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CONTROLLABILITY OF PARTIAL DIFFERENTIAL EQUATIONS ON GRAPHS

Abstract. We study boundary control problems for the wave, heat, and Schrödinger equations on a finite graph. We suppose that the graph is a tree (i.e., it does not contain cycles), and on each edge an equation is defined. The control is acting through the Dirichlet condition applied to all or all but one boundary vertices. Exact controllability in L_2-classes of controls is proved and sharp estimates of the time of controllability are obtained for the wave equation. Null controllability for the heat equation and exact controllability for the Schrödinger equation in any time interval are obtained.

1. Introduction. Controllability problems for multi-link flexible structures or, in other words, for the wave and beam equations on graphs were the subject of extensive investigations of many mathematicians (see, e.g., the review paper [1] and references therein). Lagnese, Leugering, and Schmidt [15, 16] used the method of energy estimates together with the Hilbert uniqueness method to show that the exact controllability can be achieved in optimal time for tree-like graphs consisting of homogeneous strings, when all but one exterior nodes are controlled. Independently Avdonin and Ivanov [2, Ch. VII] applied the method of moments and the theory of vector-valued exponentials to study controllability problems on graphs for the wave equation. They proved exact controllability in optimal time for the wave equation on a star-shaped graph of non-homogeneous strings. Belishev [5, 6], using the propagation of singularities method, obtained a result on boundary con-

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trollability for a tree of non-homogeneous strings with respect to the first component (shape) of the complete state.

Exact controllability results fail as soon as cycles occur within the network, even if all nodes (including the interior ones) are subjected to control [2, Sec. VII.1]. However, spectral controllability may be retained for many graphs with cycles (see [2, Ch. VII], [10, 16] for details). In [19], for a tree of homogeneous vibrating strings, the authors prove exact controllability for some special class of initial/final data. Many interesting results on spectral controllability are obtained in [10].

In this paper we prove exact controllability for the wave equation on a tree-like graph of non-homogeneous strings for controls acting through Dirichlet conditions applied to all or all but one boundary vertices. Our result generalizes the ones from [2] and [16]. Using the controllability of the wave equation and results from [2, 21, 22, 24], we also prove the null controllability of the heat equation and exact controllability of the Schrödinger equation on trees.

Controllability problems for partial differential equations on graphs have many important applications. They are also related to inverse problems on graphs [3, 5, 7] and to harmonic analysis [2, Ch. VII]. In this paper we use some known and prove several new results describing connections between controllability of distributed parameter systems and properties of exponential families.

We do not consider some important problems closely related to the topic of the paper, such as controllability of networks of beams and hybrid systems, and refer the reader to the comprehensive papers [11, 16, 17].

2. Statement of the problems and main results. Let Ω be a finite connected compact graph without cycles (a tree). The graph consists of edges $E=\{e_1,\dots,e_N\}$ connected at the vertices $V=\{v_1,\dots,v_{N+1}\}$. Every edge $e_j\in E$ is identified with an interval (a_{2j-1},a_{2j}) of the real line. The edges are connected at the vertices v_j which can be considered as equivalence classes of the edge end points $\{a_j\}$. The boundary $\Gamma=\{\gamma_1,\dots,\gamma_m\}$ of Ω is a set of vertices having multiplicity one (the exterior nodes). We suppose that the graph is equipped with a density

$$(2.1)\qquad \varrho(x)\geq \text{const}>0,\quad x\in\Omega\setminus V,\quad \varrho\in C^1(\bar e_j),\ j=1,\dots,N.$$

All the results of this paper are also valid for piecewise continuously differentiable functions ϱ. Discontinuity of ϱ or its derivative at an inner point of an edge is equivalent to an additional inner vertex of multiplicity two (see the compatibility conditions (2.3), (2.4) below).

Since the graph under consideration is a tree, for every $a,b\in\Omega$, $a\neq b$, there exists a unique path $\pi[a,b]$ connecting these points. The density determines the *optical metric* and the *optical distance*

$$d\sigma^2 = \varrho(x)|dx|^2, \qquad x \in \Omega \setminus V,$$
$$\sigma(a,b) = \int\limits_{\pi[a,b]} \sqrt{\varrho(x)}\,|dx|, \qquad a,b \in \Omega,$$

The *optical diameter* of the graph Ω is defined as

$$d(\Omega) = \max_{a,b\in\Gamma} \sigma(a,b).$$

The graph Ω and the optical metric determine the *metric graph* denoted by $\{\Omega,\varrho\}$. For a rigorous definition of the metric graph, see e.g. [12, 13, 14, 20, 23]. The space of real-valued functions on the graph, square integrable with the weight ϱ, is denoted by $L_{2,\varrho}(\Omega)$.

