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NONLOCAL ROBIN PROBLEM FOR ELLIPTIC SECOND ORDER EQUATIONS IN A PLANE DOMAIN WITH A BOUNDARY CORNER POINT

Abstract. We investigate the behavior of weak solutions to the nonlocal Robin problem for linear elliptic divergence second order equations in a neighborhood of a boundary corner point. We find an exponent of the solution's decreasing rate under minimal assumptions on the problem coefficients.

1. Introduction. Our article is devoted to the nonlocal Robin problem in a plane domain with a boundary corner point. This problem often appears in different fields of physics and engineering. For example, nonlocal elliptic boundary value problems have important applications to the theory of diffusion processes and the theory of turbulence. Various problems in this field have been studied by many mathematicians. We refer to [3, 10] for the history and extensive literature. Solvability of nonlocal elliptic value boundary problems was considered by Skubachevskiĭ [10]. He also obtained a priori estimates of solutions in Sobolev spaces, both weighted and unweighted. All results in [10] relate to equations with *infinitely differentiable* coefficients. Gurevich [3] considered the asymptotics of solutions for nonlocal elliptic problems for equations with *constant* coefficients in plane angles. A principal new feature of our work is the consideration of estimates for equations with coefficients *of minimal smoothness.*

We establish global and local estimates of weighted and unweighted Dirichlet integrals as well as the modulus of weak solutions to our problem, employing methods different from those in [3, 10]: we investigate the behavior of weak solutions in a neighborhood of the boundary corner point by means of integro-differential inequalities and Kondratiev's ring methods.

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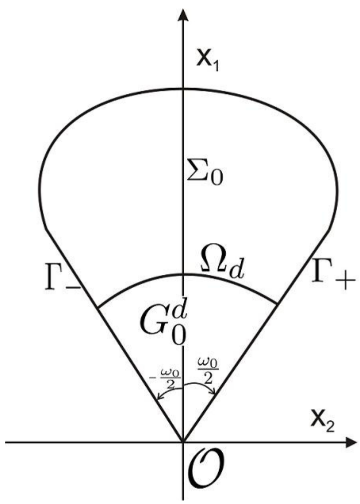
For this purpose we derive *a new Friedrichs–Wirtinger type inequality*, which is adapted to our problem.

Setting of nonlocal problem. Let $G \subset \mathbb{R}^2$ be a bounded domain whose boundary $\partial G = \overline{\Gamma_+} \cup \overline{\Gamma_-}$ is a smooth curve everywhere except at the origin $\mathcal{O} \in \partial G$, near $\mathcal{O}$ the curves $\Gamma_\pm$ are the lateral sides of an angle with measure $\omega_0 \in [0, 2\pi)$ and vertex at $\mathcal{O}$. Let $\Sigma_0 = G \cap \{x_2 = 0\}$, where $\mathcal{O} \in \overline{\Sigma_0}$.

We will use the following notation:

- S^1: the unit circle in $\mathbb{R}^2$ centered at $\mathcal{O}$;
- (r, ω): the polar coordinates of $x = (x_1, x_2) \in \mathbb{R}^2$ with pole $\mathcal{O}$: $x_1 = r\cos\omega,\ x_2 = r\sin\omega$;
- $\mathcal{C}$: the angle $\{x_1 > r\cos(\omega_0/2),\ -\infty < x_2 < \infty\}$ with vertex $\mathcal{O}$;
- $\partial\mathcal{C}$: the lateral sides of $\mathcal{C}$: $x_1 = r\cos(\omega_0/2)$, $x_2 = \pm r\sin(\omega_0/2)$;
- Ω: the arc obtained by intersecting the angle $\mathcal{C}$ with S^1: $\Omega = \mathcal{C} \cap S^1 = (-\omega_0/2, \omega_0/2)$;
- $G_a^b = \{(r, \omega) : 0 \le a < r < b; \omega \in \Omega\} \cap G$: a ring domain in $\mathbb{R}^2$;
- $\Gamma_{a\pm}^b = \{(r, \omega) : 0 \le a < r < b,\ \omega = \pm\omega_0/2\} \cap \partial G$: the lateral sides of G_a^b;
- $G_d = G \setminus G_0^d$; $\Gamma_{d\pm} = \Gamma_\pm \setminus \Gamma_{0\pm}^d,\ d > 0$;
- $\Omega_\rho = G_0^d \cap \{|x| = \varrho\}$, $0 < \varrho < d$;
- $\operatorname{meas} G$: the Lebesgue measure of the set G.



We shall consider an elliptic equation with a nonlocal boundary condition connecting the values of the unknown function u on the curve Γ_+ with its values on Σ_0:

$$
\text{(L)} \qquad \begin{cases} \mathcal{L}[u] \equiv \dfrac{\partial}{\partial x_i}(a^{ij}(x)u_{x_j}) + b^i(x)u_{x_i} + c(x)u = f(x), & x \in G; \\ \mathcal{B}_+[u] \equiv \dfrac{\partial u}{\partial \nu} + \beta_+ \dfrac{u(x)}{|x|} + \dfrac{b}{|x|}u(\gamma(x)) = g(x), & x \in \Gamma_+; \\ \mathcal{B}_-[u] \equiv \dfrac{\partial u}{\partial \nu} + \beta_- \dfrac{u(x)}{|x|} = h(x), & x \in \Gamma_-; \end{cases}
$$

here:

- $\partial/\partial\nu = a^{ij}(x)\cos(\vec{n}, x_i)\partial/\partial x_j$, and $\vec{n}$ denotes the unit vector outward with respect to G normal to $\partial G \setminus \mathcal{O}$ (summation over repeated indices from 1 to 2 is understood);
- γ is a diffeomorphism of Γ_+ onto Σ_0; we assume that there exists $d > 0$ such that in the neighborhood Γ^d_{0+} of $\mathcal{O}$ the mapping γ is the rotation by the angle $-\omega_0/2$, that is, $\gamma(\overline{\Gamma^d_{0+}}) = \overline{\Sigma^d_0} = \overline{G^d_0} \cap \Sigma_0$.

Remark 1.1. We observe that

$$
u(\gamma(x))|_{\Gamma^d_{0+}} = u(r, 0), \qquad 0 < r < d.
$$

In fact, $\gamma(x) = \gamma(x_1, x_2) = \gamma(r\cos(\omega_0/2), r\sin(\omega_0/2)) = (r, 0)$, because in Γ^d_{0+} the mapping γ is the rotation by the angle $-\omega_0/2$.

We use also standard function spaces:

- $C^k(\overline{G})$ with the norm $|u|_{k,\overline{G}}$,
- the Lebesgue space $L_p(G)$, $p \geq 1$, with the norm $\|u\|_{p,G}$,
- the Sobolev space $W^{k,p}(G)$ with the norm

$$
\|u\|_{p,k,(G)} = \Big(\int_G \sum_{|\beta|=0}^{k} |D^\beta u|^p \, dx\Big)^{1/p}.
$$

We define the weighted Sobolev space $V^k_{p,\alpha}(G)$ for integer $k \geq 0$ and real α as the space of distributions $u \in \mathcal{D}'(G)$ with the finite norm

$$
\|u\|_{V^k_{p,\alpha}(G)} = \Big(\int_G \sum_{|\beta|=0}^{k} r^{\alpha+p(|\beta|-k)}|D^\beta u|^p \, dx\Big)^{1/p},
$$

and $V^{k-1/p}_{p,\alpha}(\partial G)$ as the space of functions φ, given on ∂G, with the norm $\|\varphi\|_{V^{k-1/p}_{p,\alpha}(\partial G)} = \inf \|\Phi\|_{V^k_{p,\alpha}(G)}$, where the infimum is taken over all functions Φ such that $\Phi|_{\partial G} = \varphi$ in the sense of traces. We write $W^k(G)$ for $W^{k,2}(G)$, $\mathring{W}^k_\alpha(G)$ for $V^k_{2,\alpha}(G)$, and $\mathring{W}^{k-1/2}_\alpha(\partial G)$ for $V^{k-1/2}_{2,\alpha}(\partial G)$.

Let us recall some well known formulae related to polar coordinates (r, ω) in $\mathbb{R}^2$ centered at $\mathcal{O}$:

- $dx = r dr d\omega,\ d\Omega_\rho = \rho d\omega,$
- $|\nabla u|^2 = \left(\frac{\partial u}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial u}{\partial \omega}\right)^2,$
- $\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \omega^2},$
- ds denotes the length element on ∂G.

$C = C(\ldots)$, $c = c(\ldots)$ denote constants depending only on the quantities appearing in parentheses. In what follows, the same letters C, c will be used to denote various constants depending on the same set of arguments.

Without loss of generality we can assume that there exists $d > 0$ such that G_0^d is *an angle* with vertex $\mathcal{O}$ and measure $\omega_0 \in (0, 2\pi)$, thus

$$\Gamma_{0\pm}^d = \{(x_1, x_2) : x_1 = \pm x_2 \cot(\omega_0/2),\ |x| \le d\}.$$

By a direct calculation we obtain

Lemma 1.2.

$$\cos(\vec{n}, x_1)|_{\Gamma_{0\pm}^d} = -\sin\frac{\omega_0}{2};\quad x_i\cos(\vec{n}, x_i)|_{\Gamma_{0\pm}^d} = 0;\quad x_i\cos(\vec{n}, x_i)|_{\Omega_\varrho} = \varrho.$$

Definition 1.3. A function u is called a *weak solution* of problem (L) provided that $u \in C^0(\overline{G}) \cap \mathring{W}_0^1(G)$ and u satisfies the integral identity

$$\text{(II)}\qquad \int_G \{a^{ij}(x)u_{x_j}\eta_{x_i} - b^i(x)u_{x_i}\eta(x) - c(x)u(x)\eta(x)\}\,dx$$
$$+ \beta_+ \int_{\Gamma_+} u(x)/r\eta(x)\,ds + b\int_{\Gamma_+}\frac{1}{r}u(\gamma(x))\eta(x)\,ds + \beta_- \int_{\Gamma_-} u(x)/r\eta(x)\,ds$$
$$= \int_{\Gamma_+} g(x)\eta(x)\,ds + \int_{\Gamma_-} h(x)\eta(x)\,ds - \int_G f(x)\eta(x)\,dx$$

for all $\eta \in C^0(\overline{G}) \cap \mathring{W}_0^1(G)$.

Lemma 1.4. *Let u be a weak solution of* (L). *For any $\eta \in C^0(\overline{G}) \cap \mathring{W}_0^1(G)$, and a.e. $\varrho \in (0, d)$, we have*

$$\text{(II)}_{\text{loc}}\qquad \int_{G_0^\varrho} \{a^{ij}(x)u_{x_j}\eta_{x_i} + (f(x) - b^i(x)u_{x_i} - c(x)u(x))\eta(x)\}\,dx$$
$$= \int_{\Omega_\varrho} a^{ij}(x)u_{x_j}\eta(x)\cos(r, x_i)\,d\Omega_\varrho + \int_{\Gamma_{0-}^\varrho}\left(h(x) - \beta_-\frac{u(x)}{r}\right)\eta(x)\,ds$$
$$+ \int_{\Gamma_{0+}^\varrho}\left(g(x) - \beta_+ u(x)/r - \frac{b}{r}u(\gamma(x))\right)\eta(x)\,ds.$$

Proof. Let χ_ϱ be the characteristic function of G_0^ϱ. Replacing in (II) the function $\eta(x)$ by $\eta(x)\chi_\varrho(x)$, we obtain

$$\begin{aligned}\int\limits_{G_0^\varrho} \{a^{ij}(x)u_{x_j}\eta_{x_i} + (f(x) - b^i(x)u_{x_i} - c(x)u(x))\eta(x)\}\,dx \\ = -\int\limits_{G_0^\varrho} a^{ij}(x)u_{x_j}\eta(x)\partial\chi_\varrho/\partial x_i\,dx \\ + \int\limits_{\Gamma_{0+}}^{\varrho} \left(g(x) - \beta_+\frac{u(x)}{r} - \frac{b}{r}u(\gamma(x))\right)\eta(x)\,ds \\ + \int\limits_{\Gamma_{0-}^\varrho} \left(h(x) - \beta_-\frac{u(x)}{r}\right)\eta(x)\,ds.\end{aligned}$$

By [2, Ch. 3, §1, Subsect. 3, formula (7′)] $\frac{\partial\chi_\varrho}{\partial x_i} = -\frac{x_i}{r}\delta(\varrho - r)$, where $\delta(\varrho - r)$ is the Dirac distribution lumped on the circle $r = \varrho$, we get (see [2, Ch. 3, §1, Subsect. 3, Example 4])

$$\begin{aligned}-\int\limits_{G_0^\varrho} a^{ij}(x)u_{x_j}\eta(x)\frac{\partial\chi_\varrho}{\partial x_i}\,dx &= \int\limits_{G_0^\varrho} a^{ij}(x)u_{x_j}\eta(x)\frac{x_i}{r}\delta(\varrho - r)\,dx \\ &= \int\limits_{\Omega_\varrho} a^{ij}u_{x_j}\eta(x)\cos(r, x_i)\,d\Omega_\varrho.\end{aligned}$$

Hence the required statement follows. ∎

We will make the following *assumptions:*

(a) (uniform ellipticity)

$$\nu\xi^2 \le a^{ij}(x)\xi_i\xi_j \le \mu\xi^2, \quad \forall x \in \overline{G}, \forall \xi \in \mathbb{R}^2;\ \nu, \mu = \text{const} > 0$$

(without loss of generality we can assume that $\nu \le 1$),

$$a^{ij}(x) = a^{ji}(x), \quad \forall x \in \overline{G}, \quad a^{ij}(0) = \delta_i^j \quad (i, j = 1, 2),$$

where δ_i^j is the Kronecker symbol;

(b) $a^{ij} \in C^0(\overline{G})$, $b^i \in L_p(G)$, $c \in L_{p/2}(G) \cap L_2(G)$ for all $p > \widetilde{n} > 2$; the inequality

$$\Big(\sum_{i,j=1}^{2} |a^{ij}(x) - a^{ij}(0)|^2\Big)^{1/2} + |x|\Big(\sum_{i=1}^{2} |b^i(x)|^2\Big)^{1/2} + |x|^2|c(x)| \le \mathcal{A}(|x|)$$

holds for all $x \in \overline{G}$, where $\mathcal{A}(r)$ is an increasing nonnegative function, continuous at 0 and $\mathcal{A}(0) = 0$;

(c) $c(x) \le 0$ in G; $b \ge 0$, $\beta_\pm > 0$;

(d) $f \in L_{p/2}(G) \cap L_2(G)$, $g \in L_\infty(\Gamma_+)$, $h \in L_\infty(\Gamma_-)$;

(e) there exist numbers $f_0 \geq 0$, $g_0 \geq 0$, $h_0 \geq 0$, $s > \max\{1; 2-4/p\}$ such that

$$|f(x)| \leq f_0|x|^{s-2}, \quad |g(x)| \leq g_0|x|^{s-1}, \quad |h(x)| \leq h_0|x|^{s-1};$$

(f) $M_0 = \max_{x\in\overline{G}} |u(x)|$ (see e.g. Section 3).

Our main results are the following theorems.

THEOREM 1.5. *Let u be a weak solution of problem* (L) *and let assumptions* (a)–(f) *be satisfied with $\mathcal{A}(r)$ Dini-continuous at zero. Let $\lambda = \lambda^*$, where λ^* is defined by Lemma* 2.6. *Suppose, in addition,*

$$\begin{gathered} 0 < b < \min\left\{\sqrt{2}\cdot\frac{\frac{\pi^2}{4\omega_0^2}-\beta_+\beta_-}{\frac{\pi}{2\omega_0}+\beta_-};\ \frac{1}{\omega_0}(\nu+\sqrt{\nu^2+2\nu\omega_0\beta_+})\right\}, \\ \beta_+\beta_- < \left(\frac{\pi}{2\omega_0}\right)^2. \end{gathered} \tag{1.1}$$

Then there are $d \in (0,1/e)$, where e is the Euler number, and a constant $C > 0$ depending only on ν, μ, p, $\|\sum_{i=1}^n |b^i(\cdot)|^2\|_{p/2,G}$, ω_0, b, β_+, β_-, M_0, f_0, h_0, g_0, s, $\operatorname{meas} G$, $\operatorname{meas} \mathit{\Gamma}_+$, $\operatorname{meas} \mathit{\Gamma}_-$ *and the quantity $\int_0^{1/e} (\mathcal{A}(r)/r)\,dr$ such that for all $x \in \overline{G_0^d}$,*

$$|u(x)| \leq C \begin{cases} |x|^{\lambda k} & \text{if } s > \lambda k, \\ |x|^{\lambda k}\ln(1/|x|) & \text{if } s = \lambda k, \\ |x|^s & \text{if } s < \lambda k, \end{cases} \tag{1.2}$$

where

$$k = \frac{B+\beta_+ + b - \sqrt{(\beta_+ + b - B)^2 + Bb^2\omega_0}}{2B} \in (0,1], \tag{1.3}$$

and $B = B(\lambda)$ is as in (2.6).

REMARK 1.6. Because of (2.4), we can observe that if $b = 0$, then $k = 1$.

THEOREM 1.7. *Let $u \geq 0$ be a weak solution of problem* (L), *and let assumptions* (a)–(f) *be satisfied with $\mathcal{A}(r)$ Dini-continuous at zero. Let $\beta_- = \beta_+ = \beta$, $b > b^*$, where b^* is defined by* (2.9) *and let $\lambda \in (\pi/\omega_0, 2\pi/\omega_0)$ be a root of equation* (2.10). *Then there are $d \in (0,1/e)$ and a constant $C > 0$ depending only on ν, μ, p, $\|\sum_{i=1}^n |b^i(x)|^2\|_{p/2,G}$, ω_0, b, β, f_0, h_0, g_0, s, M_0,* $\operatorname{meas} G$, $\operatorname{meas} \mathit{\Gamma}_+$, $\operatorname{meas} \mathit{\Gamma}_-$ *and $\int_0^{1/e} (\mathcal{A}(r)/r)\,dr$ such that for all $x \in \overline{G_0^d}$,*

$$|u(x)| \leq C \begin{cases} |x|^{\lambda} & \text{if } s > \lambda, \\ |x|^{\lambda}\ln(1/|x|) & \text{if } s = \lambda, \\ |x|^s & \text{if } s < \lambda. \end{cases}$$

THEOREM 1.8. *Let u be a weak solution of problem* (L), *and let assumptions* (a)–(f) *be satisfied with $\mathcal{A}(r)$ Dini-continuous at zero. Suppose, in addition, that*

$$\beta_+ u^2(x)|_{\Gamma_+} + \beta_- u^2(x)|_{\Gamma_-} + bu(x)|_{\Gamma_+} \cdot u(\gamma(x))|_{\Gamma_+} = 0,$$
$$b = \frac{\pi}{\omega_0} \cdot \frac{\beta_+ + \beta_-}{\beta_-} \quad \text{and} \quad u^2(x)|_{\Gamma_+} = u^2(x)|_{\Gamma_-}.$$

Then there are $d \in (0, 1/e)$ and a constant $C > 0$ depending only on ν, μ, p, ω_0, b, β, f_0, h_0, g_0, s, M_0, $\operatorname{meas} G$, $\operatorname{meas} \Gamma_+$, $\operatorname{meas} \Gamma_-$ *and* $\int_0^{1/e} (\mathcal{A}(r)/r)\, dr$ *such that for all $x \in \overline{G_0^d}$,*

$$|u(x)| \le C \begin{cases} |x|^{\pi/\omega_0} & \text{if } s > \pi/\omega_0, \\ |x|^{\pi/\omega_0} \ln(1/|x|) & \text{if } s = \pi/\omega_0, \\ |x|^s & \text{if } s < \pi/\omega_0. \end{cases}$$

2. Preliminaries

2.1. Eigenvalue problem. In what follows we need some statements and inequalities. We consider the following eigenvalue problem:

$$\text{(EVP)} \qquad \begin{cases} \psi''(\omega) + \lambda^2 \psi(\omega) = 0, \quad \omega \in \Omega, \\ \psi'(\omega_0/2) + \beta_+ \psi(\omega_0/2) + b\psi(0) = 0, \\ -\psi'(-\omega_0/2) + \beta_- \psi(-\omega_0/2) = 0, \end{cases}$$

with $\beta_\pm > 0$, $b \ge 0$, which consists in determining all values λ^2 (eigenvalues) for which (EVP) has nonzero weak solutions (eigenfunctions) $\psi(\omega)$.

