APPLICATIONES MATHEMATICAE
42,1 (2015), pp. 57–81

George A. Anastassiou (Memphis, TN)

APPROXIMATION BY PERTURBED NEURAL NETWORK OPERATORS

Abstract. This article deals with the determination of the rate of convergence to the unit of each of three newly introduced perturbed normalized neural network operators of one hidden layer. These are given through the modulus of continuity of the function involved or its high order derivative that appears in the right-hand side of the associated Jackson type inequalities. The activation function is very general, in particular it can derive from any sigmoid or bell-shaped function. The right-hand sides of our convergence inequalities do not depend on the activation function. The sample functionals are of Stancu, Kantorovich or quadrature types. We give applications for the first derivative of the function involved.

1. Introduction. Feed-forward neural networks (FNNs) with one hidden layer, the type of networks we deal with in this article, are mathematically expressed as

$$N_n(x) = \sum_{j=0}^{n} c_j \sigma(\langle a_j \cdot x\rangle + b_j), \qquad x \in \mathbb{R}^s,\ s \in \mathbb{N},$$

where for $0 \le j \le n$, $b_j \in \mathbb{R}$ are the thresholds, $a_j \in \mathbb{R}^s$ are the connection weights, $c_j \in \mathbb{R}$ are the coefficients, $\langle a_j \cdot x\rangle$ is the inner product of a_j and x, and σ is the activation function of the network. In many fundamental network models, the activation function is the sigmoidal function of logistic type or other sigmoidal function or bell-shaped function.

It is well known that FNNs are universal approximators. Theoretically, any continuous function defined on a compact set can be approximated to any desired degree of accuracy by increasing the number of hidden neurons.

2010 *Mathematics Subject Classification*: 41A17, 41A25, 41A30, 41A36.
Key words and phrases: neural network approximation, perturbation of operators, modulus of continuity, Jackson type inequality.

DOI: 10.4064/am42-1-5

It was proved by Cybenko [14] and Funahashi [16] that any continuous function can be approximated on a compact set in the uniform topology by a network of the form $N_n(x)$, using any continuous, sigmoidal activation function. Hornik et al. [19] have shown that any measurable function can be approached with such a network. Furthermore, these authors proved in [20] that any function in the Sobolev spaces can be approached with all derivatives. A variety of density results on FNN approximations to multivariate functions were later established by many authors using different methods, for more or less general situations: Leshno et al. [21], Mhaskar and Micchelli [25], Chui and Li [11], Chen and Chen [10], Hahm and Hong [17], etc.

Usually these results only give theorems about the existence of an approximation. A related and important problem is that of complexity: determining the number of neurons required to guarantee that all functions belonging to a space can be approximated to the prescribed degree of accuracy ϵ.

Barron [6] showed that if the function is supposed to satisfy certain conditions expressed in terms of its Fourier transform, and if each of the neurons evaluates a sigmoidal activation function, then at most $O(\epsilon^{-2})$ neurons are needed to achieve the order of approximation ϵ. Some other authors have published similar results on the complexity of FNN approximations: Mhaskar and Micchelli [26], Suzuki [29], Maiorov and Meir [22], Makovoz [23], Ferrari and Stengel [15], Xu and Cao [30], Cao et al. [7], etc.

P. Cardaliaguet and G. Euvrard [8] were the first to describe precisely and study neural network approximation operators to the unit operator. Namely they proved: given a continuous bounded function $f : \mathbb{R} \to \mathbb{R}$ and a centered bell-shaped function b, the functions

$$F_n(x) = \sum_{k=-n^2}^{n^2} \frac{f(k/n)}{In^\alpha} b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right),$$

where $I := \int_{-T}^{T} b(t)\,dt$, $0 < \alpha < 1$, converge uniformly on compacta to f.

We see that the weights $\frac{f(k/n)}{In^\alpha}$ are explicitly given. Still [8] is qualitative and not quantitative.

