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A STABILITY RESULT FOR A CLASS OF NONLINEAR INTEGRODIFFERENTIAL EQUATIONS WITH L^1 KERNELS

Abstract. We study second order nonlinear integro-differential equations in Hilbert spaces with weakly singular convolution kernels obtaining energy estimates for the solutions, uniform in t. Then we show that the solutions decay exponentially at ∞ in the energy norm. Finally, we apply these results to a problem in viscoelasticity.

1. Introduction. In this paper we treat maximal regularity and asymptotic behaviour at ∞ for the solutions of the abstract nonlinear integro-differential equation

$$\ddot{u}(t) + F(u(t))\dot{u}(t) + Au(t) - \int_0^t \beta(t-s)Au(s)\,ds = f(t), \qquad t \geq 0, \tag{1.1}$$

where A is a positive operator on a Hilbert space X with domain $D(A)$, and F is a functional defined on $D(A^\theta)$, $0 < \theta \leq 1/2$. Our interest is mainly motivated by the fact that the above equation may be regarded as a model problem for some elastic systems with memory (see [6, 8, 9, 17, 18, 24]).

The main feature of our approach is that, unlike most of the literature on this subject, the convolution kernel β is not assumed to be absolutely continuous but just integrable. Instead of higher regularity, we shall suppose that $k(t) := \int_t^\infty \beta(s)\,ds$ is a kernel of positive type satisfying $k(0) < 1$. It is noteworthy that such an abstract assumption implies a commonly accepted thermodynamical restriction for the concrete models described by (1.1) (see, e.g., [12, 11]). This explains why, as noted in [18, 23, 13], discontinuous kernels are relevant for applications. Some typical examples covered by our

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theory are the following:

$$\beta(t) = \frac{e^{-at}\, t^{-b}}{\Gamma(1-b)}, \qquad a > 1,\ 0 \le b < 1,$$
$$\beta(t) = \frac{e^{-at}\, t^{-1/2}\, \cos(ct)}{\Gamma(1/2)}, \quad a \ge 1,\ 0 \le c \le a,$$

where Γ is the Euler gamma function (see Example 2.8).

In order to discuss well-posedness, let us integrate equation (1.1) with $F \equiv 0 \equiv f$ twice to obtain an integral equation of the form

$$u(t) + \int_0^t a(t-s)Au(s)\, ds = u(0) + \dot{u}(0)t. \tag{1.2}$$

Such equations have been extensively studied by Prüss in the classical monograph [21], where the existence of the resolvent $S(t)$ of (1.2) is established together with some regularity properties of $S(t)$. In Section 3 of this paper, we apply this well-posedness result to the linear version of (1.1), that is,

$$\ddot{u}(t) + Au(t) - \int_0^t \beta(t-s)Au(s)\, ds = f(t), \qquad t \ge 0. \tag{1.3}$$

After recalling the notions of strong and mild solutions of (1.3) with initial conditions

$$u(0) = u_0 \in D(A^{1/2}), \qquad \dot{u}(0) = u_1 \in X,$$

and the well-known representation formula for mild solutions

$$u(t) = S(t)u_0 + \int_0^t S(\tau)u_1\, d\tau + \int_0^t 1 * S(t-\tau)f(\tau)\, d\tau,$$

where $S(t)$ is the resolvent obtained using the theory of [21], we need to know that any mild solution of (1.3) with smooth data is a strong solution. This follows from further estimates for $S(t)$ that are, here, fully justified (see Propositions 3.4 and 3.5) because their derivation from the results of [21] is not straightforward. On the other hand, with such a refined linear theory at our disposal, the existence and uniqueness of solutions to equation (1.1) can be obtained, in a rather standard way, assuming F to be locally Lipschtiz continuous on bounded subsets of $D(A^\theta)$ and $F(x) \ge 0$ for all $x \in D(A^\theta)$. Therefore, we have excluded the proofs of this last part from the main body of the paper providing them in the appendix, for the reader's convenience.

Once well-posedness has been established, in Section 5 we turn to the analysis of the asymptotic behaviour of the solutions of (1.1) under stronger assumptions on β and F, namely,

$$(1.4)\qquad \begin{aligned} &\int_0^\infty e^{\alpha_0 t}|\beta(t)|\,dt<\infty, \qquad t\mapsto \int_t^\infty e^{\alpha_0 s}\beta(s)\,ds \quad \text{of positive type,}\\ &\int_0^\infty |\beta(s)|\,ds<1, \qquad F(x)\ge c_0 \qquad \forall x\in D(A^\theta) \end{aligned}$$

for some constants $\alpha_0, c_0 > 0$. Our method, in the case of $f \equiv 0$, can be easily explained: multiplying u by an exponential function $e^{\alpha t}$ for some $\alpha > 0$, we see that $u_\alpha(t) = e^{\alpha t}u(t)$ satisfies

$$\ddot{u}_\alpha(t)+(F(u(t))-2\alpha)\dot{u}_\alpha(t)+(\alpha^2-F(u(t))\alpha)u_\alpha(t)+Au_\alpha(t)-\beta_\alpha * Au_\alpha(t)=0.$$

So, uniform estimates for u_α, in the energy norm, can be derived by means of suitable multipliers. For related results on exponential decay the reader is referred to [1, 16, 22], where the case of smooth convolution kernels is considered, and to [14, 19], where the decay of the semigroup associated with (1.1) is obtained.

Finally, Section 6 is devoted to the analysis of a partial integro-differential equation arising in the theory of viscoelasticity, in the case of materials for which memory effects cannot be neglected. Applying our abstract results, we show that the exponential decay of the energy of solutions, obtained in [6] for smooth convolution kernels, also holds for integrable kernels satisfying condition (1.4).

2. Preliminaries. Let X be a real Hilbert space with scalar product $\langle\cdot\,,\cdot\rangle$ and norm $\|\cdot\|$. For any $T\in(0,\infty]$ and $p\in[1,\infty]$ we denote by $L^p(0,T;X)$ the usual space of measurable functions $v:(0,T)\to X$ such that

$$\begin{aligned} \|v\|_{p,T}^p &:= \int_0^T \|v(t)\|^p\,dt<\infty, \qquad 1\le p<\infty,\\ \|v\|_{\infty,T} &:= \operatorname*{ess\,sup}_{0\le t\le T}\|v(t)\|<\infty, \end{aligned}$$

respectively. We shall use the shorter notation $\|v\|_p$ for $\|v\|_{p,\infty}$, $1\le p\le\infty$. We denote by $L^1_{\mathrm{loc}}(0,\infty;X)$ the space of functions belonging to $L^1(0,T;X)$ for any $T\in(0,\infty)$. In the case of $X=\mathbb{R}$, we will use the abbreviations $L^p(0,T)$ and $L^1_{\mathrm{loc}}(0,\infty)$.

$C^k([0,T];X)$, $k=0,1,2$, stands for the space of continuous functions from $[0,T]$ to X having continuous derivatives up to order k in $[0,T]$. In particular, we write $C([0,T];X)$ for $C^0([0,T];X)$.

In this paper, $A:D(A)\subset X\to X$ denotes a self-adjoint positive linear operator on X with dense domain $D(A)$. In the following, we denote by A^θ, $0<\theta\le 1/2$, the fractional powers of A (see e.g. [20]). It is well-known that

$D(A^{1/2}) \hookrightarrow D(A^{\theta})$, that is, there exists a constant $c_\theta > 0$ such that

$$\|A^{\theta}x\| \le c_\theta \|A^{1/2}x\| \quad \forall x \in D(A^{1/2}). \tag{2.1}$$

For any $\psi \in L^1_{\mathrm{loc}}(0,\infty)$ and $v \in L^1_{\mathrm{loc}}(0,\infty;X)$ we define

$$\psi * v\,(t) = \int_0^t \psi(t-s)v(s)\,ds, \qquad t \ge 0.$$

Throughout the paper, for $v \in L^1_{\mathrm{loc}}(0,\infty;X)$ we denote by $\widehat{v}(\lambda)$ the Laplace transform of v, that is,

$$\widehat{v}(\lambda) = \int_0^\infty e^{-\lambda t} v(t)\,dt.$$

We recall that a function $k \in L^1_{\mathrm{loc}}(0,\infty)$ is a kernel of *positive type* if

$$\int_0^T \langle k * v(t), v(t)\rangle\,dt \ge 0 \quad \text{for any } T > 0,\ v \in L^2(0,T;X). \tag{2.2}$$

In case $k \in L^\infty(0,\infty)$, k is of positive type if and only if

$$\operatorname{Re}\widehat{k}(\lambda) \ge 0 \quad \text{for any } \lambda \in \mathbb{C} \text{ with } \operatorname{Re}\lambda > 0 \tag{2.3}$$

(see, e.g., [21, p. 38]). For $k \in L^\infty(0,\infty) \cap L^1(0,\infty)$, using the Poisson integral representation formula for harmonic functions in a half-plane (see, e.g., [10, p. 37]), one can prove that k is of positive type iff

$$\operatorname{Re}\widehat{k}(i\omega) \ge 0 \quad \text{for any } \omega \ge 0. \tag{2.4}$$

Classical results for integral equations (see, e.g., [15, Theorem 2.3.5]) ensure that, for any kernel $\beta \in L^1_{\mathrm{loc}}(0,\infty)$ and any $g \in L^1_{\mathrm{loc}}(0,\infty;X)$, the problem

$$\varphi(t) - \beta * \varphi(t) = g(t), \qquad t \ge 0, \tag{2.5}$$

admits a unique solution $\varphi \in L^1_{\mathrm{loc}}(0,\infty;X)$. In particular, there is a unique solution $\varrho \in L^1_{\mathrm{loc}}(0,\infty)$ of

$$\varrho(t) - \beta * \varrho(t) = \beta(t), \qquad t \ge 0. \tag{2.6}$$

Such a solution is called the *resolvent kernel* of β. Furthermore, the solution φ of (2.5) is given by the variation of constants formula

$$\varphi(t) = g(t) + \varrho * g(t), \qquad t \ge 0, \tag{2.7}$$

where ϱ is the resolvent kernel of β.

Now, we recall the classical Paley–Wiener theorem (see, e.g., [15, Theorem 2.4.5]), which gives a necessary and sufficient condition for the resolvent kernel of $\beta \in L^1(0,\infty)$ to belong to $L^1(0,\infty)$.

THEOREM 2.1. *Let $\beta \in L^1(0,\infty)$. Then the resolvent kernel of β belongs to $L^1(0,\infty)$ if and only if $\widehat{\beta}(\lambda) \ne 1$ for all $\lambda \in \mathbb{C}$ with* $\operatorname{Re}\lambda \ge 0$.

LEMMA 2.2. *Let $\beta \in L^1(0,\infty)$ be such that $\int_0^\infty \beta(s)\,ds < 1$ and suppose that the kernel $k(t) = \int_t^\infty \beta(s)\,ds$ is of positive type. Then $\widehat{\beta}(\lambda) \neq 1$ for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > 0$.*

Proof. Since $k(t) = \int_0^\infty \beta(s)\,ds - \int_0^t \beta(s)\,ds$, we have

$$\widehat{k}(\lambda) = \frac{1}{\lambda}\int_0^\infty \beta(s)\,ds - \frac{\widehat{\beta}(\lambda)}{\lambda}. \tag{2.8}$$

Now, for $\operatorname{Re}\lambda > 0$,

$$\frac{1}{\lambda}\int_0^\infty \beta(s)\,ds - \frac{\widehat{\beta}(\lambda)}{\lambda} \neq \frac{1}{\lambda}\Big(\int_0^\infty \beta(s)\,ds - 1\Big),$$

because the real part of the left-hand side is nonnegative on account of (2.3), whereas the real part of the right-hand side is negative. Therefore, $\widehat{\beta}(\lambda) \neq 1$ for $\operatorname{Re}\lambda > 0$. ∎

The following corollary of Lemma 2.2 and Theorem 2.1 provides uniform estimates for solutions of integral equations.

COROLLARY 2.3. *Let $\beta \in L^1(0,\infty)$ be such that $\int_0^\infty \beta(s)\,ds < 1$, $\widehat{\beta}(\lambda) \neq 1$ for all $\lambda \in \mathbb{C} - \{0\}$ with $\operatorname{Re}\lambda = 0$, and suppose that the kernel $k(t) = \int_t^\infty \beta(s)\,ds$ is of positive type. Then*

(a) *the resolvent kernel ϱ of β belongs to $L^1(0,\infty)$;*
(b) *for any $g \in L^\infty(0,\infty;X)$, the solution φ of (2.5) is in $L^\infty(0,\infty;X)$ and*

$$\|\varphi\|_\infty \le (1 + \|\varrho\|_1)\|g\|_\infty. \tag{2.9}$$

The following results are useful to study exponential decay.

