APPLICATIONES MATHEMATICAE
30,3 (2003), pp. 267–286

Karol Dziedziul (Gdańsk)

# SHIFT INVARIANT OPERATORS AND A SATURATION THEOREM

*Abstract.* The properties of shift invariant operators $Q_h$ are proved: It is shown that $Q$ has polynomial order $r$ iff $r$ is the rate of convergence of $Q_h$. A weak saturation theorem is given. If $f$ is replaced by $Q_hf$ in the weak saturation formula the asymptotics of the expression is calculated. Moreover, bootstrap approximation is introduced.

**1. Introduction.** This paper is a continuation of our earlier research [Dz2], [Dz4]. Let us present the results by means an example. We prove them for a general shift invariant operator $Q$ (see [Dz4]). Let

$$F(x) = \begin{cases} 1 & \text{for } -1/2 \le x < 1/2, \\ 0 & \text{elsewhere}, \end{cases} \qquad G(x) = \begin{cases} 1 - |x| & \text{for } |x| < 1, \\ 0 & \text{elsewhere.} \end{cases}$$

For $h > 0$ we define an operator $\widetilde{Q}_h$ by

$$\widetilde{Q}_h f(x) = \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} F(u-k) f(hu)\, du\, G(x/h - k) = \sum_{k\in\mathbb{Z}} \frac{1}{h} \int_{(k-.5)h}^{(k+.5)h} f(u)\, du\, G(x/h-k).$$

We say that an operator $Q$ has *polynomial order* $r$ if $Q(P) = P$ for all polynomials $P$ with $\deg P < r$. Note $\widetilde{Q} = \widetilde{Q}_1$ has polynomial order 2. For $f \in W_2^2$,

$$\frac{\widetilde{Q}_h f - f}{h^2} \to \frac{f''}{8},$$

where the convergence is weak in $L^2(\mathbb{R})$. Let $f(x) = \sin^9 x$. In Fig. 1 we show both $f''/8$ and $(\widetilde{Q}_h f - f)/h^2$ for $h = 1/8$.

2000 *Mathematics Subject Classification*: Primary 41A35.

*Key words and phrases*: shift invariant operators, asymptotic formula, saturation theorem.

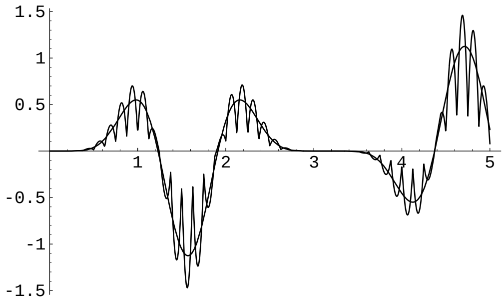


Fig. 1

We call this phenomenon the *weak saturation theorem* (see for instance Theorem 3.2 of [Dz2]). Now replacing in the above formula $f$ by $\widetilde{Q}_h f$ we prove that

$$\frac{\widetilde{Q}_h(\widetilde{Q}_h f)(x) - \widetilde{Q}_h f(x)}{h^2} = \widetilde{Q}_h\left(\frac{\widetilde{Q}_h f - f}{h^2}\right)(x) \to \frac{f''(x)}{8}$$

uniformly for $x \in \mathbb{R}$ provided $f$ is sufficiently smooth. In Fig. 2 we picture both $f''/8$ and $(\widetilde{Q}_h(\widetilde{Q}_h f) - \widetilde{Q}_h f)/h^2$ for $h = 1/8$.

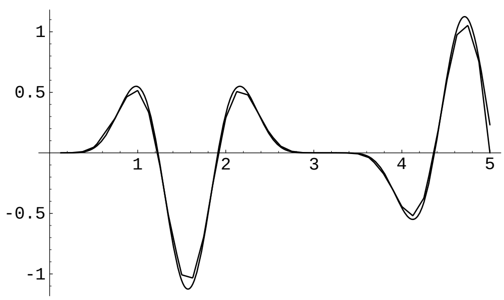


Fig. 2

We see that replacing $f$ by a sample $\widetilde{Q}_h f$ (we call it a *bootstrap*) smooths the limit. In the last section we use this procedure to increase the rate of approximation. Application of this procedure is called *bootstrap approximation*. Let $T_H$ denote the convolution operator, i.e. $T_H f = H * f$. Recall that $\widetilde{Q}$ has polynomial order $r = 2$. Then for any $N > 0$ there is a function $H_N$ such that the operator

$$T_{H_N} \circ \widetilde{Q} f(x) = \sum_{\alpha \in \mathbb{Z}^d} \int_{\mathbb{R}^d} f(u) F(u - \alpha)\, du\, (H_N * G)(x - \alpha)$$

has polynomial order $r(4N - 1)$. The function $H$ will be constructed later.

Consider the adjoint operator $\widetilde{Q}_h^*$ given by

$$\widetilde{Q}_h^* f(u) = \int_{\mathbb{R}} \sum_{k \in \mathbb{Z}} G(x - k) f(hx)\, dx\, F(u/h - k).$$

We have

$$\widetilde{Q}_h^* \left( \frac{\widetilde{Q}_h f - f}{h^2} \right)(u) \to \frac{f''(u)}{8},$$

uniformly for $u \in \mathbb{R}$, provided $f$ is sufficiently smooth (see Fig. 3).

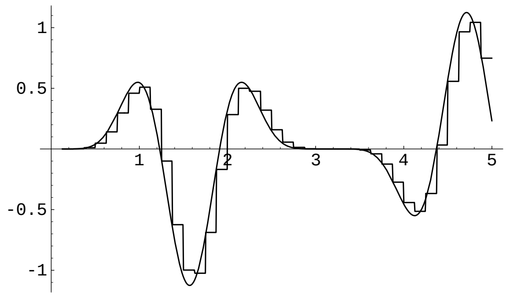


Fig. 3

The last result was stated without proof in [Dz2, Lemma 3.2]. It gives an appropriate tool for proving the central limit theorem for the square error of multivariate nonparametric estimators based on shift invariant operators (for definition see [Dz4]).

Let us introduce standard notation. For $\beta = (\beta_1, \ldots, \beta_d)$ where $\beta_j \geq 0$, and $1 \leq p < \infty$, set

$$|\beta| := \beta_1 + \ldots + \beta_d, \quad \beta! := \beta_1! \ldots \beta_d!, \quad []^\beta(x) := x^\beta = x_1^{\beta_1} \ldots x_d^{\beta_d},$$

$$|f|_{r,p} = \sum_{|\beta|=r} \|D^\beta f\|_p, \quad \|f\|_p = \Big( \int_{\mathbb{R}^d} |f|^p \Big)^{1/p}, \quad D^\beta f = \frac{\partial^{|\beta|} f}{\partial x_1^{\beta_1} \ldots \partial x_d^{\beta_d}}.$$

For $p = \infty$,

$$\|f\|_\infty = \sup_{x \in \mathbb{R}^d} |f(x)|, \quad |f|_{k,\infty} = \sup_{x \in \mathbb{R}^d} \sup_{|\beta|=k} |D^\beta f(x)|.$$

Define also

$$\|f\|_{C^k(K)} = \sup_{x \in K} \sup_{|\beta| \leq k} |D^\beta f(x)|,$$

where $K$ is a compact ball or $K = \mathbb{R}^d$. The Fourier transform of $f$ is

$$\widehat{f}(x) = \int_{\mathbb{R}^d} f(t) e^{-2\pi i t \cdot x} \, dt.$$

Set $\breve{F}(x) = F(-x)$. The convolution is defined by

$$f * g(x) = \int_{\mathbb{R}^d} f(t) g(x-t) \, dt.$$

We use the Sobolev spaces $W_p^k = W_p^k(\mathbb{R}^d)$, $1 \leq p < \infty$ (see [M]). For $p = \infty$ we can take the $k$-times differentiable bounded functions with compact support, $C_0^k$, and define $W_\infty^k = W_\infty^k(\mathbb{R}^d)$ to be the closure of $C_0^k$ in the norm $\|\cdot\|_{C^k(\mathbb{R}^d)}$. This is the space of functions whose derivatives up to order $k$ vanish at infinity.

