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ESTIMATES FOR PERTURBATIONS OF GENERAL DISCOUNTED MARKOV CONTROL CHAINS

Abstract. We extend previous results of the same authors ([11]) on the effects of perturbation in the transition probability of a Markov cost chain for discounted Markov control processes. Supposing valid, for each stationary policy, conditions of Lyapunov and Harris type, we get upper bounds for the *index of perturbations*, defined as the difference of the total expected discounted costs for the original Markov control process and the perturbed one. We present examples that satisfy our conditions.

1. Introduction. This paper deals with discounted Markov control processes (DMCPs) with discrete time, general space of states $\mathbb{X}$ and (possibly) unbounded one-step cost functions (see [6–8]). The main problem in the theory of DMCPs is to determine an optimal policy (see [6, 7]) with respect to an objective function equal to the total expected discounted cost $V_\alpha(x,\pi)$, where $\alpha \in (0,1)$ is a discount factor, x is an initial state of the process and π is a policy that we apply (see Section 2 for definitions). But in a lot of cases, the transition probability Q of the given DMCP is incompletely known because it may be obtained through estimation or approximation (see [3]). Actually, we only have a theoretical approximation $\widetilde{Q}$ of Q. In this case, if we know that the approximating process with transition probability $\widetilde{Q}$ has an optimal policy $\widetilde{\pi}^*$, we may, in a natural way, use this policy to control the original process (corresponding to Q). As a consequence, we will have an increase of the total expected discounted cost, because $\widetilde{\pi}^*$ is not an

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optimal policy for the original DMCP. We measure the increment by the following value (see [4, 5]):

$$\Delta_\alpha^{**}(x) \equiv V_\alpha(x, \widetilde{\pi}^*) - \inf_{\pi \in \Pi} V_\alpha(x, \pi) \tag{1}$$

where Π is the set of all possible policies. The value $\Delta_\alpha^{**}(\cdot)$ was called in [4] and [5] the *stability index*, but we will name it the *index of perturbations* because the word “stability” has several meanings.

Now a natural task is to obtain upper bounds for $\Delta_\alpha^{**}(x)$. Under some conditions such estimates were obtained in [1] for DMCPs with bounded one-step cost functions and in [5] for DMCPs with nonnegative unbounded one-step cost functions. Note that [5] essentially used results for uncontrolled Markov chains given by Kartashov [9] and Zolotarev [12].

The first aim of this work was to obtain bounds for $\Delta_\alpha^{**}(x)$ better than those in [1, 4, 5], using more elementary estimates, obtained earlier by the authors in [11] for uncontrolled Markov chains.

Now let $\mathbb{F}$ denote the set of all *stationary control policies* and suppose that there exists a stationary optimal policy $\widetilde{f}^*$, corresponding to $\widetilde{Q}$ (see Section 2 for definitions). In this case we may define the value

$$\Delta_\alpha^{*}(x) := V_\alpha(x, \widetilde{f}^*) - \inf_{f \in \mathbb{F}} V_\alpha(x, f). \tag{2}$$

which may be called the *stationary index of perturbations*.

Under certain assumptions proposed in the literature (see, for instance, Assumptions 1 and 2 in [4]), the following basic equalities hold:

$$\begin{aligned} \forall x \in \mathbb{X} \qquad & \inf_{\pi \in \Pi} V_\alpha(x, \pi) \equiv \inf_{f \in \mathbb{F}} V_\alpha(x, f), \\ & \widetilde{V}_\alpha(x, \widetilde{\pi}^*) \equiv \inf_{\pi \in \Pi} \widetilde{V}_\alpha(x, \pi) \equiv \inf_{f \in \mathbb{F}} \widetilde{V}_\alpha(x, f) \equiv \widetilde{V}_\alpha(x, \widetilde{f}^*), \end{aligned} \tag{3}$$

where $\widetilde{V}_\alpha(x, \pi)$ is the total expected discounted cost defined for the approximating DMCP corresponding to $\widetilde{Q}$. Thus, under (3), we have the following identity:

$$\forall x \in \mathbb{X} \qquad \Delta_\alpha^{*}(x) \equiv \Delta_\alpha^{**}(x). \tag{4}$$

The main goal of this work is to obtain estimates for the value $\Delta_\alpha^{*}(x)$, defined in (2), under more general assumptions than those used in [1, 5]. We aim at estimates which should contain only explicitly defined functions and constants, and should have simple proofs.

These goals are achieved in Theorem 1 below. Note that if the equalities (3) and (4) are valid then our estimate from Theorem 1 is sharper and simpler than the ones for $\Delta_\alpha^{*}(x) \equiv \Delta_\alpha^{**}(x)$ obtained in [1] and [5].

The paper is organized as follows. Section 2 presents several general definitions concerning DMCPs. The main results are stated in Section 3 and their proofs are gathered in Section 4. In Sections 5 and 6 we give two examples of DMCPs that satisfy our assumptions. Note that in the second example we use a condition of the Harris type with a small set (see [10] for definitions) which is not an ordinary atom.

2. General definitions

2.1. *Basic definitions.* Let

$$\mathcal{M} = (\mathbb{X},\, \mathcal{B}(\mathbb{X}),\, \mathbb{A},\, \mathcal{B}(\mathbb{A}),\, A(\cdot),\, Q(\cdot|\cdot,\cdot),\, c(\cdot,\cdot)) \tag{5}$$

be a standard *Markov control model* (see [6–8]). Here the state space $\mathbb{X}$ and the control set $\mathbb{A}$ are measurable spaces with the σ-algebras $\mathcal{B}(\mathbb{X})$ and $\mathcal{B}(\mathbb{A})$, respectively. For each $x \in \mathbb{X}$, the set $A(x) \subset \mathbb{A}$ is the measurable subset of admissible controls at state $x \in \mathbb{X}$ and, in addition, the set

$$\mathbb{K} := \{(x,a) : x \in \mathbb{X},\, a \in A(x)\} \subset \mathbb{X} \times \mathbb{A}$$

is assumed to be a measurable subset of $\mathbb{X} \times \mathbb{A}$ endowed with the product σ-algebra $\mathcal{B}(\mathbb{X}) \times \mathcal{B}(\mathbb{A})$. The transition law $Q = Q(\cdot|\cdot,\cdot)$ is a stochastic kernel on $\mathbb{X}$ given $\mathbb{K}$ (i.e., $Q(\cdot|x,a)$ is a probability measure on $\mathbb{X}$ for every $(x,a) \in \mathbb{K}$, and $Q(B|\cdot)$ is a measurable function on $\mathbb{K}$ for every $B \in \mathcal{B}(\mathbb{X})$). Finally, the cost function $c(\cdot,\cdot)$ is a *nonnegative* measurable function on $\mathbb{K}$.

Let $\mathbb{F}$ denote the collection of all measurable functions $f : \mathbb{X} \to \mathbb{A}$ such that $f(x) \in A(x)$ for all $x \in \mathbb{X}$. Every function from $\mathbb{F}$ may be called a *stationary policy*.