PROPOSITION 2.4. *Let $\beta \in L^1(0,\infty)$ be a function such that $k(t) = \int_t^\infty \beta(s)\,ds$ is of positive type. Then*

$$\int_0^\infty \sin(\omega t)\beta(t)\,dt \ge 0 \qquad \textit{for any } \omega > 0. \tag{2.10}$$

Proof. In view of (2.8), we note that for any $\tau > 0$ and $\omega \in \mathbb{R}$ we have

$$\begin{aligned}\operatorname{Re}\widehat{k}(\tau + i\omega) &= \operatorname{Re}\frac{1}{\tau + i\omega}\Big[\int_0^\infty \beta(s)\,ds - \widehat{\beta}(\tau + i\omega)\Big] \\ &= \frac{\tau}{\tau^2+\omega^2}\Big[\int_0^\infty \beta(s)\,ds - \operatorname{Re}\widehat{\beta}(\tau + i\omega)\Big] + \frac{\omega}{\tau^2+\omega^2}\int_0^\infty e^{-\tau t}\sin(\omega t)\beta(t)\,dt.\end{aligned} \tag{2.11}$$

Passing to the limit in the previous equation as $\tau \to 0^+$ and keeping in mind that k is of positive type (see (2.3)), we have

$$0 \le \lim_{\tau\to 0^+} \operatorname{Re}\widehat{k}(\tau + i\omega) = \frac{1}{\omega}\int_0^\infty \sin(\omega t)\beta(t)\,dt \qquad \text{for any } \omega \in \mathbb{R},\ \omega \ne 0,$$

that is, (2.10) holds. ∎

The converse of the previous result is true in a special case.

Proposition 2.5. *Let $\beta \in L^1(0,\infty)$ be a function such that $t\beta(t) \in L^1(0,\infty)$ and $k(t) = \int_t^\infty \beta(s)\,ds$. Then*

(a) $k \in L^1(0,\infty)$;
(b) *the kernel k is of positive type if and only if* $\operatorname{Re}\widehat{k}(i\omega) \ge 0$ *for any* $\omega > 0$;
(c) *if β satisfies* $\int_0^\infty \sin(\omega t)\beta(t)\,dt \ge 0$ *for any* $\omega > 0$, *then k is of positive type.*

Proof. Point (a) easily follows from Fubini's theorem. As for (b), thanks to (2.4), it suffices to check that if $\operatorname{Re}\widehat{k}(i\omega) \ge 0$ for any $\omega > 0$, then $\operatorname{Re}\widehat{k}(0) \ge 0$. Indeed,

$$\widehat{k}(0) = \int_0^\infty k(t)\,dt = \lim_{\omega\to 0^+} \operatorname{Re}\widehat{k}(i\omega) \ge 0.$$

Finally, since $k \in L^1(0,\infty)$ we can take $\tau = 0$ in (2.11) to obtain, for any $\omega > 0$,

$$\operatorname{Re}\widehat{k}(i\omega) = \frac{1}{\omega}\int_0^\infty \sin(\omega t)\beta(t)\,dt \ge 0.$$

So, (c) holds owing to (b) and the proof is complete. ∎

We now recall a known result (see [3, Lemma 3.4]) that will be used next.

Lemma 2.6. *If $\beta \in L^1(0,\infty)$ is a function satisfying* (2.10), *then, for any $\sigma > 0$, the perturbed function $e^{-\sigma t}\beta(t)$ satisfies* (2.10) *as well.*

Proposition 2.7. *Let $\beta \in L^1(0,\infty)$ be a function such that $k(t) = \int_t^\infty \beta(s)\,ds$ is of positive type. Then, for any $\sigma > 0$, $t \mapsto \int_t^\infty e^{-\sigma s}\beta(s)\,ds$ is a kernel of positive type as well.*

Proof. We observe that, for any $\sigma > 0$, the function $t \mapsto te^{-\sigma t}\beta(t)$ is in $L^1(0,\infty)$. Therefore, we can apply Proposition 2.5(c) to $e^{-\sigma t}\beta(t)$ and invoke Lemma 2.6 to obtain the conclusion. ∎

EXAMPLE 2.8. 1. Let $\Gamma(s) = \int_0^\infty x^{s-1}e^{-x}\,dx$ ($s > 0$) be the Euler gamma function. Then

$$\beta_{a,b}(t) = \frac{e^{-at}\,t^{-b}}{\Gamma(1-b)}, \qquad t > 0,\ a > 0,\ 0 \le b < 1,$$

satisfies the condition

$$\widetilde{\beta}_{a,b}(\omega) := \int_0^\infty \sin(\omega t)\beta_{a,b}(t)\,dt \ge 0 \qquad \text{for any } \omega > 0.$$

Indeed, since $\widehat{\beta}_{a,b}(\lambda) = 1/(a+\lambda)^{1-b}$ for $\operatorname{Re}\lambda \ge 0$, we have

$$\begin{aligned}(2.12)\quad \widetilde{\beta}_{a,b}(\omega) &= -\operatorname{Im}\widehat{\beta}_{a,b}(i\omega)\\ &= (a^2+\omega^2)^{(b-1)/2}\sin\left((1-b)\arctan\left(\frac{\omega}{a}\right)\right) > 0 \qquad \forall \omega > 0.\end{aligned}$$

By Proposition 2.5(c) the kernel $t \mapsto \int_t^\infty \beta_{a,b}(s)\,ds$ is of positive type. Moreover,

$$\int_0^\infty \beta_{a,b}(t)\,dt = \frac{1}{a^{1-b}} < 1 \qquad \text{for } a > 1.$$

2. To give an example of a function with variable sign, let us consider

$$\gamma_{a,c}(t) = \frac{e^{-at}t^{-1/2}\cos(ct)}{\Gamma(1/2)}, \qquad t > 0,\ a > 0,\ 0 \le c \le a.$$

In order to verify that

$$\widetilde{\gamma}_{a,c}(\omega) := \frac{1}{\Gamma(1/2)}\int_0^\infty e^{-at}t^{-1/2}\cos(ct)\sin(\omega t)\,dt \ge 0 \qquad \text{for any } \omega > 0,$$

let us rewrite $\widetilde{\gamma}_{a,c}(\omega)$, using the elementary identity $\cos(ct)\sin(\omega t) = \frac{1}{2}[\sin((\omega+c)t) + \sin((\omega-c)t)]$ and (2.12). We have

$$\begin{aligned}\widetilde{\gamma}_{a,c}(\omega) &= \frac{1}{2\Gamma(1/2)}\int_0^\infty e^{-at}t^{-1/2}\sin((\omega+c)t)\,dt\\ &\quad + \frac{1}{2\Gamma(1/2)}\int_0^\infty e^{-at}t^{-1/2}\sin((\omega-c)t)\,dt\\ &= \frac{1}{2}\,\widetilde{\beta}_{a,1/2}(\omega+c) + \frac{1}{2}\,\widetilde{\beta}_{a,1/2}(\omega-c))\\ &= \frac{1}{2}\,(a^2+(\omega+c)^2)^{-1/4}\sin\left(\frac{1}{2}\arctan\left(\frac{\omega+c}{a}\right)\right)\\ &\quad + \frac{1}{2}\,(a^2+(\omega-c)^2)^{-1/4}\sin\left(\frac{1}{2}\arctan\left(\frac{\omega-c}{a}\right)\right).\end{aligned}$$

Since

$$\sin\left(\frac{\arctan x}{2}\right) = \operatorname{sign}(x)\sqrt{\frac{1-\cos(\arctan x)}{2}}$$
$$= \frac{\operatorname{sign}(x)}{\sqrt{2}}\sqrt{1-\frac{1}{\sqrt{1+x^2}}}, \qquad x \in \mathbb{R},$$

we observe that

$$\widetilde{\gamma}_{a,c}(\omega) = \frac{1}{2\sqrt{2a}}\left(\left(1+\frac{(\omega+c)^2}{a^2}\right)^{-1/2} - \left(1+\frac{(\omega+c)^2}{a^2}\right)^{-1}\right)^{1/2}$$
$$+ \frac{1}{2\sqrt{2a}}\operatorname{sign}(\omega-c)\left(\left(1+\frac{(\omega-c)^2}{a^2}\right)^{-1/2} - \left(1+\frac{(\omega-c)^2}{a^2}\right)^{-1}\right)^{1/2}.$$

It is clear that, if $\omega \geq c$, then $\widetilde{\gamma}_{a,c}(\omega) > 0$. So, suppose $\omega < c$. We want to check that

$$\left(1+\frac{(\omega+c)^2}{a^2}\right)^{-1/2} - \left(1+\frac{(\omega+c)^2}{a^2}\right)^{-1}$$
$$> \left(1+\frac{(\omega-c)^2}{a^2}\right)^{-1/2} - \left(1+\frac{(\omega-c)^2}{a^2}\right)^{-1}.$$

Set $\overline{\omega} = \omega/a$ and $\overline{c} = c/a$. The previous inequality is equivalent to

$$\frac{1}{1+(\overline{\omega}-\overline{c})^2} - \frac{1}{1+(\overline{\omega}+\overline{c})^2} > \frac{1}{\sqrt{1+(\overline{\omega}-\overline{c})^2}} - \frac{1}{\sqrt{1+(\overline{\omega}+\overline{c})^2}},$$

or

$$\frac{4\overline{\omega}\,\overline{c}}{\sqrt{1+(\overline{\omega}-\overline{c})^2}\sqrt{1+(\overline{\omega}+\overline{c})^2}} > \frac{4\overline{\omega}\,\overline{c}}{\sqrt{1+(\overline{\omega}+\overline{c})^2}+\sqrt{1+(\overline{\omega}-\overline{c})^2}}.$$

Hence, after simple computations we arrive at the equivalent inequality

$$(\overline{\omega}^2-\overline{c}^2)^2 < 1+2\sqrt{1+(\overline{\omega}+\overline{c})^2}\sqrt{1+(\overline{\omega}-\overline{c})^2},$$

which holds true since, in view of the fact that $\overline{c} = c/a \leq 1$, we have

$$|\overline{\omega}^2-\overline{c}^2| = \overline{c}^2-\overline{\omega}^2 \leq 1.$$

So, $\widetilde{\gamma}_{a,c}(\omega) > 0$ for all $\omega > 0$. Again by Proposition 2.5(c), $t \mapsto \int_t^\infty \gamma_{a,c}(s)\,ds$ is of positive type. Moreover,

$$\int_0^\infty \gamma_{a,c}(t)\,dt < \frac{1}{\Gamma(1/2)}\int_0^\infty e^{-t}\,t^{-1/2}\,dt = 1 \qquad \text{for } a \geq 1.$$

3. The linear problem: well-posedness

3.1. *Existence and regularity of the resolvent.* In the following, A is a self-adjoint linear operator on X with dense domain $D(A)$ and $\beta \in L^1_{\mathrm{loc}}(0,\infty)$.

DEFINITION 3.1. A family $\{S(t)\}_{t\geq 0}$ of bounded linear operators in X is called a *resolvent* for the equation

$$\ddot{u}(t) + Au(t) - \int_0^t \beta(t-s)Au(s)\,ds = 0 \tag{3.1}$$

if the following conditions are satisfied:

(S1) $S(0) = I$ and $S(t)$ is strongly continuous on $[0,\infty)$, that is, for all $x \in X$, $S(\cdot)x$ is continuous;

(S2) $S(t)$ commutes with A, which means that $S(t)D(A) \subset D(A)$ and

$$AS(t)x = S(t)Ax, \qquad x \in D(A),\ t \geq 0;$$

(S3) for any $x \in D(A)$, $S(\cdot)x$ is twice continuously differentiable in X on $[0,\infty)$ and $\dot{S}(0)x = 0$;

(S4) for any $x \in D(A)$ and any $t \geq 0$,

$$\ddot{S}(t)x + AS(t)x - \int_0^t \beta(t-\tau)AS(\tau)x\,d\tau = 0. \tag{3.2}$$

REMARK 3.2. We note that $S * f \in C([0,T];X)$ for $f \in L^1(0,T;X)$, thanks to the uniform boundedness of $S(t)$ on compact intervals (this is, in turn, a consequence of (S1) by the principle of uniform boundedness).

The following result ensures the existence and uniqueness of the resolvent.

THEOREM 3.3. *Let the operator A be positive and $\beta \in L^1(0,\infty)$ be such that $\int_0^\infty \beta(t)\,dt < 1$. Suppose that the kernel $k(t) = \int_t^\infty \beta(s)\,ds$ is of positive type. Then there exists a unique resolvent for* (3.1). *In addition,*

$$\dot{S}(t)x + 1 * AS(t)x - 1 * \beta * AS(t)x = 0, \qquad \forall x \in D(A). \tag{3.3}$$

Proof. First, we observe that equation (3.1) is formally equivalent to a suitable integral equation. In fact, if we integrate equation (3.1) twice, then we obtain the integral equation

$$u(t) + \int_0^t a(t-s)Au(s)\,ds = u(0) + \dot{u}(0)t, \tag{3.4}$$

where

$$a(t) = t - \int_0^t (t-s)\beta(s)\,ds = t - 1 * 1 * \beta(t), \quad t \geq 0.$$

On the other hand, differentiating equation (3.4) twice we recover (3.1). Therefore, to obtain the conclusion, it suffices to show that there exists a unique resolvent $S(t)$ for (3.4) and that $S(t)$ is the resolvent for (3.1). Since uniqueness follows from a well-known result for integral equations (see [21, Corollary 1.1]), let us prove the existence of the resolvent for (3.4). For this, we note that the kernel $\dot{a}(t) = 1 - 1 * \beta(t)$ is of positive type. Indeed,

$$1 - \int_0^t \beta(s)\,ds = 1 - \int_0^\infty \beta(s)\,ds + \int_t^\infty \beta(s)\,ds = 1 - \int_0^\infty \beta(s)\,ds + k(t)$$

where $1 - \int_0^\infty \beta(s)\,ds > 0$ and k is of positive type. Since A is a self-adjoint positive operator on X, we can apply Corollary 1.2 of [21]: the resolvent $S(t)$ for (3.4) exists, $S(\cdot)x$ is differentiable for any $x \in D(A)$, and satisfies $\dot{S}(t)x + 1 * AS(t)x - 1 * \beta * AS(t)x = 0$. The remaining properties required by Definition 3.1 can be easily derived from the above equation. ■

3.2. *Uniform estimates.* We establish regularity properties and uniform estimates for the resolvent of (3.1), under an extra assumption on the operator A.

