APPLICATIONES MATHEMATICAE
28,2 (2001), pp. 125–142

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CONFIDENCE REGIONS OF MINIMAL AREA FOR THE SCALE-LOCATION PARAMETER AND THEIR APPLICATIONS

Abstract. The area of a confidence region is suggested as a quality exponent of parameter estimation. It is shown that under very mild restrictions imposed on the underlying scale-location family there exists an optimal confidence region. Explicit formulae as well as numerical results concerning the normal, exponential and uniform families are presented. The question how to estimate the quantile function is also discussed.

1. Introduction. Let $x_1, \ldots, x_n$ be independent copies of a random variable ξ. Suppose that the distribution function of ξ is known up to a parametric family $\mathcal{F} = \{F : F = F_\theta,\ \theta \in \Theta\}$, where $\Theta \subseteq \mathbb{R}^k$. The unknown parameter θ is to be estimated on the basis of the sample $x = (x_1, \ldots, x_n)$.

Let $(\mathbb{R}^n, \mathcal{B}^n, P_\theta)$ be the probability space induced by $x = (x_1, \ldots, x_n)$ and F_θ. By $\mathcal{B}^n$ we denote the σ-algebra of Borel subsets of $\mathbb{R}^n$.

There are two approaches to parameter estimation, the point estimation and confidence region estimation. A point estimator is a mapping $\theta^* : \mathbb{R}^n \to \mathbb{R}^k$. Its quality is usually characterized by the risk function

$$R(\theta^*, \theta) = E_\theta l(\theta^*, \theta) = \int l(\theta^*, \theta)\, P_\theta(dx),$$

which is, in its turn, determined by a loss function $l(u, v)$.

There exists a perfect asymptotic theory of optimal point estimation (see e.g. Ibragimov and Khas'minskii (1981)). The case of moderate size samples, which is of much greater interest, is less studied. Nevertheless, for the most popular parametric families that admit sufficient statistics the optimal point estimators are known (see e.g. Zacks (1971), Lehmann (1997)).

2000 *Mathematics Subject Classification*: Primary 62F25.

Key words and phrases: scale-location family, parameter estimation, mathematical programming, quantile function.

A *confidence region estimator* or simply a *confidence region*, in contrast to a point estimator, is defined as a mapping $B : \mathbb{R}^n \to \mathcal{B}^k$ such that for all $\theta \in \Theta$,

$$P_\theta(\theta \in B(x)) \geq \alpha \in (0, 1).$$

Here α is the confidence level. We omit the details related to the measurability of $B(x)$.

A confidence region is called *strong* if for all $\theta \in \Theta$,

$$P_\theta(\theta \in B(x)) = \alpha.$$

The quality of a confidence region may be characterized by the expectation

$$R(\theta, B) = E_\theta \lambda_k(B(x))$$

where λ_k is the Lebesgue measure on $\mathcal{B}^k$.

It should be noted that the asymptotic aspects of the confidence region theory are of small interest. In the non-asymptotic context the theory remains rather poor. There are not too many cases where the family $\mathcal{F}$ admits confidence regions. In a few of them the family admits an optimal, in a sense, confidence region.

The basic goal of the paper is to suggest a reasonable approach to the optimal confidence region problem based on the algebraic properties of the family parameterization. In the focus of our attention are the so-called scale-location families though the method can be applied to more general situations.

The paper is organized as follows. In Section 2 the notion of confidence region optimality is introduced. Here we also give general formulae for the boundary of the optimal confidence region for the scale-location vector parameter. Given the region one can easily build a confidence region for the unknown quantile function that appears as the solution of a problem of mathematical programming. The properties of the confidence regions so defined are discussed in Section 3. Sections 4–6 are devoted to the normal, exponential and uniform families, that is, to those admitting sufficient statistics with the required properties. Concluding remarks are given in Section 7.

2. General case of a scale-location family. Suppose that $F_\theta(u)$ is absolutely continuous and its density with respect to λ_1 is of the form

$$(1) \qquad p_\theta(u) = \frac{1}{\theta_1} p_{(1,0)}\left(\frac{u - \theta_2}{\theta_1}\right), \qquad \theta = (\theta_1, \theta_2),\ \theta \in \Theta = \mathbb{R}^1_+ \times \mathbb{R}^1.$$

The density $p_{(1,0)}(u) = p(u)$ is called *standard* while the parameters θ_1 and θ_2 are called, respectively, *scale* and *location*.

Set

$$t_2(x) = \overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i,$$

$$t_1(x) = s, \quad \text{where} \quad s^2 = \frac{1}{n}\sum_{j=1}^{n}(x_j - \overline{x})^2.$$

Obviously,

$$P_\theta\left(\left(\frac{\theta_1}{t_1(x)}, \frac{t_2(x)-\theta_2}{t_1(x)}\right) \in A\right) = P_{(1,0)}\left(\left(\frac{1}{t_1(x)}, \frac{t_2(x)}{t_1(x)}\right) \in A\right), \quad A \in \mathcal{B}^2.$$

If one chooses A in such a way that

$$P_{(1,0)}\left(\left(\frac{1}{t_1(x)}, \frac{t_2(x)}{t_1(x)}\right) \in A\right) = \alpha \tag{2}$$

then the random region of the form

$$B_A(x) = (0, t_2(x)) + t_1(x)A^- \tag{3}$$

with

$$A^- = \{(u,v) : (u,-v) \in A\}$$

is a strong confidence region of level α. Furthermore, its risk has the form

$$R(\theta, B_A) = \lambda_2(A)E_\theta t_1^2(x) = \theta_1^2\lambda_2(A)E_{(1,0)}t_1^2(x).$$

Note that $\lambda_2(A^-) = \lambda_2(A)$.

We say that the confidence region $B_A(x)$ is λ-*optimal* if

$$A \in \arg\min_{A\in\mathcal{A}_\alpha} \lambda_2(A)$$

where $\mathcal{A}_\alpha$ is the class of sets satisfying (2). It is clear that $\arg\min_{A\in\mathcal{A}_\alpha}\lambda_2(A)$ depends on the choice of the statistics $t_1(x)$ and $t_2(x)$.