The author [1], [2], [3, Chap. 2–5] was the first to establish neural network approximations to continuous functions with rates, by very specifically defined neural network operators of Cardaliaguet–Euvrard and "squashing" types, by employing the modulus of continuity of the function or its high order derivative, and producing very tight Jackson type inequalities. He treated both the univariate and multivariate cases. The "bell-shaped" and "squashing" functions defining these operators were assumed to be of compact support. Also in [3, Chap. 3–5] he gave the Nth order asymptotic expansion for the error of weak approximation of these two operators for a special natural class of smooth functions.

Though the work in [1]–[3] was quantitative, the rates of convergence were not precisely determined.

Finally in [4], [5], by normalizing his operators the author determined the exact rates of convergence.

In this article the author continues and completes this work, by introducing three new perturbed neural network operators of Cardaliaguet–Euvrard type.

The sample coefficients $f(k/n)$ are replaced by three suitable natural perturbations; that is what actually happens in reality of a neural network operation.

The calculation of $f(k/n)$ at the neurons is often not performed as such, but rather in a distorted way.

Next we justify why we take here the activation function to be of compact support, which of course helps us to conduct our study.

The activation function, also called the transfer function or learning rule, is connected to firing of neurons. Firing, which sends electric pulses or an output signal to other neurons, occurs when the activation level is above the threshold level set by the learning rule.

Each neural network firing is essentially of finite time duration. Essentially the firing decays in time, but in practice it sends positive energy over a finite time interval.

Thus by using an activation function of compact support, in practice we do not alter much the good results of our approximation.

To be more precise, we may take the compact support to be a large interval symmetric about the origin. This is what happens in real time with the firing of neurons.

For more about neural networks in general we refer to [9], [12], [13], [18], [24], [27].

2. Basics. Here the activation function $b : \mathbb{R} \to \mathbb{R}_+$ is of compact support $[-T, T]$, $T > 0$. That is, $b(x) > 0$ for any $x \in [-T, T]$, and clearly b may have jump discontinuities. Also the shape of the graph of b could be arbitrary. Typically in neural network approximation we take b as a sigmoidal function or bell-shaped function, of course here of compact support $[-T, T]$, $T > 0$.

EXAMPLE 1. Here are some examples of activation functions.

(i) The characteristic function of $[-1, 1]$.

(ii) The hat function over $[-1,1]$, i.e.,

$$b(x)=\begin{cases}1+x, & -1\le x\le 0,\\ 1-x, & 0<x\le 1,\\ 0, & \text{elsewhere.}\end{cases}$$

(iii) The truncated sigmoidals

$$b(x)=\begin{cases}\frac{1}{1+e^{-x}} \text{ or } \tanh x \text{ or } \operatorname{erf}(x), & x\in[-T,T], \text{ with large } T>0,\\ 0, & x\in\mathbb{R}-[-T,T].\end{cases}$$

(iv) The truncated Gompertz function

$$b(x)=\begin{cases}e^{-\alpha e^{-\beta x}}, & x\in[-T,T],\ \alpha,\beta>0, \text{ large } T>0,\\ 0, & x\in\mathbb{R}-[-T,T].\end{cases}$$

The Gompertz functions are also sigmoidal functions, with wide applications to many applied fields, e.g. demography and tumor growth modeling.

So the general function b we will be using here covers all kinds of activation functions in neural network approximations.

Here we consider functions $f:\mathbb{R}\to\mathbb{R}$ that are either continuous and bounded, or uniformly continuous.

Let the parameters be $\mu,\nu\ge 0$; $\mu_i,\nu_i\ge 0$, $i=1,\dots,r\in\mathbb{N}$; $w_i\ge 0$, $\sum_{i=1}^r w_i=1$; $0<\alpha<1$, $x\in\mathbb{R}$, $n\in\mathbb{N}$.

We use the first modulus of continuity

$$\omega_1(f,\delta):=\sup_{\substack{x,y\in\mathbb{R}\\ |x-y|\le\delta}}|f(x)-f(y)|,$$

and given that f is uniformly continuous we get $\lim_{\delta\to 0}\omega_1(f,\delta)=0$.

