightarrow 0} \beta(r) = 0;$$

and non-negative parameters α_0, β_0 with $\alpha_0 + \beta_0 \in [0, 1)$ such that

$$(31) \quad \|A(x^*)^{-1}[F'(y) - A(x)]\| \leq \alpha(\|y - x\|) + \alpha_1(\|x - x^*\|) + \alpha_0,$$

$$(32) \quad \|A(x^*)^{-1}[A(x) - A(x^*)]\| \leq \beta(\|x - x^*\|) + \beta_0$$

for all $x, y \in D$. Suppose also that the equation

$$(33) \quad \int_0^1 \alpha(tr) dt + \beta(r) + \alpha_1(r) + \alpha_0 + \beta_0 - 1 = 0$$

has a minimum non-negative zero R_1 such that

$$(34) \quad \bar{U}(x^*, R_1) \subseteq D.$$

Then the Newton-like method $\{x_n\}$ ($n \geq 0$) generated by (2) is well defined, remains in $U(x^, r)$ ($r < R_1$) for all $n \geq 0$ and converges to x^* provided that $x_0 \in U(x^*, r)$. Moreover the following error bounds hold for all $n \geq 1$:*

$$(35) \quad \|x_n - x^*\| \leq \gamma_n \|x_{n-1} - x^*\|,$$

where

$$(36) \quad \gamma_n = \frac{\int_0^1 \alpha(t\|x^* - x_{n-1}\|) dt + \alpha_1(\|x^* - x_{n-1}\|) + \alpha_0}{1 - [\beta(\|x^* - x_{n-1}\|) + \beta_0]}.$$

Proof. Clearly, $x_0 \in U(x_0, r)$. Assume $x_{k-1} \in U(x_0, r)$ for all $k < n$. Using (32) and (33) we obtain

$$(37) \quad \|A(x^*)(A(x_k) - A(x^*))\| \leq \beta(\|x_k - x^*\|) + \beta_0 \leq \beta(R_1) + \beta_0 < 1.$$

It follows from (37) and the Banach lemma on invertible operators that $A(x_k)^{-1}$ exists and

$$(38) \quad \|A(x_k)^{-1}A(x^*)\| \leq \frac{1}{1 - [\beta(\|x_k - x^*\|) + \beta_0]} \leq \frac{1}{1 - [\beta(R_1) + \beta_0]}.$$

Using (2) we get in turn

$$(39) \quad \begin{aligned} x_k - x^* &= x_{k-1} - x^* - A(x_{k-1})^{-1}F(x_{k-1}) \\ &= A(x_{k-1})^{-1}[F(x^*) - F(x_{k-1}) - A(x_{k-1})(x^* - x_{k-1})] \\ &= [A(x_{k-1})^{-1}A(x^*)]A(x^*)^{-1} \\ &\quad \cdot \int_0^1 [F'(x_{k-1} + t(x^* - x_{k-1})) - A(x_{k-1})](x^* - x_{k-1}) dt. \end{aligned}$$

Moreover, by (31), (38) and (39) we obtain

$$\begin{aligned} \|x_k - x^*\| &\leq \|A(x_{k-1})^{-1}A(x^*)\| \\ &\quad \times \int_0^1 \|A(x^*)^{-1}[F'(x_{k-1} + t(x^* - x_{k-1})) - A(x_{k-1})]\| \cdot \|x^* - x_{k-1}\| dt \\ &\leq \gamma_k \|x^* - x_{k-1}\|, \end{aligned}$$

which shows (35) for all $n \geq 1$. Furthermore, by the choice of R_1, r and by (33) there exists

$$\gamma = \frac{\int_0^1 \alpha(tr_0) dt + \alpha_1(r_0) + \alpha_0}{1 - [\beta(r_0) + \beta_0]} \in [0, 1)$$

such that

$$\gamma_k \leq \gamma \quad (k \geq 1).$$

That is,

$$(40) \quad \|x_k - x^*\| \leq \gamma \|x_{k-1} - x^*\| \leq \gamma^k \|x_0 - x^*\| \leq \gamma^k r \quad (k \geq 1).$$

It follows from (40) that $x_k \in U(x^*, r)$ and $\lim_{n \rightarrow \infty} x_n = x^*$ (since $\gamma \in [0, 1)$). ■

REMARK 2. Let $A(x) = F'(x)$ ($x \in D$). Choose:

$$\alpha(\|x - y\|) = \delta_1\|x - y\|, \quad \alpha_1(r) = 0, \quad r \in [0, R_1), \quad \alpha_0 = 0,$$

and

$$\beta(\|x - x^*\|) = \delta_2\|x - x^*\|, \quad \beta_0 = 0.$$

Consequently, equation (33) becomes

$$\frac{1}{2}\delta_1 r + \delta_2 r - 1 = 0,$$

which gives

$$R_1 = \frac{2}{\delta_1 + 2\delta_2}.$$

It was shown in [16] that the convergence radius for Newton's method under the Lipschitz condition:

$$\|F'(x^*)^{-1}[F'(y) - F'(x)]\| \leq \delta_1\|y - x\|$$

for all $x, y \in D$, is given by

$$\bar{R}_1 = \frac{2}{3\delta_1}.$$

However, since

$$\delta_2 \leq \delta_1 \quad (\text{in general}),$$

we obtain

$$\bar{R}_1 \leq R_1.$$

Let $X = Y = \mathbb{R}$, $D = U(0, 1)$ and define a function F on D by

$$F(x) = e^x - 1.$$

Then it can easily be seen that $\delta_1 = e$ and $\delta_2 = e - 1$. Consequently, we obtain

$$\bar{R}_1 = .245252961 < R_1 = .254028662.$$

That is, we obtain a wider range of initial guesses x_0 than before. This observation is very important in the construction of very efficient mesh refinement strategies and in projection methods [3], [8], [15], [18].

Note that several other local convergence theorems can be generated along the lines of Theorem 1 and Remark 1.

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