For every $f \in \mathbb{F}$ we define the following kernels:

$$\forall x \in \mathbb{X}\ \forall B \in \mathcal{B}(\mathbb{X}) \qquad Q_f^1(B|x) = Q_f(B|x) = Q_f(B|x, f(x)),$$
$$Q_f^n(B|x) = \int Q_f^{n-1}(B|y)\, Q_f(dy|x), \quad n = 2, 3, \ldots$$

For every policy $f \in \mathbb{F}$ and for all $x \in \mathbb{X}$ we now define the total expected discounted cost by the formula

$$V_\alpha(x,f) = c(x, f(x)) + \sum_{n=1}^{\infty} \alpha^n \int c(y, f(x))\, Q_f^n(dy|x), \qquad c(\cdot,\cdot) \geq 0. \tag{6}$$

Recall that the number $\alpha \in (0,1)$ is called the *discount factor*.

A stationary policy $f^* \in \mathbb{F}$ is said to be *optimal* (for the model $\mathcal{M}$) if

$$\forall x \in \mathbb{X} \qquad V_\alpha(x, f^*) = \inf_{f \in \mathbb{F}} V_\alpha(x,f). \tag{7}$$

It is possible that an optimal policy does not exist, or is unknown, or is too expensive to use. In this case we have to use another policy, say f. (For example, we may use an ε-optimal policy.) But if we use a nonoptimal policy, we will have an increase of the total expected discounted cost. We measure the increment by the following value:

$$\forall x \in \mathbb{X}\ \forall f \in \mathbb{F} \qquad \varepsilon_\alpha(x,f) := V_\alpha(x,f) - \inf_{f \in \mathbb{F}} V_\alpha(x,f). \tag{8}$$

2.2. *Approximating model.* Together with the original Markov control model $\mathcal{M}$, introduced in (5), we consider another Markov control model

$$\widetilde{\mathcal{M}} = (\mathbb{X}, \mathcal{B}(\mathbb{X}), \mathbb{A}, \mathcal{B}(\mathbb{A}), A(\cdot), \widetilde{Q}(\cdot|\cdot,\cdot), \widetilde{c}(\cdot,\cdot))$$

which will be used as approximation of $\mathcal{M}$. The models $\mathcal{M}$ and $\widetilde{\mathcal{M}}$ have the same state and action sets $\mathbb{X}$, $\mathbb{A}$ and $A(\cdot)$. Hence, they have the same set $\mathbb{F}$ of stationary policies. But they have different transition probabilities Q and $\widetilde{Q}$ and different cost functions c and $\widetilde{c}$.

Using $\widetilde{Q}$ and $\widetilde{c}$ instead of Q and c in formulas (5)–(8) we easily define the values

$$\widetilde{Q}_f(\cdot|\cdot), \qquad \widetilde{Q}_f^n(\cdot|\cdot), \qquad \widetilde{V}_\alpha(\cdot,\cdot), \qquad \widetilde{\varepsilon}_\alpha(x,f), \qquad \widetilde{f}^*.$$

For example,

$$\forall x \in \mathbb{X}\ \forall f \in \mathbb{F} \quad \widetilde{\varepsilon}_\alpha(x,f) := V_\alpha(x,f) - \inf_{f\in\mathbb{F}} \widetilde{V}_\alpha(x,f). \tag{9}$$

An optimal stationary policy $\widetilde{f}^*$ for the model $\widetilde{\mathcal{M}}$, if it exists, satisfies the condition

$$\forall x \in \mathbb{X} \quad \widetilde{V}_\alpha(x,\widetilde{f}^*) = \inf_{f\in\mathbb{F}} \widetilde{V}_\alpha(x,f),$$

which, by (9), may be rewritten as

$$\forall x \in \mathbb{X} \quad \widetilde{\varepsilon}_\alpha(x,\widetilde{f}^*) = 0. \tag{10}$$

2.3. *The stationary index of perturbations and its generalization.* If the stationary optimal policy $\widetilde{f}^*$ exists we may define the stationary index of perturbations $\Delta_\alpha^*(x)$ by (2). We remark that, by (8) and (10),

$$\Delta_\alpha^*(x) \equiv V_\alpha(x,\widetilde{f}^*) - \inf_f V_\alpha(x,f) \equiv \varepsilon_\alpha(x,\widetilde{f}^*) \equiv \varepsilon_\alpha(x,\widetilde{f}^*) - \widetilde{\varepsilon}_\alpha(x,\widetilde{f}^*) \tag{11}$$

for all $x \in \mathbb{X}$.

But it is possible that the stationary optimal policy $\widetilde{f}^*$ for the model $\widetilde{\mathcal{M}}$ does not exist or is too expensive. In this case we have to use another policy, say f. (For example, we may use an ε-optimal policy for $\widetilde{\mathcal{M}}$.) For the chosen policy f we may find the value $\widetilde{\varepsilon}_\alpha(x,f)$, using the known properties of the model $\widetilde{\mathcal{M}}$. But we are interested in evaluating the value $\varepsilon_\alpha(x,f)$. In this situation a natural way is to estimate the difference

$$\varepsilon_\alpha(x,f) - \widetilde{\varepsilon}_\alpha(x,f). \tag{12}$$

It follows from (11) that the difference in (12) is a more general value than the stationary index of perturbations $\Delta_\alpha^*(x)$.

Another reason to use this natural generalization of the index of perturbations is that the difference in (12) is well defined also in the case when the optimal policy $\widetilde{f}^*$ does not exist and so the index $\Delta_\alpha^*(x)$ is undefined.

3. Main estimates. We write $Q \in \mathcal{H}_0(W,\beta,w,\nu,h)$ if there exist a probability measure $\nu(\cdot)$, functions $W(\cdot)$ and $h(\cdot,\cdot)$, and real numbers β and w such that the following conditions hold:

$$\forall x \in \mathbb{X}\ \forall a \in A(x)\ \forall B \in \mathcal{B}(\mathbb{X}) \qquad Q(B|x,a) \ge h(x,a)\nu(B) \ge 0, \tag{13}$$

$$\int W(y)\,Q(dy|x,a) \le \beta W(x) + h(x,a)\int W(y)\,\nu(dy) < \infty, \tag{14}$$

$$W(x) \ge 1, \quad \int W(y)\,\nu(dy) \le w < \infty, \quad 0 \le \beta < \infty. \tag{15}$$

Remark 1. It is easy to see that assumption (13) is a part of a Harris type condition. For example, if $h(x,a) = \text{const} \cdot I_C(x)$, where $I_C(\cdot)$ is the indicator of the set C, then C is frequently called ([10]) a *small set*. On the other hand, assumption (14) is a natural combination of Lyapunov and Harris conditions.

Note that Propositions 3 and 4 in Section 6 contain examples where $Q \in \mathcal{H}_0(W,\beta,w,\nu,h)$ with nontrivial functions $h(\cdot,\cdot)$ and sets $A(\cdot)$.