In this article we mainly study the pointwise convergence with rates over $\mathbb{R}$, to the unit operator, of the following one hidden layer normalized neural network perturbed operators

$$H_n^*(f)(x)=\frac{\sum_{k=-n^2}^{n^2}\left(\sum_{i=1}^r w_i f\left(\frac{k+\mu_i}{n+\nu_i}\right)\right)b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{\sum_{k=-n^2}^{n^2}b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}, \tag{1}$$

Kantorovich type operators

$$K_n^*(f)(x)=\frac{\sum_{k=-n^2}^{n^2}\left(\sum_{i=1}^r w_i(n+\nu_i)\int_{\frac{k+\mu_i}{n+\nu_i}}^{\frac{k+\mu_i+1}{n+\nu_i}}f(t)\,dt\right)b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{\sum_{k=-n^2}^{n^2}b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}, \tag{2}$$

and quadrature type operators

$$M_n^*(f)(x)=\frac{\sum_{k=-n^2}^{n^2}\left(\sum_{i=1}^r w_i f\left(\frac{k}{n}+\frac{i}{nr}\right)\right)b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{\sum_{k=-n^2}^{n^2}b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}. \tag{3}$$

Similar operators defined for bell-shaped functions and sample coefficients $f(k/n)$ were studied initially in [8], [1]–[5], etc.

Here we study the generalized perturbed cases of these operators.

The operator K_n^*, in the corresponding signal processing context, represents the natural "time-jitter" error, where the sample information is calculated in a perturbed neighborhood of $\frac{k+\mu}{n+\nu}$ rather than exactly at the node k/n.

The perturbed sample coefficients $f\left(\frac{k+\mu}{n+\nu}\right)$ with $0 \le \mu \le \nu$ were first used by D. Stancu [28], in a totally different context, generalizing Bernstein operator approximation on $C([0,1])$.

The terms in the ratio of sums (1)–(3) are nonzero iff

$$\left| n^{1-\alpha}\left(x - \frac{k}{n}\right)\right| \le T, \quad \text{i.e.} \quad \left| x - \frac{k}{n}\right| \le \frac{T}{n^{1-\alpha}} \tag{4}$$

iff

$$nx - Tn^{\alpha} \le k \le nx + Tn^{\alpha}. \tag{5}$$

In order to have the desired order of the numbers

$$-n^2 \le nx - Tn^{\alpha} \le nx + Tn^{\alpha} \le n^2, \tag{6}$$

it is enough to assume that

$$n \ge T + |x|.$$

When $x \in [-T, T]$ it is enough to assume $n \ge 2T$, which implies (6).

Remark 2 ([1]). Let $a \le b$, $a, b \in \mathbb{R}$. Let *card* (≥ 0) be the maximum number of integers contained in $[a, b]$. Then

$$\max(0, (b-a)-1) \le \mathit{card} \le (b-a)+1.$$

Note 3. We would like to establish a lower bound on *card* over the interval $[nx - Tn^{\alpha}, nx + Tn^{\alpha}]$. From Remark 2 we get

$$\mathit{card} \ge \max(2Tn^{\alpha} - 1, 0).$$

We obtain $\mathit{card} \ge 1$ if

$$2Tn^{\alpha} - 1 \ge 1 \quad \text{iff} \quad n \ge T^{-1/\alpha}. \tag{7}$$

So to have the desired order (6) and $\mathit{card} \ge 1$ over $[nx - Tn^{\alpha}, nx + Tn^{\alpha}]$, we need to consider

$$n \ge \max(T + |x|, T^{-1/\alpha}). \tag{8}$$

Also notice that $\mathit{card} \to \infty$ as $n \to \infty$.

Denote by $[\cdot]$ the integral part of a number and by $\lceil \cdot \rceil$ its ceiling.