2.1. *Dirichlet spectral problem.* Let $\partial w(a_j)$ denote the derivative of w at the vertex a_j taken along the corresponding edge in the direction toward the vertex. We associate the following spectral problem to the graph:

$$-\frac{1}{\varrho}\frac{d^2w}{dx^2} = \lambda w, \tag{2.2}$$
$$w \in C(\Omega), \tag{2.3}$$
$$\sum_{a_j\in v} \partial w(a_j) = 0 \quad \text{for } v \in V \setminus \Gamma, \tag{2.4}$$
$$w = 0 \quad \text{on } \Gamma. \tag{2.5}$$

It is well known (see, e.g., [8, 20, 25]) that the problem (2.2)–(2.5) has a discrete spectrum of eigenvalues $0 < \lambda_1 \le \lambda_1 \le \cdots$, $\lambda_k \to +\infty$, and corresponding eigenfunctions $\phi_1, \phi_2, \ldots$ can be chosen so that $\{\phi_k\}_{k=1}^{\infty}$ forms an orthonormal basis in $\mathcal{H} := L_{2,\varrho}(\Omega)$:

$$(\phi_i,\phi_j)_{\mathcal{H}} = \int\limits_{\Omega} \phi_i(x)\phi_j(x)\varrho(x)\,dx = \delta_{ij}.$$

Set $\varkappa_k(\gamma) = \partial\phi_k(\gamma)$ for $\gamma \in \Gamma$. Let α_k be the m-dimensional column vector defined as $\alpha_k = \mathrm{col}\,(\varkappa_k(\gamma)/\sqrt{\lambda_k})_{\gamma\in\Gamma}$.

Definition 1. The set of pairs

$$\{\lambda_k, \alpha_k\}_{k=1}^{\infty} \tag{2.6}$$

is called the *Dirichlet spectral data* of the graph $\{\Omega,\varrho\}$.

2.2. *Initial boundary value problems. Control from the whole boundary.* We associate to the graph $\{\Omega,\varrho\}$ three dynamical systems, described correspondingly by the wave, heat and Schrödinger equations. The first one has the form

(2.7) $$\varrho u_{tt} - u_{xx} = 0 \quad \text{in } \Omega \setminus V \times [0,T],$$

(2.8) $$u|_{t=0} = u_t|_{t=0} = 0,$$

(2.9) $$u(\cdot,t) \text{ satisfies (2.3) and (2.4) for all } t \in [0,T],$$

(2.10) $$u = f \quad \text{on } \Gamma \times [0,T].$$

Here $T > 0$ and $f = f(\gamma, t)$, $\gamma \in \Gamma$, is the Dirichlet boundary control which belongs to $\mathcal{F}_\Gamma^T = L_2([0,T];\mathbb{R}^m)$. The inner product in $\mathcal{F}_\Gamma^T$ is defined by

$$(f,g)_{\mathcal{F}_\Gamma^T} = \sum_{i=1}^{m} \int_0^T f(\gamma_i,t)g(\gamma_i,t)\,dt.$$

Let $D'(\Omega)$ be the set of distributions over the graph. We introduce the space

$$\mathcal{H}_{-1} = \Big\{ g \in D'(\Omega) : g(x) = \sum_{k=1}^{\infty} g_k \phi_k(x),\ \{g_k/\sqrt{\lambda_k}\}_{k=1}^{\infty} \in l_2 \Big\}.$$

The initial boundary value problem (2.7)–(2.10) has a classical solution if $f \in C^2([0,T];\mathbb{R}^m)$, $f(0) = f'(0) = 0$. In our case when $f \in \mathcal{F}_\Gamma^T$, the solution to (2.7)–(2.10) is understood in a weak (distributional) sense. It can be proved (see [2, 7, 10, 16]) that for any $f \in \mathcal{F}_\Gamma^T$, the initial boundary value problem (2.7)–(2.10) has a unique weak solution u^f and it satisfies the inclusion

$$u^f \in C([0,T];\mathcal{H}) \cap C^1([0,T];\mathcal{H}_{-1}).$$

This means that $u^f(\cdot,t) \in \mathcal{H}$, $u_t^f(\cdot,t) \in \mathcal{H}_{-1}$ for all $t \in [0,T]$, and both functions are continuous with respect to t in the corresponding norms. In other words, the state $(u(\cdot,t), u_t(\cdot,t))$ of the dynamical system (2.7)–(2.10) is a point of $\mathcal{H} \times \mathcal{H}_{-1}$, and the trajectory of the system is a continuous curve in the state space $\mathcal{H} \times \mathcal{H}_{-1}$. This regularity result is sharp.

One of the main results of the present paper demonstrates the *exact controllability* of the system (2.7)–(2.10).

Theorem 1. *For any state $\{a,b\} \in \mathcal{H} \times \mathcal{H}_{-1}$, there exists a control function $f(\gamma,t) \in \mathcal{F}_\Gamma^T$ with $T = d(\Omega)$ such that the solution of the initial boundary value problem* (2.7)–(2.10) *satisfies the equalities $u^f(\cdot,T) = a$, $u_t^f(\cdot,T) = b$.*

Another system we associate to the graph $\{\Omega, \varrho\}$ is

(2.11) $$\varrho u_t - u_{xx} = 0 \quad \text{in } \Omega \setminus V \times [0,\tau],$$

(2.12) $$u|_{t=0} = a,$$

(2.13) $$u(\cdot,t) \text{ satisfies (2.3) and (2.4) for all } t \in [0,\tau],$$

(2.14) $$u = f \quad \text{on } \Gamma \times [0,\tau],$$

where $\tau > 0$, $f \in \mathcal{F}_\Gamma^\tau$ and $a \in \mathcal{H}_{-1}$.