The paper is organized as follows: in Section 2 we define shift invariant operators and formulate the main results. In Section 3 and 4 we give the proofs of the stated theorems. In Section 5 we introduce bootstrap approximation, i.e. methods of increasing the rate of convergence of a shift invariant operator. This method was proposed in [Dz3] for the orthogonal projection in the box spline case.

**2. Shift invariant operators.** Following the notation of [JM], let $\mathcal{E}^\infty$ denote the space of functions which decay exponentially fast, i.e. there are constants $C > 0$ and $0 < q < 1$ such that for all $x \in \mathbb{R}^d$,

$$|G(x)| < Cq^{|x|},$$

where $|x|^2 = x \cdot x$, the scalar product in $\mathbb{R}^d$. For

$$F, G \in \mathcal{E}^\infty \tag{1}$$

consider the shift invariant operators [Dz4]

$$Qf(x) = \int_{\mathbb{R}^d} K(x,y) f(y)\, dy, \tag{2}$$

where

$$K(x,y) = \sum_{\alpha \in \mathbb{Z}^d} F(y-\alpha) G(x-\alpha). \tag{3}$$

Moreover, for $h > 0$ define

$$Q_h = \sigma_h \circ Q \circ \sigma_{1/h}, \tag{4}$$

where

$$\sigma_h f(x) = f(x/h).$$

It is known [BHR, (4) Proposition, p. 63], [LC], [BDR] that $r$ is the *rate of convergence* of a local box-spline operator $Q$ if and only if $Q$ has polynomial order $r$. Below we generalize that theorem. Recall that an operator $Q$ has *polynomial order* $r$ if $Q(P) = P$ for all polynomials $P$ with $\deg P < r$.

THEOREM 2.1. *Let $Q$ be shift invariant. Let $r \geq 1$. The following conditions are equivalent*:

(i) *$Q$ has polynomial order $r$.*

(ii) *For all $1 \leq p \leq \infty$ there is a constant $C = C(p,r,Q) > 0$ such that for all $f \in W_p^r$,*

$$\|Q_h f - f\|_p \leq Ch^r |f|_{r,p}. \tag{5}$$

The implication (i)⇒(ii) was proved in [Dz4]. The remaining details, together with the proof of the corollary below, will be given in the next section.

COROLLARY 2.1. *Let $Q$ be shift invariant. Let $r \geq 1$. Assume that $Q$ has polynomial order $r$, $G \in C^n$ and $D^\beta G \in \mathcal{E}^\infty$ for all $|\beta| \leq n$. Then for all $1 \leq p \leq \infty$ there is a constant $C = C(p, r, Q, n) > 0$ such that for all $f \in W_p^r$ and $|\beta| < \min\{n+1, r\}$,*

$$\|D^\beta Q_h f - D^\beta f\|_p \leq C h^{r-|\beta|} |f|_{r,p}.$$

Theorem 2.2 below was proved in [Dz4] (see also [BD3] and [BD4]).

THEOREM 2.2. *Let $Q$ be shift invariant of polynomial order $r \geq 1$. Then for all $1 \leq p < \infty$ and $f \in W_p^r$,*

$$\lim_{h \to 0^+} \left\| \frac{Q_h f - f}{h^r} \right\|_p^p = \int_{\mathbb{R}^d} \left( \int_{[0,1]^d} \left| \sum_{|\beta|=r} \frac{1}{\beta!} D^\beta f(t) (Q([]^\beta)(x) - x^\beta) \right|^p dx \right) dt. \tag{6}$$

A similar result was recently proved for $p = \infty$.

We also have a generalization of Blu–Unser's theorem, i.e. the estimate of the error of approximation by an asymptotic constant:

THEOREM 2.3. *Assume that $Q$ has polynomial order $r > 0$. Let $1 \leq p < \infty$. Then there is a constant $C > 0$ such that for all $f \in W_p^{r+1}(\mathbb{R}^d)$,*

$$\left\| \frac{Q_h f - f}{h^r} \right\|_p \leq Ch|f|_{r+1,p} + \left( \int_{\mathbb{R}^d} \left( \int_{[0,1]^d} \left| \sum_{|\beta|=r} \frac{1}{\beta!} D^\beta f(t) (Q([]^\beta)(x) - x^\beta) \right|^p dx \right) dt \right)^{1/p}.$$

The proof will be given in a forthcoming paper.

We say that $G$ satisfies the *Strang–Fix conditions* of order $r$ (briefly $G \in \mathrm{SF}(r)$) if for all $|\beta| < r$,

$$D^\beta \widehat{G}(\alpha) = 0, \qquad \alpha \in \mathbb{Z}^d \setminus \{0\}, \tag{7}$$

and $\widehat{G}(0) \neq 0$ (see [SF]). The following is known [LC]:

THEOREM 2.4. *If a shift invariant operator $Q$ has polynomial order $r \geq 1$ then $G \in \mathrm{SF}(r)$. Moreover*

$$D^\beta (\widehat{G} \widehat{\breve{F}})(0) = D^\beta (\widehat{G * \breve{F}})(0) = \begin{cases} 0, & 0 < |\beta| < r, \\ 1, & \beta = 0. \end{cases} \tag{8}$$

A simple poof will be given in Section 4.

Z. Ciesielski [C] proved the saturation theorem for spline operators. The saturation theorem was also shown for quasi-projections in [Dz1]. The new concept of weak saturation was introduced for orthogonal projections in [BD2]. It turns out that for the shift invariant operators we also have a weak saturation theorem.

Theorem 2.5. *Assume that a shift invariant operator $Q$ has polynomial order $r \geq 1$. For every $f \in W_2^r$ we have the weak saturation formula*:

$$\frac{Q_h f - f}{h^r} \to D_Q f \qquad \text{as } h \to 0^+, \tag{9}$$

*weakly in $L^2(\mathbb{R}^d)$, where*

$$D_Q f = \frac{1}{(2\pi i)^r} \sum_{|\beta|=r} \frac{D^\beta f}{\beta!} D^\beta(\widehat{G * \check{F}})(0). \tag{10}$$

*If additionally $G \in \mathrm{SF}(r+1)$, then for all $f \in W_p^r$,*

$$\frac{Q_h f - f}{h^r} \to D_Q f \qquad \text{as } h \to 0^+ \tag{11}$$

*in $L^p$ norm provided $1 \leq p < \infty$. For all $|\beta| = r$,*

$$Q[]^\beta - []^\beta = \frac{1}{(2\pi i)^r} D^\beta(\widehat{G}\widehat{\check{F}})(0) \qquad a.e. \tag{12}$$

The poof will be given in Section 4.