PROPOSITION 3.4. *Suppose that*

$$\langle Ax, x\rangle \ge M\|x\|^2 \qquad \forall x \in D(A) \tag{3.5}$$

for some $M > 0$ and let $\beta \in L^1(0,\infty)$ be such that

$$\int_0^\infty \beta(t)\,dt < 1. \tag{3.6}$$

If the kernel $k(t) = \int_t^\infty \beta(s)\,ds$ is of positive type, then:

(i) *For any $x \in X$ and any $t > 0$, $1 * S(t)x \in D(A^{1/2})$ and*

$$\|S(t)x\|^2 + \Big(1 - \int_0^\infty \beta(t)\,dt\Big)\Big\|A^{1/2}\int_0^t S(\tau)x\,d\tau\Big\|^2 \le \|x\|^2. \tag{3.7}$$

In particular, $A^{1/2}(1 * S)(\cdot)$ is strongly continuous in X.

(ii) *For any $x \in D(A)$,*

$$\|\ddot{S}(t)x\| \le (1 + \|\beta\|_1)\|Ax\|. \tag{3.8}$$

(iii) *For any $x \in D(A^{1/2})$ and any $t > 0$, $1 * S(t)x \in D(A)$ and*

$$\Big\|A\int_0^t S(\tau)x\,d\tau\Big\| \le \Big(1 - \int_0^\infty \beta(t)\,dt\Big)^{-1/2}\|A^{1/2}x\|. \tag{3.9}$$

So, $A(1 * S)(\cdot)x$ is continuous on $[0,\infty)$.

(iv) *For any $x \in D(A^{1/2})$, $S(\cdot)x$ is continuously differentiable on $[0,\infty)$ and*

$$\dot{S}(t)x = -A(1 * S)(t)x + \beta * A(1 * S)(t)x. \tag{3.10}$$

Moreover,

$$\|\dot S(t)x\| \le (1+\|\beta\|_1)\Big(1-\int_0^\infty \beta(t)\,dt\Big)^{-1/2}\|A^{1/2}x\| \qquad \forall t\ge 0. \tag{3.11}$$

(v) *For any $x\in D(A)$ and any $t>0$, $\dot S(t)x\in D(A^{1/2})$ and $A^{1/2}\dot S(\cdot)x$ is continuous on $[0,\infty)$.*

Proof. (i) For fixed $x\in D(A)$, let us recast equation (3.3) in the form

$$\dot S(t)x + \Big(1-\int_0^\infty \beta(t)\,dt\Big) 1 * AS(t)x = -k * AS(t)x. \tag{3.12}$$

Taking the scalar product of both sides of (3.12) with $S(t)x$, we obtain

$$\frac12\frac{d}{dt}\|S(t)x\|^2 + \frac12\Big(1-\int_0^\infty \beta(t)\,dt\Big)\frac{d}{dt}\|1 * A^{1/2}S(t)x\|^2 = -\langle k * A^{1/2}S(t)x, A^{1/2}S(t)x\rangle.$$

Integrating this equality from 0 to t, we have

$$\frac12\,\|S(t)x\|^2 + \frac12\Big(1-\int_0^\infty \beta(t)\,dt\Big)\|1 * A^{1/2}S(t)x\|^2 \le \frac12\,\|x\|^2,$$

since k is of positive type. Since $D(A)$ is dense in X, the above inequality holds for any $x\in X$.

(ii) Estimate (3.8) follows from (3.2) and (i).

(iii) Since

$$A\int_0^t S(\tau)x\,d\tau = A^{1/2}\int_0^t S(\tau)A^{1/2}x\,d\tau \qquad \forall x\in D(A^{1/2}),$$

estimate (3.9) is a consequence of (i).

(iv) The conclusion follows from (3.3), (iii) and the density of $D(A)$ in $D(A^{1/2})$.

(v) (3.10) and (i) imply that $\dot S(t)x\in D(A^{1/2})$ for any $x\in D(A)$ and

$$A^{1/2}\dot S(t)x = -A^{1/2}(1*S)(t)Ax + \beta * A^{1/2}(1*S)(t)Ax.$$

Consequently, $t\mapsto A^{1/2}\dot S(t)x$ is continuous and the proof is complete. ∎

Finally, we show another regularity property of the resolvent operator.

PROPOSITION 3.5. *Assume*

(a) $\langle Ax, x\rangle \ge M\|x\|^2$ *for all* $x\in D(A)$ *and some* $M>0$;
(b) $\beta\in L^1(0,\infty)$ *satisfies* $\int_0^\infty \beta(t)\,dt<1$;
(c) *the kernel* $k(t)=\int_t^\infty \beta(s)\,ds$ *is of positive type.*

Then $1*1*S(t)x \in D(A)$ *for any* $x \in X$ *and any* $t > 0$. *Moreover*, *for any* $T > 0$,

$$\Big\| A \int_0^t (t-\tau)S(\tau)x\, d\tau \Big\| \le C_T \,\|x\| \quad \forall x \in X,\ \forall t \in [0,T] \tag{3.13}$$

for some constant $C_T > 0$ *increasing in* T. *So*, $A(1*1*S)(\cdot)$ *is strongly continuous in* X *and for any* $x \in X$ *and any* $t > 0$,

$$S(t)x + A(1*1*S)(t)x - \beta * A(1*1*S)(t)x = x. \tag{3.14}$$

If, *in addition*, $\widehat{\beta}(\lambda) \neq 1$ *for all* $\lambda \in \mathbb{C} - \{0\}$ *with* $\operatorname{Re}\lambda = 0$, *then* (3.13) *holds for a constant* C *independent of* T.

Proof. Let $x \in D(A)$. Integrating the resolvent equation (3.2) twice yields

$$Av(t) - \beta * Av(t) = -S(t)x + x, \tag{3.15}$$

where $v(t) := 1*1*S(t)x$. Denoting by $w(t)$ the right-hand side of the above equation, we see that $w \in L^\infty(0,\infty;X)$ owing to (3.7) and

$$\|w(t)\| \le 2\|x\| \quad \forall t \ge 0.$$

Let $\varrho \in L^1_{\mathrm{loc}}(0,\infty)$ be the resolvent kernel of β. Then $Av = w + \varrho * w$. So,

$$\|Av\|_{\infty,T} \le (1 + \|\varrho\|_{1,T})\|w\|_\infty \le 2(1 + \|\varrho\|_{1,T})\|x\|.$$

Therefore, (3.13) holds for $x \in D(A)$. The general case $x \in X$ follows by an approximation argument, and so (3.14) follows from (3.15).

Finally, under the extra assumption that $\widehat{\beta}(\lambda) \neq 1$ for all $\lambda \in \mathbb{C} - \{0\}$ with $\operatorname{Re}\lambda = 0$, Corollary 2.3 yields $\varrho \in L^1(0,\infty)$. This allows one to turn (3.13) into a global estimate on $[0,\infty)$. ∎

3.3. *Mild and strong solutions.* We now turn to the analysis of the equation

$$\ddot{u}(t) + Au(t) - \int_0^t \beta(t-s)Au(s)\, ds = f(t), \quad t \in [0,T], \tag{3.16}$$

where $f \in L^1(0,T;X)$ and $T > 0$ is given. To begin with, we recall two notions of solution.

Definition 3.6. We say that u is a *strong solution* of (3.16) if

$$u \in C^2([0,T];X) \cap C([0,T];D(A))$$

and u satisfies (3.16) for every $t \in [0,T]$.

Let u_0, $u_1 \in X$. The *mild solution* of (3.16) with initial conditions

$$u(0) = u_0, \quad \dot{u}(0) = u_1, \tag{3.17}$$

is the function $u \in C([0,T];X)$ defined, for every $t \in [0,T]$, by the formula

$$u(t) = S(t)u_0 + \int_0^t S(\tau)u_1\,d\tau + \int_0^t 1 * S(t-\tau)f(\tau)\,d\tau \tag{3.18}$$

(see Remark 3.2).

We now describe the connection between the mild and strong solutions.

Proposition 3.7. *A strong solution u of equation* (3.16) *is also the mild solution of* (3.16) *with initial conditions $u(0), \dot u(0)$. Consequently*, *the Cauchy problem* (3.16)–(3.17) *has at most one strong solution.*

Proof. Let u be a strong solution. Integrating (3.16) twice, we obtain

$$u(t) + \int_0^t a(t-s)Au(s)\,ds = u(0) + \dot u(0)t + 1 * 1 * f(t),$$

where $a(t) = t - 1 * 1 * \beta(t)$. Therefore, a well-known variation of parameters formula for integral equations (see [21, Proposition 1.2(i)]) yields

$$\begin{aligned} u(t) &= \frac{d}{dt}\int_0^t S(r)[u(0) + \dot u(0)(t-r) + 1 * 1 * f(t-r)]\,dr \\ &= S(t)u(0) + \int_0^t S(r)\dot u(0)\,dr + \int_0^t S(r)1 * f(t-r)\,dr. \end{aligned}$$

Thus, u is given by (3.18). ∎

Using Proposition 3.4, we want to show that the mild solution is more regular if the initial value of u belongs to $D(A^{1/2})$.

Proposition 3.8. *Let $u_0 \in D(A^{1/2})$, $u_1 \in X$, $f \in L^1(0,T;X)$, and let u be the mild solution u of* (3.16)–(3.17). *Then*

$$u \in C([0,T];D(A^{1/2})) \cap C^1([0,T];X)$$

and

$$\|A^{1/2}u\|_{\infty,T} + \|\dot u\|_{\infty,T} \le C(\|A^{1/2}u_0\| + \|u_1\| + \|f\|_{1,T}), \tag{3.19}$$

where

$$C = 1 + (1 + \|\beta\|_1)\Big(1 - \int_0^\infty \beta(t)\,dt\Big)^{-1/2}.$$

Proof. Let u be the mild solution of (3.16)–(3.17). Then $u = v + w + z$ where

$$v(t) := S(t)u_0, \quad w(t) := \int_0^t S(\tau)u_1\,d\tau, \quad z(t) := \int_0^t 1 * S(t-r)f(r)\,dr. \tag{3.20}$$

Now, in view of Definition 3.1, $v \in C([0,T];D(A^{1/2}))$ and by Proposition 3.4(iii), $v \in C^1([0,T];X)$. Moreover, owing to Proposition 3.4(i), $w \in C([0,T];D(A^{1/2}))$. As regards z, we have $z \in C([0,T];D(A^{1/2}))$ again by Proposition 3.4(i) and $z \in C^1([0,T];X)$ keeping in mind Remark 3.2.

Finally, the estimate (3.19) follows from (3.7) and (3.11). ∎

Lastly, we prove that, under suitable regularity conditions on the data, the mild solution is strong.

Proposition 3.9. *Let $u_0 \in D(A), u_1 \in D(A^{1/2})$ and $f \in W^{1,1}(0,T;X)$. Then the mild solution u of the Cauchy problem* (3.16)-(3.17) *is a strong solution. In addition, u belongs to $C^1([0,T];D(A^{1/2}))$ and there exists a constant $C_T > 0$, increasing in T, such that for any $t \in [0,T]$ we have*

$$\|\ddot{u}(t)\| + \|Au(t)\| + \|A^{1/2}\dot{u}(t)\| \le C_T(\|Au_0\| + \|A^{1/2}u_1\| + \|f(0)\| + \|\dot{f}\|_{1,T}). \tag{3.21}$$

Furthermore, if $\widehat{\beta}(\lambda) \neq 1$ for all $\lambda \in \mathbb{C} - \{0\}$ with $\operatorname{Re}\lambda = 0$, *then* (3.21) *holds for a constant C independent of T.*

Proof. We split $u = v + w + z$ where v, w, z are defined in (3.20).

Now, in view of Definition 3.1, v is a strong solution of (3.16) with $f = 0$. Moreover, owing to Proposition 3.4(ii), $w \in C([0,T];D(A))$. Furthermore, by (iii) of the same proposition, w belongs to $C^2([0,T];X)$ and is also a strong solution of (3.16) with $f = 0$.