DEFINITION 2.1. A function ψ is called a *weak solution* of problem (EVP) provided that $\psi \in W^1(\Omega) \cap C^0(\overline{\Omega})$ and

$$(2.1) \qquad \int_\Omega (\psi'(\omega)\eta'(\omega) - \lambda^2 \psi(\omega)\eta(\omega))\, d\omega + \beta_+ \psi(\omega_0/2)\eta(\omega_0/2)$$
$$+ b\psi(0)\eta(\omega_0/2) + \beta_- \psi(-\omega_0/2)\eta(-\omega_0/2) = 0 \quad \text{for all } \eta \in W^1(\Omega) \cap C^0(\overline{\Omega}).$$

We are interested in the smallest positive eigenvalue of (EVP). Solving the equation of (EVP) we find

$$(2.2) \qquad \psi(\omega) = \beta_- \sin \lambda(\omega + \omega_0/2) + \lambda \cos \lambda(\omega + \omega_0/2)$$

and λ is defined by the transcendental equation

$$(2.3) \qquad f(\lambda) := \lambda(\beta_+ + \beta_-) \cos \lambda\omega_0 + (\beta_+\beta_- - \lambda^2) \sin \lambda\omega_0$$
$$+ b\left(\lambda \cos \frac{\lambda\omega_0}{2} + \beta_- \sin \frac{\lambda\omega_0}{2}\right) = 0.$$

REMARK 2.2. From (2.2) it follows that in fact $\psi \in C^\infty(\overline{\Omega})$.

Remark 2.3. Let $\lambda = \pi/\omega_0$. Then

$$f(\pi/\omega_0) = 0 \Leftrightarrow$$
$$b = \frac{\pi}{\omega_0} \cdot \frac{\beta_+ + \beta_-}{\beta_-}, \quad \psi(0) = \beta_-, \quad \psi\left(\frac{\omega_0}{2}\right) = -\frac{\pi}{\omega_0}, \quad \psi\left(-\frac{\omega_0}{2}\right) = \frac{\pi}{\omega_0}.$$

2.2. The Friedrichs–Wirtinger type inequality

Theorem 2.4. *Let λ^2 be the smallest positive eigenvalue of problem* (EVP) *and ψ the corresponding eigenfunction. Then for any $u \in W^1(\Omega) \cap C^0(\overline{\Omega})$, $u \not\equiv \mathrm{const} \neq 0$, we have*

$$\lambda^2 \int_\Omega u^2(\omega)\,d\omega \le \int_\Omega u'^2(\omega)\,d\omega + Bu^2(\omega_0/2) + \beta_- u^2(-\omega_0/2), \tag{2.4}$$

where

$$B = b\frac{\psi(0)}{\psi(\omega_0/2)} + \beta_+.$$

Proof. At first, we assume that $u \in C^2(\Omega) \cap W^1(\Omega) \cap C^0(\overline{\Omega})$. Setting $u(\omega) = \psi(\omega)v(\omega)$ we obtain

$$\begin{aligned}
[u'(\omega)]^2 &= [(\psi(\omega)v(\omega))']^2 \\
&= \psi'^2(\omega)v^2(\omega) + 2\psi'(\omega)\psi(\omega)v(\omega)v'(\omega) + \psi^2(\omega)v'^2(\omega) \\
&= \psi^2(\omega)v'^2(\omega) + [v^2(\omega)\psi(\omega)\psi'(\omega)]' - v^2(\omega)\psi(\omega)\psi''(\omega) \\
&\ge [v^2(\omega)\psi(\omega)\psi'(\omega)]' - v^2(\omega)\psi(\omega)\psi''(\omega).
\end{aligned}$$

Integrating over Ω and recalling that ψ is an infinitely differentiable solution of (EVP) we have

$$\begin{aligned}
\int_\Omega u'^2(\omega)\,d\omega &\ge v^2(\omega)\psi(\omega)\psi'(\omega)\Big|_{\omega=-\omega_0/2}^{\omega=\omega_0/2} - \int_\Omega v^2(\omega)\psi(\omega)\psi''(\omega)\,d\omega \\
&= u^2(\omega)\frac{\psi'(\omega)}{\psi(\omega)}\bigg|_{\omega=-\omega_0/2}^{\omega=\omega_0/2} + \lambda^2\int_\Omega u^2(\omega)\,d\omega \\
&= u^2(\omega_0/2)\left(-\beta_+ - b\frac{\psi(0)}{\psi(\omega_0/2)}\right) \\
&\quad - \beta_- u^2(-\omega_0/2) + \lambda^2 \int_\Omega u^2(\omega)\,d\omega.
\end{aligned} \tag{2.5}$$

Then from (2.5) we get (2.4). The extension of (2.4) to $u \in W^1(\Omega) \cap C^0(\overline{\Omega})$ follows directly by approximation.

Further, for $\lambda = \pi/\omega_0$, by Remark 2.3, from (2.4) it follows that $B = -\beta_-$ and therefore inequality (2.4) is false for $u \equiv \mathrm{const} \neq 0$. ∎

Remark 2.5. Inequality (2.4) is the best possible, i.e. the constant λ^2 in this inequality is sharp.

In fact, putting $\eta = \psi$ in (2.1) we obtain

$$\lambda^2 \int_\Omega \psi^2(\omega)\,d\omega = \int_\Omega \psi'^2(\omega)\,d\omega + \beta_+\psi^2(\omega_0/2) + \beta_-\psi^2(-\omega_0/2) + b\psi(0)\psi(\omega_0/2)$$

for any solution (λ^2, ψ) of (EVP). Now we see that the equality sign in (2.4) is attained for $u = \psi$, i.e. for the eigenfunction of (EVP).

Now we establish under what conditions the parameter B is positive. From the first boundary condition of (EVP) and (2.2) we get

$$(2.6)\quad B = b\frac{\psi(0)}{\psi(\omega_0/2)} + \beta_+ = \frac{-\psi'(\omega_0/2) - \beta_+\psi(\omega_0/2)}{\psi(\omega_0/2)} + \beta_+ = -\frac{\psi'(\omega_0/2)}{\psi(\omega_0/2)} = \frac{\lambda(\lambda\sin\lambda\omega_0 - \beta_-\cos\lambda\omega_0)}{\beta_-\sin\lambda\omega_0 + \lambda\cos\lambda\omega_0} \equiv B(\lambda).$$

LEMMA 2.6. *Let* (λ^2, ψ) *be a weak solution of* (EVP) *and let* $f(\lambda)$ *be defined by* (2.3). *Suppose that*

$$(2.7)\qquad \beta_\pm > 0, \quad 0 < b < \sqrt{2}\cdot\frac{\frac{\pi^2}{4\omega_0^2} - \beta_+\beta_-}{\frac{\pi}{2\omega_0} + \beta_-}, \quad \beta_+\beta_- < \left(\frac{\pi}{2\omega_0}\right)^2,$$

and let $\overline{\lambda} \in (0, \pi/(2\omega_0))$ *be a solution of*

$$(2.8)\qquad \tan(\overline{\lambda}\omega_0) = \frac{\beta_-}{\overline{\lambda}}.$$

Then the interval $(\overline{\lambda}, \pi/(2\omega_0))$ *contains the least positive zero* λ^* *of the function* $f(\lambda)$ *for which* $B(\lambda^*) > 0$. *Moreover,* $(\lambda^*)^2$ *is the least eigenvalue of* (EVP) *and the corresponding eigenfunction* ψ *is nonnegative.*

Proof. Let $\lambda \in (0, \pi/(2\omega_0)]$. Then from (2.6) it follows that $B(\lambda) > 0$ if $\lambda\sin\lambda\omega_0 - \beta_-\cos\lambda\omega_0 > 0$. From (2.8) for all $\lambda \in (0, \overline{\lambda})$ we have (by the graphical method)

$$\tan\lambda\omega_0 \le \frac{\beta_-}{\lambda}, \quad \text{so} \quad \cos\lambda\omega_0 \ge \frac{\lambda}{\beta_-}\sin\lambda\omega_0.$$

Therefore from (2.3) we get

$$f(\lambda) > \left(\frac{\beta_+}{\beta_-}\lambda^2 + \beta_+\beta_-\right)\sin\lambda\omega_0 > 0, \qquad \forall\lambda \in (0, \overline{\lambda}].$$

Further,

$$f\left(\frac{\pi}{2\omega_0}\right) = \beta_+\beta_- - \left(\frac{\pi}{2\omega_0}\right)^2 + \frac{b}{\sqrt{2}}\left(\frac{\pi}{2\omega_0} + \beta_-\right) < 0,$$

by (2.7). Hence, by (2.2) and because $f(\lambda)$ is continuous, the statement of the lemma follows. ∎

Remark 2.7. $\lambda^2 = 0$ is not an eigenvalue of (EVP). In fact, the solution of problem (EVP) with $\lambda^2 = 0$ has the form $\psi(\omega) = A_1\omega + A_2$, where A_1, A_2 are unknown constants. From the boundary conditions we obtain a homogeneous algebraic system for A_1, A_2,

$$\begin{cases} A_1 + \beta_+(A_1\omega_0/2 + A_2) + bA_2 = 0, \\ A_1 + \beta_-(A_1\omega_0/2 - A_2) = 0. \end{cases}$$

The determinant of this system is

$$\begin{vmatrix} 1 + \beta_+\frac{\omega_0}{2} & \beta_+ + b \\ 1 + \beta_-\frac{\omega_0}{2} & -\beta_- \end{vmatrix} \neq 0,$$

since $\beta_+, \beta_- > 0$ and $b \geq 0$. Thus $\psi(\omega) \equiv 0$ for any $\omega \in \Omega$.

Lemma 2.8. *Let λ^2 be the smallest positive eigenvalue of problem* (EVP) *with $\beta_+ = \beta_- = \beta$ and let ψ be the corresponding eigenfunction. Let $b > b^*$, where*

$$b^* = 2\frac{\omega_0(\widetilde{\lambda}^2 + \beta^2) + 2\beta}{\sqrt{\omega_0^2(\widetilde{\lambda}^2 + \beta^2) + 4\beta\omega_0 + 4}} \tag{2.9}$$

and $\widetilde{\lambda} \in (\pi/\omega_0, 2\pi/\omega_0)$ is a root of $\tan(\lambda\omega_0/2) = -\lambda\omega_0/(2 + \beta\omega_0)$. *Then λ satisfies the transcendental equation*

$$\beta \sin\frac{\lambda\omega_0}{2} + \lambda\cos\frac{\lambda\omega_0}{2} = 0, \qquad \lambda \in \left(\frac{\pi}{\omega_0}, \frac{2\pi}{\omega_0}\right), \tag{2.10}$$

and $B(\lambda) = \beta$.

Proof. By the assumption $\beta_+ = \beta_- = \beta$ and trigonometrical properties we can rewrite (2.3) in the form

$$\begin{aligned} f(\lambda) &= 2\lambda\beta\cos\lambda\omega_0 + (\beta^2 - \lambda^2)\sin\lambda\omega_0 + b\left(\lambda\cos\frac{\lambda\omega_0}{2} + \beta\sin\frac{\lambda\omega_0}{2}\right) \\ &= 2\lambda\beta\left(\cos^2\frac{\lambda\omega_0}{2} - \sin^2\frac{\lambda\omega_0}{2}\right) + 2(\beta^2 - \lambda^2)\sin\frac{\lambda\omega_0}{2}\cos\frac{\lambda\omega_0}{2} \\ &\quad + b\left(\lambda\cos\frac{\lambda\omega_0}{2} + \beta\sin\frac{\lambda\omega_0}{2}\right) \\ &= \left(\lambda\cos\frac{\lambda\omega_0}{2} + \beta\sin\frac{\lambda\omega_0}{2}\right)\left(b + 2\beta\cos\frac{\lambda\omega_0}{2} - 2\lambda\sin\frac{\lambda\omega_0}{2}\right) = 0. \end{aligned}$$

We now establish that

$$\chi(\lambda) := b + 2\beta\cos\frac{\lambda\omega_0}{2} - 2\lambda\sin\frac{\lambda\omega_0}{2} > 0$$

for all $\lambda \in (0, 2\pi/\omega_0)$. In fact, by calculation, we find that $\chi'(\widetilde{\lambda}) = 0$ and $\chi''(\widetilde{\lambda}) > 0$ for $\widetilde{\lambda} \in (\pi/\omega_0, 2\pi/\omega_0)$ satisfying $\tan(\lambda\omega_0/2) = -\lambda\omega_0/(\beta\omega_0 + 2)$.

Therefore

$$\inf_{\lambda\in(0,2\pi/\omega_0)}\chi(\lambda)=\chi(\widetilde{\lambda})=b-2\frac{\omega_0(\widetilde{\lambda}^2+\beta^2)+2\beta}{\sqrt{\omega_0^2(\widetilde{\lambda}^2+\beta^2)+4\beta\omega_0+4}}>0,$$

by assumption. Thus (2.10) is proved.

Now, we calculate $B(\lambda)$ for λ satisfying (2.10). By (2.2),

$$\psi(0)=\lambda\cos\frac{\lambda\omega_0}{2}+\beta\sin\frac{\lambda\omega_0}{2}=0,$$

$$\begin{aligned}\psi(\omega_0/2)&=\beta\sin\lambda\omega_0+\lambda\cos\lambda\omega_0\\&=2\beta\sin\frac{\lambda\omega_0}{2}\cos\frac{\lambda\omega_0}{2}+\lambda\cos^2\frac{\lambda\omega_0}{2}-\lambda\sin^2\frac{\lambda\omega_0}{2}\\&=\beta\sin\frac{\lambda\omega_0}{2}\cos\frac{\lambda\omega_0}{2}+\lambda\cos^2\frac{\lambda\omega_0}{2}+\beta\sin\frac{\lambda\omega_0}{2}\cos\frac{\lambda\omega_0}{2}-\lambda\sin^2\frac{\lambda\omega_0}{2}\\&=-\lambda\cos^2\frac{\lambda\omega_0}{2}-\lambda\sin^2\frac{\lambda\omega_0}{2}+\cos\frac{\lambda\omega_0}{2}\left(\beta\sin\frac{\lambda\omega_0}{2}+\lambda\cos\frac{\lambda\omega_0}{2}\right)=-\lambda\neq 0.\end{aligned}$$

Hence we get the desired result: $B=b\frac{\psi(0)}{\psi(\omega_0/2)}+\beta=\beta$. ∎

Taking into account Lemmas 2.6, 2.8 and Remark 2.3 we get the following formulations of Theorem 2.4 for the Friedrichs–Wirtinger type inequality:

COROLLARY 2.9. *Let the assumptions of Lemma* 2.6 *be satisfied, and* $\lambda=\lambda^*$, *where* λ^* *is defined by that lemma. Then*

$$\int_\Omega u^2(\omega)\,d\omega\le\frac{1}{\lambda^2}\biggl\{\int_\Omega\left(\frac{\partial u}{\partial\omega}\right)^2d\omega+Bu^2(\omega_0/2)+\beta_-u^2(-\omega_0/2)\biggr\}\tag{2.11}$$

for all $u\in W^1(\Omega)\cap C^0(\overline{\Omega})$ *with* $B=B(\lambda^*)$ *defined by* (2.6).

COROLLARY 2.10. *Let* $\beta_+=\beta_-=\beta>0$ *and* $b>b^*$, *where* b^* *is defined by* (2.9). *Then*

$$\int_\Omega u^2(\omega)\,d\omega\le\frac{1}{\lambda^2}\biggl\{\int_\Omega\left(\frac{\partial u}{\partial\omega}\right)^2d\omega+\beta u^2(\omega_0/2)+\beta u^2(-\omega_0/2)\biggr\}\tag{2.12}$$

for all $u\in W^1(\Omega)\cap C^0(\overline{\Omega})$, *where* $\lambda\in(\pi/\omega_0,2\pi/\omega_0)$ *is the smallest positive root of equation* (2.10).

COROLLARY 2.11. *Let* $b=\frac{\pi}{\omega_0}\cdot\frac{\beta_++\beta_-}{\beta_-}$. *Then*

$$\frac{\pi^2}{\omega_0^2}\int_\Omega u^2(\omega)\,d\omega+\beta_-u^2(\omega_0/2)\le\int_\Omega\left(\frac{\partial u}{\partial\omega}\right)^2d\omega+\beta_-u^2(-\omega_0/2)\tag{2.13}$$

for all $u\in W^1(\Omega)\cap C^0(\overline{\Omega})$, $u\not\equiv\text{const}\neq 0$.

Now using the well known Hardy inequality (see Theorem 330 of [4]) we get:

Proposition 2.12 (The Hardy–Friedrichs–Wirtinger inequality). *Let $u \in C^0(\overline{G_0^d}) \cap \mathring{W}^1_{\alpha-2}(G_0^d)$, $\alpha \le 2$, and let λ^2 be the smallest positive eigenvalue of problem* (EVP) *and $\psi \in W^1(\Omega) \cap C^0(\overline{\Omega})$ the corresponding eigenfunction. Then*

$$\int\limits_{G_0^d} r^{\alpha-4} u^2(x)\, dx \le \frac{1}{(2-\alpha)^2/4 + \lambda^2} \Big\{ \int\limits_{G_0^d} r^{\alpha-2} |\nabla u|^2\, dx + B \int\limits_{\Gamma_{0+}^d} r^{\alpha-3} u^2(x)\, ds + \beta_- \int\limits_{\Gamma_{0-}^d} r^{\alpha-3} u^2(x)\, ds \Big\} \tag{2.14}$$

with B defined by (2.4).

Proof. For the proof we refer to [1, Theorem 2.34]. ∎

Corollary 2.13. *Let the assumptions of Lemma* 2.6 *be satisfied, and let $\lambda = \lambda^*$, where λ^* is defined by that lemma. Let $u \in C^0(\overline{G_0^d}) \cap \mathring{W}^1_{\alpha-2}(G_0^d)$, $\alpha \le 2$. Then*

$$\int\limits_{G_0^d} r^{\alpha-4} u^2(x)\, dx \le \frac{1}{(2-\alpha)^2/4 + \lambda^2} \Big\{ \int\limits_{G_0^d} r^{\alpha-2} |\nabla u|^2\, dx + B \int\limits_{\Gamma_{0+}^d} r^{\alpha-3} u^2(x)\, ds + \beta_- \int\limits_{\Gamma_{0-}^d} r^{\alpha-3} u^2(x)\, ds \Big\}$$

with $B = B(\lambda^*)$ defined by (2.6).