Thus, the problem is reduced to an extremal problem. Suppose that in the standard case there exists a joint density $q(u,v)$ of the statistics

$$T_1(x) = \frac{1}{t_1(x)}, \qquad T_2(x) = \frac{t_2(x)}{t_1(x)}. \tag{4}$$

Proposition 1. *Let*

$$A_z = \{(u,v) : q(u,v) \geq z\}$$

and assume that for all $z > 0$,

$$\int_{\{(u,v):q(u,v)=z\}} q(u,v)\,du\,dv = 0.$$

Then A_{z_α} *determined by the equation*

$$\int_{A_{z_\alpha}} q(u,v)\,du\,dv = \alpha$$

belongs to $\arg\min_{A\in\mathcal{A}_\alpha}\lambda_2(A)$. *Moreover, if* $A\in\arg\min_{A\in\mathcal{A}_\alpha}\lambda_2(A)$ *then*

$$\lambda_2(A\bigtriangleup A_{z_\alpha})=0.$$

This is a version of the well known Neyman–Pearson Lemma (see e.g. Proposition 2.1 in Einmahl and Mason (1992)).

If A_{z_α} is connected we obtain a strong confidence region $B(x)$ determined by the equation (see (3))

$$B(x)=(0,t_2(x))+t_1(x)A^-_{z_\alpha}. \tag{5}$$

Obviously, it is sufficient to assume only that

$$\int\limits_{\{(u,v):q(u,v)=z_\alpha\}} q(u,v)\,du\,dv=0.$$

But if A_{z_α} is not connected the region (5) is not convenient from the practical point of view. In such a situation one can take instead the convex hull C_α of A_{z_α}. So, we come to the confidence region of the form

$$B^*(x)=(0,t_2(x))+t_1(x)C^-_\alpha$$

for which

$$P_\theta(\theta\in B^*(x))\ge\alpha,$$

that is, $B^*(x)$ is not strong.

Note that the way in which the confidence region was built is based on the so-called *invariance* and *equivariance* of the sample variance and mean. More precisely,

$$t_1(\theta_1x+\theta_2\mathbf{1})=\theta_1t_1(x), \tag{6}$$

$$t_2(\theta_1x+\theta_2\mathbf{1})=\theta_1t_2(x)+\theta_2, \tag{7}$$

where $\mathbf{1}=(1,\dots,1)\in\mathbb{R}^n$.

It is clear that instead of s and $\overline{x}$ we could take any statistics that satisfy (6) and (7). If the family is normal then the choice $t_1(x)=\sqrt{n}\,s$, $t_2(x)=\overline{x}$ is also justified by the sufficiency of $(s,\overline{x})$. From this viewpoint, dealing with, say, the exponential or uniform family we should take, respectively, $(t_1(x),t_2(x))=(2n(\overline{x}-x_{(1)}),x_{(1)})$ and $(t_1(x),t_2(x))=(x_{(n)}-x_{(1)}, x_{(n)}+x_{(1)})$ where

$$x_{(1)}=\min_{1\le j\le n}x_j,\qquad x_{(n)}=\max_{1\le j\le n}x_j. \tag{8}$$

Such a choice is realized in Sections 4–6.

3. Confidence regions for the quantile function. Let the distribution function F correspond to the standard density $p(u)=p_{(1,0)}(u)$ in (1). The standard quantile function is defined as

$$u(p)=\inf\{u:F(u)\ge p\},\qquad 0<p<1.$$

In the case of non-standard density (1) the quantile function has the form

$$x(p) = \theta_1 u(p) + \theta_2. \tag{9}$$

Consider the following functions defined on $(0, 1)$:

$$x_+(p) = \max_{\theta \in B(x)} (\theta_1 u(p) + \theta_2), \qquad x_-(p) = \min_{\theta \in B(x)} (\theta_1 u(p) + \theta_2). \tag{10}$$

Naturally, the region

$$\widehat{B} = \{x(p) : x_-(p) \le x(p) \le x_+(p),\ 0 < p < 1\} \tag{11}$$

can serve as a confidence region for the unknown quantile function (9).

Note that in order to construct a confidence region for the quantile function $x(p)$ at a given point p one has to solve the extremal problems (10) that are typical problems of mathematical programming.

Let

$$\begin{cases} \theta_+(p) = (\theta_1^+(p), \theta_2^+(p)) = \arg \max\limits_{\theta \in A_{z_\alpha}^-} (\theta_1 u(p) + \theta_2), \\ \theta_-(p) = (\theta_1^-(p), \theta_2^-(p)) = \arg \min\limits_{\theta \in A_{z_\alpha}^-} (\theta_1 u(p) + \theta_2), \end{cases} \tag{12}$$

and

$$y_+(p) = \theta_1^+(p) u(p) + \theta_2^+(p), \qquad y_-(p) = \theta_1^-(p) u(p) + \theta_2^-(p). \tag{13}$$

It is easily seen that

$$\arg \max_{\theta \in B(x)} (\theta_1 u(p) + \theta_2) = t_1(x)\theta_+(p) + (0, t_2(x)),$$

$$\arg \min_{\theta \in B(x)} (\theta_1 u(p) + \theta_2) = t_1(x)\theta_-(p) + (0, t_2(x)).$$

Then (9), (10), (12) and (13) lead to

$$x_\pm(p) = y_\pm(p) t_1(x) + t_2(x). \tag{14}$$

So, it suffices to solve the extremal problems only within $A_{z_\alpha}^-$.

The question arises: *what is the level of confidence provided by* (11)? Assume that

$$\{u : p(u) > 0\} = (u_-, u_+) \quad \text{and} \quad \int_{u_-}^{u_+} p(u)\, du = 1.$$

Then $u(p)$ is strictly increasing and continuous. First, suppose that $u_- = u(0) = -\infty$ and $u_+ = u(1) = \infty$.