Next, since $f(t) = f(0) + \int_0^t \dot{f}(s)\,ds$, we have

$$z(t) = \int_0^t 1 * S(r)f(0)\,dr + \int_0^t 1 * 1 * S(t-r)\dot{f}(r)\,dr =: z_1(t) + z_2(t).$$

Thus, $z \in C([0,T];D(A))$ in view of Proposition 3.5. Also, $z \in C^2([0,T];X)$ by direct inspection and z is a strong solution of (3.16) with $u_0 = u_1 = 0$. Indeed, using (3.14) we have

$$\ddot{z}_1(t) + Az_1(t) - \beta * Az_1(t) = f(0),$$

$$S(t-r)\dot{f}(r) + A(1*1*S)(t-r)\dot{f}(r) - \beta * A(1*1*S)(t-r)\dot{f}(r) = \dot{f}(r),$$

and integrating the last identity from 0 to t yields

$$\ddot{z}_2(t) + Az_2(t) - \beta * Az_2(t) = \int_0^t \dot{f}(r)\,dr.$$

Therefore, u is the strong solution of problem (3.16)–(3.17).

Finally, we note that

$$\dot{u}(t) = \dot{S}(t)u_0 + S(t)u_1 + 1 * S(t)f(0) + 1 * S * \dot{f}(t), \qquad t \ge 0.$$

So, the continuity of $A^{1/2}\dot{u}$ follows from (iv) and (i) of Proposition 3.4 and from the definition of $S(t)$. Finally, estimate (3.21) follows from Propositions 3.4 and 3.5. The proof is thus complete. ∎

4. The nonlinear problem: well-posedness

4.1. *Local existence.* Let us consider the nonlinear equation

$$\ddot{u}(t) + F(u(t))\dot{u}(t) + Au(t) - \int_0^t \beta(t-s)Au(s)\,ds = f(t), \qquad t \geq 0. \tag{4.1}$$

Throughout this section we will assume that the kernel β and the operator A satisfy the following conditions:

$$\begin{aligned}
&\text{(a)}\ \langle Ax, x\rangle \geq M\|x\|^2 \quad \forall x \in D(A) \quad (M > 0),\\
&\text{(b)}\ \beta \in L^1(0,\infty) \quad \text{and} \quad k(t) := \int_t^\infty \beta(s)\,ds \quad \text{is of positive type,}\\
&\text{(c)}\ \int_0^\infty \beta(s)\,ds < 1.
\end{aligned} \tag{4.2}$$

Moreover, we shall suppose that $F : D(A^\theta) \to \mathbb{R}$, $0 < \theta \leq 1/2$, is a functional Lipschitz continuous on bounded sets of $D(A^\theta)$, that is, for any $R > 0$ there exists a constant $L_R > 0$ such that

$$|F(x) - F(y)| \leq L_R\|A^\theta x - A^\theta y\| \tag{4.3}$$

for all $x, y \in D(A^\theta)$ satisfying $\|A^\theta x\|, \|A^\theta y\| \leq R$.

Let $0 < T \leq \infty$ be given and $f \in L^1(0,T;X)$. To begin with, we recall two notions of solution.

Definition 4.1. We say that u is a *strong solution* of (4.1) on $[0,T]$ if

$$u \in C^2([0,T];X) \cap C([0,T];D(A))$$

and u satisfies (4.1) for every $t \in [0,T]$.

Let $u_0, u_1 \in X$. A function $u \in C^1([0,T];X) \cap C([0,T];D(A^{1/2}))$ is a *mild solution* of (4.1) on $[0,T]$ with initial conditions

$$u(0) = u_0, \quad \dot{u}(0) = u_1, \tag{4.4}$$

if

$$u(t) = S(t)u_0 + \int_0^t S(\tau)u_1\,d\tau + \int_0^t 1 * S(t-\tau)(f(\tau) - F(u(\tau))\dot{u}(\tau))\,d\tau, \tag{4.5}$$

where $\{S(t)\}$ is the resolvent for (3.1) (see Theorem 3.3).

Notice that the convolution term in (4.5) is well defined, thanks to (4.3) and (2.1). In view of Proposition 3.7 a strong solution is also a mild one.

Another useful notion of generalized solution of (4.1) is the so-called *weak solution*, that is, a function $u \in C^1([0,T];X) \cap C([0,T];D(A^{1/2}))$ such that, for any $v \in D(A^{1/2})$, $\langle \dot{u}(t), v\rangle \in C^1([0,T])$ and for any $t \in [0,T]$ one has

$$
(4.6) \qquad \frac{d}{dt}\langle \dot{u}(t), v\rangle + \langle F(u(t))\dot{u}(t), v\rangle + \langle A^{1/2}u(t), A^{1/2}v\rangle \\
- \Big\langle \int_0^t \beta(t-s)A^{1/2}u(s)\,ds, A^{1/2}v \Big\rangle = \langle f(t), v\rangle.
$$

Adapting a classical argument due to Ball [2], one can show that any mild solution of (4.1) is also a weak solution, and the two notions of solution are equivalent when $F \equiv 0$.

First of all, we claim the uniqueness of the mild solution.

PROPOSITION 4.2. *Let $f \in L^1(0,T;X)$. Then the Cauchy problem* (4.1)–(4.4) *possesses at most one mild solution.*

The next proposition ensures the local existence and uniqueness of mild solutions to (4.1)–(4.4). The proof is by a standard fixed point argument.

PROPOSITION 4.3. *Let $u_0 \in D(A^{1/2})$, $u_1 \in X$ and $f \in L^1(0,T;X)$. Then there exists a positive number $T_0 \le T$ such that the Cauchy problem* (4.1)–(4.4) *admits a unique mild solution on* $[0,T_0]$.

For more regular data, the mild solution is a strong one: we state that in the following result.

PROPOSITION 4.4. *Let $u_0 \in D(A), u_1 \in D(A^{1/2})$ and $f \in W^{1,1}(0,T;X)$. Then the mild solution of the Cauchy problem* (4.1)-(4.4) *in* $[0,T_0]$, $T_0 \in (0,T]$, *is a strong solution. In addition, u belongs to $C^1([0,T_0];D(A^{1/2}))$ and there exists a constant $C_T > 0$ such that for any $t \in [0,T]$ we have*

$$
(4.7) \qquad \|\ddot{u}(t)\| + \|Au(t)\| + \|A^{1/2}\dot{u}(t)\| \\
\le C_T(\|Au_0\| + \|A^{1/2}u_1\| + \|f(0)\| + \|\dot{f}\|_{1,T}) \\
+ C_T \sup_{\tau\in[0,T_0]} [\|\dot{u}(\tau)\| + \|A^{1/2}u(\tau)\|].
$$

4.2. *Global existence.* In this section we will investigate the existence in the large of the solution to the Cauchy problem

$$
(4.8) \qquad \begin{cases} \ddot{u}(t) + F(u(t))\dot{u}(t) + Au(t) - \displaystyle\int_0^t \beta(t-s)Au(s)\,ds = f(t), \\ u(0) = u_0, \quad \dot{u}(0) = u_1. \end{cases}
$$

Global existence will follow from the uniform estimates obtained previously under the extra assumption

$$F(x) \geq 0 \quad \forall x \in D(A^\theta). \tag{4.9}$$

Let $T > 0$ be given.

THEOREM 4.5. *Under assumptions* (4.2), (4.3) *and* (4.9), *for any* $u_0 \in D(A^{1/2})$, $u_1 \in X$, $f \in L^1(0,T;X)$ *problem* (4.8) *possesses a unique mild solution* u *on* $[0,T]$ *such that*

$$\frac{1}{2}\,\|\dot u(t)\|^2 + \frac{1}{2}\,\|A^{1/2}u(t)\|^2 \leq C(\|u_1\|^2 + \|A^{1/2}u_0\|^2 + \|f\|_{1,T}^2) \tag{4.10}$$

for any $t \in [0,T_0]$ *and some constant* $C > 0$ *independent of* T_0 *and* T. *If, in addition,* $u_0 \in D(A)$, $u_1 \in D(A^{1/2})$ *and* $f \in W^{1,1}(0,T;X)$, *then* u *is a strong solution of the equation in* (4.8) *on* $[0,T]$ *and* $u \in C^1([0,T];D(A^{1/2}))$.

The following a priori estimates are a crucial step in the proof of Theorem 4.5.

LEMMA 4.6. *Let* $0 < T_0 \leq T$, $u_0 \in D(A)$, $u_1 \in D(A^{1/2})$ *and* $f \in W^{1,1}(0,T;X)$. *If* u *is the strong solution of the equation in* (4.8) *on* $[0,T_0]$, *then*

$$\sup_{\tau\in[0,T_0]} [\|\dot u(\tau)\| + \|A^{1/2}u(\tau)\|] \leq C_1, \tag{4.11}$$

$$\sup_{\tau\in[0,T_0]} [\|\ddot u(t)\| + \|Au(t)\| + \|A^{1/2}\dot u(t)\|] \leq C_2, \tag{4.12}$$

where $C_1 = C_1(\|u_1\|, \|A^{1/2}u_0\|, \|f\|_{1,T})$ *and* $C_2 = C_2(T, \|Au_0\|, \|A^{1/2}u_1\|, \|f(0)\|, \|\dot f\|_{1,T})$ *are positive functions, increasing in each variable and independent of* T_0.

The next result follows at once from Theorem 4.5.

COROLLARY 4.7. *Under assumptions* (4.2), (4.3) *and* (4.9), *for any* $u_0 \in D(A^{1/2})$, $u_1 \in X$, $f \in L^1(0,\infty;X)$ *problem* (4.8) *possesses a unique mild solution* u *on* $[0,\infty)$ *such that*

$$\frac{1}{2}\,\|\dot u(t)\|^2 + \frac{1}{2}\,\|A^{1/2}u(t)\|^2 \leq C(\|u_1\|^2 + \|A^{1/2}u_0\|^2 + \|f\|_1^2) \tag{4.13}$$

for any $t \geq 0$ *and some constant* $C > 0$. *If, in addition,* $u_0 \in D(A)$, $u_1 \in D(A^{1/2})$ *and* $f \in W^{1,1}_{\mathrm{loc}}(0,\infty;X) \cap L^1(0,\infty;X)$, *then* u *is a strong solution of the equation in* (4.8) *on* $[0,\infty)$ *and* $u \in C^1([0,\infty);D(A^{1/2}))$.

5. Exponential decay of the energy. This section is devoted to the study of the asymptotic behaviour of the solution to the Cauchy problem

$$\begin{cases} \ddot{u}(t) + F(u(t))\dot{u}(t) + Au(t) - \int_0^t \beta(t-s)Au(s)\,ds = f(t), \\ u(0) = u_0, \quad \dot{u}(0) = u_1. \end{cases} \tag{5.1}$$

For any measurable function $g : [0,\infty) \to X$ and $\alpha \in \mathbb{R}$ let us set

$$g_\alpha(t) = e^{\alpha t} g(t), \qquad t > 0.$$

The exponential decay at ∞ of the energy will follow, in a rather straightforward way, from the uniform estimates obtained previously under the following assumptions:

$$\begin{aligned} &\text{(a) } F : D(A^\theta) \to \mathbb{R}, \quad 0 < \theta \le 1/2, \quad \text{locally Lipschitz continuous},\\ &\text{(b) } F(x) \ge c_0 > 0 \quad \forall x \in D(A^\theta),\\ &\text{(c) } \langle Ax, x\rangle \ge M\|x\|^2 \quad \forall x \in D(A),\\ &\text{(d) } \beta_{\alpha_0} \in L^1(0,\infty),\\ &\text{(e) } t \mapsto \int_t^\infty \beta_{\alpha_0}(s)\,ds \quad \text{is of positive type},\\ &\text{(f) } \int_0^\infty |\beta(s)|\,ds < 1,\\ &\text{(g) } u_0 \in D(A^{1/2}), \quad u_1 \in X, \quad f_{\eta_0} \in L^1(0,\infty;X), \end{aligned} \tag{5.2}$$

for some constants $M, \alpha_0, \eta_0 > 0$.

