APPLICATIONES MATHEMATICAE
28,2 (2001), pp. 169–180

ANDRZEJ CHYDZIŃSKI and BOGDAN SMOŁKA (Gliwice)

SELF-AVOIDING WALKS ON THE LATTICE $\mathbb{Z}^2$ WITH THE 8-NEIGHBOURHOOD SYSTEM

Abstract. This paper deals with the properties of self-avoiding walks defined on the lattice with the 8-neighbourhood system. We compute the number of walks, bridges and mean-square displacement for $N = 1$ through 13 (N is the number of steps of the self-avoiding walk). We also estimate the connective constant and critical exponents, and study finite memory and generating functions. We show applications of this kind of walk. In addition, we compute upper bounds for the number of walks and the connective constant.

Introduction and basic properties. The aim of this paper is to study the main properties of self-avoiding walks defined on the lattice with the 8-neighbourhood system. This new type is a natural extension of the ordinary self-avoiding walk (self-avoiding walk on $\mathbb{Z}^2$ with the 4-neighbourhood system; see [1], [4], [5], [7]–[10]) and is useful in many applications. An example of applications is digital image processing ([11], [12]). The grey-scaled image is an array of numbers between 0 and 255. The 8-neighbourhood system for the points of the image is more natural than the 4-neighbourhood. We use self-avoiding walks in new methods of image enhancement, smoothing and edge detection (see the "Applications" section). It is possible to apply them in many other fields like pattern recognition, binarization, etc.

A self-avoiding walk takes place on a graph. The most important graph is the *d-dimensional hypercubic lattice* $\mathbb{Z}^d$. The graph we use is $\mathbb{Z}^2$ with some additional edges. These edges are a collection of pairs of points $((i,j),(k,l))$ such that $|i-k| = 1$ and $|j-l| = 1$. In this situation, every site has eight

2000 *Mathematics Subject Classification*: 82B41, 05A15, 68U10.

Key words and phrases: self-avoiding walk, lattice with the 8-neighbourhood system, connective constant, critical exponent, mean-square displacement.

neighbours. This means that the walker can go not only north, south, west and east, but also in four additional directions.

DEFINITION 1. An *N-step self-avoiding walk ω on $\mathbb{Z}^2$ with the 8-neighbourhood system*, beginning at a site $\omega(0)$, is a sequence of sites $\omega(0), \omega(1), \ldots, \omega(N)$ satisfying $\varrho(\omega(j+1), \omega(j)) = 1$, and $\omega(i) \neq \omega(j)$ for all $i \neq j$. Here $\varrho((i,j),(k,l)) = \max\{|i-k|, |j-l|\}$.

Many interesting definitions and properties of this kind of walk are similar to definitions and properties of the ordinary self-avoiding walk. We denote by c_N the number of N-step self-avoiding walks starting at the origin. We are interested in finding values of c_N and their asymptotics as $N \to \infty$.

It is known that the number c_N resists rigorous analysis and most researchers think it is impossible to find a formula for it. Table I gives the values of c_N for $N = 1$ through 13.

In addition, it is known that the limit

$$\mu = \lim_{N\to\infty} \sqrt[N]{c_N} \tag{1}$$

exists for the ordinary self-avoiding walk. The existence of this limit follows from Lemma 1 below and the fact that the sequence $\log c_N$ is *subadditive*. Lemma 1 was proven by Hammersley and Morton [3].

LEMMA 1. *Let $\{a_n\}_{n\ge1}$ be a sequence of real numbers which is subadditive, i.e. $a_{n+m} \le a_n + a_m$. Then the limit $\lim_{n\to\infty} n^{-1}a_n$ exists in $[-\infty,\infty)$ and is equal to*

$$\lim_{n\to\infty} \frac{a_n}{n} = \inf_{n\ge1} \frac{a_n}{n}. \tag{2}$$

In the case of our self-avoiding walk the situation is similar. The number of concatenations of M-step and N-step walks is greater than the number of $M{+}N$-step self-avoiding walks: $c_{N+M} \le c_N c_M$. This implies that the sequence $\log c_N$ is subadditive, and in this way we have proven that for self-avoiding walks on $\mathbb{Z}^2$ with the 8-neighbourhood system the limit (1) exists. We call it the *connective constant*. Moreover, according to (2), we have $\log\mu = \inf_{N\ge1} \log c_N$. Thus

$$\mu^N \le c_N \tag{3}$$

for $N \ge 1$.