Assumptions (13)–(15) were used, for example, in [5, 11]. but with additional restrictions on the value $\int h(y)\,\nu(dy)$.

We now fix a function $W(\cdot)$ from (15) and introduce the norm

$$\|g\| := \sup_{x\in\mathbb{X}}\sup_{a\in A(x)} \frac{|g(x,a)|}{W(x)}$$

for any real-valued function g, defined on $\mathbb{X}$. We also need the notation

$$\varrho(x,a) = \int W(x)\,|Q(dy|x,a) - \widetilde{Q}(dy|x,a)|, \tag{16}$$

where the measure $|Q(\cdot|x,a) - \widetilde{Q}(\cdot|x,a)|$ is the total variation of the signed measure $Q(\cdot|x,a) - \widetilde{Q}(\cdot|x,a)$.

Let

$$C' := \alpha(w+1-\alpha\beta)\min\{\|c\|,\|\widetilde{c}\|\},$$
$$\delta_0(x) := \left(W(x) + \frac{\alpha w}{1-\alpha}\right)\left(\frac{\|(c-\widetilde{c})\|}{1-\alpha\beta} + \frac{C'\|\varrho\|}{(1-\alpha\beta)^3}\right).$$

The following theorem is the main and simplest result of the paper.

Theorem 1. *Assume that*

$$Q \in \mathcal{H}_0(W,\beta,w,\nu,h) \quad \textit{and} \quad \widetilde{Q} \in \mathcal{H}_0(W,\beta,w,\widetilde{\nu},\widetilde{h})$$

for some ν, h, $\widetilde{\nu}$, $\widetilde{h}$ with the same W, β and w. Then

$$\forall x \in \mathbb{X}\ \forall f \in \mathbb{F} \qquad |\varepsilon(x,f) - \widetilde{\varepsilon}(x,f)| \le 2\delta_0(x).$$

In addition, if $\widetilde{f}^*$ exists, then

$$\forall x \in \mathbb{X} \qquad \Delta_\alpha^*(x) \le 2\delta_0(x).$$

We now consider several generalizations of Theorem 1. We write $Q \in \mathcal{H}(W_f,\beta_f,w_f,\nu_f,h_f)$ for some $f \in \mathbb{F}$ if there exist a probability measure

$\nu_f(\cdot)$, functions $W_f(\cdot)$ and $h_f(\cdot)$, real numbers β_f and w_f such that

$$\forall x \in \mathbb{X}\ \forall B \in \mathcal{B}(\mathbb{X}) \quad Q(B|x, f(x)) \ge h_f(x)\nu_f(B) \ge 0, \tag{17}$$

$$\int W_f(y)\, Q(dy|x, f(x)) \le \beta_f W_f(x) + h_f(x) \int W_f(y)\, \nu_f(dy) < \infty, \tag{18}$$

$$W_f(x) \ge 1, \quad \int W_f(y)\, \nu_f(dy) \le w_f < \infty, \quad 0 \le \beta_f < \infty. \tag{19}$$

It is easy to see that assumptions (13)–(15) are a special case of (17)–(19) when the values $\nu(\cdot)$, $W(\cdot)$, $h(\cdot,\cdot)$, β and w do not depend on f and $h_f(\cdot) = h(\cdot\,, f(\cdot))$.

Suppose now that $f \in \mathbb{F}$ is such that

$$Q \in \mathcal{H}(W_f, \beta_f, w_f, \nu_f, h_f) \quad \text{and} \quad \widetilde{Q} \in \mathcal{H}(W_f, \beta_f, w_f, \widetilde{\nu}_f, \widetilde{h}_f) \tag{20}$$

for some ν_f, h_f, $\widetilde{\nu}_f$, $\widetilde{h}_f$ with the same W_f, β_f and w_f. In this case we introduce the following simplified notations:

$$c_f(x) = c(x, f(x)), \qquad \widetilde{c}_f(x) = \widetilde{c}(x, f(x)),$$
$$\varrho_f(x) = \int W_f(x)\, |Q(dy|x, f(x)) - \widetilde{Q}(dy|x, f(x))|.$$

For any real-valued function g defined on $\mathbb{X}$ we will use the norm

$$\|g\|_f := \sup_{x\in\mathbb{X}} \frac{|g(x)|}{W_f(x)}.$$

Let

$$C_f := \min\{\|c_f\|_f, \|\widetilde{c}_f\|_f\}, \qquad w_{\alpha,f} := \alpha(w + 1 - \alpha\beta_f),$$
$$\delta(x, f) := \left(W_f(x) + \frac{\alpha w_f}{1-\alpha}\right)\left(\frac{\|(c_f - \widetilde{c}_f)^+\|_f}{1 - \alpha\beta_f} + \frac{w_{\alpha,f} C_f \|\varrho_f\|_f}{(1-\alpha\beta_f)^3}\right),$$
$$\widetilde{\delta}(x, f) := \left(W_f(x) + \frac{\alpha w_f}{1-\alpha}\right)\left(\frac{\|(\widetilde{c}_f - c_f)^+\|_f}{1 - \alpha\beta_f} + \frac{w_{\alpha,f} C_f \|\varrho_f\|_f}{(1-\alpha\beta_f)^3}\right).$$

THEOREM 2. *Suppose that assumption* (20) *holds for all* $f \in \mathbb{F}$. *Then*

$$\forall f \in \mathbb{F}\ \forall x \in \mathbb{X} \quad \varepsilon_\alpha(x, f) - \widetilde{\varepsilon}_\alpha(x, f) \le \delta(x, f) + \sup_f \widetilde{\delta}(x, f), \tag{21}$$

$$\forall f \in \mathbb{F}\ \forall x \in \mathbb{X} \quad \widetilde{\varepsilon}_\alpha(x, f) - \varepsilon_\alpha(x, f) \le \widetilde{\delta}(x, f) + \sup_f \delta(x, f). \tag{22}$$

In addition, if $\widetilde{f}^*$ *exists, then*

$$\forall x \in \mathbb{X} \quad \Delta_\alpha^*(x) \le \delta(x, \widetilde{f}^*) + \sup_f \widetilde{\delta}(x, f).$$

THEOREM 3. *Suppose that* f^* *and* $\widetilde{f}^*$ *exist and that assumption* (20) *holds for* $f = f^*$ *and for* $f = \widetilde{f}^*$. *Then*

$$\forall x \in \mathbb{X} \quad \Delta_\alpha^*(x) \le \delta(x, \widetilde{f}^*) + \widetilde{\delta}(x, f^*).$$

Theorems 1–3 will be proved at the end of §4 as immediate corollaries of Lemmas 1–4.