Another approach to the asymptotic formula, different from the one presented in Theorem 2.2, is motivated by (9). We prove

Theorem 2.6. *Assume that a shift invariant operator $Q$ has polynomial order $r \geq 1$. Then for all $f \in W_2^r$,*

$$\lim_{h\to 0^+} \left\| \frac{Q_h f - f}{h^r} - D_Q f \right\|_2^2 = \lim_{h\to 0^+} \left\| \frac{Q_h f - f}{h^r} \right\|_2^2 - \|D_Q f\|_2^2$$
$$= \frac{1}{(4\pi^2)^r} (\widehat{F}(0))^2 \sum_{\alpha \in \mathbb{Z}^d \setminus \{0\}} \left\| \sum_{|\beta|=r} \frac{1}{\beta!} D^\beta \widehat{G}(\alpha) D^\beta f \right\|_2^2.$$

The proof will be given in Section 4 together with the proof of Theorem 2.5. A similar result for box-spline operators was proved in [Dz3, Theorem 5.1].

The following result was stated for box-spline operators in [Dz2, Lemma 3.2] without proof. Let $Q^*$ be the adjoint operator to $Q$.

Theorem 2.7. *Assume that a shift invariant operator $Q$ has polynomial order $r \geq 1$, and $Q^*$ has polynomial order 1. For $f \in W_2^r \cap W_\infty^r$, we have*

$$\frac{Q_h(Q_h f) - Q_h f}{h^r}(x) = \frac{Q_h(Q_h f - f)}{h^r}(x) \to D_Q f(x) \qquad \text{as } f \to 0^+, \tag{13}$$

*uniformly for* $x \in \mathbb{R}^d$. *Moreover*

$$\frac{Q_h^*(Q_h f - f)}{h^r}(u) \to D_Q f(u) \quad \text{as } f \to 0^+, \tag{14}$$

*uniformly for* $u \in \mathbb{R}^d$.

The proof will be given in Section 4.

**3. Proof of Theorem 2.1.** (i)⇒(ii). For $1 \le p < \infty$ this follows from [Dz4, Lemma 1.1]. The case $p = \infty$ follows easily from the lemma below.

LEMMA 3.1. *Let $Q$ be shift invariant. Let $P_x$ be the Taylor polynomial of degree $k-1$ of a function $f$ at the point $x$. There is $C$ such that for all* $f \in C_0^k$,

$$\sup_{x\in\mathbb{R}^d} |Q_h(f - P_x)(x)| \le Ch^k |f|_{k,\infty}, \tag{15}$$

*where*

$$|f|_{k,\infty} = \sup_{x\in\mathbb{R}^d} \sup_{|\beta|=k} |D^\beta f(x)|.$$

*Proof.* By Taylor's formula there is $C > 0$ independent of $f$ such that

$$\begin{aligned}|Q_h(f - P_x)(x)| &\le Ch^k |f|_{k,\infty} \sum_{\alpha\in\mathbb{Z}^d} \Big( \int_{\mathbb{R}^d} |x/h - y|^k q^{|y-\alpha|}\, dy \Big) q^{|x/h-\alpha|} \\ &\le Ch^k |f|_{k,\infty}. \ \blacksquare\end{aligned}$$

(ii)⇒(i). In this implication we use only the estimate (5) for $p = \infty$. By induction on $k$, $0 \le k < r$, we prove that

$$Q[\,]^\beta = [\,]^\beta \quad \text{for } |\beta| \le k. \tag{16}$$

Let $k = 0$. Take $f \in C_0^r$ such that $f(0) = 1$. Define $f_t(x) = f(x - t)$. Let 1 stand also for the function constantly equal to 1. Then

$$|(Q_h 1 - 1)(t)| \le |(Q_h f_t - f_t)(t)| + |Q_h(f_t - 1)(t)|.$$

By Lemma 3.1 and assumption (ii) for $p = \infty$,

$$|(Q_h 1 - 1)(t)| \le Ch^r |f_t|_{r,\infty} + Ch|f_t|_{1,\infty}.$$

Since $Q_h(1)(t) = Q(1)(t/h)$, for all $t \in \mathbb{R}^d$ and $0 < h < 1$ we have

$$|(Q1 - 1)(t/h)| \le Ch.$$

Choosing $t = hx$ we get $Q1(x) = 1$ for all $x \in \mathbb{R}^d$.

Assume (16) is true for $k < r - 1$. We prove it for $k + 1$. Fix $|\delta| = k + 1$. Take $f \in C_0^r$ such that for $|\beta| = k + 1$,

$$D^\beta f(0) = \begin{cases} 1 & \text{if } \beta = \delta, \\ 0 & \text{otherwise.} \end{cases}$$

Let $P_t$ be the Taylor polynomial of degree $k+1$ of $f_t$ at $t$. Then

$$|(Q_hP_t - P_t)(t)| \le |(Q_hf_t - f_t)(t)| + |Q_h(f_t - P_t)(t)|.$$

By Lemma 3.1 and assumption (ii) for $p = \infty$,

$$|(Q_hP_t - P_t)(t)| \le Ch^r|f_t|_{r,\infty} + Ch^{k+2}|f_t|_{n,\infty}.$$

By induction if $0 < |\beta| \le k+1$ then

$$\begin{aligned} Q_h[\cdot - t]^\beta(t) &= Q[h\cdot - t]^\beta(t/h) \\ &= \sum_{0\le\delta\le\beta} h^{|\delta|}\binom{\beta}{\delta} Q([]^\delta)(t/h)(-t)^{\beta-\delta} \\ &= \sum_{0\le\delta<\beta} h^{|\delta|}\binom{\beta}{\delta}(t/h)^\delta(-t)^{\beta-\delta} + h^{|\beta|}Q([]^\beta)(t/h) \\ &= h^{|\beta|}Q([]^\beta)(t/h) - []^\beta(t) = h^{|\beta|}(Q[]^\beta - []^\beta)(t/h). \end{aligned} \tag{17}$$

Thus

$$\begin{aligned} (Q_hP_t - P_t)(t) &= \sum_{|\beta|=k+1} \frac{D^\beta f_t(t)}{\beta!} Q_h([\cdot - t]^\beta)(t) \\ &= \frac{1}{\delta!} h^{k+1}(Q[]^\delta - []^\delta)(t/h). \end{aligned}$$

Thus for all $t \in \mathbb{R}^d$,

$$\left|\frac{1}{\delta!} h^{k+1}(Q[]^\delta - []^\delta)(t/h)\right| \le Ch^{k+2}.$$

Choosing $t = hx$ we obtain $Q[]^\delta = []^\delta$, which finishes the proof. ∎

*Proof of Corollary 2.1.* Fix $x$. If $|\beta| < \min\{r, n+1\}$ then

$$D^\beta Q_hf(x) - D^\beta f(x) = D^\beta Q_hf(x) - D^\beta P_x(x) = D^\beta Q_h(f - P_x)(x).$$

But

$$D^\beta Q_h(f - P_x)(x) = \frac{1}{h^{|\beta|}} Q_h^\beta(f - P_x)(x),$$

where $Q_h^\beta$ is the shift invariant operator of polynomial order $r - |\beta|$, given by

$$Q_h^\beta f(y) = \sum_{\alpha\in\mathbb{Z}^d} \int_{\mathbb{R}^d} f(hy)F(y-\alpha)\,dy\,D^\beta G(x/h - \alpha).$$

Lemma 1.1 in [Dz4] for $1 \le p < \infty$ and Lemma 3.1 for $p = \infty$ finish the proof.