Denote by $E = \{e \in \mathbb{R}^2 : |e| = 1\}$ the unit circle in $\mathbb{R}^2$. Consider the unit vectors

$$e^{(p)} = \left(\frac{u(p)}{\sqrt{u^2(p) + 1}}, \frac{1}{\sqrt{u^2(p) + 1}} \right), \qquad 0 < p < 1.$$

It is evident that, as p runs through $(0,1)$, $e^{(p)}$ describes the upper part of the unit circle, that is, the set

$$E_+ = \{e = (e_1, e_2) : |e| = 1,\ e_2 > 0\}.$$

Obviously, the convex $A^-_{z_\alpha}$ is uniquely determined by its support function

$$s(e) = \sup_{x \in A^-_{z_\alpha}} \langle e, x \rangle, \qquad e \in E.$$

Note that

$$y_-(p) = s(-e^{(p)})\sqrt{u^2(p)+1}, \qquad y_+(p) = s(e^{(p)})\sqrt{u^2(p)+1}, \qquad p \in (0,1).$$

Since $u(p)$ is continuous and strictly increasing, $A^-_{z_\alpha}$ is uniquely determined also by the vector-valued function $(y_-(p), y_+(p))$. This implies that the events

$$x_-(p) \le x(p) \le x_+(p), \qquad 0 < p < 1, \tag{15}$$

and

$$\left(\frac{\theta_1}{t_1(x)}, \frac{\theta_2 - t_2(x)}{t_1(x)}\right) \in A^-_{z_\alpha} \tag{16}$$

are equivalent. That is,

$$P_\theta(\widehat{B}) = \alpha.$$

If $A^-_{z_\alpha}$ is not convex then the equivalence of (15) and (16) fails. More precisely, (16) implies (15) but not vice versa, that is,

$$P_\theta(\theta \in B^*(x)) = P_\theta(\widehat{B}) \ge \alpha.$$

If $A^-_{z_\alpha}$ is convex but at least one of the values $u_- = u(0)$ and $u_+ = u(1)$ is finite then the calculation of $P(\widehat{B})$ is quite different. The problem is that the extreme points $\theta_\pm(p)$, $0 < p < 1$, defined in (12) do not cover all the boundary $\partial A^-_{z_\alpha}$. Hence, there exists a class of convex sets whose boundaries contain the segments $\theta_\pm(p)$, $0 < p < 1$. Denote by $\widehat{C}_\alpha$ the maximal convex set with this property. In accordance with (5) we define

$$B^{**}(x) = (0, t_2(x)) + t_1(x)\widehat{C}_\alpha. \tag{17}$$

Obviously,

$$P_\theta(\theta \in B^{**}(x)) = P_\theta(\widehat{B}) \ge \alpha.$$

Below we consider the exponential and uniform scale-location families that serve as good examples of such a situation.

4. The case of the normal family. In (1), set

$$p(u) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right).$$

Table 1. Normal distribution. The values of z_α, u_- and u_+ for $\alpha = 0.95$, $n = 3(1)10(2)40$

n	z_α	u_-	u_+
3	0.00347029	0.20780859	3.72270733
4	0.01854300	0.20998484	1.97710183
5	0.05035515	0.20744904	1.37653237
6	0.10172153	0.20339481	1.07484882
7	0.17398975	0.19885234	0.89322195
8	0.26594273	0.19416318	0.77214691
9	0.38145268	0.18968929	0.68412936
10	0.51601506	0.18534340	0.61792495
12	0.85716513	0.17745370	0.52293996
14	1.27972174	0.17037336	0.45839003
16	1.80384765	0.16415186	0.41075334
18	2.39749101	0.15850641	0.37449811
20	3.09927806	0.15349170	0.34539063
22	3.89135104	0.14894527	0.32160354
24	4.76667625	0.14479252	0.30177053
26	5.71558242	0.14097412	0.28495566
28	6.80983245	0.13751304	0.27028749
30	7.92727463	0.13425943	0.25764570
32	9.14289587	0.13125223	0.24646207
34	10.45933434	0.12845788	0.23648618
36	11.87922648	0.12585401	0.22752092
38	13.40521111	0.12341931	0.21941186
40	15.03992493	0.12113377	0.21203708

We recall that the statistic $(t_1(x), t_2(x)) = (\sqrt{n}\, s, \overline{x})$ is sufficient (see e.g. Ex. 2.7 in Zacks (1971)). Furthermore, the densities of $\sqrt{n}\, s$ and $\overline{x}$ are, respectively (see e.g. Ch. 18 in Cramér (1946)),

$$q_1(u) = \frac{2^{-(n-3)/2}}{\Gamma\left(\frac{n-1}{2}\right)} u^{n-2} \exp\left(-\frac{u^2}{2}\right) \quad \text{and} \quad q_2(u) = \sqrt{n}\, p(\sqrt{n}\, u).$$

Since $\overline{x}$ and s are independent (see ibid., Ch. 29) the density of (T_1, T_2), defined as in (4), has the form

$$q(u,v) = c_n \exp\left(-\frac{1+nv^2}{2u^2} - (n+1)\ln u\right), \qquad u > 0,$$

where

$$c_n = \frac{\sqrt{n}}{2^{n/2-1}\Gamma\left(\frac{n-1}{2}\right)\sqrt{\pi}}.$$

Obviously, $q(u,-v) = q(u,v)$, that is, $A^-_{z_\alpha} = A_{z_\alpha}$.

Furthermore, $q(u,v)$ is unimodal with the mode at $((n+1)^{-1/2}, 0)$. The modal value is

$$q_0 = \max_{u,v} q(u,v) = c_n \exp\left(\frac{n+1}{2}(\ln(n+1) - 1)\right).$$

Table 2. Normal distribution. The values of z_α, u_- and u_+ for $\alpha = 0.99$, $n = 3(1)10(2)40$

n	z_α	u_-	u_+
3	0.00013698	0.18031714	8.41263897
4	0.00124809	0.18452681	3.45047277
5	0.00439144	0.18392899	2.12036077
6	0.01043298	0.18163529	1.53579396
7	0.01993633	0.17863524	1.21385993
8	0.03364336	0.17545397	1.00989565
9	0.04921970	0.17184580	0.87521811
10	0.07152684	0.16871818	0.77181700
12	0.12671137	0.16250443	0.63355461
14	0.19451771	0.15669140	0.54425587
16	0.28269254	0.15156788	0.47993231
18	0.39387352	0.14699809	0.43130819
20	0.50947037	0.14267427	0.39429174
22	0.67003395	0.13896152	0.36333049
24	0.83032425	0.13539621	0.33856314
26	1.01288802	0.13213499	0.31765227
28	1.17700808	0.12898606	0.30032377
30	1.40177417	0.12621852	0.28471191
32	1.65233058	0.12364511	0.27102711
34	1.86773827	0.12112560	0.25934277
36	2.16621189	0.11888472	0.24850481
38	2.49399276	0.11678123	0.23877117
40	2.76596320	0.11469730	0.23029683

The curve of level z, $0 < z < q_0$, is determined by the equation

$$\frac{1+nv^2}{2u^2} + (n+1)\ln u = \ln(c_n/z)$$

or

$$v = \pm n^{-1/2}\sqrt{2u^2(\ln(c_n/z) - (n+1)\ln u) - 1} = \pm v(u,z).$$

Let $u_-(z)$ and $u_+(z)$ be the roots of the equation

$$2u^2(\ln(c_n/z) - (n+1)\ln u) = 1.$$

Then $v(u,z)$, $u_-(z) < u < u_+(z)$, is the upper part of the boundary of A_z while the lower one is symmetric with respect to the axis $v = 0$ and, therefore, is of the form $-v(u,z)$, $u_-(z) < u < u_+(z)$.