It is known (see, e.g., [2, 9, 16]) that the initial boundary value problem (2.11)–(2.14) has a unique weak solution u^f and

$$u^f \in C([0,\tau];\mathcal{H}_{-1}).$$

For parabolic-type dynamical systems various types of controllability are considered in the literature (see [2, 16]). The following result demonstrates the *null controllability* of the system (2.11)–(2.14).

THEOREM 2. *For any initial state* $a \in \mathcal{H}_{-1}$ *and for an arbitrary time interval* $[0,\tau]$, $\tau > 0$, *there exists a control* $f \in \mathcal{F}_\Gamma^\tau$ *such that the solution of the initial boundary value problem* (2.11)–(2.14) *satisfies the equality* $u^f(\cdot,\tau) = 0$.

The Schrödinger equation can also be associated to the graph $\{\Omega, \varrho\}$:

$$\varrho u_t + iu_{xx} = 0 \quad \text{in } \Omega \setminus V \times [0,\tau], \tag{2.15}$$

$$u|_{t=0} = a, \tag{2.16}$$

$$u(\cdot,t) \text{ satisfies (2.3) and (2.4) for all } t \in [0,\tau], \tag{2.17}$$

$$u = f \quad \text{on } \Gamma \times [0,\tau], \tag{2.18}$$

where $f \in \mathcal{F}_\Gamma^\tau$, $a \in \mathcal{H}_{-1}$. It is known (see, e.g. [4, 26]) that a unique weak solution $u^f(x,t)$ of (2.15)–(2.18) exists and satisfies the inclusion

$$u^f \in C([0,T];\mathcal{H}_{-1}).$$

For the dynamical system governed by the Schrödinger equation (2.15)–(2.18) the following exact controllability result holds. (Due to time reversibility, the exact and null controllability are equivalent for the Schrödinger equation.)

THEOREM 3. *For any initial state* $a \in \mathcal{H}_{-1}$ *and for an arbitrary time interval* $[0,\tau]$, $\tau > 0$, *there exists a control* $f \in \mathcal{F}_\Gamma^\tau$ *such that the solution to the initial boundary value problem* (2.15)–(2.18) *satisfies the equality* $u^f(\cdot,\tau) = 0$.

2.3. *Initial boundary value problems. Control from a part of the boundary.* In the case when the graph is controlled from the whole boundary but contains cycles, the system (2.7)–(2.10) is not exactly controllable (see, e.g., [2, Sec. VII.1]). Similarly, if the graph is a tree, but the system is not controlled at two or more boundary points (the Dirichlet condition $u = 0$ is imposed there), the conclusion of Theorem 1 fails; an example (in the case of homogeneous strings) is given in [10, Sec. 6.3] (see also [1]). Suppose that the graph is not controlled at one of the boundary points, say γ_1. Then one can introduce the length of the longest path from γ_1 to the rest of the boundary $\Gamma_1 = \Gamma \setminus \{\gamma_1\}$:

$$d_1(\gamma_1, \Omega) = \max_{\gamma \in \Gamma_1} \tau(\gamma_1, \gamma).$$

The boundary conditions for the system (2.7)–(2.9) have the form

$$u(\gamma_1,t)=0,\qquad u(\gamma_i,t)=f(\gamma_i,t),\qquad i=2,\dots,N,\tag{2.19}$$

where $f\in\mathcal{F}^T_{\Gamma_1}=L_2([0,T];\mathbb{R}^{m-1})$. In this situation the analog of Theorem 1 holds true:

Theorem 4. *For any state* $\{a,b\}\in\mathcal{H}\times\mathcal{H}_{-1}$, *there exists a control function* $f\in\mathcal{F}^T_{\Gamma_1}$ *with* $T=2d_1(\gamma_1,\Omega)$ *such that the solution of the initial boundary value problem* (2.7)–(2.9), (2.19) *satisfies the equalities* $u^f(\cdot,T)=a$, $u^f_t(\cdot,T)=b$.

For the parabolic and Schrödinger type systems (2.11)–(2.13), (2.15)–(2.18), we can also consider the problem of a controllability from a part of the boundary, i.e., we add the boundary conditions (2.19) to the initial-value problem (2.11)–(2.13) and to the problem (2.15)–(2.17). In this case one can prove the analogs of Theorems 2 and 3:

Theorem 5. *For any initial state* $a\in\mathcal{H}_{-1}$ *and for an arbitrary time interval* $[0,\tau]$, $\tau>0$, *there exists a control* $f\in\mathcal{F}^\tau_{\Gamma_1}$ *such that the solution of the initial boundary value problem* (2.11)–(2.13), (2.19) *satisfies the equality* $u^f(\cdot,\tau)=0$.

Theorem 6. *For any initial state* $a\in\mathcal{H}_{-1}$ *and for an arbitrary time interval* $[0,\tau]$, $\tau>0$, *there exists a control* $f\in\mathcal{F}^\tau_{\Gamma_1}$ *such that the solution of the initial boundary value problem* (2.15)–(2.17), (2.19) *satisfies the equality* $u^f(\cdot,\tau)=0$.