## 4. Other proofs

*Proof of Theorem 2.4.* Note that if $f \in L^2(\mathbb{R}^d)$ then the sum in the definition of $Qf$ converges in $L^2(\mathbb{R}^d)$ norm. Thus

$$\widehat{Q_h f}(y) = h^d \widehat{G}(hy) \sum_{\alpha\in\mathbb{Z}^d} (\sigma_{1/h} f * \breve{F})(\alpha) e^{-2\pi i hy\cdot\alpha}.$$

Moreover assume that $\widehat{f} \in L^2$ has compact support $\mathcal{C} \subset [0,N]^d$. Consequently, for $0 < h < 1/N$ and $z \in [0,1)^d$ we have the periodic function

$$\sum_{\alpha\in\mathbb{Z}^d} \widehat{\breve{F}}(z-\alpha)\widehat{f}((z-\alpha)/h) = \widehat{\breve{F}}(z)\widehat{f}(z/h).$$

Its Fourier expansion is $\sum_{\alpha\in\mathbb{Z}^d} (\sigma_{1/h} f * \breve{F})(\alpha) e^{-2\pi i z\cdot\alpha}$. Taking $z = hy$ we get (cf. [BD2, Lemma 2.3])

$$\widehat{Q_h f}(y) = \widehat{G}(hy) \sum_{\alpha\in\mathbb{Z}^d} \widehat{\breve{F}}(hy-\alpha)\widehat{f}(y-\alpha/h) \tag{18}$$

for almost every $y$. Thus if $0 < h < 1/N$ then

$$\int_{\mathbb{R}^d} \left| \frac{\widehat{Q_h(f)} - \widehat{f}}{h^r} \right|^2 = \int_{\mathcal{C}} \left( \left| \frac{\widehat{G * \breve{F}}(hx) - 1}{h^r} \right|^2 + \sum_{\delta\in\mathbb{Z}^d\setminus\{0\}} \left| \frac{\widehat{G}(hx+\delta)\widehat{\breve{F}}(hx)}{h^r} \right|^2 \right) |\widehat{f}(x)|^2\, dx. \tag{19}$$

By (5) the last expression is uniformly bounded by $|f|_{2,p}$ for $h > 0$ and any $f$, provided $\widehat{f}$ has compact support. Consequently, both expressions in large brackets are bounded. Thus we get (7) and (8). ∎

The following lemma is taken from the proof of [M, Lemma 7, p. 29].

LEMMA 4.1. *Let $K$ be a compact ball. If for each $m \in \mathbb{N}$,*

$$|g(x)| \le C_m (1+|x|)^{-m} \quad \text{for } x \in \mathbb{R}^d,$$

*then*

$$\sum_{\alpha\in\mathbb{Z}^d} \|\widehat{g}(\cdot+\alpha)\|^2_{C^k(K)} < \infty \quad \text{for all } k. \tag{20}$$

LEMMA 4.2. *Let $r \ge 1$. Let $K$ be a compact ball. Let $\phi$ satisfy the assumption of Lemma 4.1 and $\phi \in \mathrm{SF}(r)$. Then*

$$\sum_{\alpha\in\mathbb{Z}^d\setminus\{0\}} \left| \frac{\widehat{\phi}(hx+\alpha)}{h^r} \right|^2 \to \sum_{\alpha\in\mathbb{Z}^d\setminus\{0\}} \left| \sum_{|\beta|=r} \frac{1}{\beta!} D^\beta \widehat{\phi}(\alpha) x^\beta \right|^2 \tag{21}$$

*uniformly in $K$ as $h \to 0^+$.*

*Proof.* Let

$$F_\alpha(h) = \widehat{\phi}(hx+\alpha).$$

By Taylor's formula for $F_\alpha$ at 0 and since $\phi \in \mathrm{SF}(r)$, we infer that there is a point $\theta_\alpha$ such that $0 < |\theta_\alpha| < h$ and

$$F_\alpha(h) = \frac{D^r F_\alpha(\theta_\alpha)}{r!}\, h^r.$$

Then

$$\sum_{\alpha\in\mathbb{Z}^d\setminus\{0\}} \left|\frac{\widehat{\phi}(hx+\alpha)}{h^r}\right|^2 = \frac{1}{r!} \sum_{\alpha\in\mathbb{Z}^d\setminus\{0\}} |D^r F_\alpha(\theta_\alpha)|^2$$
$$= \sum_{\alpha\in\mathbb{Z}^d\setminus\{0\}} \left|\sum_{|\beta|=r} \frac{1}{\beta!} D^\beta \widehat{\phi}(\theta_\alpha x+\alpha) x^\beta\right|^2.$$

If $0 < |\theta_\alpha| < h \to 0$ then for the finite choice of $\alpha \in \mathbb{Z}^d$,

$$D^\beta \widehat{\phi}(\theta_\alpha x+\alpha) \to D^\beta \widehat{\phi}(\alpha)$$

uniformly for $x \in K$ and $|\beta| = r$. From Lemma 4.1,

$$\sum_{\alpha\in\mathbb{Z}^d\setminus\{0\}} \|D^\beta \widehat{\phi}(\cdot+\alpha)\|^2_{C^0(K)} < \infty,$$

Consequently, we get (21). ∎

*Proof of Theorem 2.5 and 2.6.* (9)–(10) are proved for orthogonal projections and cardinal interpolation in [BD1], [BD2]. Let us outline the proof.

Assume that $f \in W_2^r$ is such that $\widehat{f}$ has compact support. By (18) and Plancherel's formula we get (9)–(10). By density we get (9)–(10) for all $f \in W_2^r$.

By Lemma 4.2, Theorem 2.4 and (19) we get Theorem 2.6, namely for $f$ such that $\widehat{f}$ has compact support,

$$\lim_{h\to 0^+} \left\|\frac{Q_h f - f}{h^r} - D_Q f\right\|_2^2 = A(f), \tag{22}$$

where

$$A(f) = \frac{1}{(4\pi^2)^r} (\widehat{F}(0))^2 \sum_{\alpha\in\mathbb{Z}^d\setminus\{0\}} \left\|\sum_{|\beta|=r} \frac{1}{\beta!} D^\beta \widehat{G}(\alpha) D^\beta f\right\|_2^2. \tag{23}$$

By density this implies that (22) holds for all $f \in W_2^r$.

Now if $G \in \mathrm{SF}(r+1)$ then applying (7) we get

$$A(f) = 0. \tag{24}$$

Thus we obtain (11) for $p = 2$.