In order to get the proper level z one has to solve the equation

$$\int_{u_-(z)}^{u_+(z)} \int_0^{v(u,z)} q(u,v)\,dv\,du = \frac{\alpha}{2}.$$

Denote by z_α the root of this equation which is obviously unique.

Set $u_\pm = u_\pm(z_\alpha)$. Then

$$A_{z_\alpha} = \{(u,v) : u_- \le u \le u_+,\ |v| \le v(u,z_\alpha)\}$$

Table 3. Normal distribution. The values of $y_{\pm}(p)$, $p = 0.500(0.025)0.900(0.005)0.995$ for the sample size $n = 10, 20, 30$ and the confidence level $\alpha = 0.95$

p	$n=10$		$n=20$		$n=30$	
	$y-$	$y+$	$y-$	$y+$	$y-$	$y+$
0.500	−0.3099	0.3099	−0.1367	0.1367	−0.0877	0.0877
0.525	−0.2837	0.3367	−0.1208	0.1529	−0.0753	0.1003
0.550	−0.2580	0.3641	−0.1051	0.1693	−0.0621	0.1131
0.575	−0.2326	0.3922	−0.0895	0.1862	−0.0508	0.1261
0.600	−0.2076	0.4212	−0.0740	0.2035	−0.0386	0.1395
0.625	−0.1827	0.4513	−0.0586	0.2214	−0.0264	0.1533
0.650	−0.1580	0.4825	−0.0431	0.2399	−0.0141	0.1675
0.675	−0.1333	0.5151	−0.0274	0.2591	−0.0017	0.1823
0.700	−0.1085	0.5492	−0.0117	0.2792	0.0109	0.1977
0.725	−0.0836	0.5853	0.0044	0.3003	0.0238	0.2138
0.750	−0.0582	0.6235	0.0209	0.3227	0.0371	0.2309
0.775	−0.0324	0.6644	0.0379	0.3466	0.0509	0.2491
0.800	−0.0057	0.7087	0.0557	0.3723	0.0653	0.2687
0.825	0.0220	0.7571	0.0744	0.4003	0.0806	0.2901
0.850	0.0513	0.8109	0.0945	0.4314	0.0971	0.3137
0.875	0.0828	0.8719	0.1164	0.4666	0.1152	0.3403
0.900	0.1177	0.9431	0.1411	0.5075	0.1357	0.3713
0.905	0.1252	0.9590	0.1467	0.5167	0.1402	0.3782
0.910	0.1330	0.9756	0.1520	0.5262	0.1448	0.3854
0.915	0.1410	0.9929	0.1578	0.5361	0.1497	0.3929
0.920	0.1493	1.0110	0.1638	0.5465	0.1547	0.4008
0.925	0.1579	1.0301	0.1701	0.5574	0.1599	0.4090
0.930	0.1669	1.0503	0.1766	0.5690	0.1654	0.4177
0.935	0.1763	1.0716	0.1835	0.5813	0.1712	0.4269
0.940	0.1862	1.0944	0.1908	0.5942	0.1774	0.4367
0.945	0.1967	1.1187	0.1985	0.6081	0.1839	0.4472
0.950	0.2078	1.1450	0.2068	0.6230	0.1908	0.4585
0.955	0.2197	1.1735	0.2157	0.6391	0.1984	0.4708
0.960	0.2327	1.2048	0.2254	0.6572	0.2066	0.4843
0.965	0.2468	1.2397	0.2360	0.6770	0.2156	0.4992
0.970	0.2626	1.2790	0.2479	0.6994	0.2257	0.5160
0.975	0.2804	1.3244	0.2615	0.7252	0.2372	0.5354
0.980	0.3013	1.3784	0.2775	0.7559	0.2509	0.5585
0.985	0.3269	1.4457	0.2971	0.7941	0.2676	0.5872
0.990	0.3606	1.5366	0.3233	0.8456	0.2900	0.6259
0.995	0.4134	1.6828	0.3646	0.9284	0.3255	0.6880

and, therefore, (5) takes the form

$$B(x) = \left\{(\theta_1, \theta_2) : \left(\frac{\theta_1}{s\sqrt{n}}, \frac{\theta_2 - \overline{x}}{s\sqrt{n}}\right) \in A_{z_\alpha}\right\}$$

or

$$B(x) = \left\{(\theta_1, \theta_2) : u_- < \frac{\theta_1}{s\sqrt{n}} < u_+,\ \left|\frac{\theta_2 - \overline{x}}{s\sqrt{n}}\right| < v\left(\frac{\theta_1}{s\sqrt{n}}, z_\alpha\right)\right\}.$$

Table 4. Normal distribution. The values of $y_{\pm}(p)$, $p = 0.500(0.025)0.900(0.005)0.995$ for the sample size $n = 10, 20, 30$ and the confidence level $\alpha = 0.99$