3. Auxiliary results. In [5] (see also [6, 7]) the following result concerning the controllability with respect to the first component (shape) of the complete state $\{u,u_t\}$ of the dynamical system (2.7)–(2.10) has been proved:

Theorem 7. *Let* $T=d(\Omega)/2$. *Then for any* $a\in\mathcal{H}$, *there exists a control* $f(\gamma,t)\in\mathcal{F}^T_\Gamma$ *such that the solution of the initial boundary value problem* (2.7)–(2.10) *satisfies the equality* $u^f(x,T)=a(x)$.

In other words, the system (2.7)–(2.10) is controllable with respect to shape in time equal to half the optical diameter of the graph. Note that in general such a control is not unique.

To prove Theorem 7 the propagation of singularities method has been used and the controllability was reduced to solvability of a Volterra type equation. It was supposed in [5]–[7] that $\varrho\in C^2$ on all edges, but the method works for $\varrho\in C^1$ as well. The same technique can be applied to obtain the controllability of the system (2.7)–(2.10) with respect to the second component (velocity) of the complete state:

Proposition 1. *If $T = d(\Omega)/2$ then for any $b \in \mathcal{H}_{-1}$, there exists a control $f \in \mathcal{F}_\Gamma^T$ such that the solution of the initial boundary value problem* (2.7)–(2.10) *satisfies the equality $u_t^f(x,T) = b(x)$.*

In the following two propositions we consider the case of boundary condition (2.19) for the system (2.7)–(2.9). The proof of Proposition 2 can be extracted from the proof of Theorem 7 [7, Sec. 2]. Let us introduce the "optical center" of the graph Ω, i.e., a point $\xi \in \Omega$ such that $\max_{\gamma\in\Gamma} \tau(\xi,\gamma) = d(\Omega)/2 = T$. Since Ω is a tree, there can be only one optical center. Suppose that the final state $a(x)$ is supported in a subtree $\Omega_1 \subset \Omega$ such that $\xi \notin \Omega_1$. As shown in [5]–[7], to solve the control problem one has to use controls supported on the part of the boundary of the graph Ω which is the boundary of Ω_1. In other words, it is possible to construct a control $f \in \mathcal{F}_\Gamma^T$ such that $u^f(T,x) = a(x)$ and $f(\gamma,t) = 0$ for $\gamma \notin \Omega_1$. The authors offer an explicit procedure for constructing such a control. If instead of the "optical center" of the graph we take a boundary point γ_1 where the homogeneous Dirichlet condition $u(\gamma_1,t) = 0$ is imposed, we come to the following statements:

Proposition 2. *If $T = d_1(\gamma_1,\Omega)$ then for any $a \in \mathcal{H}$ there exists a control $f \in \mathcal{F}_{\Gamma_1}^T$ such that the solution of the boundary value problem* (2.7)–(2.9), (2.19) *satisfies the equality $u^f(x,T) = a(x)$.*

The same result holds true for controllability with respect to velocity:

Proposition 3. *If $T = d_1(\gamma_1,\Omega)$ then for any $b \in \mathcal{H}_{-1}$ there exists a control $f \in \mathcal{F}_{\Gamma_1}^T$ such that the solution of the boundary value problem* (2.7)–(2.9), (2.19) *satisfies the equality $u_t^f(x,T) = b(x)$.*

4. Proof of Theorem 1. We begin by reducing the problem of controllability of the dynamical system (2.7)–(2.10) to a moment problem in $\mathcal{F}_\Gamma^T$. Solving the initial boundary value problem (2.7)–(2.10) by the Fourier method and looking for the solution in the form

$$u^f(x,t) = \sum_{k=1}^{\infty} c_k^f(t)\phi_k(x), \tag{4.1}$$

we get the expression for the coefficients:

$$c_k^f(t) = \sum_{\gamma\in\Gamma} \frac{\varkappa_k(\gamma)}{\sqrt{\lambda_k}} \int_0^t \sin(\sqrt{\lambda_k}\,(t-s)) f(\gamma,s)\,ds.$$

Suppose that we are given the final state $\{a,b\} \in \mathcal{H}\times\mathcal{H}_{-1}$ at $t = T$, where the functions $a(x)$, $b(x)$ have the expansions

$$a(x)=\sum_{k=1}^{\infty} a_k\phi_k(x), \qquad b(x)=\sum_{k=1}^{\infty} b_k\phi_k(x),$$

for some $\{a_k\}_{k=1}^{\infty}\in l_2$ and $\{b_k/\sqrt{\lambda_k}\}_{k=1}^{\infty}\in l_2$. Then for an unknown control $f\in\mathcal{F}_\Gamma^T$, the following moment equalities should hold at time $t=T$:

$$a_k=c_k^f(T)=\sum_{\gamma\in\Gamma}\frac{\varkappa_k(\gamma)}{\sqrt{\lambda_k}}\int_0^T \sin(\sqrt{\lambda_k}\,(T-s))f(\gamma,s)\,ds, \qquad k\in\mathbb{N}, \tag{4.2}$$

$$\frac{b_k}{\sqrt{\lambda_k}}=\frac{\dot{c}_k^f(T)}{\sqrt{\lambda_k}}=\sum_{\gamma\in\Gamma}\frac{\varkappa_k(\gamma)}{\sqrt{\lambda_k}}\int_0^T \cos(\sqrt{\lambda_k}\,(T-s))f(\gamma,s)\,ds, \qquad k\in\mathbb{N}. \tag{4.3}$$