Under assumption (8), the operators H_n^*, K_n^*, M_n^* become

$$H_n^*(f)(x) = \frac{\sum_{k=\lceil nx-Tn^\alpha\rceil}^{[nx+Tn^\alpha]} \left(\sum_{i=1}^r w_i f\left(\frac{k+\mu_i}{n+\nu_i}\right)\right) b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{\sum_{k=\lceil nx-Tn^\alpha\rceil}^{[nx+Tn^\alpha]} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}, \tag{9}$$

$$K_n^*(f)(x) = \frac{\sum_{k=\lceil nx-Tn^\alpha\rceil}^{[nx+Tn^\alpha]} \left(\sum_{i=1}^r w_i (n+\nu_i) \int_{\frac{k+\mu_i}{n+\nu_i}}^{\frac{k+\mu_i+1}{n+\nu_i}} f(t)\,dt\right) b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{\sum_{k=\lceil nx-Tn^\alpha\rceil}^{[nx+Tn^\alpha]} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}, \tag{10}$$

$$M_n^*(f)(x) = \frac{\sum_{k=\lceil nx-Tn^\alpha\rceil}^{[nx+Tn^\alpha]} \left(\sum_{i=1}^r w_i f\left(\frac{k}{n}+\frac{i}{nr}\right)\right) b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{\sum_{k=\lceil nx-Tn^\alpha\rceil}^{[nx+Tn^\alpha]} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}. \tag{11}$$

Remark 4. Let k be as in (5). We observe that

$$\left|\frac{k+\mu}{n+\nu} - x\right| \le \left|\frac{k}{n+\nu} - x\right| + \frac{\mu}{n+\nu}.$$

Next we see that

$$\left|\frac{k}{n+\nu} - x\right| \le \left|\frac{k}{n+\nu} - \frac{k}{n}\right| + \left|\frac{k}{n} - x\right| \overset{(4)}{\le} \frac{\nu|k|}{n(n+\nu)} + \frac{T}{n^{1-\alpha}}$$

(as $|k| \le \max(|nx - Tn^\alpha|, |nx + Tn^\alpha|) \le n|x| + Tn^\alpha$)

$$\le \frac{\nu}{n+\nu}\left(|x| + \frac{T}{n^{1-\alpha}}\right) + \frac{T}{n^{1-\alpha}}.$$

Consequently,

$$\begin{aligned}\left|\frac{k+\mu}{n+\nu} - x\right| &\le \frac{\nu}{n+\nu}\left(|x| + \frac{T}{n^{1-\alpha}}\right) + \frac{T}{n^{1-\alpha}} + \frac{\mu}{n+\nu} \\ &= \frac{\nu|x|+\mu}{n+\nu} + \left(1 + \frac{\nu}{n+\nu}\right)\frac{T}{n^{1-\alpha}}.\end{aligned} \tag{12}$$

Hence we obtain

$$\omega_1\left(f, \left|\frac{k+\mu}{n+\nu} - x\right|\right) \overset{(12)}{\le} \omega_1\left(f, \frac{\nu|x|+\mu}{n+\nu} + \left(1 + \frac{\nu}{n+\nu}\right)\frac{T}{n^{1-\alpha}}\right), \tag{13}$$

where $\mu, \nu \ge 0$, $0 < \alpha < 1$, so that the dominant speed above is $1/n^{1-\alpha}$.

Also, by a change of variable, the operator K_n^* can be conveniently written as

$$(14)\quad K_n^*(f)(x) = \frac{\sum_{k=\lceil nx-Tn^\alpha\rceil}^{[nx+Tn^\alpha]} \left(\sum_{i=1}^r w_i(n+\nu_i)\int_0^{\frac{1}{n+\nu_i}} f\left(t+\frac{k+\mu_i}{n+\nu_i}\right)dt\right) b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{\sum_{k=\lceil nx-Tn^\alpha\rceil}^{[nx+Tn^\alpha]} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}.$$