p	$n=10$		$n=20$		$n=30$	
	y_-	y_+	y_-	y_+	y_-	y_+
0.500	−0.4252	0.4252	−0.1777	0.1777	−0.1120	0.1120
0.525	−0.3939	0.4573	−0.1604	0.1953	−0.0989	0.1253
0.550	−0.3633	0.4902	−0.1433	0.2134	−0.0859	0.1389
0.575	−0.3332	0.5241	−0.1265	0.2319	−0.0731	0.1528
0.600	−0.3036	0.5592	−0.1098	0.2509	−0.0604	0.1671
0.625	−0.2744	0.5956	−0.0933	0.2706	−0.0476	0.1818
0.650	−0.2454	0.6334	−0.0767	0.2911	−0.0349	0.1970
0.675	−0.2167	0.6730	−0.0601	0.3123	−0.0220	0.2129
0.700	−0.1880	0.7146	−0.0434	0.3347	−0.0091	0.2294
0.725	−0.1593	0.7586	−0.0265	0.3581	0.0042	0.2469
0.750	−0.1305	0.8053	−0.0093	0.3831	0.0177	0.2653
0.775	−0.1013	0.8554	0.0083	0.4097	0.0317	0.2850
0.800	−0.0715	0.9096	0.0266	0.4385	0.0463	0.3062
0.825	−0.0408	0.9690	0.0458	0.4699	0.0617	0.3294
0.850	−0.0089	1.0351	0.0662	0.5047	0.0782	0.3550
0.875	0.0250	1.1102	0.0883	0.5442	0.0961	0.3840
0.900	0.0619	1.1980	0.1130	0.5903	0.1165	0.4178
0.905	0.0698	1.2176	0.1184	0.6006	0.1209	0.4253
0.910	0.0779	1.2380	0.1239	0.6113	0.1255	0.4331
0.915	0.0862	1.2594	0.1296	0.6225	0.1302	0.4413
0.920	0.0948	1.2818	0.1355	0.6342	0.1351	0.4499
0.925	0.1037	1.3054	0.1417	0.6465	0.1403	0.4589
0.930	0.1130	1.3303	0.1482	0.6595	0.1457	0.4684
0.935	0.1226	1.3566	0.1550	0.6733	0.1513	0.4785
0.940	0.1327	1.3848	0.1621	0.6880	0.1573	0.4892
0.945	0.1433	1.4149	0.1697	0.7037	0.1637	0.5007
0.950	0.1546	1.4473	0.1778	0.7206	0.1704	0.5131
0.955	0.1666	1.4827	0.1864	0.7390	0.1778	0.5265
0.960	0.1795	1.5214	0.1959	0.7592	0.1857	0.5412
0.965	0.1936	1.5645	0.2062	0.7816	0.1944	0.5576
0.970	0.2092	1.6132	0.2177	0.8069	0.2042	0.5760
0.975	0.2268	1.6695	0.2309	0.8361	0.2154	0.5973
0.980	0.2473	1.7364	0.2463	0.8709	0.2285	0.6226
0.985	0.2721	1.8199	0.2652	0.9141	0.2447	0.6541
0.990	0.3048	1.9328	0.2902	0.9726	0.2662	0.6966
0.995	0.3553	2.1144	0.3296	1.0665	0.3001	0.7649

Numerical results related to the calculation of z_α and $u_\pm$ are given in Tables 1 and 2.

It should be noted that similar computations were first implemented in Chmielecka (1996). However, the results presented therein proved to be erroneous.

For any $p \in (0,1)$ the point $\theta_+(p)$ of maximum of the linear function $\theta_1 u(p) + \theta_2$ lies on the upper part of the boundary of A_{z_α} while that of minimum, $\theta_-(p)$, is on the lower part. Furthermore, ∂A_{z_α} contains no other points.

Let $u = a_\pm(p)$ be the values determined by the equations

$$\frac{\partial v(u,z)}{\partial u} = \mp u(p) = \Phi^{-1}(p)$$

where $\Phi(u)$ is the distribution function of the standard normal law. Then in (12),

$$\theta_2^\pm(p) = v(a_\pm(p), z_\alpha), \qquad \theta_1^\pm(p) = a_\pm(p),$$

and from (12) and (13) it follows that

$$y_\pm(p) = a_\pm(p)u(p) + v(a_\pm(p), z_\alpha). \tag{18}$$

In accordance with (14) and (11) we obtain the upper and lower functions that determine $\widehat{B}$,

$$x_\pm(p) = y_\pm(p)\sqrt{n}\, s + \overline{x}. \tag{19}$$

The numerical values of $y_\pm(p)$ for $n = 10, 20, 30$ are given in Tables 3 and 4.

Since $u(p)$ strictly increases and $u_- = u(0) = -\infty$, $u_+ = u(1) = \infty$ the events (15) and (16) are equivalent. Thus, in the case of the normal scale-location family (11), (18) and (19) determine the strong confidence region of level α. Its boundary can be evaluated with any given accuracy.

5. The case of the exponential family. Assume that the standard density in (1) is of the form

$$p(u) = e^{-u}, \qquad u > 0.$$

As noted in Section 2, we choose

$$t_1(x) = 2n(\overline{x} - x_{(1)}) \quad \text{and} \quad t_2(x) = x_{(1)}$$

where $x_{(1)}$ is as in (8). It is well known that the statistic $(t_1(x), t_2(x))$ is sufficient (see e.g. Ex. 2.3 in Zacks (1971)). Obviously, both (6) and (7) are satisfied. So, we may apply the formulae derived in Section 2.

From the cited Ex. 2.3 in Zacks (1971) it follows that $t_1(x)$ has the χ^2-distribution with $2(n-1)$ degrees of freedom while $nt_2(x)$ has the standard exponential distribution, that is, the densities of $t_1(x)$ and $t_2(x)$ are, respectively,

$$q_1(u) = \frac{1}{2^{n-1}\Gamma(n-1)} u^{n-2} e^{-u/2}, \ u > 0, \quad \text{and} \quad q_2(v) = ne^{-nv}, \ v \ge 0.$$

Hence, the joint density of the statistic $(T_1(x), T_2(x))$ given by (4) has the form

$$q(u,v) = c_n \exp\biggl(-\frac{1+2nv}{2u} - (n+1)\ln u\biggr), \quad u > 0, \ v \ge 0,$$