Using the Euler formulas for exponentials, we rewrite (4.2), (4.3) as

$$\frac{b_k}{\sqrt{\lambda_k}}\pm ia_k=\sum_{\gamma\in\Gamma}\frac{\varkappa_k(\gamma)}{\sqrt{\lambda_k}}\int_0^T e^{\pm i\sqrt{\lambda_k}(T-s)}f(\gamma,s)\,ds, \qquad k\in\mathbb{N}. \tag{4.4}$$

Definition 2. We call the moment problem (4.4) *solvable* (and $f(\gamma,t)$ a *solution* of the moment problem) in the time interval $[0,T]$ if, for arbitrary sequences $\{a_k\}_{k=1}^{\infty},\{b_k/\sqrt{\lambda_k}\}_{k=1}^{\infty}\in l_2$, there exists a function $f\in\mathcal{F}_\Gamma^T$ such that equalities (4.4) hold.

We emphasize that the solvability of the moment problem (4.4) in the time interval $[0,T]$ for some $T>0$ is equivalent to the controllability of the dynamical system (2.7)–(2.10) in the sense of Theorem 1 in the same time interval. This is a basic statement of the method of moments (see e.g. [2, Ch. III], [24]).

We need a couple of definitions concerning vector families in a Hilbert space.

Definition 3. The family $\{\xi_k\}_{k=1}^{\infty}$ in a Hilbert space H is called a *Riesz basis* if it is the image of an orthonormal basis under the action of some linear isomorphism.

Definition 4. The family $\{\xi_k\}_{k=1}^{\infty}$ in a Hilbert space H is called an $\mathcal{L}$*-basis* if it is a Riesz basis in the closure of the linear span of the family.

The controllability result in Theorem 7 implies the solvability of the moment problem (4.2) for $T=d(\Omega)/2$ for every $\{a_k\}_{k=1}^{\infty}$. The controllability result in Proposition 1 implies the solvability of the moment problem (4.3) for $T=d(\Omega)/2$ for every $\{b_k/\sqrt{\lambda_k}\}_{k=1}^{\infty}\in l_2$. Our goal is to show that the solvability of the moment problems (4.2) and (4.3) for $T=d(\Omega)/2$ implies the solvability of the moment problem (4.4) for $T=d(\Omega)$. Let us put $T_* = d(\Omega)/2$ and introduce the families of vector-valued functions

$$S_k(t) = \alpha_k \sin(\sqrt{\lambda_k}\, t), \qquad C_k(t) = \alpha_k \cos(\sqrt{\lambda_k}\, t), \qquad k \in \mathbb{N}.$$

According to Theorem III.3.3 of [2] the solvability of the moment problems (4.2) and (4.3) means that the families $\{S_k\}_{k=1}^\infty$ and $\{C_k\}_{k=1}^\infty$ form $\mathcal{L}$-bases in $L_2([0, T_*]; \mathbb{R}^m)$.

Let us introduce subspaces of $L_2([0, T_*]; \mathbb{R}^m)$:

$$\Xi_o = \bigvee \{S_k\}_{k=1}^\infty, \qquad \Xi_e = \bigvee \{C_k\}_{k=1}^\infty,$$

where $\bigvee$ denotes the closure of the linear span of a family. We extend the functions from Ξ_o to the interval $[-T_*, 0)$ in the odd way:

$$\widetilde{\varphi}(t) = \begin{cases} \varphi(t), & t \geq 0, \\ -\varphi(-t), & t < 0, \end{cases} \qquad -T_* \leq t \leq T_*,\ \varphi \in \Xi_o,$$

and the functions from Ξ_e in the even way:

$$\widetilde{\varphi}(t) = \begin{cases} \varphi(t), & t \geq 0, \\ \varphi(-t), & t < 0, \end{cases} \qquad -T_* \leq t \leq T_*,\ \varphi \in \Xi_e.$$

Let us denote the spaces of extended functions by $\widetilde{\Xi}_o$ and $\widetilde{\Xi}_e$ and notice that the extended families $\{\widetilde{S}_k\}_{k=1}^\infty$ and $\{\widetilde{C}_k\}_{k=1}^\infty$ are Riesz bases in $\widetilde{\Xi}_o$ and $\widetilde{\Xi}_e$ respectively. The orthogonality of the spaces $\widetilde{\Xi}_o$ and $\widetilde{\Xi}_e$ implies that the union

$$\{\widetilde{C}_k(t)\}_{k=1}^\infty \cup \{\widetilde{S}_k(t)\}_{k=1}^\infty$$

is a Riesz basis in $\widetilde{\Xi}_o \oplus \widetilde{\Xi}_e \subset L_2([-T_*, T_*]; \mathbb{R}^m)$. Introducing the functions