THEOREM 5.1. *Under assumptions* (5.2), *for any $R > 0$ there exist $\alpha_R \in (0, \alpha_0 \wedge \eta_0]$ and $C_R \ge 0$ such that the mild solution u of* (5.1) *with $\|u_1\|^2 + \|A^{1/2}u_0\|^2 + \|f_{\eta_0}\|_1^2 \le R$ satisfies*

$$\frac{1}{2}\|\dot{u}(t)\|^2 + \frac{1}{2}\|A^{1/2}u(t)\|^2 \le C_R e^{-2\alpha_R t} \qquad \forall t \ge 0. \tag{5.3}$$

Proof. Let us suppose, first, that $u_0 \in D(A)$, $u_1 \in D(A^{1/2})$ and $f \in C^1([0,\infty);X)$, so that the mild solution u is a strong one (see Corollary 4.7). For any $\alpha \le \alpha_0$, we note that $\beta_\alpha \in L^1(0,\infty)$ and the kernel

$$K(t) := \int_t^\infty \beta_\alpha(s)\,ds$$

is of positive type, as one can see by noting that $K(t) = \int_t^\infty e^{-(\alpha_0-\alpha)s}\beta_{\alpha_0}(s)\,ds$ and applying Proposition 2.7 to β_{α_0}. Also, let us set

$$\nu_\alpha := 1 - \int_0^\infty \beta_\alpha(r)\,dr \qquad \forall \alpha \ge 0.$$

Observe that, in particular,

$$\nu_0 = 1 - \int_0^\infty \beta(r)\,dr > 0.$$

Now, it is easy to see that, for any $\alpha \geq 0$, the function $u_\alpha(t) = e^{\alpha t}u(t)$ solves the problem

$$\tag{5.4} \begin{cases} \ddot{u}_\alpha(t) + (F(u(t)) - 2\alpha)\dot{u}_\alpha(t) + (\alpha^2 - F(u(t))\alpha)u_\alpha(t) \\ \qquad\qquad + Au_\alpha(t) - \beta_\alpha * Au_\alpha(t) = f_\alpha(t), \\ u_\alpha(0) = u_0, \quad \dot{u}_\alpha(0) = u_0 + \alpha u_1. \end{cases}$$

To proceed with the proof, we need the following two lemmas. Notice that our first lemma holds under the weaker assumption

$$\tag{5.5} \int_0^\infty \beta(s)\,ds < 1.$$

LEMMA 1. *There exists $\alpha \in (0, \alpha_0 \wedge \eta_0]$ such that, for any $R > 0$, if*

$$\|u_1\|^2 + \|A^{1/2}u_0\|^2 + \|f_{\eta_0}\|_1^2 \leq R,$$

then

$$\tag{5.6} \frac{1}{2}\,\|\dot{u}_\alpha(t)\|^2 + \frac{\nu_0}{8}\,\|A^{1/2}u_\alpha(t)\|^2 + \frac{c_0}{2}\int_0^t \|\dot{u}_\alpha(\tau)\|^2\,d\tau \leq C_{1,R} + \int_0^t (\alpha C_{1,R} + |\beta_{\alpha_0}(\tau)|)\|A^{1/2}u_\alpha(\tau)\|^2\,d\tau$$

for some constant $C_{1,R} > 0$. ■

Proof of Lemma 1. First, multiplying the equation in (5.4) by $\dot{u}_\alpha(t)$, since $u \in C^1([0,\infty); D(A^{1/2}))$ (see Corollary 4.7) we obtain

$$\frac{1}{2}\frac{d}{dt}\|\dot{u}_\alpha(t)\|^2 + (F(u(t)) - 2\alpha)\|\dot{u}_\alpha(t)\|^2 + (\alpha^2 - F(u(t))\alpha)\langle u_\alpha(t), \dot{u}_\alpha(t)\rangle + \frac{1}{2}\frac{d}{dt}\|A^{1/2}u_\alpha(t)\|^2 - \langle \beta_\alpha * A^{1/2}u_\alpha(t), A^{1/2}\dot{u}_\alpha(t)\rangle = \langle f_\alpha(t), \dot{u}_\alpha(t)\rangle.$$

Integrating the above identity from 0 to s, $0 < s \leq t$, by (5.2)(b) and the fact that $\dot{K} = -\beta_\alpha$,

$$\tag{5.7} \frac{1}{2}\,\|\dot{u}_\alpha(s)\|^2 + (c_0 - 2\alpha)\int_0^s \|\dot{u}_\alpha(\tau)\|^2\,d\tau + \frac{1}{2}\,\|A^{1/2}u_\alpha(s)\|^2 + \int_0^s \langle \dot{K} * A^{1/2}u_\alpha(\tau), A^{1/2}\dot{u}_\alpha(\tau)\rangle\,d\tau$$

$$\le \frac{1}{2}\|u_1+\alpha u_0\|^2+\frac{1}{2}\|A^{1/2}u_0\|^2+\int_0^s\langle f_\alpha(\tau),\dot u_\alpha(\tau)\rangle\,d\tau$$
$$+\int_0^s (F(u(\tau))\alpha-\alpha^2)\langle u_\alpha(\tau),\dot u_\alpha(\tau)\rangle\,d\tau.$$

The convolution term $\dot K * A^{1/2}u_\alpha(\tau)$ can be computed by integrating by parts:

$$\int_0^\tau \dot K(\tau-r)A^{1/2}u_\alpha(r)\,dr = K(\tau)A^{1/2}u_0-K(0)A^{1/2}u_\alpha(\tau)$$
$$+\int_0^\tau K(\tau-r)A^{1/2}\dot u_\alpha(r)\,dr.$$

Using the above identity to evaluate the fourth term on the left-hand side of (5.7), we have

$$\int_0^s\langle \dot K * A^{1/2}u_\alpha(\tau),A^{1/2}\dot u_\alpha(\tau)\rangle\,d\tau \tag{5.8}$$
$$=\Big\langle A^{1/2}u_0,\int_0^s K(\tau)A^{1/2}\dot u_\alpha(\tau)\,d\tau\Big\rangle-\frac{K(0)}{2}\|A^{1/2}u_\alpha(s)\|^2$$
$$+\frac{K(0)}{2}\|A^{1/2}u_0\|^2+\int_0^s\langle K * A^{1/2}\dot u_\alpha(\tau),A^{1/2}\dot u_\alpha(\tau)\rangle\,d\tau$$
$$\ge\Big\langle A^{1/2}u_0,\int_0^s K(\tau)A^{1/2}\dot u_\alpha(\tau)\,d\tau\Big\rangle-\frac{K(0)}{2}\|A^{1/2}u_\alpha(s)\|^2+\frac{K(0)}{2}\|A^{1/2}u_0\|^2$$

since $\int_0^s\langle K * A^{1/2}\dot u_\alpha(\tau),A^{1/2}\dot u_\alpha(\tau)\rangle\,d\tau \ge 0$. Another integration by parts yields

$$\int_0^s K(\tau)A^{1/2}\dot u_\alpha(\tau)\,d\tau = K(s)A^{1/2}u_\alpha(s)-K(0)A^{1/2}u_0+\int_0^s\beta_\alpha(\tau)A^{1/2}u_\alpha(\tau)\,d\tau.$$

Hence, from (5.8) it follows that

$$\int_0^s\langle \dot K * A^{1/2}u_\alpha(\tau),A^{1/2}\dot u_\alpha(\tau)\rangle\,d\tau \ge K(s)\langle A^{1/2}u_0,A^{1/2}u_\alpha(s)\rangle-\frac{K(0)}{2}\|A^{1/2}u_0\|^2$$
$$+\int_0^s\beta_\alpha(\tau)\langle A^{1/2}u_0,A^{1/2}u_\alpha(\tau)\rangle\,d\tau-\frac{K(0)}{2}\|A^{1/2}u_\alpha(s)\|^2.$$

By the above estimate and (5.7) we obtain, since $\alpha\le\eta_0$,

$$\tag{5.9} \frac{1}{2}\,\|\dot u_\alpha(s)\|^2 + (c_0 - 2\alpha)\int_0^s \|\dot u_\alpha(\tau)\|^2\,d\tau + \frac{\nu_\alpha}{2}\,\|A^{1/2}u_\alpha(s)\|^2$$
$$\le \frac{1}{2}\,\|u_1 + \alpha u_0\|^2 + \frac{1}{2}\Big(1 + \int_0^\infty \beta_\alpha(r)\,dr\Big)\|A^{1/2}u_0\|^2 - K(s)\langle A^{1/2}u_0, A^{1/2}u_\alpha(s)\rangle$$
$$- \int_0^s \beta_\alpha(\tau)\langle A^{1/2}u_0, A^{1/2}u_\alpha(\tau)\rangle\,d\tau + \int_0^s \|f_{\eta_0}(\tau)\|\,\|\dot u_\alpha(\tau)\|\,d\tau$$
$$+ \alpha\int_0^s |F(u(\tau)) - \alpha|\,\|u_\alpha(\tau)\|\,\|\dot u_\alpha(\tau)\|\,d\tau.$$

Thanks to (5.2)(d), the dominated convergence theorem and (5.5), for some $\alpha_1 \in (0, \alpha_0 \wedge \eta_0]$ we have

$$\nu_\alpha = 1 - \int_0^\infty \beta_\alpha(r)\,dr \ge \frac{1}{2}\Big(1 - \int_0^\infty \beta(r)\,dr\Big) = \frac{\nu_0}{2} > 0 \quad \text{for any } 0 \le \alpha \le \alpha_1.$$

So, by the fact that $|\beta_\alpha(\tau)| \le |\beta_{\alpha_0}(\tau)|$, from (5.9) we deduce that

$$\tag{5.10} \frac{1}{2}\,\|\dot u_\alpha(s)\|^2 + (c_0 - 2\alpha)\int_0^s \|\dot u_\alpha(\tau)\|^2\,d\tau + \frac{\nu_0}{4}\,\|A^{1/2}u_\alpha(s)\|^2$$
$$\le \|u_1\|^2 + \Big(\frac{\alpha_0^2}{M} + 1\Big)\|A^{1/2}u_0\|^2 + \|\beta_{\alpha_0}\|_1\|A^{1/2}u_0\|\,\|A^{1/2}u_\alpha(s)\|$$
$$+ \|A^{1/2}u_0\|\int_0^s |\beta_{\alpha_0}(\tau)|\,\|A^{1/2}u_\alpha(\tau)\|\,d\tau$$
$$+ \int_0^s \|f_{\eta_0}(\tau)\|\,\|\dot u_\alpha(\tau)\|\,d\tau + \alpha\int_0^s |F(u(\tau)) - \alpha|\,\|u_\alpha(\tau)\|\,\|\dot u_\alpha(\tau)\|\,d\tau.$$

The analysis of the right-hand side of (5.10) can be completed by observing that

$$\|\beta_{\alpha_0}\|_1\|A^{1/2}u_0\|\,\|A^{1/2}u_\alpha(s)\| \le \frac{\nu_0}{8}\,\|A^{1/2}u_\alpha(s)\|^2 + \frac{2\|\beta_{\alpha_0}\|_1^2}{\nu_0}\,\|A^{1/2}u_0\|^2$$

and

$$\|A^{1/2}u_0\|\int_0^s |\beta_{\alpha_0}(\tau)|\,\|A^{1/2}u_\alpha(\tau)\|\,d\tau$$
$$\le \|\beta_{\alpha_0}\|_1\|A^{1/2}u_0\|^2 + \frac{1}{4}\int_0^s |\beta_{\alpha_0}(\tau)|\,\|A^{1/2}u_\alpha(\tau)\|^2\,d\tau.$$

Moreover,

$$\|f_{\eta 0}\|_1 \sup_{0\le\tau\le t} \|\dot u_\alpha(\tau)\| \le \|f_{\eta 0}\|_1^2 + \frac14 \sup_{0\le\tau\le t} \|\dot u_\alpha(\tau)\|^2.$$

In view of the the above three inequalities, from (5.10) we deduce that

$$\begin{aligned}(5.11)\qquad & \frac12 \|\dot u_\alpha(s)\|^2 + (c_0-2\alpha)\int_0^s \|\dot u_\alpha(\tau)\|^2\,d\tau + \frac{\nu_0}{8}\|A^{1/2}u_\alpha(s)\|^2 \\ &\le \|u_1\|^2 + \left(\frac{\alpha_0^2}{M} + 1 + \frac{2\|\beta_{\alpha_0}\|_1^2}{\nu_0} + \|\beta_{\alpha_0}\|_1\right)\|A^{1/2}u_0\|^2 \\ &\quad + \frac14 \int_0^s |\beta_{\alpha_0}(\tau)|\,\|A^{1/2}u_\alpha(\tau)\|^2\,d\tau \\ &\quad + \|f_{\eta 0}\|_1^2 + \frac14 \sup_{0\le\tau\le t}\|\dot u_\alpha(\tau)\|^2 + \alpha\int_0^s |F(u(\tau))-\alpha|\,\|u_\alpha(\tau)\|\,\|\dot u_\alpha(\tau)\|\,d\tau.\end{aligned}$$

Now, for $\alpha < c_0/2$, a straightforward computation yields

$$\begin{aligned}\frac14 \sup_{0\le\tau\le t}\|\dot u_\alpha(\tau)\|^2 &\le \|u_1\|^2 + \left(\frac{\alpha_0^2}{M} + 1 + \frac{2\|\beta_{\alpha_0}\|_1^2}{\nu_0} + \|\beta_{\alpha_0}\|_1\right)\|A^{1/2}u_0\|^2 + \|f_{\eta 0}\|_1^2 \\ &\quad + \frac14 \int_0^t |\beta_{\alpha_0}(\tau)|\,\|A^{1/2}u_\alpha(\tau)\|^2\,d\tau + \alpha\int_0^t |F(u(\tau))-\alpha|\,\|u_\alpha(\tau)\|\,\|\dot u_\alpha(\tau)\|\,d\tau.\end{aligned}$$

By the above estimate and (5.11), we obtain

$$\begin{aligned}(5.12)\qquad & \frac12 \|\dot u_\alpha(t)\|^2 + (c_0-2\alpha)\int_0^t \|\dot u_\alpha(\tau)\|^2\,d\tau + \frac{\nu_0}{8}\|A^{1/2}u_\alpha(t)\|^2 \\ &\le 2\|u_1\|^2 + 2\left(\frac{\alpha_0^2}{M} + 1 + \frac{2\|\beta_{\alpha_0}\|_1^2}{\nu_0} + \|\beta_{\alpha_0}\|_1\right)\|A^{1/2}u_0\|^2 \\ &\quad + \frac12 \int_0^t |\beta_{\alpha_0}(\tau)|\,\|A^{1/2}u_\alpha(\tau)\|^2\,d\tau \\ &\quad + 2\|f_{\eta 0}\|_1^2 + 2\alpha\int_0^t |F(u(\tau))-\alpha|\,\|u_\alpha(\tau)\|\,\|\dot u_\alpha(\tau)\|\,d\tau.\end{aligned}$$