Table 5. Exponential distribution. The values of z_α, u_- and u_+ for $\alpha=0.95$, $n=3(1)10(2)40$

n	z_α	u_-	u_+
3	0.03520	0.02905	2.01952
4	0.31101	0.02844	0.85116
5	1.24702	0.02705	0.49832
6	3.41478	0.02551	0.34006
7	7.57448	0.02404	0.25272
8	14.52606	0.02268	0.19868
9	25.13685	0.02143	0.16242
10	40.68355	0.02031	0.13644
12	90.09095	0.01837	0.10237
14	173.00608	0.01678	0.08113
16	300.76911	0.01544	0.06677
18	484.26108	0.01431	0.05650
20	736.55739	0.01334	0.04881
22	1069.08859	0.01249	0.04288
24	1504.66121	0.01176	0.03814
26	2046.39568	0.01110	0.03431
28	2723.42942	0.01052	0.03114
30	3543.05390	0.01000	0.02847
32	4517.64660	0.00953	0.02621
34	5679.65144	0.00911	0.02426
36	7025.01376	0.00872	0.02258
38	8625.91237	0.00836	0.02109
40	10408.27405	0.00804	0.01979

where

$$c_n = \frac{n}{2^{n-1}\Gamma(n-1)}.$$

Furthermore, the function $q(u,v)$ takes the maximum value

$$q_0 = c_n \exp((n+1)(\ln(2n+2)-1))$$

at the point $(1/(2(n+1)), 0)$. For $z>0$, $u_- \le u \le u_+$ consider

$$v(u,z) = \frac{1}{2n}(2u(\ln(c_n/z) - (n+1)\ln u) - 1)$$

where $u_\pm = u_\pm(z)$ are the roots of the equation $q(u,0)=z$.

It is easily seen that $\partial A_z^- = \Gamma_1 \cup \Gamma_2$ where

$$\Gamma_1 = \{(u,v) : u_- \le u \le u_+,\ v=0\},$$
$$\Gamma_2 = \{(u,v) : u_- \le u \le u_+,\ v=-v(u,z)\}.$$

Solving the equation

$$\int_{u_-(z)}^{u_+(z)} \int_0^{v(u,z)} q(u,v)\,dv\,du = \alpha$$

with respect to $z=z_\alpha$ leads, in view of (5), to the confidence region

Table 6. Exponential distribution. The values of z_α, u_- and u_+ for $\alpha = 0.99$, $n = 3(1)10(2)40$

n	z_α	u_-	u_+
3	0.001287	0.02337	4.78584
4	0.018634	0.02339	1.57759
5	0.097056	0.02261	0.81371
6	0.309391	0.02158	0.51460
7	0.741417	0.02049	0.36454
8	1.556363	0.01950	0.27525
9	2.869955	0.01856	0.21833
10	4.760841	0.01767	0.17966
12	11.330457	0.01615	0.13007
14	22.940032	0.01486	0.10047
16	40.665211	0.01376	0.08124
18	67.517169	0.01282	0.06769
20	104.739811	0.01200	0.05777
22	156.105856	0.01130	0.05019
24	218.625988	0.01066	0.04433
26	306.959352	0.01011	0.03952
28	409.621498	0.00960	0.03565
30	536.398599	0.00915	0.03242
32	690.934186	0.00874	0.02969
34	877.166245	0.00837	0.02735
36	1099.334214	0.00804	0.02533
38	1329.557921	0.00772	0.02361
40	1631.147426	0.00743	0.02206

$$B(x) = \Big\{(\theta_1, \theta_2) : u_-(z_\alpha) \le \frac{\theta_1}{2n(\overline{x} - x_{(1)})} \le u_+(z_\alpha),$$
$$0 \le \frac{x_{(1)} - \theta_2}{2n(\overline{x} - x_{(1)})} \le v\Big(\frac{\theta_1}{2n(\overline{x} - x_{(1)})}, z_\alpha\Big)\Big\}.$$

The numerical values of z_α, $u_\pm = u_\pm(z_\alpha)$ are given in Tables 5 and 6.

It is easily seen that A_z is convex. Since $u(p) = -\ln(1-p)$, $0 < p < 1$, we have $u_- = u(0) = 0$, $u_+ = u(1) = \infty$, that is, the events (15) and (16) are not equivalent. This means that the confidence region built on the basis of (14) is not strong. Since both the components of the vector $(-\ln(1-p), 1)$ are positive the maximal value of $\theta_1 u(p) + \theta_2$ over $A^-_{z_\alpha}$ is

$$y^+(p) = u_+(z_\alpha)(-\ln(1-p))$$

for all $p \in (0,1)$. The minimal value, as in the case of the normal family, is

$$y_-(p) = a_-(p)(-\ln(1-p)) - v(a_-(p), z_\alpha)$$

where $a_-(p)$ is the root of the equation

$$\frac{\partial v(u, z_\alpha)}{\partial u} = -\ln(1-p).$$

Table 7. Exponential distribution. The values of $y_{\pm}(p)$, $p = 0.0(0.025)0.975$ for the sample size $n = 10, 20, 30$ and the confidence level $\alpha = 0.95$

p	$n = 10$		$n = 20$		$n = 30$	
	y_-	y_+	y_-	y_+	y_-	y_+
0.000	−0.02704	0.00000	−0.00571	0.00000	−0.00241	0.00000
0.025	−0.00253	0.00345	−0.00498	0.00124	−0.00194	0.00072
0.050	−0.02353	0.00700	−0.00425	0.00250	−0.00148	0.00146
0.075	−0.02177	0.01064	−0.00351	0.00381	−0.00102	0.00222
0.100	−0.02000	0.01438	−0.00278	0.00514	−0.00056	0.00300
0.125	−0.01823	0.01822	−0.00204	0.00652	−0.00009	0.00380
0.150	−0.01646	0.02217	−0.00131	0.00793	0.00037	0.00463
0.175	−0.01468	0.02625	−0.00057	0.00939	0.00083	0.00548
0.200	−0.01289	0.03045	0.00017	0.01089	0.00130	0.00635
0.225	−0.01111	0.03478	0.00091	0.01244	0.00176	0.00726
0.250	−0.00931	0.03925	0.00165	0.01404	0.00223	0.00819
0.275	−0.00751	0.04388	0.00239	0.01570	0.00270	0.00916
0.300	−0.00570	0.04866	0.00314	0.01741	0.00316	0.01016
0.325	−0.00389	0.05363	0.00388	0.01919	0.00363	0.01119
0.350	−0.00208	0.05877	0.00463	0.02103	0.00410	0.01227
0.375	−0.00025	0.06412	0.00537	0.02294	0.00456	0.01338
0.400	0.00158	0.06970	0.00612	0.02494	0.00503	0.01454
0.425	0.00342	0.07550	0.00687	0.02701	0.00550	0.01576
0.450	0.00526	0.08157	0.00762	0.02918	0.00597	0.01702
0.475	0.00711	0.08791	0.00838	0.03145	0.00644	0.01835
0.500	0.00897	0.09457	0.00913	0.03384	0.00693	0.01974
0.525	0.01084	0.10157	0.00989	0.03634	0.00745	0.02120
0.550	0.01272	0.10895	0.01065	0.03898	0.00799	0.02274
0.575	0.01461	0.11674	0.01141	0.04177	0.00856	0.02436
0.600	0.01651	0.12502	0.01222	0.04473	0.00916	0.02609
0.625	0.01842	0.13382	0.01308	0.04788	0.00981	0.02793
0.650	0.02034	0.14323	0.01400	0.05125	0.01050	0.02989
0.675	0.02227	0.15334	0.01499	0.05486	0.01124	0.03200
0.700	0.02422	0.16427	0.01606	0.05877	0.01204	0.03428
0.725	0.02618	0.17614	0.01722	0.06302	0.01291	0.03676
0.750	0.02816	0.18914	0.01849	0.06767	0.01387	0.03947
0.775	0.03030	0.20352	0.01989	0.07281	0.01492	0.04247
0.800	0.03269	0.21959	0.02147	0.07856	0.01608	0.04583
0.825	0.03540	0.23780	0.02325	0.08508	0.01743	0.04963
0.850	0.03853	0.25884	0.02530	0.09261	0.01897	0.05402
0.875	0.04224	0.28371	0.02773	0.10151	0.02080	0.05921
0.900	0.04677	0.31416	0.03071	0.11240	0.02303	0.06556
0.925	0.05261	0.35341	0.03455	0.12644	0.02591	0.07375
0.950	0.06085	0.40873	0.03995	0.14623	0.02996	0.08530
0.975	0.07493	0.50330	0.04920	0.18007	0.03689	0.10503