Now we prove (12). Let $f \in C_0^{r+1}$ and $P_x^r$ be the Taylor polynomial of $f$ at $x$ of degree $r$. Note that

$$\sum_{|\beta|=r} \frac{D^\beta f(x)}{\beta!} (y-x)^\beta = P_x^r(y) - P_x^{r-1}(y).$$

Consequently, by (13),

$$(25)\qquad Q_h(P_x^r - P_x^{r-1})(x) = Q_h\Big(\sum_{|\beta|=r} \frac{D^\beta f(x)}{\beta!}\,(\cdot - x)^\beta\Big)(x)$$

$$= h^r \sum_{|\beta|=r} \frac{D^\beta f(x)}{\beta!}\, Q((h\cdot - x)^\beta)(x/h) = h^r \sum_{|\beta|=r} \frac{D^\beta f(x)}{\beta!}\,(Q[]^\beta - []^\beta)(x/h).$$

Since $Q_h(P_x^{r-1}) = P_x^{r-1}$ we get

$$(26)\qquad \Big\|\frac{Q_h f - f}{h^r} - D_Q f\Big\|_p = \Big\|\frac{Q_h(f - P_x^r)(x)}{h^r} + \frac{Q_h(P_x^r - P_x^{r-1})(x)}{h^r} - D_Q f(x)\Big\|_p.$$

From Lemma 1.1 of [Dz4] there is $C = C(p,r) > 0$ such that for all $f \in C_0^{r+1}$,

$$\|Q_h(f - P_x^r)(x)\|_p \le C h^{r+1} |f|_{r+1,p}.$$

Thus if $p = 2$, from (22), (24), (26) and the above inequality we have

$$\Big\|\frac{Q_h(P_x^r - P_x^{r-1})(x)}{h^r} - D_Q f\Big\|_2 = o(1).$$

Consequently, by (25),

$$\Big\|\sum_{|\beta|=r} \frac{D^\beta f(x)}{\beta!}\Big((Q[]^\beta - []^\beta)(x/h) - \frac{1}{(2\pi i)^r}\, D^\beta(\widehat{G * \check{F}})(0)\Big)\Big\|_2 = o(1).$$

Since $Q[]^\beta - []^\beta$ is a periodic function an application of the Fejér–Mazur–Orlicz theorem (see [Dz4]) gives

$$\int_{\mathbb{R}^d}\int_{[0,1]^d}\Big|\sum_{|\beta|=r} \frac{D^\beta f(x)}{\beta!}\Big(Q([]^\beta)(t) - t^\beta - \frac{1}{(2\pi i)^r}\, D^\beta(\widehat{G}\widehat{\check{F}})(0)\Big)\Big|^2 dt\,dx = 0.$$

Since $f \in C_0^{r+1}$ is arbitrary we get (12). To prove (11) it is sufficient to consider $f \in C_0^{r+1}$. By (12) and (25),

$$\Big\|\frac{Q_h f - f}{h^r} - D_Q f\Big\|_p = \Big\|\frac{Q_h(f - P_x^r)(x)}{h^r} + \frac{Q_h(P_x^r - P_x^{r-1})(x)}{h^r} - D_Q f(x)\Big\|_p = \Big\|\frac{Q_h(f - P_x^r)(x)}{h^r}\Big\|_p.$$

By [Dz4, Lemma 1.1] (see above) we get (11) for $f \in C_0^{r+1}$. A density argument finishes the proof. ∎

REMARK. It is not true that in (12) the functions $Q[]^\beta - []^\beta$ are constant for all $x \in \mathbb{R}^d$. Consider the following example:

$$F(x) = \begin{cases} 1 & \text{if } 0 \le x < 1, \\ 0 & \text{otherwise,} \end{cases} \qquad G(x) = \begin{cases} 1 - |x| & \text{if } 0 < |x| < 1, \\ 1 & \text{if } x = 1, \\ 0 & \text{otherwise.} \end{cases}$$

The operator $Q$ corresponding to $F, G$ has polynomial order $r = 1$, since $\sum_{k\in\mathbb{Z}} G(x-k) = 1$ for all $x \in \mathbb{R}$. But

$$Q([]^1)(x) - x = \begin{cases} 1/2 & \text{for } x \notin \mathbb{Z}, \\ -1/2 & \text{for } x \in \mathbb{Z}. \end{cases}$$

Consequently, (12) is not true for all $x$ and (11) does not hold in sup norm.

*Proof of Theorem 2.7*

STEP 1. We prove the theorem for $f \in W_2^r$ such that

$$\operatorname{supp} \widehat{f} \subset [-N, N]^d = \mathcal{C}.$$

Obviously $[]^\beta \widehat{f} \in L^1$ for $|\beta| \le r$. Then from the Riemann–Lebesgue theorem (see [SW]), $f \in W_\infty^r$. Let

$$K_h(x, y) = h^{-d} K(x/h, y/h).$$

By Plancherel's theorem,

$$\int_{\mathbb{R}^d} K_h(x,y) \frac{Q_h f(y) - f(y)}{h^r}\, dy = \int_{\mathbb{R}^d} \overline{[K_h(x,\cdot)]^\wedge(y)}\, \frac{\widehat{Q_h f}(y) - \widehat{f}(y)}{h^r}\, dy,$$

where for fixed $x \in \mathbb{R}^d$,

$$[K_h(x,\cdot)]^\wedge(t) = \widehat{F}(ht) \sum_{\alpha\in\mathbb{Z}^d} e^{-2\pi i h\alpha\cdot t} G(x/h - \alpha) \in L^2(\mathbb{R}^d). \tag{27}$$

We split the last integral into integrals over $\mathcal{C}$ and $\mathbb{R}^d \setminus \mathcal{C}$. Using (18) and the fact that $\widehat{f}$ has compact support we get, for sufficiently small $h > 0$ and all $y \in \mathcal{C}$,

$$\widehat{Q_h f}(y) = \widehat{G}(hy)\widehat{\check{F}}(hy)\widehat{f}(y) \quad \text{a.e.} \tag{28}$$

Consequently,

$$\int_{\mathcal{C}} \overline{[K_h(x,\cdot)]^\wedge(y)}\, \frac{\widehat{Q_h f}(y) - \widehat{f}(y)}{h^r}\, dy$$

$$= \int_{\mathcal{C}} \overline{[K_h(x,\cdot)]^\wedge(y)}\, \frac{\widehat{G}(hy)\widehat{\check{F}}(hy)\widehat{f}(y) - \widehat{f}(y)}{h^r}\, dy$$

$$= \int_{\mathbb{R}^d} K_h(x,y) \frac{T_{Y,h} f(y) - f(y)}{h^r}\, dy = Q_h\left(\frac{T_{Y,h} f - f}{h^r}\right)(x),$$

where

$$T_{Y,h}f = \sigma_h \circ T_Y \circ \sigma_{1/h}, \qquad T_Y f = Y * f, \qquad Y = G * \breve{F}.$$

Note that $Y \in \mathcal{E}^\infty$ (see Lemma 5.1 below).

By (8), for all $|\beta| < r$,

$$\begin{aligned}
T_Y([]^\beta)(x) &= \int_{\mathbb{R}^d} (x-y)^\beta Y(y)\,dy \\
&= \sum_{0\le\delta\le\beta} \binom{\beta}{\delta} x^{\beta-\delta}(-1)^{|\delta|} \int_{\mathbb{R}^d} y^\delta Y(y)\,dy \\
&= \sum_{0\le\delta\le\beta} \binom{\beta}{\delta} x^{\beta-\delta}\left(\frac{1}{2\pi i}\right)^{|\delta|} D^\delta \widehat{Y}(0) = []^\beta(x),
\end{aligned}$$

and for $|\beta| = r$,

$$T_Y[]^\beta = []^\beta + \left(\frac{1}{2\pi i}\right)^r D^\beta \widehat{Y}(0).$$

By a similar argument to that in (17),

$$T_{Y,h}((\cdot - x)^\beta)(x) = h^{|\beta|}(T_Y[]^\beta - []^\beta)(x/h) = h^{|\beta|}\left(\frac{1}{2\pi i}\right)^r D^\beta \widehat{Y}(0).$$