Bridges and their properties. There is an important subset of self-avoiding walks, called *bridges*. The definition is the same as in the ordinary case.

DEFINITION 2. An N-step self-avoiding walk $\omega = (\omega_1, \omega_2)$ is called an N-step *bridge* if

$$\omega_1(0) < \omega_1(i) \leq \omega_1(N) \quad \text{for } 1 \leq i \leq N.$$

The number of N-step bridges starting at the origin will be denoted by b_N. Table I gives the values of b_N for $N = 1$ through 13. Concatenation of two bridges will always yield another bridge: $b_M b_N \leq b_{M+N}$. This means that the sequence $\log b_N$ is *superadditive*. It is easy to check that Lemma 1 is still true if we replace "subadditive" by "superadditive" and "inf" in (2) by "sup". Therefore the limit

$$\mu_b = \lim_{n\to\infty} \sqrt[N]{b_N}$$

exists. In the same way as in (3) we obtain

$$b_N \leq \mu_b^N \leq \mu^N. \tag{4}$$

More information about the asymptotics of c_N and b_N is given by the following theorem.

THEOREM 1. *For any constant $T > \pi\sqrt{2/3}$, there exists an $N_0(T)$ such that*

$$c_N \leq b_{N+1} e^{T\sqrt{N}} \quad \textit{for all } N \geq N_0. \tag{5}$$

The proof in the ordinary case is based on Hammersley–Welsh's method (see [4]). The method uses the fact that self-avoiding walks are subadditive while bridges are superadditive, and other properties which are the same in our case. In this way we can rewrite the classical proof with obvious changes. We will do it using the notation of [8]. First we need two definitions and two lemmas.

DEFINITION 3. An *N-step half-space walk* is an N-step self-avoiding walk ω satisfying $\omega_1(0) < \omega_1(i)$ for all $i = 1, \ldots, N$. The number of N-step half-space walks starting at the origin is denoted by h_N.

DEFINITION 4. The *span* of an N-step self-avoiding walk ω is

$$\max_{0\leq j\leq N} \omega_1(j) - \min_{0\leq j\leq N} \omega_1(j).$$

The number of N-step half-space walks (respectively, bridges) starting at the origin and having span A is denoted by $h_{N,A}$ (respectively, $b_{N,A}$).

LEMMA 2. *For each integer $A \geq 1$, let $P_D(A)$ denote the number of partitions of A into distinct integers* (*i.e. the number of ways to write $A = A_1 + \ldots + A_k$ where $A_1 > \ldots > A_k$*). *Then*

$$\log P_D(A) \sim \pi\sqrt{A/3} \quad \textit{as } A \to \infty. \tag{6}$$

This classical result of number theory is proved in [6].

LEMMA 3. *For every* $N \geq 1$,

$$h_N \leq P_D(N) b_N. \tag{7}$$

Proof. Let $N \geq 1$, and let ω be an N-step half-space walk that starts at the origin. Let $n_0 = 0$. For each $j = 1, 2, \ldots$, recursively define $A_j(\omega)$ and $n_j(\omega)$ so that

$$A_j(\omega) = \max_{n_{j-1} < i \leq N} (-1)^j (\omega_1(n_{j-1}) - \omega_1(i))$$

and n_j is the largest value of i for which the maximum is attained. The recursion is stopped at the smallest integer k such that $n_k = N$. This means that $A_{k+1}(\omega)$ and $n_{k+1}(\omega)$ are not defined. Observe that $A_1(\omega)$ is the span of ω; in general, $A_{j+1}(\omega)$ is the span of the self-avoiding walk $(\omega(n_j), \ldots, \omega(N))$, which is either a half-space walk or the reflection of one. Moreover, each of the subwalks $(\omega(n_j), \ldots, \omega(n_{j+1}))$ is either a bridge or the reflection of one. Also observe that $A_1 > \ldots > A_k > 0$.