4. Main lemmas. Introduce the notations

$$\Delta_\alpha(x,f) = V_\alpha(x,f) - \widetilde{V}_\alpha(x,f), \qquad \widetilde{\Delta}_\alpha(x,f) = \widetilde{V}_\alpha(x,f) - V_\alpha(x,f).$$

LEMMA 1. *If f^* and $\widetilde{f}^*$ exist then*

$$\Delta^*_\alpha(x) \le \Delta_\alpha(x,\widetilde{f}^*) + \widetilde{\Delta}_\alpha(x,f^*).$$

Proof. This is evident because

$$\begin{aligned}\Delta^*_\alpha(x) &= V_\alpha(x,\widetilde{f}^*) - V_\alpha(x,f^*)\\ &= \Delta_\alpha(x,\widetilde{f}^*) + (\widetilde{V}_\alpha(x,\widetilde{f}^*) - \widetilde{V}_\alpha(x,f^*)) + \widetilde{\Delta}_\alpha(x,f^*)\\ &\le \Delta_\alpha(x,\widetilde{f}^*) + \widetilde{\Delta}_\alpha(x,f^*).\end{aligned}$$

LEMMA 2. *For all $x \in \mathbb{X}$ and $f \in \mathbb{F}$,*

$$\varepsilon_\alpha(x,f) \le \widetilde{\varepsilon}_\alpha(x,f) + \Delta_\alpha(x,f) + \sup_f \widetilde{\Delta}_\alpha(x,f). \tag{23}$$

Proof. For arbitrary f_n and $\widetilde{f}_N$ we have

$$\begin{aligned}V_\alpha(x,f) - V_\alpha(x,f_n) &= V_\alpha(x,f) - \widetilde{V}_\alpha(x,f) + \widetilde{V}_\alpha(x,f) - \widetilde{V}_\alpha(x,\widetilde{f}_N)\\ &\quad + \widetilde{V}_\alpha(x,\widetilde{f}_N) - \widetilde{V}_\alpha(x,f_n) + \widetilde{V}_\alpha(x,f_n) - V_\alpha(x,f_n).\end{aligned}$$

Hence

$$\begin{aligned}V_\alpha(x,f) - V_\alpha(x,f_n) &= \Delta_\alpha(x,f) + (\widetilde{V}_\alpha(x,f) - \widetilde{V}_\alpha(x,\widetilde{f}_N))\\ &\quad + (\widetilde{V}_\alpha(x,\widetilde{f}_N) - \widetilde{V}_\alpha(x,f_n)) + \widetilde{\Delta}_\alpha(x,f_n).\end{aligned} \tag{24}$$

Let the functions $\widetilde{f}_N$ be such that

$$\widetilde{V}_\alpha(x,\widetilde{f}_N) \to \inf_f \widetilde{V}_\alpha(x,f) \quad \text{as } N \to \infty.$$

Then

$$\widetilde{V}_\alpha(x,f) - \widetilde{V}_\alpha(x,\widetilde{f}_N) \to \widetilde{V}_\alpha(x,f) - \inf_f \widetilde{V}_\alpha(x,f) = \widetilde{\varepsilon}_\alpha(x,f),$$

$$\widetilde{V}_\alpha(x,\widetilde{f}_N) - \widetilde{V}_\alpha(x,f_n) \to \inf_f \widetilde{V}_\alpha(x,f) - \widetilde{V}_\alpha(x,f_n) \le 0,$$

and so we may rewrite (24) in the form

$$V_\alpha(x,f) - V_\alpha(x,f_n) \le \Delta_\alpha(x,f) + \widetilde{\varepsilon}_\alpha(x,f) + \widetilde{\Delta}_\alpha(x,f_n). \tag{25}$$

But $\widetilde{\Delta}_\alpha(x,f_n) \le \sup_f \widetilde{\Delta}_\alpha(x,f)$, and thus the desired inequality (23) follows from (25) if we choose f_n such that

$$V_\alpha(x,f_n) \to \inf_f V_\alpha(x,f) \quad \text{as } n \to \infty. \ \blacksquare$$

LEMMA 3. *For all $x \in \mathbb{X}$ and $f \in \mathbb{F}$,*

$$\widetilde{\varepsilon}_\alpha(x,f) \le \varepsilon_\alpha(x,f) + \widetilde{\Delta}_\alpha(x,f) + \sup_f \Delta_\alpha(x,f).$$

The proof is similar to that of Lemma 2.

LEMMA 4. *For all $x \in \mathbb{X}$ and $f \in \mathbb{F}$,*

$$\Delta_\alpha(x,f) \le \delta(x,f), \qquad \widetilde{\Delta}_\alpha(x,f) \le \widetilde{\delta}(x,f).$$

This assertion is a special case of Corollary 2 in [6].

We may now prove the theorems of §3. Theorem 3 follows immediately from Lemmas 1 and 4. Lemmas 2 and 4 imply the first assertion (21) of Theorem 2 and, similarly, Lemmas 3 and 4 yield the second assertion (22). The third assertion follows from Lemma 4 and the representation (11).

Theorem 1 is an evident corollary of Theorem 2 for the case where the probability measures ν_f, functions W_f and real numbers β_f and w_f do not depend on f.

5. Example 1. In this section we present a generalization of the example provided in Section 5 of [2] (see also [5]). The version that we give here satisfies all assumptions of Theorem 1.

5.1. *The model.* Consider two controlled Markov processes $\{\Phi_n, a_n\}$ and $\{\widetilde{\Phi}_n, \widetilde{a}_n\}$, with state space $\mathbb{X} = [0,\infty)$, defined by the following recursive equations:

$$\begin{aligned} \Phi_0 &= x, \quad \Phi_{n+1} = (\Phi_n + a_n\xi_n - \eta_n)^+, \quad n = 0,1,2,\ldots, \\ \widetilde{\Phi}_0 &= x, \quad \widetilde{\Phi}_{n+1} = (\widetilde{\Phi}_n + \widetilde{a}_n\widetilde{\xi}_n - \widetilde{\eta}_n)^+, \quad n = 0,1,2,\ldots, \end{aligned}$$

where the controls a_n and $\widetilde{a}_n$ are real numbers and $x \in \mathbb{X}$ is a given state.

ASSUMPTION 1. *Each of the two sequences of vectors*

$$(\xi,\eta), (\xi_1,\eta_1), (\xi_2,\eta_2), \ldots \quad \textit{and} \quad (\widetilde{\xi},\widetilde{\eta}), (\widetilde{\xi}_1,\widetilde{\eta}_1), (\widetilde{\xi}_2,\widetilde{\eta}_2), \ldots$$

consists of independent and identically distributed random vectors with nonnegative first components $\xi \ge 0$ and $\widetilde{\xi} \ge 0$.

In this case, for all $x \in \mathbb{X}$ and $B \in \mathcal{B}$ we may define the kernels

$$\begin{aligned} Q(B|x,a) &:= \mathbf{P}(\Phi_{n+1} \in B \,|\, \Phi_n = x, a_n = a) \\ &= \mathbf{P}((x + a\xi - \eta)^+ \in B), \\ \widetilde{Q}(B\,|\,x,a) &:= \mathbf{P}(\widetilde{\Phi}_{n+1} \in B \,|\, \widetilde{\Phi}_n = x, \widetilde{a}_n = a) \\ &= \mathbf{P}((x + a\widetilde{\xi} - \widetilde{\eta})^+ \in B), \end{aligned} \tag{26}$$

where $\mathcal{B}$ denotes the Borel σ-algebra of $\mathbb{X} = [0,\infty)$.