$$E_{\pm k}(t) = C_k(t) \pm i S_k(t) = \alpha_k e^{\pm i\sqrt{\lambda_k} t}, \qquad k \in \mathbb{N}, \tag{4.5}$$

we see that the set $\{E_{\pm k}\}_{k\in\mathbb{N}}$ is an $\mathcal{L}$-basis in $L_2([-T_*, T_*]; \mathbb{C}^m)$. Shifting the argument, we come to the conclusion that the same family is an $\mathcal{L}$-basis in $L_2([0, 2T_*]; \mathbb{C}^m)$. Then according to Theorem III.3.3 of [2], the moment problem (4.4) is solvable for $T = 2T_* = d(\Omega)$. As already noticed, this implies the exact controllability of (2.7)–(2.10) in the time interval $[0, d(\Omega)]$. Theorem 1 is proved.

The proof of Theorem 4 is analogous to the previous one. We set α_k' to be the $(m-1)$-dimensional column vector defined as

$$\alpha_k' = \operatorname{col}(\varkappa(\gamma)/\sqrt{\lambda_k})_{\gamma\in\Gamma_1}. \tag{4.6}$$

There naturally arise the families of vector functions in $L_2([0, T_1]; \mathbb{R}^{m-1})$ with $T_1 = d_1(\gamma_1, \Omega)$:

$$S_k^1(t) = \alpha_k' \sin(\sqrt{\lambda_k}\, t), \qquad C_k^1(t) = \alpha_k' \cos(\sqrt{\lambda_k}\, t), \qquad k \in \mathbb{N}.$$

One should perform the same procedure (using Propositions 2 and 3 instead of Theorem 7 and Proposition 1) as in the proof of Theorem 1, construct the family of vector exponentials

$$\{E^1_{\pm k}\}_{k\in\mathbb{N}}, \quad E^1_{\pm k}(t) = \alpha'_k e^{\pm i\sqrt{\lambda_k}t}, \quad t\in(0,2T_1),\ k\in\mathbb{N}, \tag{4.7}$$

and use the connection between controllability and vector exponentials ([2, Theorem III.3.3]).

In the proofs of Theorems 1 and 4 we have obtained important results which are of independent interest in function theory.

Proposition 4. *The family* $\{E_{\pm k}\}_{k\in\mathbb{N}}$ (*see* (4.5)) *constructed using the Dirichlet spectral data* (2.6) *is an* $\mathcal{L}$*-basis in* $L_2([0,d(\Omega)];\mathbb{C}^m)$.

Suppose that we pick an arbitrary boundary point of the graph (we keep the notation γ_1 for it); then we get

Proposition 5. *The family* $\{E^1_{\pm k}\}_{k\in\mathbb{N}}$ (*see* (4.7)) *constructed using the Dirichlet spectral data* (2.6), (4.6) *is an* $\mathcal{L}$*-basis in* $L_2([0,2T_1];\mathbb{C}^{m-1})$ *for* $T_1 = d_1(\gamma_1,\Omega)$.

It seems difficult to obtain these results without using the control-theoretic approach.

5. Proof of Theorem 2. Looking for the solution of (2.11)–(2.14) in the form (4.1) for a fixed initial state $a\in\mathcal{H}_{-1}$ with the expansion

$$a(x) = \sum_{k=1}^{\infty} a_k\phi_k(x), \tag{5.1}$$

we come to the following formulas for the coefficients:

$$c^f_k(t) = a_k e^{-\lambda_k t} + \sum_{\gamma\in\Gamma}\varkappa_k(\gamma)\int_0^t e^{-\lambda_k(t-s)} f(\gamma,s)\,ds.$$

Solving the control problem associated with (2.11)–(2.14) in the time interval $[0,\tau]$, we need the equation $c^f_k(\tau)=0$, $k\in\mathbb{N}$, to be satisfied. This leads to the following moment problem:

$$0 = \frac{a_k}{\sqrt{\lambda_k}}e^{-\lambda_k\tau} + \sum_{\gamma\in\Gamma}\frac{\varkappa_k(\gamma)}{\sqrt{\lambda_k}}\int_0^\tau e^{-\lambda_k(\tau-s)} f(\gamma,s)\,ds, \quad k\in\mathbb{N}. \tag{5.2}$$

Definition 5. The moment problem (5.2) is solvable in the time interval $[0,\tau]$ for some $\tau>0$ if and only if, for any $\{a_k/\sqrt{\lambda_k}\}_{k=1}^\infty\in l_2$, there exists $f\in\mathcal{F}^\tau_\Gamma$ such that equalities (5.2) hold.

Note that solvability of the moment problem (5.2) is equivalent to the null controllability of the dynamical system (2.11)–(2.14).

Definition 6. A family $\{\xi_k\}_{k=1}^\infty$ in a Hilbert space H is called *minimal* if no ξ_k belongs to the closure of the linear span of the remaining elements.

Another equivalent characterization of a minimal family $\{\xi_k\}_{k=1}^{\infty}$ in a Hilbert space H with the scalar product $\langle\cdot,\cdot\rangle$ is the existence of a bi-orthogonal family $\{\xi'_k\}_{k=1}^{\infty} \subset H$ such that

$$\langle \xi_k, \xi'_n\rangle = \delta_{k,n}, \qquad k, n \in \mathbb{N}.$$

It is well known that if a vector family is an $\mathcal{L}$-basis in H, it is minimal in H.