For every decreasing sequence $a_1 > a_2 > \ldots > a_k > 0$ of integers, let $H_N[a_1, \ldots, a_k]$ be the set of N-step half-space walks ω with $\omega(0) = 0$, $A_1(\omega) = a_1, \ldots, A_k(\omega) = a_k$, and $n_k(\omega) = N$. In particular $H_N[a]$ is the set of N-step bridges of span a. Given an N-step half-space walk ω, define a new N-step walk ω' as follows: for $0 \leq i \leq n_1(\omega)$, set $\omega'(i) = \omega(i)$; and for $n_1(\omega) < i \leq N$, define $\omega'(i)$ to be the reflection of $\omega(i)$ in the hyperplane $x_1 = A_1(\omega)$. Observe that if ω is in $H_N[a_1, a_2, \ldots, a_k]$, then ω' is in $H_N[a_1 + a_2, a_3, \ldots, a_k]$; moreover, this transformation is one-to-one, so

$$|H_N[a_1, a_2, \ldots, a_k]| \leq |H_N[a_1 + a_2, a_3, \ldots, a_k]|.$$

Therefore, summing over all finite integer sequences $a_1 > \ldots > a_k > 0$ we get

$$h_N = \sum |H_N[a_1, \ldots, a_k]| \leq \sum |H_N[a_1 + \ldots + a_k]| = \sum b_{N, a_1 + \ldots + a_k},$$

which tells us that

$$h_N \leq \sum_{A=1}^{N} P_D(A) b_{N,A}. \tag{8}$$

Since $P_D(A) \leq P_D(N)$ for $A \leq N$, it follows from (8) that

$$h_N \leq P_D(N) \sum_{A=1}^{N} b_{N,A}, \tag{9}$$

which proves the lemma. ∎

Proof of Theorem 1. Fix $T > \pi\sqrt{2/3}$, and choose $\varepsilon > 0$ so that $T - \varepsilon > \pi\sqrt{2/3}$. By Lemma 2, there exists a constant K such that

$$P_D(A) \leq K \exp[(T - \varepsilon)(A/2)^{1/2}] \quad \text{for all } A. \tag{10}$$

For an arbitrary n-step self-avoiding walk ω, let $M = \min_i \omega_1(i)$ and let m be the largest i such that $\omega_1(i) = M$. Then $(\omega(m), \dots, \omega(n))$ is a half-space walk, as also is

$$(\omega(m) - (1,0), \omega(m), \omega(m-1), \dots, \omega(0)).$$

Using this decomposition, as well as Lemma 3, the inequality $b_i b_j \le b_{i+j}$, (10), and the inequality $x^{1/2} + y^{1/2} \le (2x+2y)^{1/2}$, we obtain

$$\begin{aligned} c_n &\le \sum_{m=0}^{n} h_{n-m} h_{m+1} \\ &\le \sum_{m=0}^{n} b_{m+1} b_{n-m} P_D(m+1) P_D(n-m) \\ &\le b_{n+1} \sum_{m=0}^{n} K^2 \exp\left((T-\varepsilon) \left[\left(\frac{m+1}{2} \right)^{1/2} + \left(\frac{n-m}{2} \right)^{1/2} \right] \right) \\ &\le b_{n+1} (n+1) K^2 \exp[(T-\varepsilon)(n+1)^{1/2}] \end{aligned}$$

for all n. Therefore, there exists an $N_0(T)$ such that

$$c_n \le b_{n+1} e^{T\sqrt{n}} \quad \text{for all } n \ge N_0. \ \blacksquare$$

Corollary 1. *For any constant $T > \pi\sqrt{2/3}$, there exists an $N_0(T)$ such that*

$$\mu^{N-1} e^{-T\sqrt{N}} \le b_N \le c_N \le \mu^{N+1} e^{T\sqrt{N}} \quad \textit{for all } N \ge N_0. \tag{11}$$

The first inequality in (11) comes from (5) and (3), the second is obvious and the last one follows from (5) and (4).

Corollary 2. *We have*

$$\mu_b = \mu. \tag{12}$$

This follows immediately from (11).

The sequences $\sqrt[N]{c_N}$ and $\sqrt[N]{b_N}$ converge very slowly to μ. We obtain $\sqrt[13]{c_{13}} = 6.335152592$ and $\sqrt[13]{b_{13}} = 5.137860853$. In the ordinary case there is a proof that

$$\lim_{N\to\infty} \frac{b_{N+1}}{b_N} = \mu \tag{13}$$

(see [7], [8]) and there is no proof that

$$\lim_{N\to\infty} \frac{c_{N+1}}{c_N} = \mu. \tag{14}$$

Although, in our case, we have not proven any of these formulas, we expect that they are true. It appears that sequences (13) and (14) converge faster than $\sqrt[N]{c_N}$ and $\sqrt[N]{b_N}$ (Table I). Using this information we conjecture that the value of μ is about 5.8.