5.2. *Main conditions*

ASSUMPTION 2. *There exists a real number θ such that*

$$\forall x \in \mathbb{X} \qquad A(x) \subset (-\infty, \theta] \equiv \mathbb{A}.$$

For real a and q put

$$\beta_{q,a} := \max\{\mathbf{E}e^{q(a\xi-\eta)}, \mathbf{E}e^{q(a\widetilde{\xi}-\widetilde{\eta})}\}.$$

ASSUMPTION 3. *There exists a real number q such that*

$$q > 0 \quad \textit{and} \quad \beta_{q,\theta} < 1. \tag{27}$$

REMARK 2. Assume that

$$\theta\mathbf{E}\xi < \mathbf{E}\eta, \quad \theta\mathbf{E}\widetilde{\xi} < \mathbf{E}\widetilde{\eta},$$
$$\mathbf{E}e^{\lambda(\theta\xi-\eta)} < \infty, \quad \mathbf{E}e^{\widetilde{\lambda}(\theta\widetilde{\xi}-\widetilde{\eta})} < \infty$$

for some $\lambda > 0$ and $\widetilde{\lambda} > 0$. It is well known (see, for example, [2, p. 232]) that in this case there exists a real number $q > 0$ such that Assumption 3 holds. On the other hand, we may obtain (27) also in the case where $\mathbf{E}\eta = \infty$ or $\mathbf{E}\widetilde{\eta} = \infty$.

ASSUMPTION 4. *The random vectors (ξ, η) and $(\widetilde{\xi}, \widetilde{\eta})$ have densities $g(\cdot,\cdot)$ and $\widetilde{g}(\cdot,\cdot)$, respectively, with respect to the Lebesgue measure in $\mathbb{R}\times\mathbb{R}$.*

For real a and q put

$$r_{q,a} = \int_0^\infty \int_0^\infty \max\{e^{q(ax-y)}, 1\}|g(x,y) - \widetilde{g}(x,y)|\,dx\,dy. \tag{28}$$

5.3. *Results*

PROPOSITION 1. *If Assumptions* 1–4 *hold then the kernels $Q(\cdot|\cdot,\cdot)$ and $\widetilde{Q}(\cdot|\cdot,\cdot)$, introduced in* (26), *satisfy all the assumptions and assertions of Theorem* 1 *with*

$$W(x) := e^{qx}, \qquad \beta := \beta_{q,\theta}, \qquad w := 1, \qquad \|\varrho\| \le r_{q,\theta}. \tag{29}$$

Suppose now that, in addition to Assumptions 1–4,

$$\forall x \ge 0\ \forall a \le \theta \qquad c(x,a) = \widetilde{c}(x,a) \quad \text{and} \quad |c(x,a)| \le Ce^{qx}.$$

In this case the assertion of Proposition 1 may by written in the following very simple form:

$$\forall x \in \mathbb{X}\ \forall f \in \mathbb{F} \qquad |\varepsilon_\alpha(x,f) - \widetilde{\varepsilon}_\alpha(x,f)| \le r_{q,\theta} C\,\frac{\alpha(2-\alpha\beta_{q,\theta})}{(1-\alpha\beta_{q,\theta})^3}\left(e^{qx} + \frac{\alpha}{1-\alpha}\right).$$

5.4. The rest of the section is devoted to the proof of Proposition 1.

LEMMA 5. *If Assumptions* 2 *and* 3 *hold then*

$$Q \in \mathcal{H}_0(W, \beta_{q,\theta}, 1, \nu_0, h) \quad \textit{and} \quad \widetilde{Q} \in \mathcal{H}_0(W, \beta_{q,\theta}, 1, \nu_0, \widetilde{h})$$

where ν_0 *is the Dirac measure concentrated at* $x = 0$, *and*

$$W(x) = e^{qx},\ h(x,a) := \mathbf{P}(x + a\xi - \eta \le 0),\ \widetilde{h}(x,a) := \mathbf{P}(x + a\widetilde{\xi} - \widetilde{\eta} \le 0).$$

Proof. This follows from Lemma 9 of [11]. ∎

For real a and q put

$$g_a(z) = \int_0^\infty g(x, ax - z)\,dx, \qquad \widetilde{g}_a(z) = \int_0^\infty \widetilde{g}(x, ax - z)\,dx, \tag{30}$$

$$r'_{q,a} = \int_{-\infty}^{\infty} \max\{e^{qz}, 1\}|g_a(z) - \widetilde{g}_a(z)|\,dz. \tag{31}$$

Lemma 6. *If Assumption* 4 *holds and* $W(x) \equiv e^{qx}$ *for some* $q > 0$ *then*

$$\forall x \ge 0\ \forall a \le \theta \qquad \varrho(x,a)/W(x) \le r'_{q,a}.$$

This assertion is a special case of Lemma 10 from [11] because the functions $g_a(z)$ and $\widetilde{g}_a(z)$, defined in (30), are the densities of the random variables $a\xi - \eta$ and $a\widetilde{\xi} - \widetilde{\eta}$, respectively.

Lemma 7. *If Assumption* 4 *holds then*

$$\forall q \in \mathbb{R}\ \forall a \in \mathbb{R} \qquad r'_{q,a} \le r_{q,a}.$$

Proof. The definitions (30) and (31) yield

$$|g_a(z) - \widetilde{g}_a(z)| \le \int_0^\infty |g(x, ax - z) - \widetilde{g}_a(x, ax - z)|\,dx.$$

Hence,

$$r'_{q,a} \le \int_0^\infty dx \int_{-\infty}^{\infty} \max\{e^{qz}, 1\}|g(x, ax - z) - \widetilde{g}_a(x, ax - z)|\,dz. \tag{32}$$

We now put $z = ax - y$ in the second integral of (32) and recall (28). ∎

Proof of Proposition 1. By Lemma 5 all assumptions of Theorem 1 are satisfied with the W, β and w given in (29). Thus, we need only prove the last inequality in (29). But it follows from (28) and Lemmas 6 and 7 because

$$\forall q > 0\ \forall x \ge 0\ \forall a \le \theta \qquad \varrho(x,a)/W(x) \le r'_{q,a} \le r_{q,a} \le r_{q,\theta}.$$

6. Example 2.

6. Example 2. In this section we present another example which illustrates Theorem 1.

6.1. *The model.* Let $\mathbb{X} = \mathbb{A} = \mathbb{R}$. For a given state $x \in \mathbb{X}$ consider the recurrence equations:

$$\Phi_0 = x, \quad \Phi_{n+1} = G(\Phi_n, a_n) + \zeta_n, \quad n = 0, 1, 2, \ldots,$$
$$\widetilde{\Phi}_0 = x, \quad \widetilde{\Phi}_{n+1} = G(\widetilde{\Phi}_n, a_n) + \widetilde{\zeta}_n, \quad n = 0, 1, 2, \ldots$$

Assumption 5. *Each of the two sequences*

$$\zeta, \zeta_1, \zeta_2, \dots \quad \textit{and} \quad \widetilde{\zeta}, \widetilde{\zeta}_1, \widetilde{\zeta}_2, \dots$$

consists of independent and identically distributed random variables with common densities g and $\widetilde{g}$, respectively.