Proof. Apply [1, Theorem 2.34] together with Corollary 2.9. ∎

Corollary 2.14. *Let $\beta_+ = \beta_- = \beta > 0$, $b > b^*$, where b^* is defined by* (2.9), *and $u \in C^0(\overline{G_0^d}) \cap \mathring{W}^1_{\alpha-2}(G_0^d)$, $\alpha \le 2$. Then we can rewrite the Hardy–Friedrichs–Wirtinger inequality* (2.14) *as*

$$\int\limits_{G_0^d} r^{\alpha-4} u^2(x)\, dx \le \frac{1}{(2-\alpha)^2/4 + \lambda^2} \Big\{ \int\limits_{G_0^d} r^{\alpha-2} |\nabla u|^2\, dx + \beta \int\limits_{\Gamma_{0+}^d} r^{\alpha-3} u^2(x)\, ds + \beta \int\limits_{\Gamma_{0-}^d} r^{\alpha-3} u^2(x)\, ds \Big\}, \tag{2.15}$$

where $\lambda \in (\pi/\omega_0, 2\pi/\omega_0)$ is the smallest positive root of equation (2.10).

Proof. Apply [1, Theorem 2.34] together with Corollary 2.10. ∎

Corollary 2.15. *Let $b = \frac{\pi}{\omega_0} \cdot \frac{\beta_+ + \beta_-}{\beta_-}$ and $u \in C^0(\overline{G_0^d}) \cap \mathring{W}^1_{\alpha-2}(G_0^d)$, $\alpha \le 2$. Then we can rewrite the Hardy–Friedrichs–Wirtinger inequality* (2.14) *as*

$$\int\limits_{G_0^d} r^{\alpha-4}u^2(x)\,dx \le \frac{1}{(2-\alpha)^2/4+\pi^2/\omega_0^2}\Big\{\int\limits_{G_0^d} r^{\alpha-2}|\nabla u|^2\,dx - \beta_-\int\limits_{\Gamma_{0+}^d} r^{\alpha-3}u^2(x)\,ds + \beta_-\int\limits_{\Gamma_{0-}^d} r^{\alpha-3}u^2(x)\,ds\Big\}. \tag{2.16}$$

Proof. Apply [1, Theorem 2.34] together with Corollary 2.11. ∎

LEMMA 2.16. *Let the assumptions of Lemma* 2.6 *be satisfied, let* $\lambda=\lambda^*$, *where* λ^* *is defined by that lemma, and let* B *be defined by* (2.6). *Let* $u\in C^0(\overline{G_0^d})\cap \mathring{W}_0^1(G_0^d)$. *Set*

$$U(\varrho)=\int\limits_{G_0^\varrho}|\nabla u|^2\,dx + B\int\limits_{\Gamma_{0+}^\varrho}\frac{u^2(x)}{r}\,ds + \beta_-\int\limits_{\Gamma_{0-}^\varrho}\frac{u^2(x)}{r}\,ds<\infty \tag{2.17}$$

for $\varrho\in(0,d)$. *Then*

$$\varrho\int\limits_{\Omega}\left(u\frac{\partial u}{\partial r}\right)\bigg|_{r=\varrho}d\omega \le \frac{\varrho}{2\lambda}U'(\varrho).$$

Proof. Writing $U(\varrho)$ in polar coordinates,

$$U(\varrho)=\int_0^\varrho r\int\limits_{\Omega}\left(\left|\frac{\partial u}{\partial r}\right|^2+\frac{1}{r^2}\left|\frac{\partial u}{\partial\omega}\right|^2\right)d\omega\,dr + B\int_0^\varrho\frac{u^2(r,\omega_0/2)}{r}\,dr + \beta_-\int_0^\varrho\frac{u^2(r,-\omega_0/2)}{r}\,dr$$

and differentiating with respect to ϱ we obtain

$$U'(\varrho)=\int\limits_{\Omega}\left(\varrho\left|\frac{\partial u}{\partial r}\right|^2+\frac{1}{\varrho}\left|\frac{\partial u}{\partial\omega}\right|^2\right)\bigg|_{r=\varrho}d\omega + B\frac{u^2(\varrho,\omega_0/2)}{\varrho}+\beta_-\frac{u^2(\varrho,-\omega_0/2)}{\varrho}. \tag{2.18}$$

Moreover, by Cauchy's inequality, we have

$$\rho u\frac{\partial u}{\partial r}\le\frac{\varepsilon}{2}u^2+\frac{1}{2\varepsilon}\rho^2\left(\frac{\partial u}{\partial r}\right)^2$$

for all $\varepsilon>0$. Thus, choosing $\varepsilon=\lambda$ we obtain, by the Friedrichs–Wirtinger inequality (2.11),

$$\varrho\int\limits_{\Omega}\left(u\frac{\partial u}{\partial r}\right)\bigg|_{r=\varrho}d\omega$$
$$\le\frac{\varepsilon}{2\lambda^2}\Big\{\int\limits_{\Omega}\left|\frac{\partial u}{\partial\omega}\right|^2_{r=\varrho}d\omega + Bu^2(\varrho,\omega_0/2)+\beta_-u^2(\varrho,-\omega_0/2)\Big\}+\frac{\varrho^2}{2\varepsilon}\int\limits_{\Omega}\left|\frac{\partial u}{\partial r}\right|^2_{r=\varrho}d\omega$$

$$= \frac{\varrho}{2\lambda}\left\{B\frac{u^2(\varrho,\omega_0/2)}{\varrho} + \beta_-\frac{u^2(\varrho,-\omega_0/2)}{\varrho} + \int_\Omega \left(\frac{1}{\varrho}\left|\frac{\partial u}{\partial \omega}\right|^2 + \varrho\left|\frac{\partial u}{\partial r}\right|^2\right)\Big|_{r=\varrho} d\omega\right\}$$

$$= \frac{\varrho}{2\lambda}U'(\varrho). \ \blacksquare$$

Applying Corollaries 2.10, 2.11 and repeating word for word the proof of Lemma 2.16 we derive the following corollaries:

Corollary 2.17. *Let $\beta_+ = \beta_- = \beta > 0$, $b > b^*$, where b^* is defined by* (2.9), *and $u \in C^0(\overline{G_0^d}) \cap \mathring{W}_0^1(G_0^d)$. Set*

$$U_+(\varrho) = \int_{G_0^\varrho} |\nabla u|^2\,dx + \beta \int_{\Gamma_{0+}^\varrho} \frac{u^2(x)}{r}\,ds + \beta \int_{\Gamma_{0-}^\varrho} \frac{u^2(x)}{r}\,ds < \infty \tag{2.19}$$

for $\varrho \in (0,d)$. Then

$$\varrho \int_\Omega \left(u\frac{\partial u}{\partial r}\right)\Big|_{r=\varrho} d\omega \le \frac{\varrho}{2\lambda}U_+'(\varrho),$$

where $\lambda \in (\pi/\omega_0, 2\pi/\omega_0)$ is the smallest positive root of (2.10).

Corollary 2.18. *Let $b = \frac{\pi}{\omega_0}\cdot\frac{\beta_+ + \beta_-}{\beta_-}$ and $u \in C^0(\overline{G_0^d}) \cap \mathring{W}_0^1(G_0^d)$. Set*

$$U_-(\varrho) = \int_{G_0^\varrho} |\nabla u|^2\,dx - \beta_- \int_{\Gamma_{0+}^\varrho} \frac{u^2(x)}{r}\,ds + \beta_- \int_{\Gamma_{0-}^\varrho} \frac{u^2(x)}{r}\,ds < \infty \tag{2.20}$$

for $\varrho \in (0,d)$. Then

$$\varrho \int_\Omega \left(u\frac{\partial u}{\partial r}\right)\Big|_{r=\varrho} d\omega \le \varrho\frac{\omega_0}{2\pi}U_-'(\varrho).$$

We also need the well known inequalities (see e.g. [5, Chapter I, (6.23), (6.24)] or [7, Lemma 6.36])

$$\int_\Gamma v\,ds \le C\int_G (|v| + |\nabla v|)\,dx, \qquad \forall v \in W^{1,1}(G),\ \forall \Gamma \subseteq \partial G,$$

$$\int_{\partial G} v^2\,ds \le \int_G \left(\delta|\nabla v|^2 + \frac{1}{\delta}c_0 v^2\right)dx, \qquad \forall v \in W^{1,2}(G),\ \forall \delta > 0, \tag{2.21}$$

and the following lemma.

Lemma 2.19. *Let $u \in C^0(\overline{G_0^d}) \cap \mathring{W}_{\alpha-2}^1(G_0^d)$. Then*

$$\int_{\Gamma_{0+}^d} r^{\alpha-3}u(x)u(\gamma(x))\,ds = \int_{\Gamma_{0+}^d} r^{\alpha-3}u^2(x)\,ds - \int_0^d r^{\alpha-3}u(r,\omega_0/2)\left(\int_0^{\omega_0/2} \frac{\partial u(r,\omega)}{\partial \omega}\,d\omega\right)dr \tag{2.22}$$

and

$$\int_0^d r^{\alpha-3}u(r,\omega_0/2)\Big(\int_0^{\omega_0/2}\frac{\partial u(r,\omega)}{\partial\omega}\,d\omega\Big)\,dr \le \frac{\varepsilon}{2}\int_{G_0^d} r^{\alpha-2}|\nabla u|^2\,dx + \frac{\omega_0}{2\varepsilon}\int_{\Gamma_{0+}^d} r^{\alpha-3}u^2(x)\,ds,\ \forall\varepsilon>0.$$

Proof. Because $u(x)|_{\Gamma_{0+}^d} = u(r,\omega_0/2)$, and $u(\gamma(x))|_{\Gamma_{0+}^d} = u(r,0)$ by Remark 1.1, using the representation $u(r,0) = u(r,\omega_0/2) - \int_0^{\omega_0/2}\frac{\partial u(r,\omega)}{\partial\omega}\,d\omega$ we obtain (2.22).

Next, by the Cauchy inequality, we have

$$\begin{aligned}
\int_0^d r^{\alpha-3}u(r,\omega_0/2)&\Big(\int_0^{\omega_0/2}\frac{\partial u(r,\omega)}{\partial\omega}\,d\omega\Big)\,dr\\
&\le \int_{G_0^d} r^{\alpha-4}\Big|\frac{\partial u(r,\omega)}{\partial\omega}\Big|\,|u(r,\omega_0/2)|\,dx\\
&\le \frac{\varepsilon}{2}\int_{G_0^d} r^{\alpha-4}\Big|\frac{\partial u(r,\omega)}{\partial\omega}\Big|^2\,dx + \frac{1}{2\varepsilon}\int_{G_0^d} u^2(r,\omega_0/2)\,dx\\
&\le \frac{\varepsilon}{2}\int_{G_0^d} r^{\alpha-2}|\nabla u|^2\,dx + \frac{1}{2\varepsilon}\int_0^d\int_{-\omega_0/2}^{\omega_0/2} r^{\alpha-3}u^2(r,\omega_0/2)\,d\omega\,dr\\
&\le \frac{\varepsilon}{2}\int_{G_0^d} r^{\alpha-2}|\nabla u|^2\,dx + \frac{\omega_0}{2\varepsilon}\int_{\Gamma_{0+}^d} r^{\alpha-3}u^2(x)\,ds, \qquad \forall\varepsilon>0. \ \blacksquare
\end{aligned}$$

2.3. The Cauchy problem for a differential inequality

THEOREM 2.20. *Let U be an increasing, nonnegative differentiable function defined on $[0,d]$ and satisfying*

$$\text{(CP)}\qquad \begin{cases} U'(\varrho) - \mathcal{P}(\varrho)U(\varrho) + \mathcal{Q}(\varrho) \ge 0, & 0<\varrho<d,\\ U(d)\le U_0, \end{cases}$$

where $\mathcal{P},\mathcal{Q}$ are nonnegative continuous functions defined on $[0,d]$, and U_0 is a constant. Then

$$U(\varrho) \le U_0\exp\Big(-\int_\varrho^d \mathcal{P}(\tau)\,d\tau\Big) + \int_\varrho^d \mathcal{Q}(\tau)\exp\Big(-\int_\varrho^\tau \mathcal{P}(\sigma)\,d\sigma\Big)\,d\tau. \tag{2.23}$$

Proof. For the proof we refer to [1, §1.10, Theorem 1.57]. ■

3. Maximum principle. The goal of this section is to derive an *a priori* $L_\infty(G)$-estimate of a weak solution to problem (L).

THEOREM 3.1. *Let u be a weak solution of* (L) *and let assumptions* (a)–(c) *be satisfied. In addition, suppose that $j > 1$, $t > p/2$, $p > 2$, $h^2 \in L_{j/(j-1)}(\Gamma_+)$, $g^2 \in L_{j/(j-1)}(\Gamma_-)$, $f^2 \in L_t(G)$, and $c(x) \le -c_0 < 0$ for all $x \in G$, where c_0 is large enough, positive and depends only on ν, p, and $\|\sum_{i=1}^2 |b^i(\cdot)|^2\|_{p/2,G}$. Then there exists a constant $M_0 > 0$, depending only on* $\operatorname{meas} G$, $\operatorname{meas}\Gamma_+, \operatorname{meas}\Gamma_-$, *$\nu$, p, $\|h^2\|_{j/(j-1),\Gamma_+}$, $\|g^2\|_{j/(j-1),\Gamma_-}$, $\|f^2\|_{t,G}$, b, β_+, β_-, ω_0, such that $\|u\|_{\infty,G} \le M_0$.*

Proof. Set $A(k) = \{x \in \overline{G} : u > k\}$ for $k \ge k_0 > 0$ (without loss of generality, we can assume $k_0 \ge 1$). We note that $A(k+d) \subseteq A(k)$ for all $d > 0$. Taking $\eta(x) = \max(u(x) - k, 0)$ as a test function in (II), by assumption we get

$$\begin{aligned}
(3.1)\qquad & \nu \int_{A(k)} |\nabla u|^2\,dx + c_0 \int_{A(k)} u(x)(u(x)-k)\,dx \\
&+ \beta_+ \int_{\Gamma_+\cap A(k)} \frac{u(x)(u(x)-k)}{r}\,ds \\
&+ b \int_{\Gamma_+\cap A(k)} \frac{1}{r} u(\gamma(x))(u(x)-k)\,ds + \beta_- \int_{\Gamma_-\cap A(k)} \frac{u(x)(u(x)-k)}{r}\,ds \\
\le & \int_{A(k)} \sqrt{\sum_{i=1}^2 |b^i(x)|^2} \cdot |\nabla u|(u(x)-k)\,dx + \int_{A(k)} |f(x)|(u(x)-k)\,dx \\
&+ \int_{\Gamma_+\cap A(k)} |g(x)|(u(x)-k)\,ds + \int_{\Gamma_-\cap A(k)} |h(x)|(u(x)-k)\,ds.
\end{aligned}$$

Now, we estimate the first integral on the right of (3.1). By assumption (b), the Cauchy inequality and the Hölder inequality with exponents $q = p/2$ and $q' = \frac{p}{p-2}$, $p > 2$,

$$\begin{aligned}
(3.2)\qquad & \int_{A(k)} \sqrt{\sum_{i=1}^2 |b^i(x)|^2} \cdot |\nabla u|(u(x)-k)\,dx \\
\le & \frac{\nu}{4} \int_{A(k)} |\nabla u|^2\,dx + \frac{1}{\nu} \int_{A(k)} \sum_{i=1}^2 |b^i(x)|^2 (u(x)-k)^2\,dx \\
\le & \frac{\nu}{4} \int_{A(k)} |\nabla u|^2\,dx + \frac{1}{\nu} \Big(\int_{A(k)} \Big(\sum_{i=1}^2 |b^i(x)|^2\Big)^{p/2} dx\Big)^{2/p} \cdot \|u(x)-k\|^2_{2p/(p-2),A(k)}.
\end{aligned}$$

Next we apply the inequality

$$\|u\|^2_{\frac{2p}{p-2},G} \le \delta\|\nabla u\|^2_{2,G} + c(\delta,p,G)\|u\|^2_{2,G}, \qquad p>2,\ \forall\delta>0,$$

(see for example [6, Ch. II, §2, (2.19)]). From (3.2) it follows that

$$(3.3)\qquad \int\limits_{A(k)} \sqrt{\sum_{i=1}^{2}|b^i(x)|^2}\cdot|\nabla u|(u(x)-k)\,dx \le \frac{\nu}{4}\int\limits_{A(k)}|\nabla u|^2\,dx$$
$$+\frac{1}{\nu}\Big\|\sum_{i=1}^{2}|b^i(\cdot)|^2\Big\|_{p/2,G}\int\limits_{A(k)}(\delta|\nabla u|^2 + c(\delta,p,G)(u(x)-k)^2)\,dx, \qquad \forall\delta>0.$$

We choose

$$\delta = \frac{\nu^2}{4\|\sum_{i=1}^{2}|b^i(\cdot)|^2\|_{p/2,G}}.$$

Since $b\int_{\Gamma_+\cap A(k)}(1/r)u(\gamma(x))(u(x)-k)\,ds>0$, from (3.1)–(3.3) it follows that

$$\frac{\nu}{2}\int\limits_{A(k)}|\nabla u|^2\,dx + \Big[c_0 - c\Big(\nu,p,G,\Big\|\sum_{i=1}^{2}|b^i(\cdot)|^2\Big\|_{p/2,G}\Big)\Big]$$
$$\times\int\limits_{A(k)}u(x)(u(x)-k)\,dx + \beta_+\int\limits_{\Gamma_+\cap A(k)}\frac{u(x)(u(x)-k)}{r}\,ds$$
$$+\beta_-\int\limits_{\Gamma_-\cap A(k)}\frac{u(x)(u(x)-k)}{r}\,ds$$
$$\le \int\limits_{A(k)}|f(x)|(u(x)-k)\,dx$$
$$+\int\limits_{\Gamma_+\cap A(k)}|g(x)|(u(x)-k)\,ds + \int\limits_{\Gamma_-\cap A(k)}|h(x)|(u(x)-k)\,ds.$$

Next, since c_0 is large enough and positive, we can rewrite the above inequality as

$$(3.4)\qquad \frac{\nu}{2}\int\limits_{A(k)}|\nabla u|^2\,dx + \beta_+\int\limits_{\Gamma_+\cap A(k)}\frac{u(x)(u(x)-k)}{r}\,ds$$
$$+\beta_-\int\limits_{\Gamma_-\cap A(k)}\frac{u(x)(u(x)-k)}{r}\,ds$$
$$\le \int\limits_{A(k)}|f(x)|(u(x)-k)\,dx$$
$$+\int\limits_{\Gamma_+\cap A(k)}|g(x)|(u(x)-k)\,ds + \int\limits_{\Gamma_-\cap A(k)}|h(x)|(u(x)-k)\,ds.$$

Now we estimate every term on the right hand side of (3.4) by the Cauchy inequality:

$$\int\limits_{A(k)} |f(x)|(u(x)-k)\,dx = \int\limits_{A(k)} \left(\frac{u(x)-k}{r}\right)(r|f(x)|)\,dx$$
$$\le \frac{\varepsilon}{2}\int\limits_{A(k)} \frac{(u(x)-k)^2}{r^2}\,dx + \frac{(\operatorname{diam} G)^2}{2\varepsilon}\int\limits_{A(k)} f^2(x)\,dx, \qquad \forall\varepsilon>0,$$

$$\int\limits_{\Gamma_+\cap A(k)} |g(x)|(u(x)-k)\,ds \le \int\limits_{\Gamma_+\cap A(k)} \left(\frac{u(x)-k}{\sqrt{r}}\right)(\sqrt{r}|g(x)|)\,ds$$
$$\le \frac{\operatorname{diam} G}{2\varepsilon_1}\int\limits_{\Gamma_+\cap A(k)} g^2(x)\,ds + \frac{\varepsilon_1}{2}\int\limits_{\Gamma_+\cap A(k)} \frac{(u(x)-k)^2}{r}\,ds$$
$$\le \frac{\operatorname{diam} G}{2\varepsilon_1}\int\limits_{\Gamma_+\cap A(k)} g^2(x)\,ds + \frac{\varepsilon_1}{2}\int\limits_{\Gamma_+\cap A(k)} \frac{u(x)(u(x)-k)}{r}\,ds$$

for all $\varepsilon_1>0$. In the same way

$$\int\limits_{\Gamma_-\cap A(k)} |h(x)|(u(x)-k)\,ds \le \frac{\operatorname{diam} G}{2\varepsilon_2}\int\limits_{\Gamma_-\cap A(k)} h^2(x)\,ds + \frac{\varepsilon_2}{2}\int\limits_{\Gamma_-\cap A(k)} \frac{u(x)(u(x)-k)}{r}\,ds$$

for all $\varepsilon_2>0$. Then if we choose $\varepsilon_1=\beta_+$ and $\varepsilon_2=\beta_-$, inequality (3.4) takes the form

$$\begin{aligned}(3.5)\qquad & \frac{\nu}{2}\int\limits_{A(k)} |\nabla u|^2\,dx + \frac{1}{2}\beta_+\int\limits_{\Gamma_+\cap A(k)} \frac{u(x)(u(x)-k)}{r}\,ds \\ &+ \frac{1}{2}\beta_-\int\limits_{\Gamma_-\cap A(k)} \frac{u(x)(u(x)-k)}{r}\,ds \\ \le{}& \frac{\varepsilon}{2}\int\limits_{A(k)} \frac{(u(x)-k)^2}{r^2}\,dx \\ &+ \frac{(\operatorname{diam} G)^2}{2\varepsilon}\int\limits_{A(k)} f^2(x)\,dx + c_1\int\limits_{\Gamma_+\cap A(k)} g^2(x)\,ds + c_2\int\limits_{\Gamma_-\cap A(k)} h^2(x)\,ds\end{aligned}$$

for all $\varepsilon>0$, with $c_1=\operatorname{diam} G/(2\beta_+)$, $c_2=\operatorname{diam} G/(2\beta_-)$.