Thus if $P_x^r$ is the Taylor polynomial of degree $\le r$ of $f$ at $x$ then by (10),

$$\begin{aligned}
T_{Y,h}(P_x^r)(x) &= P_x^{r-1}(x) + \frac{1}{(2\pi i)^r}\sum_{|\beta|=r} h^{|\beta|}\,\frac{D^\beta f(x)}{\beta!}\,D^\beta\widehat{Y}(0) \\
&= f(x) + h^r D_Q f(x).
\end{aligned} \tag{29}$$

Now by a similar argument to that in [Dz1, Theorem 2.23],

$$\frac{T_{Y,h}f - f}{h^r} \to D_Q f \tag{30}$$

uniformly in $\mathbb{R}^d$. For the convenience of the reader we give the argument. It is sufficient prove this for $f \in C_0^{r+1}$. Let

$$f(y) = P_x^r(y) + R_x(y),$$

where $P_x^r$ is the Taylor polynomial of degree $r$ at $x$. By (29),

$$\begin{aligned}
\frac{T_{Y,h}f(x) - f(x)}{h^r} - D_Q f(x) &= \frac{T_{Y,h}(P_x^r + R_x)(x) - f(x)}{h^r} - D_Q f(x) \\
&= \frac{T_{Y,h}(R_x)(x) + h^r D_Q f(x)}{h^r} - D_Q f(x) \\
&= \frac{T_{Y,h}(R_x)(x)}{h^r},
\end{aligned}$$

where

$$R_x(y) = \sum_{|\beta|=r+1} \frac{D^\beta f(\theta_{x,y})}{\beta!}\,(y-x)^\beta.$$

It is sufficient to estimate the last expression for any $|\beta| = r+1$:

$$\begin{aligned}\Bigg|\frac{1}{h^{d+r}}\int_{\mathbb{R}^d} Y((x-y)/h)D^\beta f(\theta_{x,y})(y-x)^\beta\,dy\Bigg| \\ &\le |f|_{r+1,\infty}\,\frac{1}{h^{d+r}}\int_{\mathbb{R}^d}|Y((x-y)/h)|\,|(y-x)^\beta|\,dy \\ &\le |f|_{r+1,\infty}\,\frac{1}{h^r}\int_{\mathbb{R}^d}|Y(z)|\,|(hz)^\beta|\,dy \\ &\le C_\beta h|f|_{r+1,\infty}.\end{aligned}$$

We get (30) for $f \in C_0^{r+1}$, and by density for all $W_\infty^r$. Consequently, from (30) and (5),

$$Q_h\bigg(\frac{T_{Y,h}f - f}{h^r}\bigg) \to D_Q f$$

uniformly in $\mathbb{R}^d$. Now consider the second integral. Since the function

$$\Big|\sum_{\alpha\in\mathbb{Z}^d} G(x/h-\alpha)e^{2\pi i h\alpha\cdot y}\Big| \le \sum_{\alpha\in\mathbb{Z}^d} q^{|x/h-\alpha|} \le C$$

is uniformly bounded, (18) and (27) imply that for $0 < h < 1/N$,

$$\begin{aligned}\Bigg|\int_{\mathbb{R}^d\setminus\mathcal{C}} \overline{[K_h(x,\cdot)]^\wedge(y)}\,\frac{\widehat{Q_h f}(y) - \widehat{f}(y)}{h^r}\,dy\Bigg| \\ &= \Bigg|\int_{\mathbb{R}^d\setminus\mathcal{C}} \frac{1}{h^r}\,\overline{\widehat{F}(hy)\sum_{\alpha\in\mathbb{Z}^d} G(x/h-\alpha)e^{-2\pi i h\alpha\cdot y}} \\ &\quad\times \widehat{G}(hy)\sum_{\alpha\in\mathbb{Z}^d}\widehat{\check{F}}(hy-\alpha)\widehat{f}(y-\alpha/h)\,dy\Bigg| \\ &\le C\sum_{\delta\in\mathbb{Z}^d\setminus\{0\}}\int_{\mathcal{C}}\frac{1}{h^r}\,|\widehat{G}(hy+\delta)\widehat{\check{F}}(hy+\delta)\widehat{\check{F}}(hy)\widehat{f}(y)|\,dy \\ &= C\int_{\mathcal{C}}\bigg(\sum_{\delta\in\mathbb{Z}^d\setminus\{0\}}\frac{1}{h^r}\,|\widehat{Y}(hy+\delta)|\bigg)|\widehat{\check{F}}(hy)\widehat{f}(y)|\,dy.\end{aligned}$$

We use the fact that $\overline{\widehat{F}} = \widehat{\check{F}}$. From Schwarz's inequality,

$$\sum_{\delta\in\mathbb{Z}^d\setminus\{0\}} \frac{1}{h^r}|\widehat{Y}(hy+\delta)|$$
$$\le h\Big(\sum_{\delta\in\mathbb{Z}^d\setminus\{0\}} \frac{|\widehat{G}(hy+\delta)|^2}{h^{2r}}\Big)^{1/2}\Big(\sum_{\delta\in\mathbb{Z}^d\setminus\{0\}} \frac{|\widehat{\breve{F}}(hy+\delta)|^2}{h^2}\Big)^{1/2}.$$

From the assumption on $Q$ and $Q^*$ and Theorem 2.4 we see that $G \in \mathrm{SF}(r)$ and $F \in \mathrm{SF}(1)$. By Lemmas 4.1 and 4.2,

$$\sum_{\delta\in\mathbb{Z}^d\setminus\{0\}} \left|\frac{\widehat{Y}(hx+\delta)}{h^r}\right| \to 0 \quad \text{as } h\to 0^+,$$

uniformly for $x \in \mathcal{C}$. This finishes the first step.

Step 2. Assume that $f \in W_2^r$ and $[]^\beta \widehat{f} \in L^1(\mathbb{R}^d)$ for $|\beta| \le r$. Thus $f \in W_\infty^r$. We use the $\varepsilon$-approximation of $f$ defined by

$$\widehat{f_\varepsilon}(x) = \begin{cases} \widehat{f}(x) & \text{if } \|x\| < 1/\varepsilon,\\ 0 & \text{otherwise.}\end{cases}$$

Note that the functions $f_\varepsilon$ satisfy the conditions of Step 1. Moreover $f_\varepsilon$ converges to $f$ in $W_\infty^r$ as $\varepsilon \to 0$, since

$$\sup_{x\in\mathbb{R}^d} |D^\beta f_\varepsilon(x) - D^\beta f(x)| \le C \int_{\|t\|>1/\varepsilon} |t^\beta \widehat{f}(t)|\,dt.$$

The triangle inequality and the estimate (5) give (13).

Step 3. If $f \in W_2^r \cap C_0^k$ for $k$ large enough then the Riemann–Lebesgue theorem shows that $[]^\beta \widehat{f} \in L^1(\mathbb{R}^d)$ for $|\beta| \le r$. Note that these functions are dense in $f \in W_2^r \cap W_\infty^r$. This finishes the proof. The proof for $Q_h^*$ is quite similar. ■

Remark. We can easily prove convergence in the $L^2$ norm in formula (13) for local operators, i.e. under the assumption that both $F$ and $G$ have compact support. We believe that the same is true for all our operators.

Generally $Q$ does not transform all sequences weakly convergent in the Hilbert space $L^2(\mathbb{R}^d)$ to sequences converging in the $L^2$ norm, since it would be compact.

**5. Bootstrap approximation.** Now we generalize Theorem 4.1 of [Dz3]. For simplicity we consider the shift invariant operators $Q$ which are orthogonal projections.