In our case, some properties of our self-avoiding walks lead to the supposition that (14) is easier to prove than in the ordinary case. The reason for the failure of the proof of (14) is generally speaking that in the ordinary case there does not exist a pair of self-avoiding walks having the same endpoints whose lengths differ by 1. In our case this disadvantage disappears.

More precisely, the best result in the ordinary case is

$$\lim_{N\to\infty} \frac{c_{N+2}}{c_N} = \mu^2 \tag{15}$$

(see Kesten [7]). The main idea of the proof is the following. Consider two self-avoiding walks U and V:

$$\begin{aligned} U &= ((0,0),(0,1),(0,2),(0,3),(1,3),(2,3),(3,3),(3,2),(3,1),(3,0)),\\ V &= ((0,0),(0,1),(0,2),(0,3),(1,3),(1,2),(2,2),(2,3),(3,3),(3,2),\\ &\qquad (3,1),(3,0)), \end{aligned}$$

Kesten proved that U and V as patterns must both occur many times on almost all self-avoiding walks, and changing a U to a V increases the length of a walk by two; this gives us a way to transform N-step walks into $(N+2)$-step walks, and $(N+2)$-step walks into $(N+4)$-step walks. To finish the proof he made calculations based on all the possibilities for these transformations.

On the lattice with the 8-neighbourhood system, instead of V we can use

$$V' = ((0,0),(0,1),(0,2),(0,3),(1,3),(2,2),(2,3),(3,3),(3,2),(3,1),(3,0)).$$

This gives us a method to transform N-step walks into $(N+1)$-step walks, and $(N+1)$-step walks into $(N+2)$-step walks, and opens a way to prove (14).

Finite memory. Upper bounds for μ. Since there is no formula for c_N and the exponential growth of c_N (which makes it difficult to count the number of self-avoiding walks for N relatively small by computer) it is interesting to find sharp bounds for μ. The best upper bound for μ in the ordinary case was obtained by Noonan [9] using Goulden–Jackson's method [2]. We use this method in our case as well.

DEFINITION 5. A walk is said to be self-avoiding *with memory* r if every subwalk of r-steps is self-avoiding. We denote by $c_{N,r}$ the number of N-step walks with memory r beginning at the origin.

The sequence $\log c_{N,r}$ is subadditive for every r. Thus, the limit

$$\mu_r = \lim_{n\to\infty} \sqrt[N]{c_{N,r}}$$

exists. It is easily seen that the following lemma is true (see [8]).

LEMMA 4. *$\mu_r \searrow \mu$ as $r \to \infty$.*

Definition 6. Denote by $\lambda(\omega)$ the number of steps of ω. The *generating function* for a set D of walks is defined by

$$g(D) = \sum_{\omega \in D} s^{\lambda(\omega)}.$$

A trivial verification shows that the generating function for all walks (not only self-avoiding) is $1/(1-8s)$.

Definition 7. A *mistake* is a walk that begins and ends at the same point and is otherwise self-avoiding. A *mistake with memory* r is a mistake which has at most r steps.

Definition 8. A *marked walk* is a walk in which a subset of its mistakes is marked.

Marked walks may have no mistakes marked. If a walk has all of its mistakes marked, then we say that it is *fully marked*. (See [9] for more details.)

We denote by $\overline{W}$ the set of all marked walks. For $\omega \in \overline{W}$ let $\gamma(\omega)$ be the number of mistakes of ω that are marked.

Definition 9. For $\omega \in \overline{W}$ let $\overline{g}(\omega) = (-1)^{\gamma(\omega)} s^{\lambda(\omega)}$.

Definition 10. For a set $S \subset \overline{W}$ let $\overline{g}(S) = \sum_{\omega \in S} \overline{g}(\omega)$.

Lemma 5. *The generating function for self-avoiding walks is* $\overline{g}(\overline{W})$.

The proof is the same as in the ordinary case (see [9]).