Now we estimate the first integral on the right of (3.5). First we use the representation $G=G_0^d\cup G_d$. The integral over G_0^d is estimated by (2.14) with $\alpha=2$; to estimate the integral over G_d we use the Friedrichs inequality (see [9, (30.5)])

$$\int_G \eta^2(x)\,dx \le K_1\Big\{\int_G |\nabla\eta|^2\,dx + \int_{\partial G}\eta^2(x)\,ds\Big\}, \tag{3.6}$$

where K_1 depends on $\operatorname{meas} G$ and $\operatorname{diam} G$. Then from (3.5) and the definition of η we obtain

$$\begin{aligned}
&\frac{\nu}{2}\int_{A(k)}|\nabla\eta|^2\,dx + \frac{\beta_+}{2}\int_{\Gamma_+\cap A(k)}\frac{1}{r}\eta^2(x)\,ds + \frac{\beta_-}{2}\int_{\Gamma_-\cap A(k)}\frac{1}{r}\eta^2(x)\,ds\\
&\quad\le \frac{\varepsilon}{2\lambda^2}\Big\{\int_{A(k)}|\nabla\eta|^2\,dx + B\int_{A(k)\cap\Gamma_+}\frac{1}{r}\eta^2(x)\,ds + \beta_-\int_{A(k)\cap\Gamma_-}\frac{1}{r}\eta^2(x)\,ds\Big\}\\
&\qquad + \frac{\varepsilon}{2}K_1 d^{-2}\Big\{\int_{A(k)}|\nabla\eta|^2\,dx + \operatorname{diam} G\int_{\partial G\cap A(k)}\frac{1}{r}\eta^2(x)\,ds\Big\}\\
&\qquad + \frac{(\operatorname{diam} G)^2}{2\varepsilon}\int_{A(k)} f^2(x)\,dx + c_1\int_{\Gamma_+\cap A(k)} g^2(x)\,ds + c_2\int_{\Gamma_-\cap A(k)} h^2(x)\,ds
\end{aligned}$$

for all $\varepsilon > 0$, where B, λ are defined according to Proposition 2.12. Now, if we choose

$$0 < \varepsilon \le \min\Big\{\frac{\nu}{2(\frac{1}{\lambda^2} + K_1 d^{-2})};\ \frac{1}{2}\cdot\frac{\beta_+}{\frac{B}{\lambda^2} + \frac{K_1 \operatorname{diam} G}{d^2}};\ \frac{1}{2}\cdot\frac{\beta_-}{\frac{\beta_-}{\lambda^2} + \frac{K_1 \operatorname{diam} G}{d^2}}\Big\},$$

then we get

$$\begin{aligned}
&\int_{A(k)}|\nabla\eta|^2\,dx + \int_{\partial G\cap A(k)}\frac{\eta^2(x)}{r}\,ds\\
&\qquad\le C\Big\{\int_{A(k)} f^2(x)\,dx + \int_{\Gamma_+\cap A(k)} g^2(x)\,ds + \int_{\Gamma_-\cap A(k)} h^2(x)\,ds\Big\},
\end{aligned}\tag{3.7}$$

where C depends only on $\lambda, b, \beta_+, \beta_-, \omega_0, d, \nu, \operatorname{meas} G$ and $\operatorname{diam} G$. Further, because $\int_{\partial G}\eta^2(x)\,ds \le \operatorname{diam} G\cdot\int_{\partial G}(\eta^2(x)/r)\,ds$, from (3.6) and (3.7) it follows that

$$\begin{aligned}
&\int_{A(k)}(|\nabla\eta|^2 + \eta^2(x))\,dx\\
&\qquad\le \widetilde{C}\Big\{\int_{A(k)} f^2(x)\,dx + \int_{\Gamma_+\cap A(k)} g^2(x)\,ds + \int_{\Gamma_-\cap A(k)} h^2(x)\,ds\Big\}.
\end{aligned}\tag{3.8}$$

By the Sobolev embedding theorem (see [6, §2, Ch. II] or [11])

$$\Big(\int_{A(k)}\eta^{\frac{2p}{p-2}}\,dx\Big)^{\frac{p-2}{p}} + \Big(\int_{\partial G\cap A(k)}\eta^{j^*}\,ds\Big)^{2/j^*} \le \widetilde{c}\int_{A(k)}(|\nabla\eta|^2 + \eta^2(x))\,dx$$

for all $j^* > 1$ and $p > 2$; from (3.8) we obtain

$$\Big(\int_{A(k)} \eta^{\frac{2p}{p-2}}\,dx\Big)^{\frac{p-2}{p}} + \Big(\int_{\partial G\cap A(k)} \eta^{j^*}\,ds\Big)^{2/j^*} \le \widetilde{c}\widetilde{C}\Big\{\int_{A(k)} f^2(x)\,dx + \int_{\Gamma_+\cap A(k)} g^2(x)\,ds + \int_{\Gamma_-\cap A(k)} h^2(x)\,ds\Big\}, \quad \forall j^* > 1,\ p > 2. \tag{3.9}$$

Let now $l > k > k_0$. By the definitions of $\eta(x)$ and $A(k)$ we have

$$\int_{A(k)} \eta^{\frac{2p}{p-2}}\,dx \ge \int_{A(l)} \eta^{\frac{2p}{p-2}}\,dx \ge \operatorname{meas}[A(l)]\cdot(l-k)^{\frac{2p}{p-2}},$$

$$\int_{\partial G\cap A(k)} \eta^{j^*}\,ds \ge \int_{\partial G\cap A(l)} \eta^{j^*}\,ds \ge \operatorname{meas}[\partial G\cap A(l)]\cdot(l-k)^{j^*}.$$

Further by the Hölder inequality we get

$$\int_{A(k)} f^2(x)\,dx \le (\operatorname{meas}[A(k)])^{1-1/t}\cdot\|f^2\|_{t,A(k)}, \quad t > 1,$$

$$\int_{\Gamma_+\cap A(k)} g^2(x)\,dx \le (\operatorname{meas}[\Gamma_+\cap A(k)])^{1/j}\cdot\|g^2\|_{j',\Gamma_+\cap A(k)},$$

$$\int_{\Gamma_-\cap A(k)} h^2(x)\,dx \le (\operatorname{meas}[\Gamma_-\cap A(k)])^{1/j}\cdot\|h^2\|_{j',\Gamma_-\cap A(k)},$$

for all $j, j' > 1$ (with $1/j + 1/j' = 1$). From these inequalities and (3.9) we get

$$\begin{aligned}(\operatorname{meas}[A(l)])^{\frac{p-2}{p}}\cdot(l-k)^2 + (\operatorname{meas}[\partial G\cap A(l)])^{2/j^*}\cdot(l-k)^2 \\ \le \widetilde{c}\widetilde{C}\{\|f^2\|_{t,A(k)}\cdot(\operatorname{meas}[A(k)])^{1-1/t} + (\operatorname{meas}[\Gamma_+\cap A(k)])^{1/j} \\ \times\|g^2\|_{j',\Gamma_+\cap A(k)} + (\operatorname{meas}[\Gamma_-\cap A(k)])^{1/j}\cdot\|h^2\|_{j',\Gamma_-\cap A(k)}\},\ \forall p > 2.\end{aligned} \tag{3.10}$$

Now, by the Jensen inequality ([4, Theorem 65]), from (3.10) it follows that

$$\begin{aligned}\operatorname{meas}[A(l)] + (\operatorname{meas}[\partial G\cap A(l)])^{\frac{2p}{j^*(p-2)}} \\ \le \frac{\overline{\widetilde{C}}}{(l-k)^{\frac{2p}{p-2}}}\{(\operatorname{meas} A(k))^{\frac{(t-1)p}{t(p-2)}} \\ + (\operatorname{meas}[\Gamma_+\cap A(k)])^{\frac{p}{j(p-2)}} + (\operatorname{meas}[\Gamma_-\cap A(k)])^{\frac{p}{j(p-2)}}\},\ \forall p > 2,\end{aligned} \tag{3.11}$$

where $\overline{\widetilde{C}}$ depends only on $b, \beta_+, \beta_-, \omega_0, \nu, t, p, \|f^2\|_{t,G}, \|g^2\|_{j',\Gamma_+}, \|h^2\|_{L_{j'}(\Gamma_-)}$, λ, d, $\operatorname{meas} G$ and $\operatorname{diam} G$. Now we set

$$\psi(k) = \operatorname{meas}[A(k)] + (\operatorname{meas}[A(k)\cap\partial G])^{\frac{2p}{j^*(p-2)}}.$$

Then from (3.11) we obtain

$$\psi(l) \le \widetilde{C}\frac{1}{(l-k)^{\frac{2p}{p-2}}}\left([\psi(k)]^{\frac{(t-1)p}{t(p-2)}} + [\psi(k)]^{\frac{j^*}{2j}}\right) \tag{3.12}$$

for all $l > k \ge k_0$, $p > 2$ and $t > 1$. Choosing $j^* > 2j$, we observe that $\min\left(\frac{p(t-1)}{t(p-2)}, \frac{j^*}{2j}\right) > 1$, since $t > p/2$ by assumption. Then from (3.12) we get

$$\psi(l) \le \frac{C}{(l-k)^{\frac{2p}{p-2}}}[\psi(k)]^{\beta}, \quad \beta > 1,\ l > k \ge k_0, p > 2,$$

and therefore, by the Stampacchia Lemma (see Lemma 3.11 of [8]), $\psi(k_0+\delta) = 0$ with δ depending only on the quantities in the formulation of Theorem 3.1. This means that $u(x) < k_0 + \delta$ for almost all $x \in G$.

Similarly, we derive $u(x) > -k_0-\delta$ if we set $A(k) = \{x \in \overline{G} : u(x) < -k\}$ for all $k \ge k_0 > 0$ and choose in (II) $\eta(x) = \min(u(x)+k, 0)$ as a test function. Thus, Theorem 3.1 is proved. ■

4. Local estimate at the boundary

THEOREM 4.1. *Let u be a weak solution of problem* (L) *and let assumptions* (a)–(d) *be satisfied. Suppose, in addition, that either*

(i) $0 < b < \frac{1}{\omega_0}(\nu + \sqrt{\nu^2 + 2\nu\omega_0\beta_+})$, *or*
(ii) $u(x) \ge 0$ *for* $x \in \overline{G}$, *or*
(iii) $\beta_+ u^2(x)|_{\Gamma_+} + \beta_- u^2(x)|_{\Gamma_-} + bu(x)|_{\Gamma_+} \cdot u(\gamma(x))|_{\Gamma_+} = 0.$

Then

$$\sup_{\varkappa \in G_0^{\varkappa\varrho}} |u(x)| \le \frac{C}{(1-\varkappa)^{\widetilde{n}/2}}\{\varrho^{-1}\|u\|_{2,G_0^\varrho} + \varrho^{2(1-2/p)}\|f\|_{p/2,G_0^\varrho} + \varrho(\|g\|_{\infty,\Gamma_{0+}^\varrho} + \|h\|_{\infty,\Gamma_{0-}^\varrho})\} \tag{4.1}$$

for any $\widetilde{n} > 2$, $p > \widetilde{n}$, $\varkappa \in (0,1)$ and $\varrho \in (0,d)$, where C is a constant depending on μ, ν, p, $\|\sum_{i=1}^2 |b^i(\cdot)|^2\|_{p/2,G}$ and the domain G.

Proof. We apply the Moser iteration method. We consider the integral identity (II) and make the coordinate transformation $x = \varrho x'$. Let G' be the image of G, Γ'_+ the image of Γ_+, and Γ'_- the image of Γ_-. Then $dx = \varrho^2 dx'$, $ds = \varrho ds'$. In addition, we denote

$$v(x') = u(\varrho x'), \quad v(\gamma(x')) = u(\gamma(\varrho x')), \quad \eta(x') = \eta(\varrho x'),$$
$$\mathcal{F}(x') = \varrho^2 f(\varrho x'), \quad \mathcal{G}(x') = \varrho g(\varrho x'), \quad \mathcal{H}(x') = \varrho h(\varrho x').$$

Then from (II) we get

$$\text{(II)}' \quad \int\limits_{G'} \{a^{ij}(\varrho x')v_{x'_j}\eta_{x'_i} - \varrho b^i(\varrho x')v_{x'_i}\eta(x') - \varrho^2 c(\varrho x')v(x')\eta(x')\}\,dx' + \int\limits_{\Gamma'_+} \left(\frac{\beta_+}{|x'|}v(x') + \frac{b}{|x'|}v(\gamma(x'))\right)\eta(x')\,ds' + \beta_- \int\limits_{\Gamma'_-} \frac{v(x')}{|x'|}\eta(x')\,ds'$$
$$= \int\limits_{\Gamma'_+} \mathcal{G}(x')\eta(x')\,ds' + \int\limits_{\Gamma'_-} \mathcal{H}(x')\eta(x')\,ds' - \int\limits_{G'} \mathcal{F}(x')\eta(x')\,dx'$$

for all $\eta \in C^0(\overline{G'}) \cap \mathring{W}_0^1(G')$. We define

$$m = m(\varrho) = \frac{1}{\nu}(\|\mathcal{F}\|_{p/2,G_0^1} + \|\mathcal{G}\|_{\infty,\Gamma_{0+}^1} + \|\mathcal{H}\|_{\infty,\Gamma_{0-}^1}) \tag{4.2}$$

and

$$\overline{v}(x') = |v(x')| + m. \tag{4.3}$$

We observe that

$$\begin{aligned} |\mathcal{F}(x')|\overline{v}(x') &= \frac{1}{m}|\mathcal{F}(x')| \cdot m\overline{v}(x') = \frac{1}{m}|\mathcal{F}(x')|(\overline{v}(x') - |v(x')|) \cdot \overline{v}(x') \\ &= \frac{1}{m}|\mathcal{F}(x')| \cdot \overline{v}^2(x') - \frac{1}{m}|\mathcal{F}(x')| \cdot |v(x')|\overline{v}(x') \\ &\le \frac{1}{m}|\mathcal{F}(x')| \cdot \overline{v}^2(x'); \\ |\mathcal{H}(x')|\overline{v}(x') &\le \frac{1}{m}|\mathcal{H}(x')| \cdot \overline{v}^2(x'); \qquad |\mathcal{G}(x')|\overline{v}(x') \le \frac{1}{m}|\mathcal{G}(x')| \cdot \overline{v}^2(x') \end{aligned} \tag{4.4}$$

in the same way. As a test function in (II)$'$ we choose $\eta(x') = \zeta^2(|x'|)v(x')$, where $\zeta(|\cdot|) \in C_0^\infty([0,1])$ is a nonnegative function to be further specified. By the chain and product rules, η is a valid test function in (II)$'$ and also $\eta_{x'_i} = v_{x'_i}\zeta^2(|x'|) + 2\zeta(|x'|)\zeta_{x'_i}v(x')$, so that by substitution into (II)$'$, in view of $c(\varrho x') \le 0$ in G' and $v \le |v| \le \overline{v}$, we obtain

$$\int\limits_{G_0^1} a^{ij}(\varrho x')v_{x'_i}v_{x'_j}\zeta^2(|x'|)\,dx' + \beta_+ \int\limits_{\Gamma_{0+}^1} \frac{v^2(x')}{|x'|}\zeta^2(|x'|)\,ds' + b\int\limits_{\Gamma_{0+}^1} \frac{v(x')}{|x'|}v(\gamma(x'))\zeta^2(|x'|)\,ds' + \beta_- \int\limits_{\Gamma_{0-}^1} \frac{v^2(x')}{|x'|}\zeta^2(|x'|)\,ds'$$
$$\le \varrho \int\limits_{G_0^1} |b^i(\varrho x')v_{x_i}|\overline{v}(x')\zeta^2(|x'|)\,dx' + 2\int\limits_{G_0^1} |a^{ij}(\varrho x')\zeta_{x'_i}v_{x'_j}|\overline{v}(x')\zeta(|x'|)\,dx' + \int\limits_{\Gamma_{0-}^1} \mathcal{H}(x')\overline{v}(x')\zeta^2(|x'|)\,ds' + \int\limits_{\Gamma_{0+}^1} \mathcal{G}(x')\overline{v}(x')\zeta^2\,ds' + \int\limits_{G_0^1} \mathcal{F}(x')\overline{v}(x')\zeta^2\,dx'.$$