We say that the integer translates of $G$ are $l^2$ *stable* (see [JM]) if there is $C > 0$ such that for all sequences $a = \{a_\alpha\} \in l^2$,

$$C\|a\|_{l^2} \le \|G *' a\|_2,$$

where the *semi-discrete convolution* $*'$ is defined as follows:

$$G *' a = \sum_{\alpha\in\mathbb{Z}^d} a_\alpha G(\cdot - \alpha).$$

If $f$ is a continuous function then

$$G *' f = \sum_{\alpha\in\mathbb{Z}^d} f(\alpha) G(\cdot - \alpha).$$

We say that a sequence $b = \{b_\alpha\}$ *decays exponentially fast* if there are $C > 0$ and $0 < q < 1$ such that

$$|b_\alpha| < Cq^{|\alpha|} \quad \text{for all } \alpha \in \mathbb{Z}^d.$$

From [JM, Theorems 3.3 and 3.4] we get

THEOREM 5.1. *Let $G \in \mathcal{E}^\infty$. Then the following conditions are equivalent*:

(i) *the integer translates of $G$ are $l^2$ stable*,

(ii) *for all $\xi \in \mathbb{R}^d$*,

$$\sum_{\alpha\in\mathbb{Z}^d} |\widehat{G}(\xi+\alpha)|^2 > 0,$$

(iii) *for all $\xi \in \mathbb{R}^d$*,

$$\varPi_G(\xi) = \sum_{\alpha\in\mathbb{Z}^d} G * \breve{G}(\alpha) e^{2\pi i\alpha\cdot\xi} > 0,$$

(iv) *there is a function $G^* \in \mathcal{E}^\infty$ such that*

$$\breve{G} * G^*(\alpha) = \delta_{0,\alpha} \quad \textit{for all } \alpha \in \mathbb{Z}^d.$$

*Moreover*

$$G^*(x) = \sum_{\alpha\in\mathbb{Z}^d} b_\alpha G(x-\alpha),$$

*and the sequence $b = \{b_\alpha\}$ decays exponentially fast.*

By [JM, Theorem 3.2] we know that if $G \in \mathcal{E}^\infty$ and the integer translates of $G \in \mathcal{E}^\infty$ are $l^2$ stable then

$$\varPi_G(\xi) = \sum_{\alpha\in\mathbb{Z}^d} |\widehat{G}(\xi+\alpha)|^2 \quad \text{for all } \xi \in \mathbb{R}^d, \tag{31}$$

which gives equivalence of (2) and (3). Moreover ([JM, Theorem 3.4])

$$\varPi_G(x) \sum_{\alpha\in\mathbb{Z}^d} b_\alpha e^{2\pi i\alpha\cdot x} = 1 \quad \text{for all } x \in \mathbb{R}^d. \tag{32}$$

Note that if the integer translates of $G \in \mathcal{E}^\infty$ are $l^2$ stable then we can construct the fundamental function $\varPhi_G$ corresponding to $G$ by

$$\varPhi_G(x) = G * \breve{G}^*(x).$$

By definition, condition (iv) from the above theorem and (32) we get

$$\widehat{\Phi}_G(x) = \widehat{G}\overline{\widehat{G^*}} = \frac{|\widehat{G}(x)|^2}{\Pi_G(x)}, \tag{33}$$

and since $\Pi_G$ is even, so is $\Phi_G$. Also Theorem 5.1(iv) yields $\Phi_G(\alpha) = \delta_{0,\alpha}$ for $\alpha \in \mathbb{Z}^d$.

Let us formulate an easy technical lemma.

LEMMA 5.1. (i) *If* $f, g \in \mathcal{E}^\infty$ *then* $f * g \in \mathcal{E}^\infty$ *and* $f * g$ *is continuous.*
(ii) *If a* $f \in \mathcal{E}^\infty$ *sequence* $b$ *decays exponentially fast then* $f *' b \in \mathcal{E}^\infty$.
(iii) *If* $f, g \in \mathcal{E}^\infty$ *and* $f \in \mathrm{SF}(r_1)$, $g \in \mathrm{SF}(r_2)$ *then* $f * g \in \mathrm{SF}(r_1 + r_2)$.

*Proof.* (i) Note that

$$|x - y| + |y| \geq |x/2| + |y/2|.$$

Consequently,

$$|f * g(x)| \leq C \int_{\mathbb{R}^d} q^{|x-y|} q^{|y|}\, dy \leq C \int_{\mathbb{R}^d} q^{|x|/2} q^{|y|/2}\, dy \leq C q^{|x|/2}.$$

To prove that $f * g$ is continuous we use the $L^1$ modulus of continuity.

We prove (ii) by the same arguments as (i). To calculate (iii) we apply Leibniz's formula. ∎

Let $N > 0$. Define $\Psi_N = \underbrace{\Phi_G * \ldots * \Phi_G}_{N}$. From Lemma 5.1, Theorem 5.1, (31), (33) we get

LEMMA 5.2. *Fix* $N > 0$. *If the integer translates of* $G \in \mathcal{E}^\infty$ *are* $l^2$ *stable then the integer translates of both* $\Phi_G \in \mathcal{E}^\infty$ *and* $\Psi = \Psi_N \in \mathcal{E}^\infty$ *are* $l^2$ *stable. Moreover there is a fundamental function* $\Phi_\Psi$ *corresponding to* $\Psi$ *such that*

$$\widehat{\Phi}_\Psi = \frac{(\widehat{\Psi})^2}{\Pi_\Psi} = \frac{(\widehat{\Phi}_G)^{2N}}{\Pi_\Psi}, \tag{34}$$

*where*

$$\Pi_\Psi(x) = \sum_{\alpha \in \mathbb{Z}^d} \Psi * \breve{\Psi}(\alpha) e^{2\pi i \alpha \cdot x}.$$

*Proof.* By definition of the fundamental function $\Phi_\Psi$ and since $\Psi$ is even we get (34). Stability follows from (34), (33), (31) and Theorem 5.1(iii). ∎

Note that if the integer translates of $G$ are $l^2$ stable then we can construct the orthogonal projection

$$Pf(x) = \sum_{\alpha \in \mathbb{Z}^d} \int_{\mathbb{R}^d} f(u) G^*(u - \alpha)\, du\, G(x - \alpha).$$

Let $T_H$ denote the convolution operator, i.e. $T_H f = H * f$ .

Let us formulate the main theorem which generalizes Theorem 4.1 of [3].

THEOREM 5.2. *If the integer translates of $G \in \mathcal{E}^\infty$ are $l^2$ stable and an orthogonal projection $P$ has order $r > 0$ then for any $N > 0$ the operator*

$$T_{H_N} \circ Pf(x) = \sum_{\alpha\in\mathbb{Z}^d} \int_{\mathbb{R}^d} f(u)G^*(u-\alpha)\,du\,(H_N * G)(x-\alpha)$$

*has order $r(4N-1)$, where*

$$\widehat{H}_N = \text{“}\frac{\widehat{\Phi}_\Psi}{\widehat{\Phi}_G}\text{”} = \frac{(\widehat{\Phi}_G)^{2N-1}}{\Pi_\Psi}, \qquad \Pi_\Psi(x) = \sum_{\alpha\in\mathbb{Z}^d} \Psi * \breve{\Psi}(\alpha)e^{2\pi i\alpha\cdot x}.$$

To prove this theorem we need two lemmas. The first is de Boor–Ron's formula [BR, Lemma 2.8]. It was proved for compactly supported functions but the proof works in our situation for $G \in \mathcal{E}^\infty$.