Proposition 4 states that the "hyperbolic" family $\{E_{\pm k}\}_{k\in\mathbb{N}}$ defined by (4.5) is an $\mathcal{L}$-basis in $L_2([0, d(\Omega)]; \mathbb{C}^m)$. Let us denote by $\{E'_{\pm k}\}_{k\in\mathbb{N}}$ the family bi-orthogonal to $\{E_{\pm k}\}_{k\in\mathbb{N}}$. There are connections between the "hyperbolic" family (4.5) and the "parabolic" one,

$$\{Q_k\}_{k=1}^{\infty}, \qquad Q_k(t) = \alpha_k e^{-\lambda_k t}, \qquad k \in \mathbb{N}, \tag{5.3}$$

first established by D. L. Russell [24]. We use his result in a slightly more general form, stated in Theorem II.5.20 of [2], from which it follows that the "parabolic" family $\{Q_k\}_{k=1}^{\infty}$ is minimal in $L_2([0,\tau]; \mathbb{C}^m)$ for every $\tau > 0$ and for the members of the "parabolic" bi-orthogonal family $\{Q'_k\}_{k=1}^{\infty}$ the following estimates hold:

$$\|Q'_k\|_{L_2([0,\tau];\mathbb{C}^m)} \le C(\tau)\|E'_k\|_{L_2([0,d(\Omega)];\mathbb{C}^m)} e^{\beta\sqrt{|\lambda_n|}}, \quad k \in \mathbb{N}, \tag{5.4}$$

with positive constants $C(\tau)$ and β.

To prove Theorem 2, one needs to show the solvability of the moment problem (5.2) which can be rewritten as

$$-\frac{a_k}{\sqrt{\lambda_k}}\, e^{-\lambda_k \tau} = \sum_{\gamma\in\Gamma} \frac{\varkappa_k(\gamma)}{\sqrt{\lambda_k}} \int_0^{\tau} e^{-\lambda_k t} f(\gamma, \tau - t)\, dt, \qquad k \in \mathbb{N},$$

or, briefly, as

$$-\frac{a_k}{\sqrt{\lambda_k}}\, e^{-\lambda_k \tau} = (Q_k, f^{\tau})_{\mathcal{F}_\Gamma^\tau}, \qquad k \in \mathbb{N}, \tag{5.5}$$

where $f^{\tau}(\gamma, t) = f(\gamma, \tau - t)$. One can check that a formal solution of (5.5) has the form

$$f^{\tau}(\gamma, t) = -\sum_{k=1}^{\infty} a_k e^{-\lambda_k \tau} Q'_k(t). \tag{5.6}$$

Estimates (5.4) imply that $f^{\tau}(\gamma, t)$ defined by (5.6) belongs to $\mathcal{F}_\Gamma^\tau$, and therefore the moment problem (5.2) is solvable. This completes the proof of Theorem 2.

The proof of Theorem 5 is similar. The corresponding family of exponentials that arise when reducing the control problem to the moment problem has the form

$$\{Q_k^1\}_{k=1}^{\infty}, \qquad Q_k^1(t) = \alpha'_k e^{-\lambda_k t}, \qquad k \in \mathbb{N}. \tag{5.7}$$

We conclude this section with results about families of vector exponentials that naturally appeared in the proofs.

PROPOSITION 6. *The family* $\{Q_k\}_{k=1}^{\infty}$ (*see* (5.3)) *constructed using the Dirichlet spectral data* (2.6) *is minimal in* $L_2([0,T];\mathbb{C}^m)$ *for any* $T > 0$.

If we pick an arbitrary boundary point of the graph (we keep the notation γ_1 for it), then the following statement is true.

PROPOSITION 7. *The family* $\{Q_k^1\}_{k=1}^{\infty}$ (*see* (5.7)) *constructed using the Dirichlet spectral data* (2.6), (4.6) *is minimal in* $L_2([0,T];\mathbb{C}^{m-1})$ *for any* $T > 0$.

We emphasize that an independent proof of Propositions 6, 7 without using the control-theoretic approach would be difficult.

6. Proof of Theorem 3. To prove Theorem 3 we use the scheme proposed in [21]. We restate the initial boundary value problems (2.7)–(2.10) and (2.15)–(2.18) in the operator form. Results concerning the dependence of solutions to systems dual to (2.7)–(2.10), (2.15)–(2.18) on the initial data allow us to use Theorem 3.1 of [21] that derives the exact controllability of the first-order system (2.15)–(2.18) in any time interval from the exact controllability of the second-order system (2.7)–(2.10) in some time interval.