In this case, for all $x \in \mathbb{X}$ and $B \in \mathcal{B}$ we may define the kernels

$$\begin{aligned} Q(B|x,a) &:= \mathbf{P}(\varPhi_{n+1} \in B \,|\, \varPhi_n = x,\, a_n = a) = \int_B g(s - G(x,a))\, ds, \\ \widetilde{Q}(B|x,a) &:= \mathbf{P}(\widetilde{\varPhi}_{n+1} \in B \,|\, \widetilde{\varPhi}_n = x,\, a_n = a) = \int_B \widetilde{g}(s - G(x,a))\, ds, \end{aligned} \tag{33}$$

where $\mathcal{B}$ denotes the Borel σ-algebra of $\mathbb{X} = [0,\infty)$.

For all real $m \geq 0$ and $s \in \mathbb{R}$ define

$$g_m(s) = \inf_{s-m \leq t \leq m+s} g(t), \qquad \widetilde{g}_m(s) = \inf_{s-m \leq t \leq m+s} \widetilde{g}(t). \tag{34}$$

Later on we use the following natural assumption:

Assumption 6. *A real number $m > 0$ is such that*

$$\tau_m := \int_{-\infty}^{\infty} g_m(s)\, ds > 0 \quad \textit{and} \quad \widetilde{\tau}_m := \int_{-\infty}^{\infty} \widetilde{g}_m(s)\, ds > 0.$$

Remark 3. Suppose that the density function $g(\cdot)$ is continuous at some point s_0 such that $g(s_0) > 0$. Then $\tau_m > 0$ for some $m > 0$. If, in addition, the function $g(\cdot)$ has at most countably many points of discontinuity, then

$$\lim_{m \to 0} \tau_m = \lim_{m \to 0} \int_{-\infty}^{\infty} g_m(s)\, ds = \int_{-\infty}^{\infty} g(s)\, ds = 1. \tag{35}$$

6.2. *The simplest result*

Assumption 7. *Suppose that $G(\cdot,\cdot)$ is a measurable function and that Borel sets $A(x) \in \mathcal{B}$ are such that*

$$\forall x \in \mathbb{X}\ \forall a \in A(x) \qquad |G(x,a)| \leq m,$$

where m is a positive number.

Proposition 2. *If Assumptions* 5 *and* 7 *are satisfied then the kernels $Q(\cdot|\cdot,\cdot)$ and $\widetilde{Q}(\cdot|\cdot,\cdot)$, introduced in* (33), *satisfy all the assumptions and assertions of Theorem* 1 *with*

$$W(x) \equiv 1, \quad \beta := \min\{1 - \tau_m, 1 - \widetilde{\tau}_m\}, \qquad w := 1$$

and, in addition,

$$\|\varrho\| = \int_{-\infty}^{\infty} |g(s) - \widetilde{g}(s)|\, ds. \tag{36}$$

REMARK 4. In this case due to the condition $w(x) \equiv 1$, we need the cost functions to be bounded.

6.3. Consider now a more complicated situation.

ASSUMPTION 8. *Suppose that* $G(\cdot,\cdot)$ *is a measurable function and that Borel sets* $A(x) \in \mathcal{B}$ *are such that*

$$\forall x \in \mathbb{X}\ \forall a \in A(x) \qquad |G(x,a)| \le \gamma |x|,$$

where γ *is a positive number.*

Introduce the functions

$$W_l(x) \equiv W_{k,l}(x) := 1 + k|x|^l, \qquad x \in \mathbb{R},\ k \ge 0,\ l = 1,2. \tag{37}$$

Fix k in (37) and set, for all $m > 0$ and $l = 1,2$,

$$\tau_{l,m} := \int_{-\infty}^{\infty} |s|^l g_m(s)\,ds, \qquad \widetilde{\tau}_{l,m} := \int_{-\infty}^{\infty} |s|^l \widetilde{g}_m(s)\,ds,$$

$$\beta_{l,m} = \max\left\{(1-\tau_m) + k(\mathbf{E}|\zeta|^l - \tau_{l,m}),\ \frac{1 + k\mathbf{E}|\zeta|^l + km^l}{\gamma^l + km^l}\gamma^l,\ \gamma^l\right\}, \tag{38}$$

$$\widetilde{\beta}_{l,m} = \max\left\{(1-\widetilde{\tau}_m) + k(\mathbf{E}|\widetilde{\zeta}|^l - \widetilde{\tau}_{l,m}),\ \frac{1 + k\mathbf{E}|\widetilde{\zeta}|^l + km^l}{\gamma^l + km^l}\gamma^l,\ \gamma^l\right\}, \tag{39}$$

$$w_{l,m} = 1 + k\tau_{l,m}/\tau_m, \qquad \widetilde{w}_{l,m} = 1 + k\widetilde{\tau}_{l,m}/\widetilde{\tau}_m. \tag{40}$$

Moreover, let

$$r_l := \int_{-\infty}^{\infty} |s|^l |g(s) - \widetilde{g}(s)|\,ds, \qquad l = 0,1,2. \tag{41}$$

PROPOSITION 3. *Assume that Assumptions* 5 *and* 8 *hold and that*

$$\mathbf{E}|\zeta| + \mathbf{E}|\widetilde{\zeta}| < \infty. \tag{42}$$

Then the kernels $Q(\cdot|\cdot,\cdot)$ *and* $\widetilde{Q}(\cdot|\cdot,\cdot)$ *introduced in* (33) *satisfy all the assumptions and assertions of Theorem* 1 *with*

$$W(\cdot) := W_1(\cdot), \qquad \beta := \max\{\beta_{1,m}, \widetilde{\beta}_{1,m}\}, \qquad w := \max\{w_{1,m}, \widetilde{w}_{1,m}\} \tag{43}$$

for all $m > 0$, *where* $k \ge 0$ *is a fixed number. In addition,*

$$\|\varrho\| \le \max\{r_0 + kr_1, \gamma r_0\}. \tag{44}$$

PROPOSITION 4. *Assume that Assumptions* 5 *and* 8 *hold and that, in addition,*

$$\mathbf{E}\zeta = \mathbf{E}\widetilde{\zeta} = 0, \qquad \mathbf{E}\zeta^2 + \mathbf{E}\widetilde{\zeta}^2 < \infty. \tag{45}$$

Then the kernels $Q(\cdot|\cdot,\cdot)$ *and* $\widetilde{Q}(\cdot|\cdot,\cdot)$ *introduced in* (33) *satisfy all the assumptions and assertions of Theorem* 1 *with*

$$W(\cdot) := W_2(\cdot), \qquad \beta := \max\{\beta_{2,m}, \widetilde{\beta}_{2,m}\}, \qquad w := \max\{w_{2,m}, \widetilde{w}_{2,m}\}, \tag{46}$$

for all $m > 0$, where $k \geq 0$ is fixed. In addition,

$$\|\varrho\| \leq \max\{r_0 + 2kr_2, 2\gamma^2 r_0\}. \tag{47}$$

REMARK 5. Suppose that the functions $g(\cdot)$ and $\widetilde{g}(\cdot)$ are continuous or have at most countably many points of discontinuity and

$$\mathbf{E}|\zeta|^l + \mathbf{E}|\widetilde{\zeta}|^l < \infty \quad \text{for some } l \geq 0. \tag{48}$$

Then for every fixed $k \geq 0$ we have

$$\forall \beta > 0 \; \exists m > 0 \; \exists \gamma > 0 \quad \beta_{l,m} \leq \beta \quad \text{and} \quad \widetilde{\beta}_{l,m} \leq \beta. \tag{49}$$

Thus, the numbers β in (43) and (46) may be sufficiently small.