3. Main results. We present our first approximation result.

THEOREM 5. *Let $x \in \mathbb{R}$, $T > 0$ and $n \in \mathbb{N}$ be such that $n \geq \max(T+|x|, T^{-1/\alpha})$. Then*

$$(15)\quad |H_n^*(f)(x) - f(x)| \leq \sum_{i=1}^r w_i\omega_1\left(f, \frac{\nu_i|x|+\mu_i}{n+\nu_i} + \left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right) \leq \max_{i\in\{1,\dots,r\}} \omega_1\left(f, \frac{\nu_i|x|+\mu_i}{n+\nu_i} + \left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right).$$

Proof. We notice that

$$H_n^*(f)(x) - f(x) = \frac{\sum_{k=\lceil nx-Tn^\alpha\rceil}^{[nx+Tn^\alpha]} \left(\sum_{i=1}^r w_i f\left(\frac{k+\mu_i}{n+\nu_i}\right)\right) b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{\sum_{k=\lceil nx-Tn^\alpha\rceil}^{[nx+Tn^\alpha]} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)} - f(x)$$

$$= \frac{\sum_{k=\lceil nx-Tn^\alpha\rceil}^{[nx+Tn^\alpha]} \left(\sum_{i=1}^r w_i f\left(\frac{k+\mu_i}{n+\nu_i}\right)\right) b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right) - f(x)\sum_{k=\lceil nx-Tn^\alpha\rceil}^{[nx+Tn^\alpha]} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{\sum_{k=\lceil nx-Tn^\alpha\rceil}^{[nx+Tn^\alpha]} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}$$

$$= \frac{\sum_{k=\lceil nx-Tn^\alpha\rceil}^{[nx+Tn^\alpha]} \left(\sum_{i=1}^r w_i f\left(\frac{k+\mu_i}{n+\nu_i}\right) - f(x)\right) b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{\sum_{k=\lceil nx-Tn^\alpha\rceil}^{[nx+Tn^\alpha]} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}$$

$$= \frac{\sum_{k=\lceil nx-Tn^\alpha\rceil}^{[nx+Tn^\alpha]} \left(\sum_{i=1}^r w_i \left(f\left(\frac{k+\mu_i}{n+\nu_i}\right) - f(x)\right)\right) b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{\sum_{k=\lceil nx-Tn^\alpha\rceil}^{[nx+Tn^\alpha]} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}.$$

Hence

$$|H_n^*(f)(x) - f(x)| \leq \frac{\sum_{k=\lceil nx-Tn^\alpha\rceil}^{[nx+Tn^\alpha]} \left(\sum_{i=1}^r w_i \left|f\left(\frac{k+\mu_i}{n+\nu_i}\right) - f(x)\right|\right) b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{\sum_{k=\lceil nx-Tn^\alpha\rceil}^{[nx+Tn^\alpha]} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}$$

$$\leq \frac{\sum_{k=\lceil nx-Tn^\alpha\rceil}^{[nx+Tn^\alpha]} \left(\sum_{i=1}^r w_i \omega_1\left(f, \left|\frac{k+\mu_i}{n+\nu_i} - x\right|\right)\right) b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{\sum_{k=\lceil nx-Tn^\alpha\rceil}^{[nx+Tn^\alpha]} b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}$$

$$\overset{(13)}{\leq} \frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\big(\sum_{i=1}^{r} w_i\omega_1\big(f, \frac{\nu_i|x|+\mu_i}{n+\nu_i} + \big(1+\frac{\nu_i}{n+\nu_i}\big)\frac{T}{n^{1-\alpha}}\big)\big) b\big(n^{1-\alpha}\big(x-\frac{k}{n}\big)\big)}{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} b\big(n^{1-\alpha}\big(x-\frac{k}{n}\big)\big)}$$

$$= \frac{\big[\sum_{i=1}^{r} w_i\omega_1\big(f, \frac{\nu_i|x|+\mu_i}{n+\nu_i} + \big(1+\frac{\nu_i}{n+\nu_i}\big)\frac{T}{n^{1-\alpha}}\big)\big]\big(\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} b\big(n^{1-\alpha}\big(x-\frac{k}{n}\big)\big)\big)}{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} b\big(n^{1-\alpha}\big(x-\frac{k}{n}\big)\big)}$$