The numerical values of $y_{\pm}(p)$ are given in Tables 7 and 8.

Table 8. Exponential distribution. The values of $y_{\pm}(p)$, $p = 0.0(0.025)0.975$ for the sample size $n = 10, 20, 30$ and the confidence level $\alpha = 0.99$

p	$n = 10$		$n = 20$		$n = 30$	
	y_-	y_+	y_-	y_+	y_-	y_+
0.000	−.04363	0.00000	−.00867	0.00000	−0.00360	0.00000
0.025	−.04150	0.00455	−.00790	0.00146	−0.00311	0.00082
0.050	−.03937	0.00922	−.00709	0.00296	−0.00262	0.00166
0.075	−.03722	0.01401	−.00629	0.00450	−0.00213	0.00253
0.100	−.03508	0.01893	−.00548	0.00609	−0.00164	0.00342
0.125	−.03293	0.02399	−.00467	0.00771	−0.00115	0.00433
0.150	−.03077	0.02920	−.00387	0.00939	−0.00065	0.00527
0.175	−.02861	0.03456	−.00306	0.01111	−0.00016	0.00624
0.200	−.02644	0.04009	−.00225	0.01289	0.00033	0.00724
0.225	−.02427	0.04579	−.00144	0.01473	0.00083	0.00826
0.250	−.02208	0.05168	−.00062	0.01662	0.00132	0.00933
0.275	−.01990	0.05777	.00019	0.01858	0.00182	0.01043
0.300	−.01770	0.06408	.00101	0.02061	0.00231	0.01156
0.325	−.01550	0.07061	.00182	0.02271	0.00281	0.01274
0.350	−.01329	0.07739	.00264	0.02489	0.00331	0.01397
0.375	−.01107	0.08444	.00346	0.02715	0.00380	0.01524
0.400	−.00885	0.09177	.00428	0.02951	0.00430	0.01656
0.425	−.00662	0.09942	.00511	0.03197	0.00480	0.01794
0.450	−.00437	0.10740	.00593	0.03454	0.00530	0.01938
0.475	−.00212	0.11576	.00676	0.03728	0.00580	0.02089
0.500	.00014	0.12453	.00759	0.04005	0.00630	0.02247
0.525	.00241	0.13374	.00842	0.04301	0.00680	0.02414
0.550	.00469	0.14346	.00925	0.04613	0.00731	0.02589
0.575	.00699	0.15372	.01008	0.04944	0.00783	0.02774
0.600	.00929	0.16462	.01092	0.05294	0.00839	0.02971
0.625	.01161	0.17621	.01176	0.05667	0.00898	0.03180
0.650	.01395	0.18861	.01260	0.06065	0.00961	0.03404
0.675	.01630	0.20192	.01349	0.06494	0.01029	0.03644
0.700	.01866	0.21630	.01445	0.06956	0.01102	0.03904
0.725	.02105	0.23193	.01550	0.07459	0.01182	0.04186
0.750	.02345	0.24905	.01664	0.08009	0.01269	0.04495
0.775	.02587	0.26798	.01791	0.08618	0.01365	0.04836
0.800	.02783	0.28914	.01932	0.09299	0.01473	0.05218
0.825	.03080	0.31313	.02092	0.10070	0.01595	0.05651
0.850	.03353	0.34083	.02277	0.10961	0.01736	0.06151
0.875	.03675	0.37358	.02496	0.12014	0.01903	0.06743
0.900	.04069	0.41367	.02764	0.13303	0.02107	0.07466
0.925	.04578	0.46535	.03110	0.14965	0.02371	0.08398
0.950	.05294	0.53820	.03596	0.17308	0.02742	0.09713
0.975	.06519	0.66273	.04428	0.21312	0.03376	0.11960

It remains to find in (17) the shape of $\widehat{C}_\alpha$ and, therefore, that of $B^{**}(x)$. Let

$$u_0 = u_0(z) = \arg \max_{u_-(z) \le u \le u_+(z)} v(u, z)$$

and $v_0 = v_0(z) = v(u_0(z), z)$. It is readily seen that

$$\partial \widehat{C}_\alpha = \varGamma_1 \cup \varGamma_2' \cup \varGamma_3 \cup \varGamma_4$$

where

$$\begin{aligned} \varGamma_1 &= \{(u,v) : u_- \le u \le u_+,\ v = 0\}, \\ \varGamma_2' &= \{(u,v) : u_- \le u \le u_0,\ v = -v(u,z)\}, \\ \varGamma_3 &= \{(u,v) : u_0 \le u \le u_+,\ v = -v_0\}, \\ \varGamma_4 &= \{(u,v) : u = u_+,\ -v_0 \le v \le 0\}. \end{aligned}$$

Obviously

$$P_\theta(\widehat{B}) = \int_{\widehat{C}_\alpha} q(u,v)\,du\,dv.$$

For $n = 10,\ 20,\ 30$ the numerical values of $P_\theta(\widehat{B})$ are respectively $0.9645\ldots$, $0.9652\ldots$, $0.9641\ldots$

6. The case of the uniform family. Consider the uniform scale-location family where the standard density is chosen as

$$p(u) = \begin{cases} 1 & \text{if } |u| \le 1/2, \\ 0 & \text{otherwise.} \end{cases}$$

As noted in Section 2 here we choose

$$t_1(x) = x_{(n)} - x_{(1)}, \qquad t_2(x) = x_{(n)} + x_{(1)}$$

where $x_{(1)}$ and $x_{(n)}$ are as in (8).