By ellipticity and (4.4) it follows that

$$
\begin{aligned}
(4.5)\qquad &\int\limits_{G_0^1} \nu|\nabla' v|^2\zeta^2(|x'|)\,dx' + \beta_+ \int\limits_{\Gamma_{0+}^1} \frac{v^2(x')}{|x'|}\zeta^2(|x'|)\,ds' \\
&+ b\int\limits_{\Gamma_{0+}^1} \frac{v(x')}{|x'|} v(\gamma(x'))\zeta^2(|x'|)\,ds' + \beta_- \int\limits_{\Gamma_{0-}^1} \frac{v^2(x')}{|x'|}\zeta^2(|x'|)\,ds' \\
\le{}& \int\limits_{G_0^1} \varrho\Big(\sum_{i=1}^{2}|b^i(\varrho x')|^2\Big)^{1/2}|\nabla' v|\overline{v}(x')\zeta^2(|x'|)\,dx' \\
&+ 2\mu\int\limits_{G_0^1} |\nabla' v|\cdot|\nabla'\zeta|\overline{v}(x')\zeta(|x'|)\,dx' + \frac{1}{m}\|\mathcal{G}\|_{\infty,\Gamma_{0+}^1}\int\limits_{\Gamma_{0+}^1}\overline{v}^2(x')\zeta^2(|x'|)\,ds' \\
&+ \frac{1}{m}\|\mathcal{H}\|_{\infty,\Gamma_{0-}^1}\int\limits_{\Gamma_{0-}^1}\overline{v}^2(x')\zeta^2(|x'|)\,ds' + \frac{1}{m}\int\limits_{G_0^1}|\mathcal{F}(x')|\overline{v}^2(x')\zeta^2(|x'|)\,dx'.
\end{aligned}
$$

It is obvious that if assumption (ii) or (iii) is satisfied, then

$$
\beta_+ \int\limits_{\Gamma_{0+}^1} \frac{v^2(x')}{|x'|}\zeta^2(|x'|)\,ds' + \beta_- \int\limits_{\Gamma_{0-}^1} \frac{v^2(x')}{|x'|}\zeta^2(|x'|)\,ds' + b\int\limits_{\Gamma_{0+}^1} \frac{v(x')}{|x'|}v(\gamma(x'))\zeta^2(|x'|)\,ds' \ge 0.
$$

We now estimate the last integral on the left hand side of (4.5) in case (i). Because $v(x')|_{\Gamma_{0+}^1} = v(r',\omega_0/2)$ and, by Remark 1.1, $v(\gamma(x'))|_{\Gamma_{0+}^1} = v(r',0)$, using the representation $v(r',0) = v(r',\omega_0/2) - \int_0^{\omega_0/2}\frac{\partial v(r',\omega)}{\partial\omega}\,d\omega$ we obtain

$$
\begin{aligned}
(4.6)\qquad \int\limits_{\Gamma_{0+}^1} \frac{v(x')}{|x'|}v(\gamma(x'))\zeta^2(|x'|)\,ds' ={}& \int_0^1 \frac{v^2(r',\omega_0/2)}{r'}\zeta^2(r')\,dr' \\
&- \int_0^1 \frac{v(r',\omega_0/2)}{r'}\zeta^2(r')\Big(\int_0^{\omega_0/2}\frac{\partial v(r',\omega)}{\partial\omega}\,d\omega\Big)\,dr'.
\end{aligned}
$$

Next, by the Cauchy inequality,

$$
\begin{aligned}
(4.7)\qquad &\int_0^1 \frac{v(r',\omega_0/2)}{r'}\zeta^2(r')\Big(\int_0^{\omega_0/2}\frac{\partial v(r',\omega)}{\partial\omega}\,d\omega\Big)\,dr' \\
&\le \int\limits_{G_0^1}\frac{\zeta^2(r')}{r'^2}\left|\frac{\partial v(r',\omega)}{\partial\omega}\right||v(r',\omega_0/2)|\,dx' \\
&\le \frac{\varepsilon}{2}\int\limits_{G_0^1}\frac{\zeta^2(r')}{r'^2}\left|\frac{\partial v(r',\omega)}{\partial\omega}\right|^2 dx' + \frac{1}{2\varepsilon}\int\limits_{G_0^1} v^2(r',\omega_0/2)\,dx'
\end{aligned}
$$

$$\le \frac{\varepsilon}{2}\int\limits_{G_0^1}|\nabla' v|^2\zeta^2(|x'|)\,dx' + \frac{1}{2\varepsilon}\int\limits_0^1\frac{\zeta^2(r')}{r'}\int\limits_{-\omega_0/2}^{\omega_0/2} v^2(r',\omega_0/2)\,d\omega\,dr'$$

$$\le \frac{\varepsilon}{2}\int\limits_{G_0^1}|\nabla' v|^2\zeta^2(|x'|)\,dx' + \frac{\omega_0}{2\varepsilon}\int\limits_{\Gamma_{0+}^1}\frac{v^2(x')}{|x'|}\zeta^2(|x'|)\,ds', \qquad \forall\varepsilon>0.$$

Choosing $\varepsilon=\nu/b$ in (4.7), from (4.5)–(4.7) it follows that

$$\begin{aligned}\frac{1}{2}\nu\int\limits_{G_0^1}|\nabla' v|^2\zeta^2(|x'|)\,dx' &+ \left(\beta_+ + b - \frac{b^2\omega_0}{2\nu}\right)\int\limits_{\Gamma_{0+}^1}\frac{v^2(x')}{|x'|}\zeta^2(|x'|)\,ds'\\ &+\beta_-\int\limits_{\Gamma_{0-}^1}\frac{v^2(x')}{|x'|}\zeta^2(|x'|)\,ds'\\ \le \int\limits_{G_0^1}\varrho\Big(\sum_{i=1}^2|b^i(\varrho x')|^2\Big)^{1/2}&|\nabla' v|\overline{v}(x')\zeta^2(|x'|)\,dx'\\ &+2\mu\int\limits_{G_0^1}|\nabla' v|\cdot|\nabla'\zeta|\overline{v}(x')\zeta(|x'|)\,dx' + \frac{1}{m}\|\mathcal{G}\|_{\infty,\Gamma_{0+}^1}\int\limits_{\Gamma_{0+}^1}\overline{v}^2(x')\zeta^2(|x'|)\,ds'\\ &+\frac{1}{m}\|\mathcal{H}\|_{\infty,\Gamma_{0-}^1}\int\limits_{\Gamma_{0-}^1}\overline{v}^2(x')\zeta^2(|x'|)\,ds' + \frac{1}{m}\int\limits_{G_0^1}|\mathcal{F}(x')|\overline{v}^2(x')\zeta^2(|x'|)\,dx'.\end{aligned}$$

By (i), we can easily verify that $\beta_+ + b - \frac{b^2\omega_0}{2\nu} >$. Therefore, in any case, from (4.5) we get

$$\begin{aligned}(4.8)\qquad &\frac{1}{2}\nu\int\limits_{G_0^1}|\nabla' v|^2\zeta^2(|x'|)\,dx'\\ \le \int\limits_{G_0^1}\varrho\Big(\sum_{i=1}^2|b^i(\varrho x')|^2\Big)^{1/2}&|\nabla' v|\overline{v}(x')\zeta^2(|x'|)\,dx'\\ &+2\mu\int\limits_{G_0^1}|\nabla' v|\cdot|\nabla'\zeta|\overline{v}(x')\zeta(|x'|)\,dx' + \frac{1}{m}\|\mathcal{G}\|_{\infty,\Gamma_{0+}^1}\int\limits_{\Gamma_{0+}^1}\overline{v}^2(x')\zeta^2(|x'|)\,ds'\\ &+\frac{1}{m}\|\mathcal{H}\|_{\infty,\Gamma_{0-}^1}\int\limits_{\Gamma_{0-}^1}\overline{v}^2(x')\zeta^2(|x'|)\,ds' + \frac{1}{m}\int\limits_{G_0^1}|\mathcal{F}(x')|\overline{v}^2(x')\zeta^2(|x'|)\,dx'.\end{aligned}$$

We estimate every term by the Cauchy inequality for any $\varepsilon>0$:

$$\begin{aligned}2\mu|\nabla' v|\,|\nabla'\zeta|\zeta(|x'|)\overline{v}(x') &= 2(|\nabla' v|\cdot\zeta(|x'|))(\mu\overline{v}(x')|\nabla'\zeta|)\\ &\le \varepsilon|\nabla' v|^2\zeta^2(|x'|) + \frac{\mu^2}{\varepsilon}\overline{v}^2(x')|\nabla'\zeta|^2;\end{aligned}$$

$$
\begin{aligned}
&\varrho\Big(\sum_{i=1}^{2}|b^i(\varrho x')|^2\Big)^{1/2}|\nabla' v|\overline{v}(x')\zeta^2(|x'|)\\
&\qquad = \zeta^2(|x'|)\Big(\varrho\overline{v}(x')\Big(\sum_{i=1}^{2}|b^i(\varrho x')|^2\Big)^{1/2}\Big)|\nabla' v|\\
&\qquad \le \frac{\varrho^2}{2\varepsilon}\overline{v}^2(x')\zeta^2(|x'|)\cdot\Big(\sum_{i=1}^{2}|b^i(\varrho x')|^2\Big) + \frac{\varepsilon}{2}|\nabla' v|^2\zeta^2(|x'|).
\end{aligned}
$$

To estimate the boundary integrals on the right in (4.8) we apply (2.21) to get

$$
\begin{aligned}
(4.9)\qquad &\frac{1}{2}\nu\int_{G_0^1}|\nabla' v|^2\zeta^2(|x'|)\,dx'\\
&\le \frac{3\varepsilon}{2}\int_{G_0^1}|\nabla' v|^2\zeta^2(|x'|)\,dx' + \frac{\mu^2}{\varepsilon}\int_{G_0^1}|\nabla'\zeta|^2\overline{v}^2(x')\,dx'\\
&\quad + \frac{\varrho^2}{2\varepsilon}\int_{G_0^1}\Big(\sum_{i=1}^{2}|b^i(\varrho x')|^2\Big)\overline{v}^2(x')\zeta^2(|x'|)\,dx'\\
&\quad + \frac{1}{m}\int_{G_0^1}|\mathcal{F}(|x'|)|\overline{v}^2(x')\zeta^2(|x'|)\,dx'\\
&\quad + \frac{1}{m}(\|\mathcal{G}\|_{\infty,\Gamma_{0-}^1} + \|\mathcal{H}\|_{\infty,\Gamma_{0+}^1})\int_{G_0^1}\Big(\delta|\nabla'(\zeta\overline{v})|^2 + \frac{1}{\delta}c_0\overline{v}^2(x')\zeta^2(|x'|)\Big)\,dx'
\end{aligned}
$$

for all $\varepsilon,\delta>0$. From

$$
(4.10)\qquad |\nabla'(\zeta\overline{v})|^2 \le 2(\zeta^2|\nabla'\overline{v}|^2 + \overline{v}^2(x')|\nabla'\zeta|^2),\qquad |\nabla'\overline{v}|^2 = |\nabla' v|^2
$$

it follows that

$$
(4.11)\qquad |\nabla'(\zeta\overline{v})|^2 \le 2|\nabla' v|^2\zeta^2 + 2\overline{v}^2(x')|\nabla'\zeta|^2.
$$

Now, by (4.9)–(4.11), choosing $\varepsilon=\nu/6$ in (4.9) and using (4.2), we find that

$$
\begin{aligned}
&\frac{\nu}{4}\int_{G_0^1}|\nabla' v|^2\zeta^2(|x'|)\,dx' \le \frac{6\mu^2}{\nu}\int_{G_0^1}|\nabla'\zeta|^2\overline{v}^2(x')\,dx'\\
&\qquad + \frac{3\varrho^2}{\nu}\int_{G_0^1}\Big(\sum_{i=1}^{2}|b^i(\varrho x')|^2\Big)\overline{v}^2(x')\zeta^2(|x'|)\,dx' + 2\delta\nu\int_{G_0^1}|\nabla' v|^2\zeta^2(|x'|)\,dx'\\
&\qquad + 2\delta\nu\int_{G_0^1}\overline{v}^2(x')|\nabla'\zeta|^2\,dx' + \frac{c_0\nu}{\delta}\int_{G_0^1}\overline{v}^2(x')\zeta^2(|x'|)\,dx'\\
&\qquad + \frac{1}{m}\int_{G_0^1}|\mathcal{F}(x')|\overline{v}^2(x')\zeta^2(|x'|)\,dx',\qquad \forall\delta>0.
\end{aligned}
$$

Now we choose $\delta = 1/16$. Then by (4.10), the last estimate yields

$$\int_{G_0^1} |\nabla'\overline{v}|^2\zeta^2(|x'|)\,dx' \le \frac{48\mu^2}{\nu^2}\int_{G_0^1} |\nabla'\zeta|^2\overline{v}^2(x')\,dx'$$
$$+\frac{24\varrho^2}{\nu^2}\int_{G_0^1}\Big(\sum_{i=1}^2|b^i(\varrho x')|^2\Big)\overline{v}^2(x')\zeta^2(|x'|)\,dx' + \int_{G_0^1}\overline{v}^2(x')|\nabla'\zeta|^2\,dx'$$
$$+128c_0\int_{G_0^1}\overline{v}^2(x')\zeta^2(|x'|)\,dx' + \frac{8}{m\nu}\int_{G_0^1}|\mathcal{F}(x')|\overline{v}^2(x')\zeta^2(|x'|)\,dx'.$$

The above inequality can be rewritten as

$$\int_{G_0^1} |\nabla'\overline{v}|^2\zeta^2(|x'|)\,dx' \le C_1\int_{G_0^1}(|\nabla'\zeta|^2+\zeta^2(|x'|))\overline{v}^2(x')\,dx'$$
$$+C_2\int_{G_0^1}\bigg(\varrho^2\sum_{i=1}^2|b^i(\varrho x')|^2+\frac{|\mathcal{F}(x')|}{m}\bigg)\overline{v}^2(x')\zeta^2(|x'|)\,dx', \tag{4.12}$$

where the constants $C_1,\ C_2$ depend only on $c_0,\ \mu,\ \nu$. The desired iteration process can now be developed from (4.12). By the Sobolev imbedding theorem (see [6, Ch. II, §2]) we have

$$\|\zeta\overline{v}\|^2_{\frac{2\widetilde{n}}{\widetilde{n}-2},G_0^1} \le C^*\int_{G_0^1}((|\nabla'\zeta|^2+\zeta^2)\overline{v}^2(x')+\zeta^2|\nabla'\overline{v}|^2)\,dx',\qquad \widetilde{n}>2, \tag{4.13}$$

where the constant C^* depends only on $\widetilde{n}$ and the domain G. The Hölder inequality yields

$$\int_{G_0^1}\bigg(\varrho^2\sum_{i=1}^2|b^i(\varrho x')|^2+\frac{|\mathcal{F}(x')|}{m}\bigg)\cdot\overline{v}^2(x')\zeta^2(x')\,dx'$$
$$\le\bigg\|\varrho^2\sum_{i=1}^2|b^i(\varrho\,\cdot)|^2+\frac{|\mathcal{F}(\cdot)|}{m}\bigg\|_{p/2,G_0^1}\|\zeta\overline{v}\|^2_{\frac{2p}{p-2},G_0^1},\quad p>2, \tag{4.14}$$

and from (4.12)–(4.14) we get

$$\|\zeta\overline{v}\|^2_{\frac{2\widetilde{n}}{\widetilde{n}-2},G_0^1}\le C_3\int_{G_0^1}(|\nabla'\zeta|^2+\zeta^2(|x'|))\overline{v}^2(x')\,dx'$$
$$+C_4\bigg\|\varrho^2\sum_{i=1}^2|b^i(\varrho\,\cdot)|^2+\frac{|\mathcal{F}(\cdot)|}{m}\bigg\|_{p/2,G_0^1}\cdot\|\zeta\overline{v}\|^2_{\frac{2p}{p-2},G_0^1},\quad p>\widetilde{n}>2. \tag{4.15}$$

By the interpolation inequality for L_p-norms,

$$\|\zeta\overline{v}\|_{\frac{2p}{p-2},G_0^1}\le\varepsilon\|\zeta\overline{v}\|_{\frac{2\widetilde{n}}{\widetilde{n}-2},G_0^1}+\tilde{c}\varepsilon^{\frac{\widetilde{n}}{\widetilde{n}-p}}\|\zeta\overline{v}\|_{2,G_0^1},\quad p>\widetilde{n}>2,\ \forall\varepsilon>0,$$

where $\widetilde{c} = \frac{p-\widetilde{n}}{p}\left(\frac{\widetilde{n}}{p}\right)^{\frac{\widetilde{n}}{p-\widetilde{n}}}$, and by (4.2) from (4.15) it follows that

$$\text{(4.16)}\qquad \|\zeta\overline{v}\|_{\frac{2\widetilde{n}}{\widetilde{n}-2},G_0^1} \le \sqrt{C_3}\cdot\|(\zeta+|\nabla'\zeta|)\overline{v}\|_{2,G_0^1}$$
$$+\sqrt{C_4}\Big(\Big\|\varrho^2\sum_{i=1}^2|b^i(\varrho\,\cdot)|^2\Big\|_{p/2,G_0^1}+\nu\Big)^{1/2}\big(\varepsilon\|\zeta\overline{v}\|_{\frac{2\widetilde{n}}{\widetilde{n}-2},G_0^1}+\widetilde{c}\varepsilon^{\frac{\widetilde{n}}{\widetilde{n}-p}}\|\zeta\overline{v}\|_{2,G_0^1}\big)$$

for all $p > \tilde{n}$ and $\varepsilon > 0$. Choosing

$$\varepsilon = \frac{1}{2\sqrt{C_4}}\Big(\Big\|\varrho^2\sum_{i=1}^2|b^i(\varrho\,\cdot)|^2\Big\|_{p/2,G_0^1}+\nu\Big)^{-1/2}$$

in (4.16) we obtain

$$\text{(4.17)}\qquad \|\zeta\overline{v}\|_{\frac{2\widetilde{n}}{\widetilde{n}-2},G_0^1} \le C\|(\zeta+|\nabla'\zeta|)\overline{v}\|_{2,G_0^1},\qquad \widetilde{n}>2,$$

where C depends only on c_0, μ, ν, p, $\operatorname{diam} G$, $\|\sum_{i=1}^2|b^i(\cdot)|^2\|_{p/2,G}$. This inequality can now be iterated to yield the desired estimate.