$$= \sum_{i=1}^{r} w_i\omega_1\left(f, \frac{\nu_i|x|+\mu_i}{n+\nu_i} + \left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right)$$

$$\leq \max_{i\in\{1,\dots,r\}} \omega_1\left(f, \frac{\nu_i|x|+\mu_i}{n+\nu_i} + \left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right),$$

proving the claim. ∎

Corollary 6. *Let $x \in [-T^*, T^*]$, $T^* > 0$, $n \in \mathbb{N}$, $n \geq \max(T+T^*, T^{-1/\alpha})$, $T > 0$. Then*

$$\|H_n^*(f) - f\|_{\infty,[-T^*,T^*]} \leq \sum_{i=1}^{r} w_i\omega_1\left(f, \frac{\nu_i T^*+\mu_i}{n+\nu_i} + \left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right) \leq \max_{i\in\{1,\dots,r\}} \omega_1\left(f, \frac{\nu_i T^*+\mu_i}{n+\nu_i} + \left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right). \tag{16}$$

Proof. By (15). ∎

Theorem 7. *Let $x \in \mathbb{R}$, $T > 0$ and $n \in \mathbb{N}$ be such that $n \geq \max(T+|x|, T^{-1/\alpha})$. Then*

$$|K_n^*(f)(x) - f(x)| \leq \max_{i\in\{1,\dots,r\}} \omega_1\left(f, \frac{\nu_i|x|+\mu_i+1}{n+\nu_i} + \left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right). \tag{17}$$

Proof. Set

$$\delta_{n,k}(f) = \sum_{i=1}^{r} w_i(n+\nu_i)\int_0^{\frac{1}{n+\nu_i}} f\left(t+\frac{k+\mu_i}{n+\nu_i}\right)dt. \tag{18}$$

We observe that

$$\delta_{n,k}(f) - f(x) = \sum_{i=1}^{r} w_i(n+\nu_i)\int_0^{\frac{1}{n+\nu_i}} f\left(t+\frac{k+\mu_i}{n+\nu_i}\right)dt - f(x)$$
$$= \sum_{i=1}^{r} w_i(n+\nu_i)\int_0^{\frac{1}{n+\nu_i}} \left(f\left(t+\frac{k+\mu_i}{n+\nu_i}\right) - f(x)\right)dt.$$

Hence

$$
\begin{aligned}
(19)\qquad |\delta_{n,k}(f)-f(x)| &\le \sum_{i=1}^{r} w_i(n+\nu_i)\int_0^{\frac{1}{n+\nu_i}}\left|f\left(t+\frac{k+\mu_i}{n+\nu_i}\right)-f(x)\right|dt\\
&\le \sum_{i=1}^{r} w_i(n+\nu_i)\int_0^{\frac{1}{n+\nu_i}}\omega_1\left(f,|t|+\left|\frac{k+\mu_i}{n+\nu_i}-x\right|\right)dt\\
&\le \sum_{i=1}^{r} w_i\omega_1\left(f,\frac{1}{n+\nu_i}+\left|\frac{k+\mu_i}{n+\nu_i}-x\right|\right)\\
&\overset{(12)}{\le} \sum_{i=1}^{r} w_i\omega_1\left(f,\frac{\nu_i|x|+\mu_i+1}{n+\nu_i}+\left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right)\\
&\le \max_{i\in\{1,\dots,r\}}\omega_1\left(f,\frac{\nu_i|x|+\mu_i+1}{n+\nu_i}+\left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right).
\end{aligned}
$$