Therefore, to complete the proof we only need to bound the last term in (5.12). In view of (5.2)(b), $F(u(\tau)) - \alpha \ge c_0 - \alpha > 0$ for any $\tau \ge 0$. Since F is Lipschitz continuous on bounded subsets of $D(A^{1/2})$ and $\|A^{1/2}u(t)\|$ is bounded on $[0,\infty)$ on account of (4.13), for any $R > 0$ we see that, if $\|u_1\|^2 + \|A^{1/2}u_0\|^2 + \|f_{\eta 0}\|_1^2 \le R$, then there is a constant $C_R^* \ge 0$ such that $F(u(\tau)) \le C_R^*$ for all $\tau \ge 0$. Hence, by (5.2)(c) we obtain

$$2\alpha\int_0^t (F(u(\tau))-\alpha)\|u_\alpha(\tau)\|\,\|\dot u_\alpha(\tau)\|\,d\tau \le 2C_R^*\alpha\int_0^t \|u_\alpha(\tau)\|\,\|\dot u_\alpha(\tau)\|\,d\tau$$
$$\le \alpha\frac{(C_R^*)^2}{M}\int_0^t \|A^{1/2}u_\alpha(\tau)\|^2\,d\tau + \alpha\int_0^t \|\dot u_\alpha(\tau)\|^2\,d\tau.$$

Finally, for $\alpha < c_0/6$, the above inequality can be used to obtain (5.6) from (5.12). ■

LEMMA 2. *For any $R > 0$ there are constants $\alpha_R^* \in (0, \alpha_0 \wedge \eta_0]$ and $C_{2,R} \ge 0$ such that, if*

$$\|u_1\|^2 + \|A^{1/2}u_0\|^2 + \|f_{\eta_0}\|_1^2 \le R,$$

then

$$\int_0^t \|A^{1/2}u_\alpha(\tau)\|^2\,d\tau \le C_{2,R}\Big(1+\int_0^t \|\dot u_\alpha(\tau)\|^2\,d\tau\Big) + C\Big(\|\dot u_\alpha(t)\|\,\|A^{1/2}u_\alpha(t)\| + \int_0^t \|f_{\eta_0}(\tau)\|\,\|A^{1/2}u_\alpha(\tau)\|^2\,d\tau\Big) \tag{5.13}$$

for any $\alpha \in (0, \alpha_R^*]$, where $C \ge 0$ is another constant, independent of R.

Proof of Lemma 2. We will follow the same method as in the above proof, now using $u_\alpha(t)$ as a multiplier. Taking the scalar product of both sides of the equation in (5.4) with $u_\alpha(t)$ and integrating on $[0,t]$ we obtain

$$\langle \dot u_\alpha(t), u_\alpha(t)\rangle - \int_0^t \|\dot u_\alpha(\tau)\|^2\,d\tau + \int_0^t (F(u(\tau))-2\alpha)\langle \dot u_\alpha(\tau), u_\alpha(\tau)\rangle\,d\tau$$
$$+\int_0^t (\alpha^2 - F(u(\tau))\alpha)\|u_\alpha(\tau)\|^2\,d\tau + \int_0^t \|A^{1/2}u_\alpha(\tau)\|^2\,d\tau$$
$$-\int_0^t \langle \beta_\alpha * A^{1/2}u_\alpha(\tau), A^{1/2}u_\alpha(\tau)\rangle\,d\tau$$
$$= \langle \dot u_\alpha(0), u_\alpha(0)\rangle + \int_0^t \langle f_\alpha(\tau), u_\alpha(\tau)\rangle\,d\tau.$$

Hence, for any $\alpha \le \eta_0$,

$$\int_0^t \|A^{1/2}u_\alpha(\tau)\|^2\,d\tau \le \|u_1+\alpha u_0\|\,\|u_0\| + \|\dot u_\alpha(t)\|\,\|u_\alpha(t)\| \tag{5.14}$$

$$+\int_0^t \|f_{\eta_0}(\tau)\|\,\|u_\alpha(\tau)\|\,d\tau + \int_0^t \|\dot u_\alpha(\tau)\|^2\,d\tau + \int_0^t \langle \beta_\alpha * A^{1/2}u_\alpha(\tau), A^{1/2}u_\alpha(\tau)\rangle\,d\tau$$
$$+\int_0^t |F(u(\tau))-2\alpha|\,\|\dot u_\alpha(\tau)\|\,\|u_\alpha(\tau)\|\,d\tau + \alpha\int_0^t (F(u(\tau))-\alpha)\|u_\alpha(\tau)\|^2\,d\tau.$$

Now, recalling that $\|u_1\|^2 + \|A^{1/2}u_0\|^2 + \|f_{\eta_0}\|_1^2 \le R$, by (5.2)(c) we have

$$\|u_1+\alpha u_0\|\,\|u_0\| + \|\dot u_\alpha(t)\|\,\|u_\alpha(t)\| + \int_0^t \|f_{\eta_0}(\tau)\|\,\|u_\alpha(\tau)\|\,d\tau \tag{5.15}$$
$$\le C_R + \frac{1}{\sqrt{M}}\|\dot u_\alpha(t)\|\,\|A^{1/2}u_\alpha(t)\| + \frac{1}{M}\int_0^t \|f_{\eta_0}(\tau)\|\,\|A^{1/2}u_\alpha(\tau)\|^2\,d\tau$$

for some constant $C_R \ge 0$. Moreover,

$$\int_0^t \langle \beta_\alpha * A^{1/2}u_\alpha(\tau), A^{1/2}u_\alpha(\tau)\rangle\,d\tau \le \frac{\|\beta_\alpha\|_1^2+1}{2}\int_0^t \|A^{1/2}u_\alpha(\tau)\|^2\,d\tau, \tag{5.16}$$

where, in view of (5.2)(f), we can assume that, for some $\alpha_1 \in (0, \alpha_0 \wedge \eta_0]$,

$$\int_0^\infty |\beta_\alpha(t)|\,dt < 1 \qquad \text{for any } 0 \le \alpha \le \alpha_1.$$

Finally, in order to estimate the last two integrals in (5.14) recall that, as we already observed in the proof of Lemma 1, there is a constant $C_R^* \ge 0$ such that $F(u(\tau)) \le C_R^*$ for all $\tau \ge 0$. So, for all $0 < \alpha < c_0/2$,

$$\int_0^t |F(u(\tau))-2\alpha|\,\|\dot u_\alpha(\tau)\|\,\|u_\alpha(\tau)\|\,d\tau + \alpha\int_0^t (F(u(\tau))-\alpha)\|u_\alpha(\tau)\|^2\,d\tau \tag{5.17}$$
$$\le \frac{C_R^*}{\sqrt{M}}\int_0^t \|\dot u_\alpha(\tau)\|\,\|A^{1/2}u_\alpha(\tau)\|\,d\tau + \frac{\alpha C_R^*}{M}\int_0^t \|A^{1/2}u_\alpha(\tau)\|^2\,d\tau$$
$$\le \left(\frac{1-\|\beta_\alpha\|_1^2}{8} + \frac{\alpha C_R^*}{M}\right)\int_0^t \|A^{1/2}u_\alpha(\tau)\|^2\,d\tau + C_R\int_0^t \|\dot u_\alpha(\tau)\|^2\,d\tau$$

for some constant $C_R \ge 0$. Thus, taking $\alpha_R^* = M(1-\|\beta_\alpha\|_1^2)/8C_R^*$ and using (5.15), (5.16) and (5.17) to bound the right-hand side of (5.14), we can easily obtain the conclusion. ∎

Proof of Theorem 5.1 (continued). Using (5.13) to bound the right-hand side of (5.6) we obtain, for all $\alpha \in (0, \alpha_R^*]$,

$$\frac{1}{2}\,\|\dot u_\alpha(t)\|^2+\frac{\nu_0}{8}\,\|A^{1/2}u_\alpha(t)\|^2+\frac{c_0}{2}\int_0^t\|\dot u_\alpha(\tau)\|^2\,d\tau$$
$$\le C_R\Big[1+\frac{\alpha}{2}\,(\|\dot u_\alpha(t)\|^2+\|A^{1/2}u_\alpha(t)\|^2)\Big]+\alpha C_R\int_0^t\|\dot u_\alpha(\tau)\|^2\,d\tau$$
$$+\int_0^t(C_R\|f_{\eta_0}(\tau)\|+|\beta_{\alpha_0}(\tau)|)\|A^{1/2}u_\alpha(\tau)\|^2\,d\tau$$

for some constant $C_R\ge 0$. Hence, taking

$$\alpha_R=\min\Big\{\alpha_R^*,\,\frac{1}{2C_R},\,\frac{c_0}{2C_R},\,\frac{\nu_0}{8C_R}\Big\}, \tag{5.18}$$

we have

$$\frac{1}{2}\|\dot u_\alpha(t)\|^2+\frac{1}{2}\|A^{1/2}u_\alpha(t)\|^2\le C_R+C_R\int_0^t(\|f_{\eta_0}(\tau)\|+|\beta_{\alpha_0}(\tau)|)\|A^{1/2}u_\alpha(\tau)\|^2\,d\tau.$$

Since β_{α_0} and f_{η_0} are summable, by Gronwall's lemma we conclude that

$$\frac{1}{2}\,\|\dot u_\alpha(t)\|^2+\frac{1}{2}\,\|A^{1/2}u_\alpha(t)\|^2\le C_R\quad\text{ for any } t\ge 0. \tag{5.19}$$

Now, recalling that $u(t)=e^{-\alpha t}u_\alpha(t)$ and $\dot u(t)=e^{-\alpha t}\dot u_\alpha(t)-\alpha e^{-\alpha t}u_\alpha(t)$, from (5.19) we obtain (5.3) for strong solutions. Since none of the above constants depend on the regularity of u, an approximation argument suffices to extend such a conclusion to mild solutions. ■

Remark 5.2. Analysing the proofs of Lemmas 1 and 2, one can easily realize that, when F is bounded above, all the constants in (5.6) and (5.13) are independent of the size of the initial conditions and forcing term. Consequently, in this case, the decay rate α in (5.3) is independent of R. This observation applies, in particular, to linear problems.

6. Application to a problem in viscoelasticity. In this section we will use our abstract results to obtain an exponential decay estimate for a partial differential equation that models the vibrations of viscoelastic beams and plates. In a bounded domain $\Omega\subset\mathbb{R}^n$, with sufficiently smooth boundary, let us consider the viscoelastic Euler–Bernoulli equation

$$\begin{cases}\partial_t^2u+\mu(t)\partial_tu+\Delta^2u-\int_0^t\beta(t-s)\Delta^2u(s)\,ds=0 & \text{in } \Omega\times(0,\infty),\\ u=\partial_\nu u=0 & \text{on } \partial\Omega\times(0,\infty),\\ u(0,\xi)=u_0(\xi)\quad\text{and}\quad\partial_tu(0,\xi)=u_1(\xi) & \xi\in\Omega,\end{cases} \tag{6.1}$$

where $\mu(t)\partial_t u$ is a frictional damping term. The damping coefficient $\mu(t)$ is a nonlinear term of nonlocal type, indeed we take

$$\mu(t) = M\Big(\int_\Omega [\lambda_1|\nabla u(t,\xi)|^2 + \lambda_2|\Delta u(t,\xi)|^2]\,d\xi\Big), \tag{6.2}$$

where $\lambda_1, \lambda_2 \geq 0$ and $M : [0,\infty) \to [0,\infty)$ is a given locally Lipschitz function.

For $\lambda_2 = 0$, problem (6.1) has been addressed in [6] in the case of a C^2-smooth convolution kernel. By the theory of the previous section, we can easily obtain exponential stability for the above problem in the case of singular kernels.

Theorem 6.1. *Assume that:*

(i) $\beta \in L^1(0,\infty)$ *is such that, for some constant* $\alpha_0 > 0$,

$$\int_0^\infty |\beta(s)|\,ds < 1, \qquad \beta_{\alpha_0} \in L^1(0,\infty),$$

$$t \mapsto \int_t^\infty \beta_{\alpha_0}(s)\,ds \quad \textit{is of positive type},$$

where $\beta_{\alpha_0}(t) = e^{\alpha_0 t}\beta(t)$;

(ii) $M : [0,\infty) \to [0,\infty)$ *is a Lipschitz continuous function satisfying*

$$\exists c_0 > 0 \quad \textit{such that} \quad M(s) \geq c_0 \quad \forall s \geq 0. \tag{6.3}$$

Then for any $R > 0$ *there are* $\alpha_R \in (0,\alpha_0]$ *and* $C_R \geq 0$ *such that for every* $(u_0,u_1) \in H_0^2(\Omega) \times L^2(\Omega)$ *with* $\|u_0\|_{H_0^2(\Omega)} + \|u_1\|_{L^2(\Omega)} \leq R$, *the solution* u *of problem* (6.1) *satisfies*

$$\int_\Omega [|\partial_t u(t,\xi)|^2 + \|\nabla^2 u(t,\xi)\|^2]\,d\xi \leq C_R e^{-2\alpha_R t} \qquad \forall t \geq 0,$$

where $\|\nabla^2 u(t,\xi)\|$ *denotes the* (*operator*) *norm of the Hessian of* u *with respect to* ξ.