For $\varkappa\in(0,1)$ we define $G'_{(j)} \equiv G_0^{\varkappa+(1-\varkappa)2^{-j}}$, $j = 0,1,2\dots$. It is easy to verify that $G_0^{\varkappa} \equiv G'_{(\infty)} \subset \cdots \subset G'_{(j+1)} \subset G'_j \subset \cdots \subset G'_{(0)} \equiv G_0^1$. Now we consider the sequence of cut-off functions $\zeta_j \in C^\infty(G'_{(j)})$ such that $0\le\zeta_j\le 1$ in $G'_{(j)}$, $\zeta_j\equiv 1$ in $G'_{(j+1)}$, and $\zeta_j(x')\equiv 0$ for $|x'|>\varkappa+2^{-j}(1-\varkappa)$. Hence

$$|\nabla'\zeta_j(x')| \le \frac{2^{j+1}}{1-\varkappa}\qquad \text{for } \varkappa+2^{-j-1}(1-\varkappa)<|x'|<\varkappa+2^{-j}(1-\varkappa).$$

We also define $t_j = 2\left(\frac{\widetilde{n}}{\widetilde{n}-2}\right)^j$, $j=0,1,\dots$. Now we rewrite (4.17) replacing $\zeta(|x'|)$ by $\zeta_j(x')$ to obtain

$$\text{(4.18)}\qquad \|\overline{v}\|_{\frac{2\widetilde{n}}{\widetilde{n}-2},G'_{(j+1)}} \le C\frac{2^{j+2}}{1-\varkappa}\|\overline{v}\|_{2,G'_{(j)}}.$$

Putting $w = |\overline{v}|^{(\frac{\widetilde{n}}{\widetilde{n}-2})^j}$, by (4.18) and the definition of t_j, we get

$$\|\overline{v}\|_{t_{j+1},G'_{(j+1)}} = \Big(\int\limits_{G_{(j+1)'}} w^{\frac{2\widetilde{n}}{\widetilde{n}-2}}\,dx'\Big)^{\frac{\widetilde{n}-2}{2\widetilde{n}}\cdot(\frac{\widetilde{n}-2}{\widetilde{n}})^j}$$
$$\le\Big(C\frac{2^{j+2}}{1-\varkappa}\Big)^{(\frac{\widetilde{n}-2}{\widetilde{n}})^j}\|w\|_{2,G'_{(j)}}^{(\frac{\widetilde{n}-2}{\widetilde{n}})^j} = \Big(\frac{C}{1-\varkappa}\Big)^{2/t_j}4^{\frac{j+2}{t_j}}\|\overline{v}\|_{t_j,G'_{(j)}}.$$

After iteration, we find that

$$\text{(4.19)}\qquad \|\overline{v}\|_{t_{j+1},G'_{(j+1)}} \le \Big\{\frac{C}{1-\varkappa}\Big\}^{2\sum_{j=0}^\infty\frac{1}{t_j}}\cdot 4^{\sum_{j=0}^\infty\frac{j+2}{t_j}}\|\overline{v}\|_{2,G_0^1}.$$

Notice that the series $\sum_{j=0}^\infty(j+2)/t_j$ is convergent by the d'Alembert ratio test, and $\sum_{j=0}^\infty 1/t_j = \widetilde{n}/4$ as a geometric series. Therefore from (4.19) we get

$$\|\overline{v}\|_{t_{j+1},G'_{(j+1)}} \le \frac{C}{(1-\varkappa)^{\widetilde{n}/2}}\|\overline{v}\|_{2,G_0^1}.$$

Consequently, letting $j \to \infty$, we have

$$\sup_{x'\in G_0^{\varkappa}} |\overline{v}(x')| \le \frac{C}{(1-\varkappa)^{\widetilde{n}/2}}\|\overline{v}\|_{2,G_0^1}.$$

Hence, by the definitions (4.3) and (4.2), we get

$$\sup_{x'\in G_0^{\varkappa}} |v(x')| \le \frac{C}{(1-\varkappa)^{\widetilde{n}/2}}(\|v\|_{2,G_0^1} + \|\mathcal{F}\|_{p/2,G_0^1} + \|\mathcal{G}\|_{\infty,\Gamma_{0+}^1} + \|\mathcal{H}\|_{\infty,\Gamma_{0-}^1}).$$

Returning to the variables x and u we obtain the required estimate (4.1). ■

5. Global integral estimate. Now we shall obtain a global estimate for the weighted Dirichlet integral.

THEOREM 5.1. *Let u be a weak solution of problem* (L), *λ^2 be the smallest positive eigenvalue of problem* (EVP) *and let assumptions* (a)–(d), (f) *be satisfied. Suppose, in addition, that $0 < b < (1/\omega_0)(\nu + \sqrt{\nu^2 + 2\nu\omega_0\beta_+})$. Then*

$$\int_G |\nabla u|^2\,dx + \int_G \frac{u^2(x)}{r^2}\,dx + \int_{\partial G}\frac{u^2(x)}{r}\,ds \le C\Big\{|u|^2_{0,\overline{G}} + \int_G f^2(x)\,dx + \int_{\Gamma_+} g^2(x)\,ds + \int_{\Gamma_-} h^2(x)\,ds\Big\}, \tag{5.1}$$

where the constant $C > 0$ depends only on b, ω_0, $\beta_\pm$, $\|\sum_{i=1}^2 |b^i(\cdot)|^2\|_{L_{p/2}(G)}$, p, ν and the domain G.

Proof. Setting $\eta(x) = u(x)$ in (II) and using the Hölder inequality, by assumptions (a), (c) we get

$$\nu\int_G |\nabla u|^2\,dx + \int_{\Gamma_+}\Big(\beta_+\frac{u^2(x)}{r} + b\frac{u(x)}{r}u(\gamma(x))\Big)\,ds + \beta_-\int_{\Gamma_-}\frac{u^2(x)}{r}\,ds \le \int_G \sqrt{\sum_{i=1}^2 |b^i(x)|^2}\,|u|\,|\nabla u|\,dx + \int_{\Gamma_+}|u|\,|g(x)|\,ds + \int_{\Gamma_-}|u|\,|h(x)|\,ds + \int_G |u|\,|f(x)|\,dx. \tag{5.2}$$

Now, by assumption (b), the Cauchy inequality with $\varepsilon = \nu/2$ and the Hölder inequality with $q = p/2$, $q' = p/(p-2)$, $p > 2$ we have

$$\int_G \sqrt{\sum_{i=1}^{2} |b^i(x)|^2}\, |u|\, |\nabla u|\, dx = \int_G |\nabla u| \Big(\sqrt{\sum_{i=1}^{2} |b^i(x)|^2}\, |u| \Big)\, dx$$
$$\le \frac{\nu}{4} \int_G |\nabla u|^2\, dx + \frac{1}{\nu} \int_G \sum_{i=1}^{2} |b_i(x)|^2 u^2\, dx$$
$$\le \frac{\nu}{4} \int_G |\nabla u|^2\, dx + \frac{1}{\nu} \Big(\int_G \Big(\sum_{i=1}^{2} |b^i(x)|^2 \Big)^{p/2} dx \Big)^{2/p} \cdot \|u\|^2_{\frac{2p}{p-2},G}.$$

Further, we apply the Sobolev inequality

$$\|u\|^2_{\frac{2p}{p-2},G} \le \delta \|\nabla u\|^2_{2,G} + c(\delta, p, G) \|u\|^2_{2,G}, \qquad p > 2,\ \forall \delta > 0$$

(see for example [6, Ch. II, §2, (2.19)]); hence

$$\int_G \sqrt{\sum_{i=1}^{2} |b^i(x)|^2}\, |u|\, |\nabla u|\, dx \le \frac{\nu}{4} \int_G |\nabla u|^2\, dx + \frac{1}{\nu} \Big\| \sum_{i=1}^{2} |b^i(\cdot)|^2 \Big\|_{p/2,G} \times \int_G (\delta |\nabla u|^2 + c(\delta, p, G) u^2(x))\, dx, \qquad \forall \delta > 0. \tag{5.3}$$

We choose $\delta = \nu^2/(8\|\sum_{i=1}^{2} |b^i(\cdot)|^2\|_{p/2,G})$. As a result from (5.2)–(5.3) we obtain

$$\frac{5\nu}{8} \int_G |\nabla u|^2\, dx + \beta_+ \int_{\Gamma_+} \frac{u^2(x)}{r}\, ds + b \int_{\Gamma_+} \frac{u(x)}{r} u(\gamma(x))\, ds + \beta_- \int_{\Gamma_-} \frac{u^2(x)}{r}\, ds$$
$$\le C \int_G u^2(x)\, dx + \int_{\Gamma_+} |u|\, |g(x)|\, ds + \int_{\Gamma_-} |u|\, |h(x)|\, ds + \int_G |u|\, |f(x)|\, dx, \tag{5.4}$$

where $C = C(p, \nu, \|\sum_{i=1}^{2} |b^i(\cdot)|^2\|_{p/2,G}, G)$. Now we consider $\Gamma_+ = \Gamma^d_{0+} \cup \Gamma_{d+}$ and estimate the third integral on the left hand side of (5.4). We estimate the integral over Γ^d_{0+} by Lemma 2.19 with $\alpha = 2$ and $\varepsilon = \nu/b$. And, by assumption (f), we estimate the integral over Γ_{d+} as follows:

$$b \int_{\Gamma_{d+}} \frac{u(x)}{r} u(\gamma(x))\, ds \le b \frac{\operatorname{meas} \Gamma_+}{d} |u|^2_{0,\overline{G}}.$$

Thus from (5.4) we get

$$\frac{\nu}{8} \int_G |\nabla u|^2\, dx + \Big(\beta_+ - \frac{b^2 \omega_0}{2\nu} + b \Big) \int_{\Gamma_+} \frac{u^2(x)}{r}\, ds + \beta_- \int_{\Gamma_-} \frac{u^2(x)}{r}\, ds$$
$$\le C \Big\{ |u|^2_{0,\overline{G}} + \int_{\Gamma_+} |u|\, |g(x)|\, ds + \int_{\Gamma_-} |u|\, |h(x)|\, ds + \int_G |u|\, |f(x)|\, dx \Big\}. \tag{5.5}$$

From $0 < b < \frac{1}{\omega_0}(\nu + \sqrt{\nu^2 + 2\nu\omega_0\beta_+})$, we can easily verify that $\beta_+ - \frac{b^2\omega_0}{2\nu} + b > 0$. Now, by the Cauchy inequality with $\varepsilon = \beta_+ - \frac{b^2\omega_0}{2\nu} + b$ and assumption (c) we obtain

$$\int\limits_{\Gamma_+} |u|\,|g(x)|\,ds = \int\limits_{\Gamma_+} \left(\frac{|u|}{\sqrt{r}}\right)(\sqrt{r}\,|g(x)|)\,ds$$
$$\le \frac{1}{2}\left(\beta_+ - \frac{b^2\omega_0}{2\nu} + b\right) \int\limits_{\Gamma_+} \frac{u^2(x)}{r}\,ds + \frac{\operatorname{diam} G}{2(\beta_+ - \frac{b^2\omega_0}{2\nu} + b)} \int\limits_{\Gamma_+} g^2(x)\,ds;$$

in the same way we have

$$\int\limits_{\Gamma_-} |u|\,|h(x)|\,ds \le \frac{1}{2}\beta_- \int\limits_{\Gamma_-} \frac{u^2(x)}{r}\,ds + \frac{\operatorname{diam} G}{2\beta_-} \int\limits_{\Gamma_-} h^2(x)\,ds;$$
$$\int\limits_{G} |u|\,|f(x)|\,dx \le \frac{1}{2} \int\limits_{G} |u|^2\,dx + \frac{1}{2} \int\limits_{G} |f|^2\,dx.$$

Hence and from (5.5) we get the inequality

$$(5.6)\qquad \int\limits_{G} |\nabla u|^2\,dx + \int\limits_{\partial G} \frac{u^2(x)}{r}\,ds \le C\Big\{|u|^2_{0,\overline{G}} + \int\limits_{G} f^2(x)\,dx + \int\limits_{\Gamma_+} g^2(x)\,ds + \int\limits_{\Gamma_-} h^2(x)\,ds\Big\}.$$

Finally, by the Hardy–Friedrichs–Wirtinger inequality (2.14) with $\alpha = 2$, we get the desired estimate (5.1). ∎

THEOREM 5.2. *Let $u \ge 0$ be a weak solution of problem* (L), *let $\lambda \in (\pi/\omega_0, 2\pi/\omega_0)$ be the smallest positive root of* (2.10) *and let assumptions* (a)–(d), (f) *be satisfied. Suppose, in addition, that $\beta_+ = \beta_- = \beta$ and $b > b^*$, where b^* is defined by* (2.9). *Then*

$$(5.7)\qquad \int\limits_{G} |\nabla u|^2\,dx + \int\limits_{G} \frac{u^2(x)}{r^2}\,dx + \int\limits_{\partial G} \frac{u^2(x)}{r}\,ds \le C\Big\{|u|^2_{0,\overline{G}} + \int\limits_{G} f^2(x)\,dx + \int\limits_{\Gamma_+} g^2(x)\,ds + \int\limits_{\Gamma_-} h^2(x)\,ds\Big\},$$

where the constant $C > 0$ depends only on b, ω_0, β, $\|\sum_{i=1}^2 |b^i(\cdot)|^2\|_{L_{p/2}(G)}$, p, ν and the domain G.

Proof. As in Theorem 5.1, we get (5.4) with $\beta_+ = \beta_- = \beta$. By assumption $u(x) \ge 0$ and estimating integrals on the right (5.4), by the Cauchy inequality with $\varepsilon = 1$, we obtain (5.6). Next, by the Hardy–Friedrichs–Wirtinger inequality (2.15) with $\alpha = 2$, we get (5.7). ∎

THEOREM 5.3. *Let u be a weak solution of problem* (L) *and let assumptions* (a)–(d), (f) *be satisfied. Suppose, in addition, that $b = \frac{\pi}{\omega_0} \cdot \frac{\beta_+ + \beta_-}{\beta_-}$ and $\beta_+ u^2(x)|_{\Gamma_+} + \beta_- u^2(x)|_{\Gamma_-} + bu(x)|_{\Gamma_+} \cdot u(\gamma(x))|_{\Gamma_+} = 0$, $u^2(x)|_{\Gamma_+} = u^2(x)|_{\Gamma_-}$. Then*

$$\tag{5.8} \int_G |\nabla u|^2\,dx + \int_G \frac{u^2(x)}{r^2}\,dx \le C\Big\{|u|^2_{0,\overline{G}} + \int_G f^2(x)\,dx + \int_{\Gamma_+} g^2(x)\,ds + \int_{\Gamma_-} h^2(x)\,ds\Big\},$$

where the constant $C > 0$ depends only on b, ω_0, $\beta_\pm$, $\|\sum_{i=1}^2 |b^i(\cdot)|^2\|_{L_{p/2}(G)}$, p, ν and the domain G.

Proof. As in Theorem 5.1, we get (5.4). Further, by $\beta_+ u^2(x)|_{\Gamma_+} + \beta_- u^2(x)|_{\Gamma_-} + bu(x)|_{\Gamma_+} \cdot u(\gamma(x))|_{\Gamma_+} = 0$ and estimating the integrals on the right of (5.4) using the Cauchy inequality with $\varepsilon = 1$ we obtain

$$\tag{5.9} \int_G |\nabla u|^2\,dx \le C\Big\{|u|^2_{0,\overline{G}} + \int_G f^2(x)\,dx + \int_{\Gamma_+} g^2(x)\,ds + \int_{\Gamma_-} h^2(x)\,ds\Big\}.$$

Next, by $u^2(x)|_{\Gamma_+} = u^2(x)|_{\Gamma_-}$ and the Hardy–Friedrichs–Wirtinger inequality (2.16) with $\alpha = 2$, we get (5.8). ∎

6. Local integral weighted estimates

THEOREM 6.1. *Let u be a weak solution of problem* (L), *let $\lambda = \lambda^*$, where λ^* is defined in Lemma 2.6, and let B be defined by* (2.6). *Let assumptions* (a)–(f) *be satisfied with $\mathcal{A}(r)$ Dini-continuous at zero. Suppose, in addition, that* (1.1) *is satisfied. Then there are $d \in (0, 1/e)$ and a constant $C > 0$ depending only on s, λ, ν, b, β_+, β_-, d, the domain G and $\int_0^{1/e} (\mathcal{A}(r)/r)\,dr$ such that for a.e. $\varrho \in (0, d)$,*

$$\int_{G_0^\varrho} \Big(|\nabla u|^2 + \frac{u^2(x)}{r^2}\Big)\,dx + B\int_{\Gamma_{0+}^\varrho} \frac{u^2(x)}{r}\,ds + \beta_- \int_{\Gamma_{0-}^\varrho} \frac{u^2(x)}{r}\,ds \le C\Big(|u|^2_{0,\overline{G}} + \omega_0 f_0^2 + \frac{1}{\beta_+} g_0^2 + \frac{1}{\beta_-} h_0^2 + \|f\|^2_{2,G} + \|g\|^2_{\infty,\Gamma_+} + \|h\|^2_{\infty,\Gamma_-}\Big) \times \begin{cases} \varrho^{2\lambda k} & \text{if } s > \lambda k, \\ \varrho^{2\lambda k}\ln^2(1/\varrho) & \text{if } s = \lambda k, \\ \varrho^{2s} & \text{if } s < \lambda k, \end{cases}$$

where k is defined by (1.3).

Proof. Setting $\eta(x) = u(x)$ in $(\mathrm{II})_{\mathrm{loc}}$, we obtain

$$\int\limits_{G_0^\varrho} |\nabla u|^2\,dx + \beta_+ \int\limits_{\Gamma_{0+}^\varrho} \frac{u^2(x)}{r}\,ds + \beta_- \int\limits_{\Gamma_{0-}^\varrho} \frac{u^2(x)}{r}\,ds \tag{6.1}$$
$$= \varrho \int\limits_{\Omega} \left(u(x)\frac{\partial u}{\partial r}\right)\bigg|_{r=\varrho} d\omega + \int\limits_{\Omega_\varrho} (a^{ij}(x) - a^{ij}(0))u(x)u_{x_j}\cos(r, x_i)\,d\Omega_\varrho$$
$$+ \int\limits_{\Gamma_{0+}^\varrho} u(x)g(x)\,ds - b \int\limits_{\Gamma_{0+}^\varrho} \frac{u(x)}{r}u(\gamma(x))\,ds + \int\limits_{\Gamma_{0-}^\varrho} u(x)h(x)\,ds$$
$$+ \int\limits_{G_0^\varrho} \Big\{-(a^{ij}(x)-a^{ij}(0))u_{x_i}u_{x_j} + b^i(x)u(x)u_{x_i} + c(x)u^2(x) - u(x)f(x)\Big\}\,dx.$$

We estimate the integral $\int_{\Gamma_{0+}^\varrho} (u(x)/r)u(\gamma(x))\,ds$ by Lemma 2.19 with $\alpha = 2$. Thus we get

$$\left(1 - \frac{b\varepsilon}{2}\right) \int\limits_{G_0^\varrho} |\nabla u|^2\,dx \tag{6.2}$$
$$+ B\left(\frac{\beta_+ + b - \frac{b\omega_0}{2\varepsilon}}{B}\right) \int\limits_{\Gamma_{0+}^\varrho} \frac{u^2(x)}{r}\,ds + \beta_- \int\limits_{\Gamma_{0-}^\varrho} \frac{u^2(x)}{r}\,ds$$
$$\le \varrho \int\limits_{\Omega} \left(u(x)\frac{\partial u}{\partial r}\right)\bigg|_{r=\varrho} d\omega + \int\limits_{\Omega_\varrho} (a^{ij}(x) - a^{ij}(0))u(x)u_{x_j}\cos(r, x_i)d\Omega_\varrho$$
$$+ \int\limits_{\Gamma_{0+}^\varrho} u(x)g(x)\,ds + \int\limits_{\Gamma_{0-}^\varrho} u(x)h(x)\,ds$$
$$+ \int\limits_{G_0^\varrho} \{-(a^{ij}(x)-a^{ij}(0))u_{x_i}u_{x_j} + b^i(x)u(x)u_{x_i} + c(x)u^2(x) - u(x)f(x)\}\,dx.$$

Now, if we choose

$$\varepsilon = \frac{\sqrt{(\beta_+ + b - B)^2 + Bb^2\omega_0} - \beta_+ - b + B}{Bb}$$

in (6.2), then since $\nu \le 1$ we can verify that

$$1 - \frac{b\varepsilon}{2} = \frac{\beta_+ + b - \frac{b\omega_0}{2\varepsilon}}{B} > 0 \quad \text{for } 0 < b < \frac{1}{\omega_0}(\nu + \sqrt{\nu^2 + 2\nu\omega_0\beta_+}).$$