To prove (49) we first need to remark that

$$\lim_{m\to 0} \tau_{l,m} = \lim_{m\to 0} \int_{-\infty}^{\infty} |s|^l g_m(s)\,ds = \int_{-\infty}^{\infty} |s|^l g(s)\,ds = \mathbf{E}|\zeta|^l < \infty \tag{50}$$

by (48). Using (35) and (50) we obtain

$$\beta_{l,m}^{(1)} := (1 - \tau_m) + k(\mathbf{E}|\zeta|^l - \tau_{l,m}) \to 0 \quad \text{as } m \to 0. \tag{51}$$

Hence, by (51),

$$\forall \beta > 0 \; \exists m_\beta > 0 \quad \beta_{l,m_\beta}^{(1)} \leq \beta. \tag{52}$$

It now follows from (38) and (52) that

$$\beta_{l,m_\beta} \leq \beta \quad \text{for} \quad \gamma^l \leq \frac{\beta k m^l}{1 + k\mathbf{E}|\zeta|^l + km^l}. \tag{53}$$

Similarly

$$\widetilde{\beta}_{l,\widetilde{m}_\beta} \leq \beta \quad \text{for some } \widetilde{m}_\beta > 0 \quad \text{and} \quad \gamma^l \leq \frac{\beta k \widetilde{m}^l}{1 + k\mathbf{E}|\widetilde{\zeta}|^l + k\widetilde{m}^l}. \tag{54}$$

Assertion (49) is an immediate consequence of (53) and (54).

6.4. *Key lemmas.* Let

$$I_m(x,a) = \begin{cases} 1 & \text{if } |G(x,a)| \leq m, \\ 0 & \text{if } |G(x,a)| > m, \end{cases} \tag{55}$$

be the indicator of the set $\{(x,a) : |G(x,a)| \leq m\}$. In this subsection we consider a function W and a number $m \geq 1$ such that

$$\forall x \in \mathbb{R} \quad W(x) \geq 1 \quad \text{and} \quad w_m := \int_{-\infty}^{\infty} W(x) g_m(x)\,dx < \infty. \tag{56}$$

For some $\beta \geq 0$ put

$$R(x,a) := \mathbf{E}W(G(x,a) + \zeta) - w_m I_m(x,a) - \beta W(x). \tag{57}$$

We will use the following assumption:

$$\forall x \in \mathbb{R} \; \forall a \in A(x) \quad R(x,a) \leq 0. \tag{58}$$

LEMMA 8. *If Assumption* 6 *and conditions* (56) *and* (58) *hold then there exist a function* $h_m(\cdot,\cdot)$ *and a probability measure* $\nu_m(\cdot)$ *such that*

$$Q \in \mathcal{H}_0(W, \beta, w_m/\tau_m, h_m, \nu_m). \tag{59}$$

Proof. First of all we remark that

$$\int W(y)\, Q(dy|x,a) = \mathbf{E}W(G(x,a)+\zeta).$$

Using (34) and (55) we deduce from (33) that

$$Q(B|x,a) \geq I_m(x,a) \int_B g(s - G(x,a))\, ds \geq I_m(x,a) \int_B g_m(s)\, ds \tag{60}$$

because

$$g(s - G(x,a)) \geq g_m(s) \quad \text{if} \quad |G(x,a)| \leq m.$$

Denote by $\nu_m(\cdot)$ the measure on $(\mathbb{R}, \mathcal{B})$ with density $g_m(s)/\tau_m$, and let

$$\forall x \in \mathbb{R}\ \forall a \in A(x) \qquad h_m(x,a) := \tau_m I_m(x,a).$$

It is easy to see that

$$I_m(x,a) \int_B g_m(s)\, ds \equiv h_m(x,a)\nu(B), \tag{61}$$

$$\int W(y)\, \nu(dy) = w_m/\tau_m, \tag{62}$$

$$h(x,a) \int W(y)\, \nu(dy) \equiv w_m \cdot I_m(x,a). \tag{63}$$

To prove (59) we now need to verify conditions (13)–(15). But (15) follows from (56) with w from (62), (13) is a special case of (60) and (61), and, finally, (14) may be rewritten in the simple form (58) by using (59) and (63). ∎

LEMMA 9. *If Assumption* 5 *holds then*

$$\varrho(x,a) = \int_{-\infty}^{\infty} W(s + G(x,a))|g(x) - \widetilde{g}(s)|\, ds. \tag{64}$$

Proof. It follows from definitions (16) and (33) that

$$\varrho(x,a) = \int_{-\infty}^{\infty} W(y)|g(y - G(x,a)) - \widetilde{g}(y - G(x,a))|\, dy. \tag{65}$$

It is evident that (65) may be rewritten in the form (64). ∎

6.5. *Proof of Proposition 2*

LEMMA 10. *If* $W(\cdot) \equiv 1$ *then there exist functions* h_m, $\widetilde{h}_m$ *and probability measures* ν_m *and* $\widetilde{\nu}_m$ *such that*

$$Q \in \mathcal{H}_0(1, 1 - \tau_m, 1, h_m, \nu_m), \tag{66}$$

$$\widetilde{Q} \in \mathcal{H}_0(1, 1 - \widetilde{\tau}_m, 1, \widetilde{h}_m, \widetilde{\nu}_m). \tag{67}$$

Proof. It follows from the definition of the class $\mathcal{H}_0$ that always

$$Q \in \mathcal{H}_0(1, 1, 1, 0, \nu) \quad \text{with } W \equiv 1 \quad \text{and } h \equiv 0 \tag{68}$$

for all measures ν. Hence, (68) yields (66) in the case when $\tau_m = 0$, whereas for $\tau_m > 0$, (66) is a consequence of Lemma 8. Similar arguments imply (67). ∎

Thus, to obtain Proposition 2 we need only prove equality (36). But it follows immediately from Lemma 9 with $W \equiv 1$.