The generating function for self-avoiding walks with memory r is also $\overline{g}(\overline{W})$, but in this situation we have to use mistakes with memory r.

Definition 11. A *cluster of mistakes* is a fully marked walk with two properties: (a) every step contributes to at least one mistake and (b) the mistakes are fully overlapping (this means that the walk begins and ends at the same point).

Definition 12. The *suffix* of length k of a mistake is its last k steps. The *prefix* of length k of a mistake is its first k steps.

Definition 13. Let $S[\sigma]$ denote the set of all clusters of mistakes which terminate with the mistake σ.

With obvious changes, we can rewrite from [9] the generating function for the self-avoiding walks (self-avoiding walks with memory if we use mistakes with memory):

$$\overline{g}(\overline{W}) = \frac{1}{1 - 8s - A(s)} \tag{16}$$

with

$$A(s) = \sum_{\text{mistakes } \sigma} \overline{g}(S[\sigma]). \tag{17}$$

It is very difficult to calculate $A(s)$ for the self-avoiding walks because the set of all mistakes is infinite. For the self-avoiding walks with memory r the set of all mistakes is finite and we can calculate $A(s)$ using the following equations (see [9]) for all mistakes σ with memory r:

$$\overline{g}(S[\sigma]) = \overline{g}(\sigma) + \sum_{\substack{p \\ p \text{ is a prefix of } \sigma}} \sum_{\substack{\sigma' \\ p \text{ is a suffix of } \sigma'}} (-1)s^{\lambda(\sigma)-\lambda(p)}\overline{g}(S[\sigma']). \tag{18}$$

Of course all mistakes which can be obtained from one particular mistake by using vertical or horizontal or oblique ($\pi/4$) reflection or a composite of these transformations have identical $\overline{g}(S[\sigma])$. (Thus, we need only one such equation for every similarity class.)

For memory 2, there are eight mistakes in two classes. After solving a system of two linear equations (18) we obtain

$$A_2(s) = \frac{-8s^2}{1+s}$$

and the generating function is

$$g_2(s) = \frac{1+s}{1-7s}. \tag{19}$$

It is known that in general μ_r is equal to the inverse of the smallest positive root of the denominator. Thus, $\mu_2 = 7$.

For memory 3 there are 32 mistakes in 5 classes. After solving a system of five equations we obtain

$$A_3(s) = \frac{-8s^2 - 32s^3 + 8s^4 + 48s^5 + 8s^6}{1 + 2s + 3s^2 - 3s^3 - 6s^4 - s^5}$$

and the generating function

$$g_3(s) = \frac{-1 - 2s - 3s^2 + 3s^3 + 6s^4 + s^5}{-1 + 6s + 5s^2 - 5s^3 - 10s^4 + s^5}. \tag{20}$$

Thus, $\mu_3 = 6.608018511$.

For memory 4 there are 176 mistakes in 17 classes and we have to symbolically solve 17 equations (18). We obtain

$$\begin{aligned} g_4(s) = (&1 + 4s + 13s^2 + 30s^3 + 23s^4 - 6s^5 - 113s^6 - 240s^7 - 300s^8 \\ &- 367s^9 - 291s^{10} - 236s^{11} - 75s^{12} - 54s^{13} - 39s^{14} - 30s^{15} \\ &- 63s^{16} - 14s^{17} - 3s^{18} - s^{19} + s^{20})/(1 - 4s - 11s^2 - 26s^3 - 17s^4 \\ &+ 42s^5 + 95s^6 + 160s^7 + 148s^8 + 113s^9 + 45s^{10} + 20s^{11} + 21s^{12} \\ &+ 18s^{13} + 25s^{14} + 10s^{15} + 9s^{16} + 2s^{17} - 3s^{18} - s^{19} + s^{20}). \end{aligned}$$

Thus, $\mu_4 = 6.387630137$.

Note that our self-avoiding walks have more complicated structure. In the ordinary self-avoiding walks with memory 2, we have 4 mistakes in 1 class. For memory 4 we have 12 mistakes in 2 classes. For memory 6, we have only 5 classes.