LEMMA 5.3. *If $G \in \mathcal{E}^\infty$ and $p$ is a polynomial then*

$$G *' p \text{ is a polynomial } \Leftrightarrow G *' p = G * p.$$

The second lemma is taken from [LC]:

LEMMA 5.4. *Let $G \in \mathcal{E}^\infty$ and suppose $G$ is continuous. Then $G \in \mathrm{SF}(r) \Leftrightarrow$ there is $c \in \mathbb{R}$ such that for all $|\beta| < r$, $[]^\beta - cG *' []^\beta$ is a polynomial of degree $\le |\beta| - 1$.*

*Proof of Theorem 5.2.* By definition the orthogonal projection $P$ has polynomial order $r > 0$. Thus $G \in \mathrm{SF}(r)$ by Theorem 2.4. Consequently, $\Phi_G \in \mathrm{SF}(2r)$ by (33). By Lemma 5.1, $H_N \in \mathrm{SF}(r(4N-2))$, $H_N * G \in \mathrm{SF}(r(4N-1))$ and $H_N * G \in \mathcal{E}^\infty$.

Let $p$ be a polynomial of total degree $(4N-1)r$. Then $p * \breve{G}^*$ is a polynomial of the same degree. Then from Lemma 5.4 (by Lemma 5.1, $H_N * G$ is continuous) $H_N * G *' (p * \breve{G}^*)$ is a polynomial of the same degree. By Lemma 5.3 and the definition of $\Phi_\Psi$,

$$H_N * G *' (\breve{G}^* * p) = H_N * G * \breve{G}^* * p = \Phi_\Psi * p.$$

Since $\Psi \in \mathcal{E}^\infty$ and $\Psi \in \mathrm{SF}(4Nr)$, by Lemmas 5.3 and 5.4 ($\Phi_\Psi$ is continuous) we have $\Phi_\Psi * p = \Phi_\Psi *' p$. But $\Phi_\Psi$ is a fundamental function, i.e. $\Phi_\Psi(\alpha) = \delta_{\alpha,0}$ for all $\alpha \in \mathbb{Z}^d$. Thus we have a polynomial $\Phi_\Psi *' p$ which is equal to $p$ on $\mathbb{Z}^d$. Consequently, $\Phi_\Psi *' p = p$. This finishes the proof. ∎

The proof of Theorem 5.2 is similar to the proof of [Dz3, Theorem 4.1]. In [Dz3] we take $N = 1$, $H = \Phi_G$, which implies that $T_H \circ P$ has polynomial order $2r$. No matter what approach we use, the operator $Q = T_{H_N} \circ P$ is shift invariant. Thus we can apply earlier results. For example let $N = 1$, $H = \Phi_G$. By Theorem 2.5,

$$\frac{Q_h f - f}{h^r} \to \sum_{|\beta|=2r} \frac{D^\beta f}{\beta!}\, 2D^\beta \widehat{\Phi}_G(0) \quad \text{as } h \to 0^+$$

in $L^p$ norm since $G*H \in \mathrm{SF}(3r)$. This was announced in [Dz3, Theorem 4.2] for $p = 2$.

**References**


[BD1] M. Beśka and K. Dziedziul, *Saturation theorems for interpolation, and the Bernstein–Schnabl operator*, Math. Comp. 70 (2001), 705–717.

[BD2] —, —, *The saturation theorem for box spline orthogonal projection*, in: *Advances in Multivariate Approximation*, W. Haussmann *et al.* (eds.), Wiley-VCH, 1999, 73–83.

[BD3] M. Beśka and K. Dziedziul, *Asymptotic formula for the error in cardinal interpolation*, Numer. Math. 89 (2001), 445–456.

[BD4] —, —, *Asymptotic formula for the error in orthogonal projection*, Math. Nachr. 233 (2002), 47–53.

[BU1] T. Blu and M. Unser, *Approximation error for quasi-interpolators and (multi-) wavelet expansions*, Appl. Comput. Harmon. Anal. 6 (1999), 219–251.

[BU2] —, —, *Quantitative Fourier analysis of approximation techniques. I. Interpolators and projectors*, IEEE Trans. Signal Process. 47 (1999), 2783–2795.

[BDR] C. de Boor, R. A. DeVore and A. Ron, *Approximation from shift-invariant subspaces of* $L^2(\mathbb{R}^d)$, Trans. Amer. Math. Soc. 341 (1994), 787–806.

[BHR] C. de Boor, K. Höllig and S. Riemenschneider, *Box Splines*, Springer, 1993.

[BR] C. de Boor and A. Ron, *The exponentials in the span of the multi-integer translates of a compactly supported function; quasi-interpolation and approximation order*, J. London Math. Soc. (2) 45 (1992), 519–535.

[C] Z. Ciesielski, *Asymptotic nonparametric spline density estimation in several variables*, in: Internat. Ser. Numer. Math. 94, Birkhäuser, Basel, 1990, 25–53.

[DU] I. Daubechies and M. Unser, *On the approximation power of convolution-based least squares versus interpolation*, IEEE Trans. Signal. Process. 45 (1997), 1697–1711.

[Dz1] K. Dziedziul, *Saturation theorem for quasi-projections*, Studia Sci. Math. Hungar. 35 (1999), 99–111.

[Dz2] —, *Central limit theorem for square error of multivariate nonparametric box spline density estimators*, Appl. Math. (Warsaw) 28 (2001), 437–456.

[Dz3] —, *Asymptotic formulas in cardinal interpolation and orthogonal projection*, in: *Recent Progress in Multivariate Approximation*, W. Haussmann *et al.* (eds.), Internat. Ser. Numer. Math. 137, Birkhäuser, Basel, 2001, 139–157.

[Dz4] —, *Application of Mazur–Orlicz's theorem in AMISE calculation*, Appl. Math. (Warsaw) 29 (2002), 33–41.

[JM] R. Q. Jia and C. A. Micchelli, *Using the refinement equations for the construction of pre-wavelets. II. Powers of two*, in: *Curves and Surfaces*, P. J. Laurent *et al.* (eds.), Academic Press, New York, 1991, 209–246.

[JZ] K. Jetter and D.-X. Zhou, *Order of linear approximation from shift-invariant spaces*, Constr. Approx. 11 (1995), 423–438.

[LC] W. A. Light and E. W. Cheney, *Quasi-interpolation with translates of a function having noncompact support*, ibid. 8 (1992), 35–48.

[MO] S. Mazur et W. Orlicz, *Sur quelques propriétés de fonctions périodiques et presque-périodiques*, Studia Math. 9 (1940), 1–16.

[M] Y. Meyer, *Wavelets and Operators*, Cambridge Univ. Press, Cambridge, 1992.

[SW] E. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton Univ. Press, 1971.

[SF] G. Strang and G. Fix, *A Fourier analysis of the finite element variational method*, in: *Constructive Aspects of Functional Analysis*, G. Geymonat (ed.), Ed. Cremonese, Roma, 1973, 793–840.

Faculty of Applied Mathematics
Gdańsk University of Technology
G. Narutowicza 11/12
80-952 Gdańsk, Poland
E-mail: kdz@mifgate.pg.gda.pl

*Received on 13.6.2002;*
*revised version on 3.2.2003* (1636)