The proof of Theorem 6.1 is a straightforward application of Theorem 5.1. Indeed, we can rewrite (6.1) as an abstract problem of type (5.1) in the Hilbert space $X = L^2(\Omega)$ endowed with the usual inner product and norm. Let $A : D(A) \subset X \to X$ be the operator defined by

$$\begin{aligned} &D(A) = H^4(\Omega) \cap H_0^2(\Omega), \\ &Ax(\xi) = \Delta^2 x(\xi), \qquad x \in D(A),\ \xi \in \Omega \text{ a.e.} \end{aligned}$$

It is well-known that A satisfies assumption (5.2)(c) and $D(A^{1/2}) = H_0^2(\Omega)$ (see, e.g., [7, pp. 28–29]). Moreover, owing to assumption (ii), the nonlinear functional $F : H_0^2(\Omega) \to \mathbb{R}$ defined by

$$F(x) = M\Big(\int_\Omega [\lambda_1|\nabla x(\xi)|^2 + \lambda_2|\Delta x(\xi)|^2]\,d\xi\Big) \quad \forall x \in H_0^2(\Omega)$$

satisfies conditions (5.2)(a), (b). So, the conclusion follows from Theorem 5.1.

A. Appendix

Proof of Proposition 4.2. Assume that there exist two mild solutions u and v of (4.1)–(4.4). Taking the difference, we get

$$u(t) - v(t) = \int_0^t 1 * S(t-\tau)[F(v(\tau))\dot v(\tau) - F(u(\tau))\dot u(\tau)]\,d\tau, \quad t \in [0,T],$$

$$\dot u(t) - \dot v(t) = \int_0^t S(t-\tau)[F(v(\tau))\dot v(\tau) - F(u(\tau))\dot u(\tau)]\,d\tau, \quad t \in [0,T].$$

Set

$$R = c_\theta \sup_{\tau\in[0,T]} [\|\dot u(\tau)\| + \|A^{1/2}u(\tau)\| + \|\dot v(\tau)\| + \|A^{1/2}v(\tau)\|],$$

where c_θ is the constant in (2.1). In view of (3.7), (4.3) and (2.1), we have, for any $t \in [0,T]$,

$$\begin{aligned}
&\|A^{1/2}u(t) - A^{1/2}v(t)\| + \|\dot u(t) - \dot v(t)\| \\
&\quad\le \Big[\Big(1 - \int_0^\infty \beta(t)\,dt\Big)^{-1/2} + 1\Big] \\
&\qquad\times \Big[\int_0^t |F(v(\tau)) - F(u(\tau))|\,\|\dot v(\tau)\|\,d\tau + \int_0^t |F(u(\tau))|\,\|\dot v(\tau) - \dot u(\tau)\|\,d\tau\Big] \\
&\quad\le (2RL_R + |F(0)|)\Big[\Big(1 - \int_0^\infty \beta(t)\,dt\Big)^{-1/2} + 1\Big] \\
&\qquad\times \int_0^t [\|A^{1/2}u(\tau) - A^{1/2}v(\tau)\| + \|\dot u(\tau) - \dot v(\tau)\|]\,d\tau.
\end{aligned}$$

Hence, by Gronwall's lemma, $\|A^{1/2}u(t) - A^{1/2}v(t)\| \equiv 0$. ■

Proof of Proposition 4.4. Let u be the mild solution of (3.16)–(3.17) in $[0,T_0]$. Thanks to the assumptions on data, we shall show that $t \mapsto F(u(t))\dot u(t)$ is Lipschitz continuous, proving that $\dot u$ and $A^{1/2}u$ are Lipschitz continuous functions as well.

First of all, we observe that $\dot u$ is given by

$$\dot u(t) = \dot S(t)u_0 + S(t)u_1 + \int_0^t S(t-\tau)(f(\tau) - F(u(\tau))\dot u(\tau))\,d\tau. \tag{A.1}$$

To show that $\dot u$ is Lipschitz continuous, we fix $0 < t < t+h < T_0$; using also formula $f(t) = f(0) + \int_0^t \dot f(s)\,ds$ we have

$$\begin{aligned}
&\dot u(t+h) - \dot u(t)\\
&\quad = \int_t^{t+h} \ddot S(\tau)u_0\,d\tau + \int_t^{t+h} \dot S(\tau)u_1\,d\tau\\
&\qquad + \int_t^{t+h} S(r)f(0)\,dr + \int_0^t S(\tau)\Big(\int_{t-\tau}^{t-\tau+h} \dot f(r)\,dr\Big)\,d\tau\\
&\qquad + \int_t^{t+h} S(\tau)\Big(\int_0^{t-\tau+h} \dot f(r)\,dr\Big)\,d\tau\\
&\qquad - \int_0^t S(\tau)[F(u(t-\tau+h)) - F(u(t-\tau))]\dot u(t-\tau+h)\,d\tau\\
&\qquad - \int_t^{t+h} S(\tau)F(u(t-\tau+h))\dot u(t-\tau+h)\,d\tau\\
&\qquad - \int_0^t S(\tau)F(u(t-\tau))[\dot u(t-\tau+h) - \dot u(t-\tau)]\,d\tau.
\end{aligned}$$

Set

$$R = (1+c_\theta)\sup_{\tau\in[0,T_0]}[\|\dot u(\tau)\| + \|A^{1/2}u(\tau)\|].$$

In view of (3.8), (3.11), (3.7), (4.3) and (2.1) we have

$$\begin{aligned}
&\|\dot u(t+h) - \dot u(t)\|\\
&\quad \le h(1+\|\beta\|_1)\Big(\|Au_0\| + \Big(1 - \int_0^\infty \beta(t)\,dt\Big)^{-1/2}\|A^{1/2}u_1\|\Big)\\
&\qquad + h(\|f(0)\| + 2\|\dot f\|_{1,T})\\
&\qquad + L_R R\int_0^t \|A^{1/2}u(\tau+h) - A^{1/2}u(\tau)\|\,d\tau + h(L_R R + |F(0)|)R\\
&\qquad + (L_R R + |F(0)|)\int_0^t \|\dot u(\tau+h) - \dot u(\tau)\|\,d\tau.
\end{aligned}$$

Applying Gronwall's lemma we get

$$\|\dot u(t+h)-\dot u(t)\| \le hC_1 + C_2\int_0^t \|A^{1/2}u(\tau+h)-A^{1/2}u(\tau)\|\,d\tau, \tag{A.2}$$

where the constants C_1, C_2 are given respectively by

$$\begin{aligned} C_1 &= e^{(L_R R+|F(0)|)T}\Big[(1+\|\beta\|_1)\Big(\|Au_0\| + \Big(1-\int_0^\infty \beta(t)\,dt\Big)^{-1/2}\|A^{1/2}u_1\|\Big)\Big] \\ &\qquad + e^{(L_R R+|F(0)|)T}[\|f(0)\| + 2\|\dot f\|_{1,T} + (L_R R + |F(0)|)R], \\ C_2 &= e^{(L_R R+|F(0)|)T} L_R R. \end{aligned}$$

As regards $A^{1/2}u(t)$, in a similar way we can write

$$\begin{aligned} &A^{1/2}u(t+h) - A^{1/2}u(t) \\ &\quad = \int_t^{t+h} \dot S(\tau)A^{1/2}u_0\,d\tau + \int_t^{t+h} S(\tau)A^{1/2}u_1\,d\tau \\ &\qquad + \int_t^{t+h} A^{1/2}(1*S)(r)f(0)\,dr + \int_0^t A^{1/2}(1*S)(\tau)\Big(\int_{t-\tau}^{t-\tau+h} \dot f(r)\,dr\Big)\,d\tau \\ &\qquad + \int_t^{t+h} A^{1/2}(1*S)(\tau)\Big(\int_0^{t-\tau+h} \dot f(r)\,dr\Big)\,d\tau \\ &\qquad - \int_0^t A^{1/2}(1*S)(\tau)[F(u(t-\tau+h)) - F(u(t-\tau))]\dot u(t-\tau+h)\,d\tau \\ &\qquad - \int_t^{t+h} A^{1/2}(1*S)(\tau)F(u(t-\tau+h))\dot u(t-\tau+h)\,d\tau \\ &\qquad - \int_0^t A^{1/2}(1*S)(\tau)F(u(t-\tau))[\dot u(t-\tau+h) - \dot u(t-\tau)]\,d\tau. \end{aligned}$$

Using again (3.11), (3.7), (4.3) and (2.1) we have

$$\begin{aligned} &\|A^{1/2}u(t+h) - A^{1/2}u(t)\| \\ &\qquad \le h(1+\|\beta\|_1)\Big(1-\int_0^\infty \beta(t)\,dt\Big)^{-1/2}\|Au_0\| + h\|A^{1/2}u_1\| \\ &\qquad\quad + h\Big(1-\int_0^\infty \beta(t)\,dt\Big)^{-1/2}(\|f(0)\| + 2\|\dot f\|_{1,T}) \end{aligned}$$

$$+\Big(1-\int_0^\infty \beta(t)\,dt\Big)^{-1/2} L_R R\int_0^t \|A^{1/2}u(\tau+h)-A^{1/2}u(\tau)\|\,d\tau$$
$$+h\Big(1-\int_0^\infty \beta(t)\,dt\Big)^{-1/2}(L_R R+|F(0)|)R$$
$$+\Big(1-\int_0^\infty \beta(t)\,dt\Big)^{-1/2}(L_R R+|F(0)|)\int_0^t \|\dot u(\tau+h)-\dot u(\tau)\|\,d\tau.$$

Applying Gronwall's lemma we get

$$\|A^{1/2}u(t+h)-A^{1/2}u(t)\| \le hC_1'+C_2'\int_0^t \|\dot u(\tau+h)-\dot u(\tau)\|\,d\tau, \tag{A.3}$$

where the constants C_1', C_2' are given respectively by

$$\begin{aligned} C_1' &= e^{(1-\int_0^\infty \beta(t)\,dt)^{-1/2}L_R RT} \\ &\quad \times \Big[(1+\|\beta\|_1)\Big(\Big(1-\int_0^\infty \beta(t)\,dt\Big)^{-1/2}\|Au_0\|+\|A^{1/2}u_1\|\Big)\Big] \\ &\quad + e^{(1-\int_0^\infty \beta(t)\,dt)^{-1/2}L_R RT} \\ &\quad \times \Big(1-\int_0^\infty \beta(t)\,dt\Big)^{-1/2}[\|f(0)\|+2\|\dot f\|_{1,T}+(L_R R+|F(0)|)R], \\ C_2' &= e^{(1-\int_0^\infty \beta(t)\,dt)^{-1/2}L_R RT}\Big(1-\int_0^\infty \beta(t)\,dt\Big)^{-1/2}(L_R R+|F(0)|). \end{aligned}$$

Combining (A.2) and (A.3), we obtain

$$\|\dot u(t+h)-\dot u(t)\| \le h(C_1+C_2C_1'T)+C_2C_2'T\int_0^t \|\dot u(\tau+h)-\dot u(\tau)\|\,d\tau,$$

whence applying again Gronwall's lemma it follows that

$$\|\dot u(t+h)-\dot u(t)\| \le e^{C_2C_2'T^2}(C_1+C_2C_1'T)h, \tag{A.4}$$

and by (A.3),

$$\|A^{1/2}u(t+h)-A^{1/2}u(t)\| \le (C_1'+C_2'e^{C_2C_2'T^2}(C_1+C_2C_1'T)T)h. \tag{A.5}$$

Therefore, $\dot u$ and $A^{1/2}u$ are Lipschitz continuous on $[0,T_0]$, because the positive constants in (A.4) and (A.5) are independent of t and h. It follows that $F(u)\dot u$ is also Lipschitz continuous, since by (4.3), (2.1), (A.5) and (A.4) we have, for any $0<t<t+h<T_0$,

$$
\begin{aligned}
&\|F(u(t+h))\dot{u}(t+h) - F(u(t))\dot{u}(t)\| \\
&\qquad \le \|(F(u(t+h)) - F(u(t)))\dot{u}(t+h)\| + \|F(u(t))(\dot{u}(t+h) - \dot{u}(t))\| \\
&\qquad \le L_R R\|A^{1/2}u(t+h) - A^{1/2}u(t)\| + (L_R R + |F(0)|)\|\dot{u}(t+h) - \dot{u}(t)\| \\
&\qquad \le [L_R R(C_1' + C_2' e^{C_2 C_2' T^2}(C_1 + C_2 C_1' T)T) \\
&\qquad\quad + (L_R R + |F(0)|)e^{C_2 C_2' T^2}(C_1 + C_2 C_1' T)]h;
\end{aligned}
$$

consequently, $F(u)\dot{u}$ has bounded derivative on $[0, T_0]$ and for any $t \in [0, T_0]$,

$$
\begin{aligned}
\left\|\frac{d}{dt}(F(u(t))\dot{u}(t))\right\| &\le L_R R(C_1' + C_2' e^{C_2 C_2' T^2}(C_1 + C_2 C_1' T)T) \\
&\qquad + (L_R R + |F(0)|)e^{C_2 C_2' T^2}(C_1 + C_2 C_1' T).
\end{aligned}
$$

Therefore, if we consider u as the mild solution of the linear Cauchy problem (3.16)–(3.17) with non-homogeneous term given by $f - F(u)\dot{u}$, then we can conclude thanks to Theorem 3.9. ■

Proof of Theorem 4.5. We note that the uniqueness of the mild solution to (4.8) follows from Proposition 4.2. Therefore, we only have to show that such a solution exists and is a strong solution of the equation for smooth data. We will obtain these conclusions in reverse order.