It is well known that the statistic $(t_1(x), t_2(x))$ is sufficient (see e.g. Ex. 2.5 in Zacks (1971)). Obviously, both (6) and (7) are satisfied and, therefore, we may apply the formulae derived in Section 2.

It is well known that the joint density of $(x_{(1)}, x_{(n)})$ is

$$r(x,y) = n(n-1)(y-x)^{n-2}, \qquad -1/2 < x < y < 1/2.$$

Therefore, that of $(t_1(x),\ t_2(x))$ is

$$\widehat{r}(t_1,t_2) = \tfrac{1}{2}n(n-1)t_1^{n-2}, \quad t_1 > 0,\ t_1 + |t_2| < 1.$$

It is easily seen that the joint density of $(T_1(x),\ T_2(x))$ given by (4) is

$$q(u,v) = \tfrac{1}{2}n(n-1)u^{-n-1}, \qquad u > 1,\ |v| < u - 1.$$

Notice that here $A_z = A_z^-$ is the triangle (cf. Podraza (1996))

$$1 < u < u(z) = \left(\frac{n(n-1)}{2z}\right)^{1/(n+1)}, \qquad |v| < u - 1.$$

In order to calculate the required level z_α one should solve the equation

$$n(n-1)\int_1^{u(z)} u^{-n-1}(u-1)\,du = \alpha$$

or

$$1 - nu(z)^{-n+1} + (n-1)u(z)^{-n} = \alpha.$$

Of course, the root u_α of this equation can be easily computed.

Although the quantile function is not of great interest we give some remarks on it just for completeness.

Obviously, $u(p) = p - 1/2$, $u_- = u(0) = -1/2$, $u_+ = u(1) = 1/2$. Therefore, the confidence region (14) for the quantile function is not strong. In accordance with (13) we obtain

$$y_+(p) = (p + 1/2)u_\alpha - 1, \qquad y_-(p) = (p - 3/2)u_\alpha + 1.$$

In order to build the upper and lower functions in (11) it remains to take advantage of (14). The $\widehat{C}_\alpha$ in (17) takes the form

$$\widehat{C}_\alpha = A_{z_\alpha} \cup \left\{(u,v) : u_\alpha < u < 3u_\alpha - 2,\ |v| < -\tfrac{1}{2}u + \tfrac{3}{2}u_\alpha - 1\right\}.$$

Obviously, $P(\widehat{C}_\alpha) > \alpha$. All the computations here are also feasible.

7. Concluding remarks. As noted in Section 2, λ_2-optimality is closely related to specific properties of the statistics $t_1(x)$ and $t_2(x)$ such as invariance and equivariance. Since the class of vector statistics $(t_1(x), t_2(x))$ which satisfy (6) and (7) is rather rich the question arises how to choose them.

In the examples of the normal, exponential or uniform family there exist sufficient statistics with the required properties (6) and (7). Moreover, explicit formulae for the joint density of $t_1(x)$ and $t_2(x)$ are known and, therefore, the boundary of the optimal region A_{z_α} can be evaluated with any given accuracy. Intuitively, we expect that the sufficiency leads to the best possible confidence regions.

According to (1) a scale-location family is determined by a standard density $p(u)$. Consider the class $\mathcal{T}$ of vector statistics $\mathbf{t}(x) = (t_1(x), t_2(x))$ satisfying (6) and (7). Assume also that $\mathbf{t}(x)$ satisfies the conditions of Proposition 1. Since the optimal region depends on $\mathbf{t}(x)$ we denote it by $A_{z_\alpha}(\mathbf{t})$.

It is natural to call a statistic $\mathbf{t}^*(x)$ *optimal* for the scale-location family (1) if

$$\mathbf{t}^* \in \mathcal{T}^* = \arg\min_{\mathbf{t}\in\mathcal{T}} \lambda_2(A_{z_\alpha}(\mathbf{t})).$$

Our conjecture is that, say, in the case of the normal family,

$$(\sqrt{n}\, s, \overline{x}) \in \mathcal{T}^*.$$

So far it is not quite clear how to verify this conjecture.

Another problem of great interest is the stability of statistical inference about the scale-location parameter with respect to possible uncertainty of the model choice. Assume that a sample x is consistent with the class of scale-location families determined by a class $\mathcal{P}$ of standard densities. Assume also that $\mathcal{T}$ and $\mathcal{P}$ are such that the conditions of Proposition 1 are satisfied for any pair $(\mathbf{t}, p) \in \mathcal{T} \times \mathcal{P}$. Denote by $A_{z_\alpha}(\mathbf{t}, p)$ the λ_2-optimal region that corresponds to $(\mathbf{t}, p)$.

Following the minimax principle we should try to find a statistic that belongs to $\arg\min_{\mathbf{t}\in\mathcal{T}} \max_{p\in\mathcal{P}} \lambda_2(A_{z_\alpha}(\mathbf{t}, p))$. The theoretical solution of this problem seems to be extremely difficult. However, the Monte Carlo method can be applied at least in the case where $\mathcal{T}$ and $\mathcal{P}$ contain not too many elements. Furthermore, one should remember that as a rule the joint density of $t_1(x)$ and $t_2(x)$ is unknown. So, the boundary of $A_{z_\alpha}(\mathbf{t}, p)$ has to be estimated by the sample. In principle, this can be realized by means of the procedures suggested, say, in Tsybakov (1997) or Polonik (1997).

Acknowledgements. The authors are grateful to the referee whose numerous comments and suggestions have helped to improve the presentation of this paper.

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Received on 2.2.1999;
revised version on 21.12.2000 (1470)