Another method of computing an upper bound for μ uses the formula (see [1])

$$\mu \leq \sqrt[N-1]{c_N/c_1}, \tag{21}$$

which is true under the conditions determined by Lemma 1. For $N = 13$ we obtain the upper bound 6.213161909. In the same paper we can find the improvement that μ is bounded above by the unique positive root of the polynominal

$$c_1 x^{N-1} = [c_N - (c_1 - 2)c_{N-1}]x + (c_1 - 2)[(c_1 - 1)c_{N-1} - c_N], \tag{22}$$

which is true when $c_N/c_{N-1} > c_1 - 2$. The numerical solution of (22) for $N = 12$ gives the bound 6.233908491.

Mean-square displacement. Critical exponents

DEFINITION 14. Set

$$d_N = \frac{1}{c_N} \sum_{\omega} |\omega(N)|^2, \tag{23}$$

where the sum is over all N-step self-avoiding walks. We call d_N the *mean-square displacement.*

Table I gives some values of d_N. Conjectured behaviour of c_N is

$$c_N \sim E\mu^N N^{\gamma-1}, \tag{24}$$

and conjectured behaviour of d_N is

$$d_N \sim F N^{2\nu} \tag{25}$$

(see [8] for a deeper discussion). The numbers γ and ν are examples of *critical exponents*. The critical exponents are believed to be independent of the lattice structure. Applying (24) yields

$$\frac{c_N}{c_{N-1}} \approx \mu \left(\frac{N}{N-1} \right)^{\gamma-1}$$

and

$$\gamma \approx \frac{\log \frac{c_N}{\mu c_{N-1}}}{\log \frac{N}{N-1}} + 1.$$

For $N = 13$ and $\mu = 5.8$ we obtain $\gamma \approx 1.399$, which is close to the classical value $\gamma = 43/32$. Likewise we have

$$\nu \approx \frac{\log \frac{d_N}{d_{N-1}}}{2 \log \frac{N}{N-1}}$$

and for $N = 13$ we get $\nu \approx 0.724$. In the classical case $\nu = 3/4$. The fact that ν seems to be the same for our self-avoiding walks could be described in the following way. Flory's arguments that the classical ν is 3/4 (see [8]) are based on the following simplification: the distance between the starting and ending point of a self-avoiding walk has Gaussian behaviour. This means that for every $x \in \mathbb{Z}^2$ and fixed $\omega(0) = 0$ we have

$$\Pr\{\omega(N) = x\} \approx N^{-1} e^{-|x|^2/N}.$$

We expect that the same formula is correct for our walks.

Table I. Numerical results (c_N—number of walks, b_N—number of bridges, d_N—mean-squared displacements)

N	c_N	b_N	d_N	c_N/c_{N-1}	b_N/b_{N-1}
1	8	3	1.500000		
2	56	15	3.428571	7.00000	5.00000
3	368	69	5.771739	6.57142	4.60000
4	2336	357	8.506849	6.34782	5.17391
5	14576	1923	11.567234	6.23972	5.38655
6	89928	10431	14.921982	6.16959	5.42434
7	550504	57093	18.543567	6.12160	5.47340
8	3349864	315129	22.415740	6.08508	5.51957
9	20290360	1750659	26.522458	6.0570	5.55537
10	122445504	9773955	30.851395	6.03466	5.58301
11	736685008	54790011	35.391505	6.01643	5.60572
12	4421048016	308185371	40.133410	6.00127	5.62484
13	26475370088	1738482795	45.068787	5.98848	5.64103

Applications. An interesting example of applications of our walks is digital image processing. We demonstrate how to use self-avoiding random walks in image smoothing.

Treating a digital image as a 2-dimensional lattice with an 8-neighbourhood system, a virtual particle performing a self-avoiding random walk can be introduced. The transition probability between two lattice points is modelled by a median distribution, which can be viewed as a modification of the classical Gibbs distribution.

Let U be a smoothing operator defined as

$$U(i,j) = \sum_{(k,l)} P[n,(i,j),(k,l)] \cdot F(k,l), \tag{26}$$

where n is the number of steps, $F(k,l)$ is the degree of greyness of point (k,l), (i,j) is the starting point, (k,l) is the end point of a self-avoiding walk, and the transition probability between two neighbouring points (k,l) and (m,n) is given by

$$(27)\qquad P[1,(k,l),(m,n)] = \frac{\exp\{-\beta|F(k,l)-F(m,n)|\}}{Z},$$
$$Z = \sum_{(u,v)} \exp\{-\beta|F(k,l)-F(u,v)|\},$$

where Z is the statistical sum and β is the temperature coefficient.