From definitions (2.17), (1.3) we obtain

$$kU(\varrho) \le \varrho \int\limits_{\Omega} \left(u(x)\frac{\partial u}{\partial r}\right)\bigg|_{r=\varrho} d\Omega \tag{6.3}$$
$$+ \int\limits_{\Omega_\varrho} (a^{ij}(x) - a^{ij}(0))u(x)u_{x_j}\cos(r, x_i)\,d\Omega_\varrho$$

$$+\int_{G_0^\varrho}\{-(a^{ij}(x)-a^{ij}(0))u_{x_i}u_{x_j}+b^i(x)u(x)u_{x_i}+c(x)u^2(x)\}\,dx$$
$$-\int_{G_0^\varrho}u(x)f(x)\,dx+\int_{\Gamma_{0+}^\varrho}u(x)g(x)\,ds+\int_{\Gamma_{0-}^\varrho}u(x)h(x)\,ds.$$

Now we estimate the integrals on the right hand side of (6.3). The first one is estimated by Lemma 2.16; and the next, by assumption (b) and the Cauchy inequality:

$$\tag{6.4}\begin{aligned}&\int_{\Omega_\varrho}(a^{ij}(x)-a^{ij}(0))u(x)u_{x_j}\cos(r,x_i)\,d\Omega_\varrho\le\varrho\mathcal{A}(\varrho)\int_{\Omega}|u(x)|\,|\nabla u|\,d\omega,\\&\int_{G_0^\varrho}\{(a^{ij}(x)-a^{ij}(0))u_{x_i}u_{x_j}+b^i(x)u_{x_i}u(x)+c(x)u^2(x)\}\,dx\\&\qquad\le\mathcal{A}(\varrho)\int_{G_0^\varrho}\left\{|\nabla u|^2+\frac{u^2(x)}{r^2}\right\}dx.\end{aligned}$$

Thus, from (6.3)–(6.4) it follows that

$$\tag{6.5}\begin{aligned}kU(\varrho)\le{}&\frac{\varrho}{2\lambda}U'(\varrho)+\varrho\mathcal{A}(\varrho)\int_{\Omega}|u(x)|\,|\nabla u|\,d\omega\\&+\mathcal{A}(\varrho)\int_{G_0^\varrho}\left(|\nabla u|^2+\frac{u^2(x)}{r^2}\right)dx\\&+\int_{\Gamma_{0+}^\varrho}|u(x)|\,|g(x)|\,ds+\int_{\Gamma_{0-}^\varrho}|u(x)|\,|h(x)|\,ds+\int_{G_0^\varrho}|u(x)|\,|f(x)|\,dx.\end{aligned}$$

Further we bound the integrals on the right of (6.5). First, applying the Cauchy and Friedrichs–Wirtinger inequalities (see (2.11)) with the use of (2.18), we have

$$\tag{6.6}\begin{aligned}\mathcal{A}(\varrho)\int_{\Omega}\varrho|u(x)|\,|\nabla u|\,d\omega&\le\frac12\mathcal{A}(\varrho)\int_{\Omega}(\varrho^2|\nabla u|^2+|u(x)|^2)\,d\omega\\&\le\frac12\mathcal{A}(\varrho)\int_{\Omega}\varrho^2\left[\left|\frac{\partial u}{\partial r}\right|^2+\frac{1}{\varrho^2}\left|\frac{\partial u}{\partial\omega}\right|^2\right]\bigg|_{r=\varrho}d\omega\\&\quad+\frac12\mathcal{A}(\varrho)\frac{1}{\lambda^2}\left\{\int_{\Omega}\left|\frac{\partial u}{\partial\omega}\right|^2d\omega+Bu^2\left(\varrho,\frac{\omega_0}{2}\right)+\beta_-u^2\left(\varrho,-\frac{\omega_0}{2}\right)\right\}\\&\le\frac12\varrho\mathcal{A}(\varrho)\left(1+\frac{1}{\lambda^2}\right)\left\{\int_{\Omega}\left[\varrho\left|\frac{\partial u}{\partial r}\right|^2+\frac{1}{\varrho}\left|\frac{\partial u}{\partial\omega}\right|^2\right]\bigg|_{r=\varrho}d\omega\right.\\&\quad\left.+B\frac{u^2(\varrho,\omega_0/2)}{\varrho}+\beta_-\frac{u^2(\varrho,-\omega_0/2)}{\varrho}\right\}\\&\le c_1(b,\beta_\pm,\omega_0,\lambda)\varrho\mathcal{A}(\varrho)U'(\varrho).\end{aligned}$$

Next, using the Cauchy and the Hardy–Friedrichs–Wirtinger inequalities (see (2.14) for $\alpha = 2$), by (2.17) we obtain

$$
\begin{aligned}
(6.7)\qquad &\mathcal{A}(\varrho)\int\limits_{G_0^\varrho}\left(|\nabla u|^2+\frac{|u|^2}{r^2}\right)dx\\
&\le\left(1+\frac{1}{\lambda^2}\right)\mathcal{A}(\varrho)\Big\{\int\limits_{G_0^\varrho}|\nabla u|^2\,dx+B\int\limits_{\Gamma_{0+}^\varrho}\frac{u^2(x)}{r}\,ds+\beta_-\int\limits_{\Gamma_{0-}^\varrho}\frac{u^2(x)}{r}\,ds\Big\}\\
&\le c_2(b,\beta_\pm,\omega_0,\lambda)\mathcal{A}(\varrho)U(\varrho),
\end{aligned}
$$

and for all $\delta > 0$ we get

$$
(6.8)\qquad
\begin{aligned}
\int\limits_{\Gamma_{0+}^\varrho}|u(x)|\,|g(x)|\,ds&=\int\limits_{\Gamma_{0+}^\varrho}\left(\sqrt{\frac{\beta_+}{r}}\,|u(x)|\right)\left(\sqrt{\frac{r}{\beta_+}}\,|g(x)|\right)ds\\
&\le\frac{\delta\beta_+}{2}\int\limits_{\Gamma_{0+}^\varrho}\frac{u^2(x)}{r}\,ds+\frac{1}{2\delta\beta_+}\int\limits_{\Gamma_{0+}^\varrho}rg^2(x)\,ds;\\
\int\limits_{\Gamma_{0-}^\varrho}|u(x)|\,|h(x)|\,ds&=\int\limits_{\Gamma_{0-}^\varrho}\left(\sqrt{\frac{\beta_-}{r}}\,|u(x)|\right)\left(\sqrt{\frac{r}{\beta_-}}\,|h(x)|\right)ds\\
&\le\frac{\delta\beta_-}{2}\int\limits_{\Gamma_{0-}^\varrho}\frac{u^2(x)}{r}\,ds+\frac{1}{2\delta\beta_-}\int\limits_{\Gamma_{0-}^\varrho}rh^2(x)\,ds;
\end{aligned}
$$

$$
\begin{aligned}
(6.9)\qquad \int\limits_{G_0^\varrho}|u(x)|\,|f(x)|\,dx&\le\frac{\delta}{2}\int\limits_{G_0^\varrho}\frac{u^2(x)}{r^2}\,dx+\frac{1}{2\delta}\int\limits_{G_0^\varrho}r^2f^2(x)\,dx\\
&\le\frac{\delta}{2}c_3(b,\beta_\pm,\omega_0,\lambda)U(\varrho)+\frac{1}{2\delta}\int\limits_{G_0^\varrho}r^2f^2(x)\,dx
\end{aligned}
$$

by (2.14) and (2.17). Thus from (6.5)–(6.9) we get

$$
\begin{aligned}
(6.10)\qquad &\langle k-c_4(\delta+\mathcal{A}(\varrho))\rangle U(\varrho)\le\frac{\varrho}{2\lambda}(1+c_5\mathcal{A}(\varrho))U'(\varrho)\\
&+\frac{1}{2\delta}\Big\{\int\limits_{G_0^\varrho}r^2f^2(x)\,dx+\frac{1}{\beta_+}\int\limits_{\Gamma_{0+}^\varrho}rg^2(x)\,ds+\frac{1}{\beta_-}\int\limits_{\Gamma_{0-}^\varrho}rh^2(x)\,ds\Big\},\qquad\forall\delta>0.
\end{aligned}
$$

But, by condition (e),

$$
\begin{aligned}
(6.11)\qquad &\int\limits_{G_0^\varrho}r^2f^2(x)\,dx+\frac{1}{\beta_+}\int\limits_{\Gamma_{0+}^\varrho}rg^2(x)\,ds+\frac{1}{\beta_-}\int\limits_{\Gamma_{0-}^\varrho}rh^2(x)\,ds\\
&\le\frac{1}{2s}\left(\omega_0f_0^2+\frac{1}{\beta_+}g_0^2+\frac{1}{\beta_-}h_0^2\right)\cdot\varrho^{2s}.
\end{aligned}
$$

Now we take into account that, by (1.3), we have $0 < k < 1$ and therefore

$$\frac{k - c_4(\delta + \mathcal{A}(\varrho))}{1 + c_5\mathcal{A}(\varrho)} = 1 - \frac{1 - k + c_4(\delta + \mathcal{A}(\varrho)) + c_5\mathcal{A}(\varrho)}{1 + c_5\mathcal{A}(\varrho)} \geq k[1 - c_6\delta - c_7\mathcal{A}(\varrho)], \quad \forall \delta > 0.$$

Thus, from (6.10), we have the differential inequality (CP) of Subsection 2.3 with

$$\begin{aligned} \mathcal{P}(\varrho) &= \frac{2\lambda k}{\varrho} \cdot [1 - c_6\delta - c_7\mathcal{A}(\varrho)], \\ \mathcal{Q}(\varrho) &= \frac{\lambda}{2s}\left(\omega_0 f_0^2 + \frac{1}{\beta_+}g_0^2 + \frac{1}{\beta_-}h_0^2\right) \cdot \delta^{-1}\varrho^{2s-1}, \qquad \forall \delta > 0, \\ U_0 &= C(1 + B + \beta_-) \\ &\quad \times \left\{|u|^2_{0,\overline{G}} + \int_G f^2(x)\,dx + \int_{\Gamma_+} g^2(x)\,ds + \int_{\Gamma_-} h^2(x)\,ds\right\}, \end{aligned} \tag{6.12}$$

by (2.17) and (5.1). We shall consider three cases:

Case 1: $s > \lambda k$. Choosing $\delta = \varrho^\varepsilon$ with $\varepsilon > 0$ we obtain

$$\begin{aligned} \mathcal{P}(\varrho) &= \frac{2\lambda k}{\varrho} \cdot [1 - c_6\varrho^\varepsilon - c_7\mathcal{A}(\varrho)], \\ \mathcal{Q}(\varrho) &= \frac{\lambda}{2s}\left(\omega_0 f_0^2 + \frac{1}{\beta_+}g_0^2 + \frac{1}{\beta_-}h_0^2\right) \cdot \varrho^{2s-1-\varepsilon}. \end{aligned}$$

Since $\mathcal{P}(\varrho) = 2\lambda k/\varrho - \mathcal{K}(\varrho)/\varrho$, where $\mathcal{K}(\varrho)$ satisfies the Dini condition at zero, we have

$$-\int_\varrho^\tau \mathcal{P}(s)\,ds = -2\lambda k \ln\left(\frac{\tau}{\varrho}\right) + \int_\varrho^\tau \frac{\mathcal{K}(s)}{s}\,ds \leq \ln\left(\frac{\varrho}{\tau}\right)^{2\lambda k} + \int_0^d \frac{\mathcal{K}(s)}{s}\,ds,$$

so

$$\begin{aligned} \exp\left(-\int_\varrho^\tau \mathcal{P}(\sigma)\,d\sigma\right) &\leq \left(\frac{\varrho}{\tau}\right)^{2\lambda k} \exp\left(\int_0^d \frac{\mathcal{K}(\sigma)}{\sigma}\,d\sigma\right) = K_0\left(\frac{\varrho}{\tau}\right)^{2\lambda k}; \\ \exp\left(-\int_\varrho^d \mathcal{P}(\tau)\,d\tau\right) &\leq \left(\frac{\varrho}{d}\right)^{2\lambda k} \exp\left(\int_0^d \frac{\mathcal{K}(\tau)}{\tau}\,d\tau\right) = K_0\left(\frac{\varrho}{d}\right)^{2\lambda k}. \end{aligned}$$

Moreover,

$$\begin{aligned} &\int_\varrho^d \mathcal{Q}(\tau)\exp\left(-\int_\varrho^\tau \mathcal{P}(\sigma)\,d\sigma\right)d\tau \\ &\qquad \leq \frac{\lambda K_0}{2s}\left(\omega_0 f_0^2 + \frac{1}{\beta_+}g_0^2 + \frac{1}{\beta_-}h_0^2\right)\varrho^{\lambda k}\int_\varrho^d \tau^{2s-2\lambda k-\varepsilon-1}\,d\tau \\ &\qquad \leq \frac{\lambda K_0}{2s}\left(\omega_0 f_0^2 + \frac{1}{\beta_+}g_0^2 + \frac{1}{\beta_-}h_0^2\right) \cdot \frac{d^{s-\lambda k}}{s - \lambda k}\varrho^{2\lambda k}, \end{aligned}$$

since $s > \lambda k$ and we can choose $\varepsilon = s - \lambda k$.

Now we apply Theorem 2.20; then from (2.23), by the above inequalities and (2.14) for $\alpha = 2$, we obtain the statement of Theorem 6.1 for $s > \lambda k$.

CASE 2: $s = \lambda k$. Taking in (6.12) any function $\delta(\varrho) > 0$ instead of $\delta > 0$, we obtain problem (CP) with

$$\mathcal{P}(\varrho) = \frac{2\lambda k(1 - c_6\delta(\varrho))}{\varrho} - c_8\frac{\mathcal{A}(\varrho)}{\varrho},$$
$$\mathcal{Q}(\varrho) = \frac{\lambda}{2s}\left(\omega_0 f_0^2 + \frac{1}{\beta_+}g_0^2 + \frac{1}{\beta_-}h_0^2\right) \cdot \delta^{-1}(\varrho)\varrho^{2\lambda k-1}.$$

We choose $\delta(\varrho) = 1/(2c_6\lambda k \ln(ed/\varrho))$, $0 < \varrho < d$, to obtain

$$-\int_\varrho^\tau \mathcal{P}(\sigma)\,d\sigma \le -2\lambda k \ln\frac{\tau}{\varrho} + \int_\varrho^\tau \frac{d\sigma}{\sigma\ln(ed/\sigma)} + c_8\int_0^d \frac{\mathcal{A}(\sigma)}{\sigma}\,d\sigma$$
$$= \ln\left(\frac{\varrho}{\tau}\right)^{2\lambda k} + \ln\left(\frac{\ln\frac{ed}{\varrho}}{\ln\frac{ed}{\tau}}\right) + c_8\int_0^d \frac{\mathcal{A}(\sigma)}{\sigma}\,d\sigma.$$

Then

$$\exp\left(-\int_\varrho^\tau \mathcal{P}(\sigma)\,d\sigma\right) \le \left(\frac{\varrho}{\tau}\right)^{2\lambda k} \cdot \frac{\ln\frac{ed}{\varrho}}{\ln\frac{ed}{\tau}} \cdot \exp\left(c_8\int_0^d \frac{\mathcal{A}(\sigma)}{\sigma}\,d\sigma\right),$$
$$\exp\left(-\int_\varrho^d \mathcal{P}(\tau)\,d\tau\right) \le \left(\frac{\varrho}{d}\right)^{2\lambda k} \cdot \ln\frac{ed}{\varrho} \cdot \exp\left(c_8\int_0^d \frac{\mathcal{A}(\tau)}{\tau}\,d\tau\right).$$

In this case we also have

$$\int_\varrho^d \mathcal{Q}(\tau)\exp\left(-\int_\varrho^\tau \mathcal{P}(\sigma)\,d\sigma\right)d\tau \le \frac{\lambda}{2s}\left(\omega_0 f_0^2 + \frac{1}{\beta_+}g_0^2 + \frac{1}{\beta_-}h_0^2\right)$$
$$\times\varrho^{2\lambda k}\exp\left(c_8\int_0^d \frac{\mathcal{A}(\sigma)}{\sigma}\,d\sigma\right)\ln\frac{ed}{\varrho}\cdot\int_\varrho^d \delta^{-1}(\tau)\tau^{-1}\frac{1}{\ln(\frac{ed}{\tau})}\,d\tau$$
$$\le c_9\left(\omega_0 f_0^2 + \frac{1}{\beta_+}g_0^2 + \frac{1}{\beta_-}h_0^2\right)\cdot\varrho^{2\lambda k}\ln^2\left(\frac{ed}{\varrho}\right).$$

Now we apply Theorem 2.20, and from (2.23), by the above inequalities, we obtain

$$U(\varrho) \le c_{10}\left(U_0 + \omega_0 f_0^2 + \frac{1}{\beta_+}g_0^2 + \frac{1}{\beta_-}h_0^2\right)\varrho^{2\lambda k}\ln^2\frac{1}{\varrho}, \quad 0 < \varrho < d < \frac{1}{e}.$$

Thus we have proved the statement of Theorem 6.1 for $s = \lambda k$.

CASE 3: $0 < s < \lambda k$. Now as in Case 1, using (6.12) we have

$$\exp\Big(-\int_\varrho^\tau \mathcal{P}(\sigma)\,d\sigma\Big) \le c_{11}\Big(\frac{\varrho}{\tau}\Big)^{2\lambda k(1-c_6\delta)} \exp\Big(\int_0^d \frac{\mathcal{A}(\sigma)}{\sigma}\,d\tau\Big) = c_{12}\Big(\frac{\varrho}{\tau}\Big)^{2\lambda k(1-c_6\delta)}$$

and

$$\exp\Big(-\int_\varrho^d \mathcal{P}(\tau)\,d\tau\Big) \le c_{13}\Big(\frac{\varrho}{d}\Big)^{2\lambda k(1-c_6\delta)} \exp\Big(\int_0^d \frac{\mathcal{A}(\tau)}{\tau}\,d\tau\Big) = c_{14}\Big(\frac{\varrho}{d}\Big)^{2\lambda k(1-c_6\delta)}.$$

In this case we also have

$$\begin{aligned}
&\int_\varrho^d \mathcal{Q}(\tau)\exp\Big(-\int_\varrho^\tau \mathcal{P}(\sigma)\,d\sigma\Big)\,d\tau \\
&\quad\le \frac{\lambda}{2s}\Big(\omega_0 f_0^2 + \frac{1}{\beta_+}g_0^2 + \frac{1}{\beta_-}h_0^2\Big)\cdot \delta^{-1}\varrho^{2\lambda k(1-c_6\delta)}\int_\varrho^d \tau^{2s-2\lambda k(1-c_6\delta)-1}\,d\tau \\
&\quad\le c_{15}\Big(\omega_0 f_0^2 + \frac{1}{\beta_+}g_0^2 + \frac{1}{\beta_-}h_0^2\Big)\cdot \varrho^{2s},
\end{aligned}$$

if we choose $\delta \in (0, \frac{1}{c_6}(1 - \frac{s}{\lambda k}))$.