Therefore by (14) and (18) we get

$$
K_n^*(f)(x)-f(x)=\frac{\sum_{k=\lceil nx-Tn^\alpha\rceil}^{[nx+Tn^\alpha]}\delta_{n,k}(f)b\big(n^{1-\alpha}\big(x-\frac{k}{n}\big)\big)}{\sum_{k=\lceil nx-Tn^\alpha\rceil}^{[nx+Tn^\alpha]}b\big(n^{1-\alpha}\big(x-\frac{k}{n}\big)\big)}-f(x)
$$

$$
=\frac{\sum_{k=\lceil nx-Tn^\alpha\rceil}^{[nx+Tn^\alpha]}\delta_{n,k}(f)b\big(n^{1-\alpha}\big(x-\frac{k}{n}\big)\big)-f(x)\sum_{k=\lceil nx-Tn^\alpha\rceil}^{[nx+Tn^\alpha]}b\big(n^{1-\alpha}\big(x-\frac{k}{n}\big)\big)}{\sum_{k=\lceil nx-Tn^\alpha\rceil}^{[nx+Tn^\alpha]}b\big(n^{1-\alpha}\big(x-\frac{k}{n}\big)\big)}
$$

$$
=\frac{\sum_{k=\lceil nx-Tn^\alpha\rceil}^{[nx+Tn^\alpha]}(\delta_{n,k}(f)-f(x))b\big(n^{1-\alpha}\big(x-\frac{k}{n}\big)\big)}{\sum_{k=\lceil nx-Tn^\alpha\rceil}^{[nx+Tn^\alpha]}b\big(n^{1-\alpha}\big(x-\frac{k}{n}\big)\big)}.
$$

Consequently, we obtain

$$
\begin{aligned}
&|K_n^*(f)(x)-f(x)|\\
&\quad\le \frac{\sum_{k=\lceil nx-Tn^\alpha\rceil}^{[nx+Tn^\alpha]}|\delta_{n,k}(f)-f(x)|b\big(n^{1-\alpha}\big(x-\frac{k}{n}\big)\big)}{\sum_{k=\lceil nx-Tn^\alpha\rceil}^{[nx+Tn^\alpha]}b\big(n^{1-\alpha}\big(x-\frac{k}{n}\big)\big)}\\
&\quad\overset{(19)}{\le}\left(\sum_{k=\lceil nx-Tn^\alpha\rceil}^{[nx+Tn^\alpha]}b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)\right)\\
&\qquad\cdot\frac{\max_{i\in\{1,\dots,r\}}\omega_1\big(f,\frac{\nu_i|x|+\mu_i+1}{n+\nu_i}+\big(1+\frac{\nu_i}{n+\nu_i}\big)\frac{T}{n^{1-\alpha}}\big)}{\sum_{k=\lceil nx-Tn^\alpha\rceil}^{[nx+Tn^\alpha]}b\big(n^{1-\alpha}\big(x-\frac{k}{n}\big)\big)}\\
&\quad=\max_{i\in\{1,\dots,r\}}\omega_1\left(f,\frac{\nu_i|x|+\mu_i+1}{n+\nu_i}+\left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right),
\end{aligned}
$$

proving the claim. ∎

COROLLARY 8. *Let* $x \in [-T^*, T^*]$, $T^* > 0$, $n \in \mathbb{N}$, $n \geq \max(T + T^*, T^{-1/\alpha})$, $T > 0$. *Then*

$$\|K_n^*(f) - f\|_{\infty,[-T^*,T^*]} \leq \max_{i\in\{1,\dots,r\}} \omega_1\left(f, \frac{\nu_i T^* + \mu_i + 1}{n+\nu_i} + \left(1 + \frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right). \tag{20}$$

Proof. By (17). ∎

THEOREM 9. *Let* $x \in \mathbb{R}$, $T > 0$ *and* $n \in \mathbb{N}$, $n \geq \max(T + |x|, T^{-1/\alpha})$. *Then*

$$|M_n^*(f)(x) - f(x)| \leq \omega_1\left(f, \frac{1}{n} + \frac{T}{n^{1-\alpha}}\right). \tag{21}$$

Proof. Let k be as in (5). Set

$$\lambda_{nk}(f) = \sum_{i=1}^{r} w_i f\left(\frac{k}{n} + \frac{i}{nr}\right),$$

so

$$\lambda_{nk}(f) - f(x) = \sum_{i=1}^{r} w_i \left(f\left(\frac{k}{n} + \frac{i}{nr}\right) - f(x)\right).$$