The smoothing operator has to be applied in an iterative way. Starting with a low value of β in (27) enables the smoothing of the image noise components. At each iteration, the parameter β is increased, as in the known operation of "simulated annealing" ($\beta(k) = \beta(k-1)\cdot\delta$). After a few iterations the image becomes frozen, and further iterations do not produce visible changes.

The assumption that the random walk is self-avoiding leads to several interesting features of the operator U. It enables the elimination of small image objects, which consist of fewer than n pixels, where n is the number of steps of a self-avoiding walk. This feature makes the new filtering technique similar to mathematical morphology filters. However, it does not have the drawbacks of dilation or erosion as the larger objects are not changed and the connectivity of objects is always preserved. Unlike in mathematical morphology no structuring element is needed. Instead, the only necessary input is the number of steps, which makes this method attractive. (See [11], [12] for more details.)

Conclusions. Many of the basic properties of self-avoiding walks on the lattice $\mathbb{Z}^2$ with the 8-neighbourhood system are the same as in the ordinary case, namely: the behaviour of the number of walks, bridges, mean-square displacement; existence of the connective constant for the number of walks and bridges; some combinatorial bounds for the number of walks, etc.

Some properties of these walks (which are the consequence of their more complicated structure) lead to the supposition that a few difficult problems which are unsolved in the ordinary case may have a simpler solution in our case. For example, we expect that it is easier to prove that $\mu = \lim_{N\to\infty} c_{N+1}/c_N$ for the lattice with the 8-neighbourhood system.

Because of this and the applications of this kind of walk, further studies on this subject would be very interesting.

Though this kind of walk helps us solve some problems, the more complicated structure makes some of the numerical calculations (like finding an upper bound for μ) more difficult.

References

[1] S. E. Alm, *Upper bounds for the connective constant of self-avoiding walks*, Combin. Probab. Comput. 2 (1993), 115–136.
[2] I. P. Goulden and D. M. Jackson, *Combinatorial Enumeration*, Wiley, New York, 1983.
[3] J. M. Hammersley and K. W. Morton, *Poor man's Monte Carlo*, J. Roy. Statist. Soc. Ser. B 16 (1954), 23–38.
[4] J. M. Hammersley and D. J. A. Welsh, *Further results on the rate of convergence to the connective constant of the hypercubical lattice*, Quart. J. Math. Oxford (2) 13 (1962), 108–110.
[5] T. Hara, G. Slade and A. D. Sokal, *New lower bounds on the self-avoiding-walk connective constant*, J. Statist. Phys. 72 (1993), 479–517; erratum, 78 (1995), 1187–1188.
[6] G. H. Hardy and S. Ramanujan, *Asymptotic formulae for the distribution of integers of various types*, Proc. London Math. Soc. (2) 16 (1917), 112–132.
[7] H. Kesten, *On the number of self-avoiding walks*, J. Math. Phys. 4 (1963), 960–969.
[8] N. Madras and S. Gordon, *The Self-Avoiding Walk*, Birkhäuser, Boston, 1993.
[9] J. Noonan, *New upper bounds for the connective constants of self-avoiding walks*, J. Statist. Phys. 91 (1998), 871–888.
[10] L. G. O'Brien, *Monotonicity of the number of self-avoiding walks*, ibid. 59 (1990), 969–979.
[11] B. Smołka, A. Chydziński, K. Wojciechowski, A. Bal and A. Swierniak, *Self-avoiding random walk approach to image enhancement*, in: Proc. EUSIPCO 2000 (Tampere, September 2000), to appear.
[12] B. Smołka, A. Swierniak, K. Wojciechowski, A. Bal and A. Chydziński, *A new method of the enhancement of gel electrophoresis images*, ibid., to appear.

Andrzej Chydziński
Department of Mathematics
Silesian Technical University
Kaszubska 23
44-101 Gliwice, Poland
E-mail: andych@zeus.polsl.gliwice.pl

Bogdan Smołka
Department of Computer Science
Silesian Technical University
Akademicka 16
44-101 Gliwice, Poland
E-mail: bsmolka@peach.ia.polsl.gliwice.pl

Received on 10.11.1999;
revised version on 12.7.2000 (1510)