Now we apply Theorem 2.20, and then from (2.23), by the above inequalities,

$$\begin{aligned}
U(\varrho) &\le c_{16}\Big\{U_0\varrho^{2\lambda k(1-c_5\delta)} + \Big(\omega_0 f_0^2 + \frac{1}{\beta_+}g_0^2 + \frac{1}{\beta_-}h_0^2\Big)\cdot\varrho^{2s}\Big\} \\
&\le c_{17}\Big(U_0 + f_0^2 + \frac{1}{\beta_+}g_0^2 + \frac{1}{\beta_-}h_0^2\Big)\varrho^{2s}.
\end{aligned}$$

Thus we have proved the statement of Theorem 6.1 for $0 < s < \lambda k$. ∎

THEOREM 6.2. *Let $\beta_+ = \beta_- = \beta$ and $b > b^*$, where b^* is defined by* (2.9), *and let $u \ge 0$ be a weak solution of problem* (L). *Let assumptions* (a)–(f) *be satisfied with $\mathcal{A}(r)$ Dini-continuous at zero. Then there are $d \in (0, 1/e)$ and a constant $C > 0$ depending only on s, ω_0, ν, b, β, d, G and $\int_0^{1/e}(\mathcal{A}(r)/r)\,dr$ such that for a.e. $\varrho \in (0, d)$,*

$$\begin{aligned}
&\int_{G_0^\varrho}\Big(|\nabla u|^2 + \frac{u^2(x)}{r^2}\Big)\,dx + \beta\int_{\partial G_0^\varrho}\frac{u^2(x)}{r}\,ds \\
&\quad\le C\Big(|u|^2_{0,\overline{G}} + \omega_0 f_0^2 + \frac{1}{\beta}(g_0^2 + h_0^2) + \|f\|^2_{2,G} + \|g\|^2_{\infty,\Gamma_+} + \|h\|^2_{\infty,\Gamma_-}\Big) \\
&\quad\quad\times\begin{cases}\varrho^{2\lambda} & \text{if } s > \lambda,\\ \varrho^{2\lambda}\ln^2(1/\varrho) & \text{if } s = \lambda,\\ \varrho^{2s} & \text{if } s < \lambda,\end{cases}
\end{aligned}$$

where $\lambda \in (\pi/\omega_0, 2\pi/\omega_0)$ is the smallest positive root of (2.10).

Proof. As in Theorem 6.1, we get equality (6.1) with $\beta = \beta_+ = \beta_-$. By the assumption $u(x) \ge 0$ and definition (2.19) we obtain

$$
\begin{aligned}
(6.13)\quad & U_+(\varrho) \\
& \le \varrho \int_{\Omega} \left(u(x)\frac{\partial u}{\partial r} \right)\bigg|_{r=\varrho} d\Omega + \int_{\Omega_\varrho} (a^{ij}(x) - a^{ij}(0)) u(x) u_{x_j} \cos(r, x_i)\, d\Omega_\varrho \\
& \quad + \int_{G_0^\varrho} \{-(a^{ij}(x) - a^{ij}(0)) u_{x_i} u_{x_j} + b^i(x) u(x) u_{x_i} + c(x) u^2(x)\}\, dx \\
& \quad - \int_{G_0^\varrho} u(x) f(x)\, dx + \int_{\Gamma_{0+}^\varrho} u(x) g(x)\, ds + \int_{\Gamma_{0-}^\varrho} u(x) h(x)\, ds.
\end{aligned}
$$

Now as in Theorem 6.1 we estimate the integrals on the right of (6.13). The first one is estimated by Corollary 2.17; the next one, by (6.4). Thus from (6.13) it follows that

$$
\begin{aligned}
(6.14)\quad U_+(\varrho) \le \frac{\varrho}{2\lambda} U'_+(\varrho) + \varrho \mathcal{A}(\varrho) \int_{\Omega} |u(x)|\, |\nabla u|\, d\omega + \int_{\Gamma_{0+}^\varrho} |u(x)|\, |g(x)|\, ds \\
+ \int_{\Gamma_{0-}^\varrho} |u(x)|\, |h(x)|\, ds + \mathcal{A}(\varrho) \int_{G_0^\varrho} \left(|\nabla u|^2 + \frac{u^2(x)}{r^2} \right) dx \\
+ \int_{G_0^\varrho} |u(x)|\, |f(x)|\, dx.
\end{aligned}
$$

Further we bound each integral on the right of (6.14). First, applying the Cauchy and Friedrichs–Wirtinger inequalities (see (2.12)) similarly to (6.6), we have

$$
(6.15)\qquad \mathcal{A}(\varrho) \int_{\Omega} \varrho |u(x)|\, |\nabla u|\, d\omega \le c_1(b, \beta, \omega_0, \lambda) \varrho \mathcal{A}(\varrho) U'_+(\varrho).
$$

Next, using the Cauchy and the Hardy–Friedrichs–Wirtinger inequalities (see (2.15) for $\alpha = 2$), by (2.19), similarly to (6.7) we obtain

$$
(6.16)\qquad \mathcal{A}(\varrho) \int_{G_0^\varrho} \left(|\nabla u|^2 + \frac{|u|^2}{r^2} \right) dx \le c_2(b, \beta, \omega_0, \lambda) \mathcal{A}(\varrho) U_+(\varrho).
$$

Thus from (6.14)–(6.16) and (6.8), we get

$$
\begin{aligned}
(6.17)\quad \langle 1 - c_4(\delta + \mathcal{A}(\varrho)) \rangle U_+(\varrho) \le \frac{\varrho}{2\lambda} (1 + c_5 \mathcal{A}(\varrho)) U'_+(\varrho) \\
+ \frac{1}{2\delta} \left\{ \int_{G_0^\varrho} r^2 f^2(x)\, dx + \frac{1}{\beta} \int_{\Gamma_{0+}^\varrho} r g^2(x)\, ds + \frac{1}{\beta} \int_{\Gamma_{0-}^\varrho} r h^2(x)\, ds \right\}, \qquad \forall \delta > 0.
\end{aligned}
$$

Then by (6.11) from (6.17) we deduce the differential inequality (CP) of Subsection 2.3 for the function $U_+(\varrho)$, with

$$
\begin{aligned}
\mathcal{P}(\varrho) &= \frac{2\lambda}{\varrho} \cdot [1 - c_6\delta - c_7\mathcal{A}(\varrho)];\\
\mathcal{Q}(\varrho) &= \frac{\lambda}{2s}\left(\omega_0 f_0^2 + \frac{1}{\beta}(g_0^2 + h_0^2)\right) \cdot \delta^{-1}\varrho^{2s-1}, \qquad \forall \delta > 0;\\
U_0 &= C(1+\beta)\Big\{|u|^2_{0,\overline{G}} + \int_G f^2(x)\,dx + \int_{\Gamma_+} g^2(x)\,ds + \int_{\Gamma_-} h^2(x)\,ds\Big\},
\end{aligned}
$$

by (2.19) and (5.7). Next, repeating the proof of Theorem 6.1 for the Cauchy problem (CP) with the function $U_+(\varrho)$ we get the desired result. ∎

THEOREM 6.3. *Let $b = \frac{\pi}{\omega_0} \cdot \frac{\beta_+ + \beta_-}{\beta_-}$, $\beta_+ u^2(x)|_{\Gamma_+} + \beta_- u^2(x)|_{\Gamma_-} + bu(x)u(\gamma(x))|_{\Gamma_+} = 0$, $u^2(x)|_{\Gamma_+} = u^2(x)|_{\Gamma_-}$ and let u be a weak solution of problem* (L). *Let assumptions* (a)–(f) *be satisfied with $\mathcal{A}(r)$ Dini-continuous at zero. Then there are $d \in (0, 1/e)$ and a constant $C > 0$ depending only on s, ω_0, ν, b, β_+, β_-, d, G and $\int_0^{1/e} (\mathcal{A}(r)/r)\,dr$ such that for a.e. $\varrho \in (0, d)$,*

$$
\begin{aligned}
&\int_{G_0^\varrho} \left(|\nabla u|^2 + \frac{u^2(x)}{r^2}\right) dx\\
&\quad\le C\left(|u|^2_{0,\overline{G}} + \omega_0 f_0^2 + \frac{1}{\beta_+}g_0^2 + \frac{1}{\beta_-}h_0^2 + \|f\|^2_{2,G} + \|g\|^2_{\infty,\Gamma_+} + \|h\|^2_{\infty,\Gamma_-}\right)\\
&\quad\times \begin{cases} \varrho^{2\lambda} & \text{if } s > \lambda,\\ \varrho^{2\lambda}\ln^2(1/\varrho) & \text{if } s = \lambda,\\ \varrho^{2s} & \text{if } s < \lambda, \end{cases}
\end{aligned}
$$

where $\lambda = \pi/\omega_0$.

Proof. As in Theorem 6.1, we get (6.1). By our assumptions and (2.20) we obtain

$$
\begin{aligned}
U_-(\varrho) \le{}& \varrho \int_{\Omega} \left(u(x)\frac{\partial u}{\partial r}\right)\Big|_{r=\varrho} d\Omega\\
&+ \int_{\Omega_\varrho} (a^{ij}(x) - a^{ij}(0))u(x)u_{x_j}\cos(r, x_i)\,d\Omega_\varrho\\
&+ \int_{G_0^\varrho} \{-(a^{ij}(x) - a^{ij}(0))u_{x_i}u_{x_j} + b^i(x)u(x)u_{x_i} + c(x)u^2(x)\}\,dx\\
&- \int_{G_0^\varrho} u(x)f(x)\,dx + \int_{\Gamma_{0+}^\varrho} u(x)g(x)\,ds + \int_{\Gamma_{0-}^\varrho} u(x)h(x)\,ds.
\end{aligned}
\tag{6.18}
$$

Now similar to Theorem 6.1 we estimate the integrals on the right hand side of (6.18). The first integral is estimated by Corollary 2.18; the next one, by (6.4). Thus, from (6.18) it follows that

$$U_-(\varrho) \le \frac{\varrho}{2\lambda} U'_-(\varrho) + \varrho\mathcal{A}(\varrho) \int\limits_{\Omega} |u(x)|\,|\nabla u|\,d\omega + \int\limits_{\Gamma^{\varrho}_{0+}} |u(x)|\,|g(x)|\,ds \tag{6.19}$$

$$+ \int\limits_{\Gamma^{\varrho}_{0-}} |u(x)|\,|h(x)|\,ds + \mathcal{A}(\varrho) \int\limits_{G^{\varrho}_0} \left(|\nabla u|^2 + \frac{u^2(x)}{r^2}\right) dx + \int\limits_{G^{\varrho}_0} |u(x)|\,|f(x)|\,dx,$$

where $\lambda = \pi/\omega_0$. Further we bound the integrals on the right of (6.19). First, applying the Cauchy and Friedrichs–Wirtinger inequalities (see (2.13)) similarly to (6.6), we have

$$\mathcal{A}(\varrho) \int\limits_{\Omega} \varrho |u(x)|\,|\nabla u|\,d\omega \le c_1(b, \beta_{\pm}, \omega_0)\varrho\mathcal{A}(\varrho)U'_-(\varrho). \tag{6.20}$$

Next, using the Cauchy and the Hardy–Friedrichs–Wirtinger inequalities (see (2.16) for $\alpha = 2$), by (2.20), similarly to (6.7) we obtain

$$\mathcal{A}(\varrho) \int\limits_{G^{\varrho}_0} \left(|\nabla u|^2 + \frac{|u|^2}{r^2}\right) dx \le c_2(b, \beta_{\pm}, \omega_0)\mathcal{A}(\varrho)U_-(\varrho). \tag{6.21}$$

Thus from (6.19)–(6.21) and (6.8) we get

$$\langle 1 - c_4(\delta + \mathcal{A}(\varrho))\rangle U_-(\varrho) \le \frac{\varrho}{2\lambda}(1 + c_5\mathcal{A}(\varrho))U'_-(\varrho) \tag{6.22}$$

$$+ \frac{1}{2\delta}\Bigg\{ \int\limits_{G^{\varrho}_0} r^2 f^2(x)\,dx + \frac{1}{\beta_+} \int\limits_{\Gamma^{\varrho}_{0+}} r g^2(x)\,ds + \frac{1}{\beta_-} \int\limits_{\Gamma^{\varrho}_{0-}} r h^2(x)\,ds \Bigg\}, \qquad \lambda = \frac{\pi}{\omega_0},$$

for all $\delta > 0$. Then by (6.11) from (6.22) we have the differential inequality (CP) of Subsection 2.3 for the function $U_-(\varrho)$ with

$$\mathcal{P}(\varrho) = \frac{2\lambda}{\varrho} \cdot [1 - c_6\delta - c_7\mathcal{A}(\varrho)], \qquad \lambda = \frac{\pi}{\omega_0},$$

$$\mathcal{Q}(\varrho) = \frac{\lambda}{2s}\left(\omega_0 f_0^2 + \frac{1}{\beta_+} g_0^2 + \frac{1}{\beta_-} h_0^2\right) \cdot \delta^{-1}\varrho^{2s-1}, \qquad \lambda = \frac{\pi}{\omega_0},\ \delta > 0,$$

$$U_0 = C\Bigg\{ |u|^2_{0,\overline{G}} + \int\limits_{G} f^2(x)\,dx + \int\limits_{\Gamma_+} g^2(x)\,ds + \int\limits_{\Gamma_-} h^2(x)\,ds \Bigg\},$$

by (2.20) and (5.8). Next, repeating the proof of Theorem 6.1 for the Cauchy problem (CP) with the function $U_-(\varrho)$ we get the desired result. ∎

7. The power modulus of continuity at a conical point for weak solutions

Proof of Theorem 1.5. We define

$$\psi(\varrho) = \begin{cases} \varrho^{\lambda k} & \text{if } s > \lambda k, \\ \varrho^{\lambda k}\ln(1/\varrho) & \text{if } s = \lambda k, \\ \varrho^{s} & \text{if } 1 < s < \lambda k, \end{cases}$$

for $0 < \varrho < d$.

By Theorem 4.1, we have

$$\sup_{x\in G_0^{\varrho/2}} |u(x)| \le C\{\varrho^{-1}\|u\|_{2,G_0^\varrho} + \varrho^{2(1-2/p)}\|f\|_{p/2,G_0^\varrho} + \varrho(\|g\|_{\infty,\Gamma_{0+}^\varrho} + \|h\|_{\infty,\Gamma_{0-}^\varrho})\}, \tag{7.1}$$

where $C = C(\nu,\mu,p,\|\sum_{i=1}^2 |b^i(\cdot)|^2\|_{p/2,G},G)$ and $p > 2$. Then, by Theorem 6.1,

$$\varrho^{-1}\|u\|_{2,G_0^\varrho} \le \Big(\int_{G_0^\varrho} \frac{u^2(x)}{r^2}\,dx\Big)^{1/2} \le C\Big(\|f\|_{2,G} + \|g\|_{\infty,\Gamma_+} + \|h\|_{\infty,\Gamma_-} + \sqrt{\omega_0}f_0 + \frac{1}{\sqrt{\beta_+}}g_0 + \frac{1}{\sqrt{\beta_-}}h_0\Big)\psi(\varrho). \tag{7.2}$$

Further, by assumption (e) and $s > 2 - 4/p$, we get

$$\varrho^{2(1-2/p)}\|f\|_{p/2,G_0^\varrho} + \varrho(\|g\|_{\infty,\Gamma_{0+}^\varrho} + \|h\|_{\infty,\Gamma_{0-}^\varrho}) \le c\Big(f_0 + \frac{1}{\sqrt{\beta_+}}g_0 + \frac{1}{\sqrt{\beta_-}}h_0\Big)\psi(\varrho) \tag{7.3}$$

for $p > \widetilde{n} > 2$. From (7.1)–(7.3) it follows that

$$\sup_{x\in G_{\varrho/4}^{\varrho/2}} |u(x)| \le C\Big(\|f\|_{2,G} + \|g\|_{\infty,\Gamma_+} + \|h\|_{\infty,\Gamma_-} + f_0 + \frac{1}{\sqrt{\beta_+}}g_0 + \frac{1}{\sqrt{\beta_-}}h_0\Big)\psi(\varrho).$$

Then putting $|x| = \frac{1}{3}\varrho$ we obtain the desired estimate (1.2). ∎

Proof of Theorem 1.7. We repeat the proof of Theorem 1.5 by taking

$$\psi(\varrho) = \begin{cases} \varrho^\lambda & \text{if } s > \lambda,\\ \varrho^\lambda \ln(1/\varrho) & \text{if } s = \lambda,\\ \varrho^s & \text{if } 1 < s < \lambda, \end{cases}$$

and applying Theorem 6.2 instead of Theorem 6.1. ∎

Proof of Theorem 1.8. We repeat the proof of Theorem 1.7, applying Theorem 6.3 instead of Theorem 6.2. ∎

8. Example. We consider the corner

$$G_0 = \{(r,\omega) : r > 0,\ -\omega_0/2 < \omega < \omega_0/2,\ \omega_0 \in (0,\pi)\} \subset \mathbb{R}^2$$

with $\partial G = \mathcal{O}\cup\Gamma_+\cup\Gamma_-$, where $\Gamma_\pm = \{r > 0,\ \omega = \pm\omega_0/2\}$. Let $b = \frac{\pi}{\omega_0}\cdot\frac{\beta_+ + \beta_-}{\beta_-}$. Then the function

$$u(r,\omega) = r^\lambda\Big(\ln\frac{1}{r}\Big)^{\frac{\lambda-1}{\lambda+1}}\psi(\omega),$$

where $\psi(\omega) = \beta_- \cos(\lambda\omega) - \lambda \sin(\lambda\omega)$ and $\lambda = \pi/\omega_0$, is a solution of the problem

$$\begin{cases} \dfrac{\partial}{\partial x_i}(a^{ij}(x)u_{x_j}) + c(x)u = 0, & x \in G_0, \\ \dfrac{\partial u}{\partial \nu} + \beta_+ \dfrac{u(x)}{|x|} + \dfrac{b}{|x|}u(\gamma(x)) = g(x), & x \in \Gamma_+, \\ \dfrac{\partial u}{\partial \nu} + \beta_- \dfrac{u(x)}{|x|} = h(x), & x \in \Gamma_-, \end{cases}$$

where

$$\begin{aligned} a^{11}(x) &= 1 - \frac{2}{\lambda+1} \cdot \frac{x_2^2}{r^2 \ln \frac{1}{r}}, \\ a^{12}(x) &= a^{21}(x) = \frac{2}{\lambda+1} \cdot \frac{x_1 x_2}{r^2 \ln \frac{1}{r}}, \\ a^{22}(x) &= 1 - \frac{2}{\lambda+1} \cdot \frac{x_1^2}{r^2 \ln \frac{1}{r}}, \\ a^{ij}(0) &= \delta_i^j \qquad (i, j = 1, 2), \\ c(x) &= -\frac{2}{1+\lambda} \cdot \frac{1}{r^2 \ln^2 \frac{1}{r}} \left(\lambda \ln \frac{1}{r} - \frac{\lambda - 1}{\lambda + 1} \right), \\ g(x) &= h(x) = \frac{2\pi\beta_-}{(\lambda+1)\omega_0} \frac{r^{\lambda-1}}{\ln^{\frac{2}{\lambda+1}} \frac{1}{r}} \end{aligned}$$

for $r > 0$. In the domain G_0^d, $d < e^{-4}$, the equation is uniformly elliptic with ellipticity constants $\mu = 1$ and $\nu = 1 - \frac{4}{\ln(1/d)}$. Further, $\mathcal{A}(r) = \frac{2}{\lambda+1} \ln^{-1}(1/r)$, i.e., $\mathcal{A}(r)$ *does not* satisfy the Dini condition at zero. Moreover, $a^{ij}(x)$ are continuous at the point $\mathcal{O}$, $c(x) \le 0$ and the conditions $\beta_+ u^2(x)|_{\Gamma_+} + \beta_- u^2(x)|_{\Gamma_-} + bu(x)|_{\Gamma_+} \cdot u(\gamma(x))|_{\Gamma_+} = 0$, $u^2(x)|_{\Gamma_+} = u^2(x)|_{\Gamma_-}$ of Theorem 1.8 are fulfilled. This example shows that the Dini-continuity condition in Theorem 1.8 is essential.

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