Let us introduce the operator

$$A = -\frac{1}{\varrho}\frac{d^2}{dx^2} \quad \text{in } H_0 := \mathcal{H} = L_{2,\varrho}(\Omega).$$

If the density ϱ satisfies (2.1), the operator A is self-adjoint, positive definite and boundedly invertible with the domain

$$D(A) = \{a \in H_0 : a|_{e_i} \in H^2(e_i),\ a \text{ satisfies } (2.3),\ (2.4),\ a|_{\Gamma} = 0\}.$$

This operator defines the scale H_p, $p \in \mathbb{Z}$, of Hilbert spaces. For $p > 0$, $H_p = D(A^{p/2})$ with the norm $\|x\|_p = \|A^{p/2}x\|$, and H_{-p} is dual to H_p with respect to the scalar product in H_0. Another characterization of $H_{-p}(\Omega)$ is that it is the completion of H_0 with respect to the norm $\|x\|_{-p} = \|A^{-p/2}x\|$. We denote by A' the operator dual to A: it is the extension of A to H_{-2} with the domain H_0. Let $Y = \mathbb{R}^m$ and let $C : H_2 \to Y$ be defined by

$$Ca = \operatorname{col}\left(\partial a(\gamma)\right)_{\gamma\in\Gamma}.$$

Let $B : Y \to H_{-2}$ be the operator dual to C. In this notation we can rewrite the dynamical system (2.15)–(2.18) as

$$u_t(t) - iA'u(t) = Bf(t), \qquad u(0) = a \in H_0. \tag{6.1}$$

The dual observation system with output function y is defined by

$$u_t(t) - iAu(t) = 0, \qquad u(0) = u_0 \in H_0, \qquad y(t) = Cu(t). \tag{6.2}$$

The smoothness of the solution of (6.2) (see [4] for the case of one interval) guarantees that for the *observation* operator $\mathcal{C}_s : u_0 \mapsto y(t)$ the following estimate holds:

$$\|\mathcal{C}_s u_0\|_{\mathcal{F}^T} \le K_T \|u_0\|_{H_0}, \qquad u_0 \in H_2, \tag{6.3}$$

with $K_T > 0$.

System (2.7)–(2.10) can be rewritten as

$$u_{tt}(t) + A' u(t) = B f(t), \qquad u(0) = 0,\ u_t(0) = 0. \tag{6.4}$$

The dual observation system with the output function z has the form

$$u_{tt}(t) + A u(t) = 0, \qquad u(0) = u_0 \in H_1, \qquad u_t(0) = u_1 \in H_0, \qquad z(t) = C u(t).$$

The *observation* operator $\mathcal{C}_w : \{u_0, u_1\} \mapsto z(t)$ satisfies the estimate

$$\|\mathcal{C}_w \{u_0, u_1\}\|_{\mathcal{F}^T} \le K_T^1 (\|u_0\|_{H_1} + \|u_1\|_{H_0}) \tag{6.5}$$

with $K_T^1 > 0$ (see [18]). Now we can use Theorem 3.1 of [21], which says that if the dynamical system (6.4) is exactly controllable in some time interval (in our case it is controllable in the time interval $(0, d(\Omega))$), then the system (6.1) is exactly controllable in any time interval, provided the observation operators satisfy inequalities (6.3), (6.5). This completes the proof of Theorem 3.

Remark 1. The proof of Theorem 6 is similar: one should refer to Theorem 4 for the controllability of the corresponding second order dynamical system.

Looking for the solution of (2.15)–(2.18) in the form (4.1) for a fixed initial state $a \in \mathcal{H}_{-1}$ with the expansion (5.1), we come to the following formulas for the coefficients:

$$c_k^f(t) = a_k e^{i\lambda_k t} + \sum_{\gamma \in \Gamma} \varkappa_k(\gamma) \int_0^t e^{i\lambda_k (t-s)} f(\gamma, s)\, ds.$$

Solving the control problem associated with (2.15)–(2.18) in the time interval $[0, \tau]$, we obtain the following moment problem:

$$0 = \frac{a_k}{\sqrt{\lambda_k}} e^{i\lambda_k \tau} + \sum_{\gamma \in \Gamma} \frac{\varkappa_k(\gamma)}{\sqrt{\lambda_k}} \int_0^\tau e^{i\lambda_k (\tau - s)} f(\gamma, s)\, ds, \qquad k \in \mathbb{N}. \tag{6.6}$$

Theorem 3 implies that the moment problem (6.6) is solvable for any $\tau > 0$. Using Theorem III.3.3 of [2] we deduce the following result about the family of vector-valued exponentials that appeared in the moment problem (6.6).

Corollary 1. *The family*

$$\{D_k\}_{k=1}^\infty, \qquad D_k(t) = \alpha_k e^{i\lambda_k t}, \qquad k \in \mathbb{N},$$

constructed using the Dirichlet spectral data (2.6) *is an $\mathcal{L}$-basis in the space $L_2([0,\tau];\mathbb{C}^m)$ for any $\tau > 0$.*

Picking an arbitrary boundary point of the graph (we keep the notation γ_1 for it) and using Theorem 6, we get

COROLLARY 2. *The family*

$$\{D_k^1\}_{k=1}^{\infty}, \qquad D_k^1(t) = \alpha_k' e^{i\lambda_k t},$$

constructed using the Dirichlet spectral data (2.6), (4.6) *is an $\mathcal{L}$-basis in $L_2([0,\tau];\mathbb{C}^{m-1})$ for any $\tau > 0$.*

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