6.6. *Auxiliary lemmas.* In the rest of the section we consider the case when Assumptions 5, 6, 8 and condition (48) hold with

$$W(x) \equiv 1 + k|x|^l \quad \text{and} \quad w_m = \tau_m + k\tau_{l,m}. \tag{69}$$

In this case

$$\forall x \in \mathbb{X}\ \forall a \in A(x) \quad I_m(x, a) \geq I_{m/\gamma}(x) = \begin{cases} 1 & \text{if } |x| \leq \gamma/m, \\ 0 & \text{if } |x| > \gamma/m, \end{cases} \tag{70}$$

because

$$|G(x, a)| \leq \gamma|x| \leq m \quad \text{if } |x| \leq \gamma/m.$$

For $l \geq 0$ introduce

$$R_l(x) = 1 + k\mathbf{E}|\zeta|^l - I_{m/\gamma}(x)(\tau_m + k\tau_{l,m}) - (\beta - \gamma^l)k|x|^l - \beta. \tag{71}$$

Our first aim is to prove the following inequality:

$$\forall x \in \mathbb{X}\ \forall a \in A(x) \quad R(x, a) \leq R_l(x). \tag{72}$$

LEMMA 11. *If condition* (42) *holds then inequality* (72) *is valid for* $l = 1$.

Proof. This follows immediately from (57), (69)–(71) and

$$\mathbf{E}W(G(x, a) + \zeta) = 1 + k\mathbf{E}|G(x, a) + \zeta| \leq 1 + k\gamma|x| + k\mathbf{E}|\zeta|. \ ∎ \tag{73}$$

LEMMA 12. *If conditions* (45) *are true then inequality* (72) *holds for* $l = 2$.

Proof. Use again (57), (69)–(73) and the following fact:

$$\begin{aligned}\mathbf{E}W(G(x, a) + \zeta) &= 1 + kG^2(x, a) + k\mathbf{E}\zeta^2 \\ &\leq 1 + k\gamma^2 x^2 + k\mathbf{E}\zeta^2. \ ∎\end{aligned}$$

LEMMA 13. *If* $\beta \leq \beta_{l,m}$ *then*

$$\forall x \in \mathbb{X}\ \forall a \in A(x) \quad R_l(x) \leq 0.$$

Recall that the value $\beta_{l,m}$ was defined in (38).

Proof. If $|x| \le m/\gamma$ then $I_{m/\gamma}(x) = 1$ and $(\beta - \gamma^l)|x| \ge 0$. Hence, in this case we need to verify the inequality

$$R_l(x) \le R_l(0) = (1 - \tau_m) + k(\mathbf{E}|\zeta|^l - \tau_{l,m}) - \beta \le 0. \tag{74}$$

On the other hand, if $|x| > m/\gamma$ then $(\beta - \gamma^l)|x|^l \ge (\beta - \gamma^l)m^l/\gamma^l$ and hence we must have

$$R_l(x) \le R_l(m/\gamma + 0) = 1 + k\mathbf{E}|\zeta|^l + km^l - \beta(1 + km^l/\gamma^l) \le 0. \tag{75}$$

Thus, the value $\beta_{l,m}$, defined in (38), is the minimal value of the β for which we have, simultaneously, inequalities (74), (75) together with $\beta \ge \gamma^l$. ∎

6.7. *Proof of Proposition 3*

LEMMA 14. *If all the conditions of Proposition* 3 *hold then there exist* h_m, $\widetilde{h}_m$, ν_m *and* $\widetilde{\nu}$ *such that*

$$Q \in \mathcal{H}_0(W_1, \beta_{1,m}, w_{1,m}, h_m, \nu_m), \tag{76}$$

$$\widetilde{Q} \in \mathcal{H}_0(W_1, \widetilde{\beta}_{1,m}, \widetilde{w}_{1,m}, \widetilde{h}_m, \widetilde{\nu}_m), \tag{77}$$

where the values $\beta_{1,m}$, $\widetilde{\beta}_{1,m}$, $w_{1,m}$ *and* $\widetilde{w}_{1,m}$ *were defined in* (38)–(40).

Proof. It follows from Lemmas 11 and 13 that

$$\forall x \in \mathbb{X}\ \forall a \in A(x) \quad R(x,a) \le R_1(x) \quad \text{if } \beta = \beta_{1,m}. \tag{78}$$

Now (78) and Lemma 8 yield (76). Similar arguments imply (77). ∎

Thus, to obtain Proposition 3 we need only prove

LEMMA 15. *If Assumption* 8 *and conditions* (43) *hold then inequality* (44) *is valid.*

Proof. Assumption 8 and Lemma 9 with $W(x) \equiv 1 + k(x)$ yield

$$\varrho(x,a) \le \int_{-\infty}^{\infty} (1 + k\gamma|x| + k|s|)|g(s) - \widetilde{g}(s)|\, ds = (1 + k\gamma|x|)r_0 + kr + 1,$$

where the values r_0 and r_1 were defined in (41). Hence,

$$\|\varrho\| = \sup_{x \in \mathbb{X}} \sup_{a \in A(x)} \frac{\varrho(x,a)}{W(x)} \le \sup_{x \in \mathbb{R}} \frac{(1 + k\gamma|x|)r_0 + kr_1}{1 + k|x|}. \tag{79}$$

It is easy to verify that the right hand sides in (44) and (79) are equal. ∎

6.8. *Proof of Proposition 4*

LEMMA 16. *If all the assumptions of Proposition* 4 *hold then*

$$Q \in \mathcal{H}_0(W_2, \beta_{2,m}, w_{1,m}, h_m, \nu_m), \qquad \widetilde{Q} \in \mathcal{H}_0(W_2, \widetilde{\beta}_{2,m}, \widetilde{w}_{2,m}, \widetilde{h}_m, \widetilde{\nu}_m),$$

for some h_m, $\widetilde{h}_m$, ν_m *and* $\widetilde{\nu}_m$.

The proof is similar to that of Lemma 14 and is based on Lemmas 8, 12 and 13 for $l = 2$. Hence, to obtain Proposition 4 we need only prove

LEMMA 17. *If Assumption* 8 *and conditions* (46) *are valid then inequality* (47) *is true.*

Proof. Assumption 8 and Lemma 9 with $W(x) = 1 + kx^2$ yield

$$\varrho(x,a) \le \int_{-\infty}^{\infty} (1 + k(\gamma|x| + |s|)^2)|g(s) - \widetilde{g}(s)|\,ds$$
$$\le \int_{-\infty}^{\infty} (1 + 2k(\gamma^2 x^2 + s^2))|g(s) - \widetilde{g}(s)|\,ds = (1 + 2k\gamma^2 x^2)r_0 + 2kr_2,$$

where the values r_0 and r_2 were defined in (41). Thus,

$$\|\varrho\| = \sup_{x\in\mathbb{X}} \sup_{a\in A(x)} \frac{\varrho(x,a)}{W(x)} \le \sup_{x\in\mathbb{R}} \frac{(1 + 2k\gamma^2 x^2)r_0 + 2kr_2}{1 + kx^2}. \tag{80}$$

The right hand in (80) is the same as that in (47). ∎

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