Suppose first that $u_0 \in D(A)$, $u_1 \in D(A^{1/2})$, $f \in W^{1,1}(0, T; X)$ and let u be the strong solution of (4.8) on $[0, T_0]$ given by Proposition 4.4. Then, a standard continuation argument applying estimate (4.12) implies that $T_0 = T$.

We now proceed to show the existence of a mild solution to (4.8) for $u_0 \in D(A^{1/2})$, $u_1 \in X$ and $f \in L^1(0, T; X)$. Take sequences $\{u_{0n}\} \subset D(A)$, $\{u_{1n}\} \subset D(A^{1/2})$ and $\{f_n\} \subset C^1([0, T]; X)$ such that $u_{0n} \to u_0$ in $D(A^{1/2})$, $u_{1n} \to u_1$ in X and $f_n \to f$ in $L^1(0, T; X)$. We have just proved that problem (4.8) with u_0 replaced by u_{0n}, u_1 by u_{1n} and f by f_n has a unique strong solution for any $n \ge 1$. Let u_n be such a solution. Then

$$
\begin{aligned}
\text{(A.6)} \qquad u_n(t) &= S(t)u_{0n} + \int_0^t S(\tau)u_{1n}\, d\tau \\
&\quad + \int_0^t 1 * S(t-\tau)(f_n(\tau) - F(u_n(\tau))\dot{u}_n(\tau))\, d\tau,
\end{aligned}
$$

for any $t \in [0, T]$. Moreover, estimate (4.11) applied to u_n yields

$$
\text{(A.7)} \qquad \sup_{\tau \in [0,T]} [\|\dot{u}_n(\tau)\| + \|A^{1/2}u_n(\tau)\|] \le C,
$$

where C is independent of n, since $\{u_{0n}\}$, $\{u_{1n}\}$ and $\{f_n\}$ are bounded sequences.

We claim that $\{u_n\}$ is a Cauchy sequence in $C^1([0,T];X)\cap C([0,T];D(A^{1/2}))$.

Finally, let us denote by u the limit of $\{u_n\}$ in $C^1([0,T];X)\cap C([0,T];D(A^{1/2}))$. Again by assumption (4.3) and (2.1) one can easily pass to the limit in (A.6) and (A.7) and deduce that u is the mild solution of (4.8), satisfying (4.10). ■

Proof of Lemma 4.6. If we multiply the equation in (4.8) by $\dot u(t)$, then we obtain

$$\frac{1}{2}\frac{d}{dt}\|\dot u(t)\|^2+F(u(t))\|\dot u(t)\|^2+\frac{1}{2}\frac{d}{dt}\|A^{1/2}u(t)\|^2-\langle\beta*A^{1/2}u(t),A^{1/2}\dot u(t)\rangle = \langle f(t),\dot u(t)\rangle,$$

keeping in mind that $u\in C^1([0,T_0];D(A^{1/2}))$ (see Proposition 4.4). Integrating the above identity from 0 to s, $0<s\le t$, yields

$$\begin{aligned}\text{(A.8)}\quad &\frac{1}{2}\|\dot u(s)\|^2+\int_0^s F(u(\tau))\|\dot u(\tau)\|^2\,d\tau+\frac{1}{2}\|A^{1/2}u(s)\|^2\\ &\quad+\int_0^s\langle\dot k*A^{1/2}u(\tau),A^{1/2}\dot u(\tau)\rangle\,d\tau\\ &=\frac{1}{2}\|u_1\|^2+\frac{1}{2}\|A^{1/2}u_0\|^2+\int_0^s\langle f(\tau),\dot u(\tau)\rangle\,d\tau,\end{aligned}$$

in view of the fact that $\dot k=-\beta$. The convolution term $\dot k*A^{1/2}u(\tau)$ can be estimated integrating by parts:

$$\int_0^\tau \dot k(\tau-r)A^{1/2}u(r)\,dr=k(\tau)A^{1/2}u_0-k(0)A^{1/2}u(\tau)+\int_0^\tau k(\tau-r)A^{1/2}\dot u(r)\,dr.$$

Plugging this identity in (A.8), we have

$$\begin{aligned}\text{(A.9)}\quad &\frac{1}{2}\|\dot u(s)\|^2+\int_0^s F(u(\tau))\|\dot u(\tau)\|^2\,d\tau+\frac{1}{2}\Big(1-\int_0^\infty\beta(r)\,dr\Big)\|A^{1/2}u(s)\|^2\\ &\le\frac{1}{2}\|u_1\|^2+\frac{1}{2}\Big(1-\int_0^\infty\beta(r)\,dr\Big)\|A^{1/2}u_0\|^2\\ &\quad-\Big\langle A^{1/2}u_0,\int_0^s k(\tau)A^{1/2}\dot u(\tau)\,d\tau\Big\rangle+\int_0^s\|f(\tau)\|\,\|\dot u(\tau)\|\,d\tau\end{aligned}$$

since, k being of positive type,

$$\int_0^s\langle k*A^{1/2}\dot u(\tau),A^{1/2}\dot u(\tau)\rangle\,d\tau\ge 0.$$

Another integration by parts yields

$$\int_0^s k(\tau)A^{1/2}\dot u(\tau)\,d\tau = k(s)A^{1/2}u(s) - k(0)A^{1/2}u_0 + \int_0^s \beta(\tau)A^{1/2}u(\tau)\,d\tau.$$

If we use the last identity, from (A.9) it follows that

$$\begin{aligned}\frac{1}{2}\,\|\dot u(s)\|^2 + \int_0^s F(u(\tau))\|\dot u(\tau)\|^2\,d\tau + \frac{1}{2}\Big(1 - \int_0^\infty \beta(r)\,dr\Big)\|A^{1/2}u(s)\|^2 \\ \le \frac{1}{2}\,\|u_1\|^2 + \frac{1}{2}\Big(1 + \int_0^\infty \beta(r)\,dr\Big)\|A^{1/2}u_0\|^2 - k(s)\langle A^{1/2}u_0, A^{1/2}u(s)\rangle \\ - \int_0^s \beta(\tau)\langle A^{1/2}u_0, A^{1/2}u(\tau)\rangle\,d\tau + \int_0^s \|f(\tau)\|\,\|\dot u(\tau)\|\,d\tau.\end{aligned}$$

Since $\int_0^\infty \beta(r)\,dr < 1$, from the previous inequality we also deduce that

$$\begin{aligned}\frac{1}{2}\,\|\dot u(s)\|^2 + \int_0^s F(u(\tau))\|\dot u(\tau)\|^2\,d\tau + \frac{1}{2}\Big(1 - \int_0^\infty \beta(r)\,dr\Big)\|A^{1/2}u(s)\|^2 \\ \le \frac{1}{2}\|u_1\|^2 + \|A^{1/2}u_0\|^2 + \|\beta\|_1\|A^{1/2}u_0\|\,\|A^{1/2}u(s)\| \\ + \|A^{1/2}u_0\|\int_0^s |\beta(\tau)|\,\|A^{1/2}u(\tau)\|\,d\tau + \int_0^s \|f(\tau)\|\,\|\dot u(\tau)\|\,d\tau \\ \le \|u_1\|^2 + \left(1 + \frac{4\|\beta\|_1^2}{1 - \int_0^\infty \beta(r)\,dr} + 4\|\beta\|_1\right)\|A^{1/2}u_0\|^2 \\ + \frac{1}{4}\Big(1 - \int_0^\infty \beta(r)\,dr\Big)\|A^{1/2}u(s)\|^2 \\ + \frac{1}{4}\int_0^s |\beta(\tau)|\,\|A^{1/2}u(\tau)\|^2\,d\tau + \sup_{0\le\tau\le t}\|\dot u(\tau)\|\,\|f\|_{1,T}.\end{aligned} \tag{A.10}$$

As $F(u(t)) \ge 0$, a straightforward computation yields

$$\begin{aligned}\frac{1}{2}\,\|\dot u(s)\|^2 + \frac{1}{4}\Big(1 - \int_0^\infty \beta(r)\,dr\Big)\|A^{1/2}u(s)\|^2 \\ \le \|u_1\|^2 + \left(1 + \frac{4\|\beta\|_1^2}{1 - \int_0^\infty \beta(s)\,ds} + 4\|\beta\|_1\right)\|A^{1/2}u_0\|^2 \\ + \frac{1}{4}\int_0^s |\beta(\tau)|\,\|A^{1/2}u(\tau)\|^2\,d\tau + \frac{1}{4}\sup_{0\le\tau\le t}\|\dot u(\tau)\|^2 + 4\|f\|_{1,T}^2, \qquad 0 < s \le t.\end{aligned} \tag{A.11}$$

Thus, by an easy computation,

$$\frac{1}{4}\sup_{0\le\tau\le t}\|\dot u(\tau)\|^2 \le \|u_1\|^2 + \left(1 + \frac{4\|\beta\|_1^2}{1-\int_0^\infty \beta(s)\,ds} + 4\|\beta\|_1\right)\|A^{1/2}u_0\|^2 + 4\|f\|_{1,T}^2 + \frac{1}{4}\int_0^t |\beta(\tau)|\,\|A^{1/2}u(\tau)\|^2\,d\tau.$$

By the above estimate and (A.11), we obtain

$$\begin{aligned}\text{(A.12)}\qquad \frac{1}{2}\|\dot u(t)\|^2 &+ \frac{1}{4}\Big(1-\int_0^\infty \beta(r)\,dr\Big)\|A^{1/2}u(t)\|^2 \\ &\le 2\|u_1\|^2 + 2\left(1 + \frac{4\|\beta\|_1^2}{1-\int_0^\infty \beta(s)\,ds} + 4\|\beta\|_1\right)\|A^{1/2}u_0\|^2 \\ &\quad + 8\|f\|_{1,T}^2 + \frac{1}{2}\int_0^t |\beta(\tau)|\,\|A^{1/2}u(\tau)\|^2\,d\tau,\end{aligned}$$

whence

$$\begin{aligned}\frac{1}{2}\|A^{1/2}u(t)\|^2 &\le \frac{4}{1-\int_0^\infty \beta(s)\,ds} \\ &\quad\times\left[\|u_1\|^2 + \left(1 + \frac{4\|\beta\|_1^2}{1-\int_0^\infty \beta(s)\,ds} + 4\|\beta\|_1\right)\|A^{1/2}u_0\|^2 + 4\|f\|_{1,T}^2\right] \\ &\quad + \frac{1}{1-\int_0^\infty \beta(s)\,ds}\int_0^t |\beta(\tau)|\,\|A^{1/2}u(\tau)\|^2\,d\tau.\end{aligned}$$

Thus, by Gronwall's lemma,

$$\begin{aligned}\frac{1}{2}\|A^{1/2}u(t)\|^2 &\le \frac{4e^{\|\beta\|_1/(1-\int_0^\infty \beta(s)\,ds)}}{1-\int_0^\infty \beta(s)\,ds} \\ &\quad\times\left[\|u_1\|^2 + \left(1 + \frac{4\|\beta\|_1^2}{1-\int_0^\infty \beta(s)\,ds} + 4\|\beta\|_1\right)\|A^{1/2}u_0\|^2 + 4\|f\|_{1,T}^2\right].\end{aligned}$$

Now, (A.12) and the above inequality yield

$$\begin{aligned}\frac{1}{2}\|\dot u(t)\|^2 &+ \frac{1}{2}\|A^{1/2}u(t)\|^2 \\ &\le \left(2 + \frac{4e^{\|\beta\|_1/(1-\int_0^\infty \beta(s)\,ds)}}{1-\int_0^\infty \beta(s)\,ds}(\|\beta\|_1+1)\right) \\ &\quad\times\left[\|u_1\|^2 + \left(1 + \frac{4\|\beta\|_1^2}{1-\int_0^\infty \beta(s)\,ds} + 4\|\beta\|_1\right)\|A^{1/2}u_0\|^2 + 4\|f\|_{1,T}^2\right]\end{aligned}$$

$$\leq 8\Bigg(1 + \frac{2e^{\|\beta\|_1/(1-\int_0^\infty \beta(s)\,ds)}}{1-\int_0^\infty \beta(s)\,ds}\,(\|\beta\|_1 + 1)\Bigg)$$
$$\times\Bigg(1 + \frac{\|\beta\|_1^2}{1-\int_0^\infty \beta(s)\,ds} + \|\beta\|_1\Bigg)(\|u_1\|^2 + \|A^{1/2}u_0\|^2 + \|f\|_{1,T}^2).$$

So, (4.11) holds, and (4.12) follows from (4.7) and (4.11). ∎

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