Then

$$\begin{aligned} |\lambda_{nk}(f) - f(x)| &\leq \sum_{i=1}^{r} w_i \left| f\left(\frac{k}{n} + \frac{i}{nr}\right) - f(x)\right| \\ &\leq \sum_{i=1}^{r} w_i \omega_1\left(f, \left|\frac{k}{n} - x\right| + \frac{i}{nr}\right) \\ &\leq \sum_{i=1}^{r} w_i \omega_1\left(f, \frac{T}{n^{1-\alpha}} + \frac{1}{n}\right) = \omega_1\left(f, \frac{1}{n} + \frac{T}{n^{1-\alpha}}\right). \end{aligned} \tag{22}$$

By (11) we can write

$$M_n^*(f)(x) = \frac{\sum_{k=\lceil nx - Tn^\alpha\rceil}^{[nx+Tn^\alpha]} \lambda_{nk}(f) b\big(n^{1-\alpha}\big(x - \frac{k}{n}\big)\big)}{\sum_{k=\lceil nx - Tn^\alpha\rceil}^{[nx+Tn^\alpha]} b\big(n^{1-\alpha}\big(x - \frac{k}{n}\big)\big)}.$$

That is, we have

$$M_n^*(f)(x) - f(x) = \frac{\sum_{k=\lceil nx - Tn^\alpha\rceil}^{[nx+Tn^\alpha]} (\lambda_{nk}(f) - f(x)) b\big(n^{1-\alpha}\big(x - \frac{k}{n}\big)\big)}{\sum_{k=\lceil nx - Tn^\alpha\rceil}^{[nx+Tn^\alpha]} b\big(n^{1-\alpha}\big(x - \frac{k}{n}\big)\big)}.$$

Hence we easily derive by (22), as before, that

$$|M_n^*(f)(x) - f(x)| \leq \omega_1\left(f, \frac{1}{n} + \frac{T}{n^{1-\alpha}}\right),$$

proving the claim. ∎

COROLLARY 10. *Let $x \in [-T^*, T^*]$, $T^* > 0$, $n \in \mathbb{N}$, $n \geq \max(T + T^*, T^{-1/\alpha})$, $T > 0$. Then*

$$\|M_n^*(f) - f\|_{\infty,[-T^*,T^*]} \leq \omega_1\left(f, \frac{1}{n} + \frac{T}{n^{1-\alpha}}\right). \tag{23}$$

Proof. By (21). ∎

Theorems 5, 7, 9 and Corollaries 6, 8, 10, given that f is uniformly continuous, produce the pointwise and uniform convergences with speed $1/n^{1-\alpha}$ of the neural network operators H_n^*, K_n^*, M_n^* to the unit operator. Notice that the right-hand sides of inequalities (15)–(17), (20), (21) and (23) do not depend on b.

We proceed to the following results where we use the smoothness of a derivative of f.

THEOREM 11. *Let $x \in \mathbb{R}$, $T > 0$ and $n \in \mathbb{N}$, $n \geq \max(T + |x|, T^{-1/\alpha})$. Let $f \in C^N(\mathbb{R})$, $N \in \mathbb{N}$, be such that $f^{(N)}$ is uniformly continuous or is continuous and bounded. Then*

$$\begin{aligned} &|H_n^*(f)(x) - f(x)| \\ &\quad \leq \sum_{j=1}^{N} \frac{|f^{(j)}(x)|}{j!} \left\{ \sum_{i=1}^{r} w_i \left[\frac{\nu_i |x| + \mu_i}{n + \nu_i} + \left(1 + \frac{\nu_i}{n + \nu_i}\right) \frac{T}{n^{1-\alpha}} \right]^j \right\} \\ &\quad + \sum_{i=1}^{r} w_i \omega_1\left(f^{(N)}, \frac{\nu_i |x| + \mu_i}{n + \nu_i} + \left(1 + \frac{\nu_i}{n + \nu_i}\right) \frac{T}{n^{1-\alpha}}\right) \\ &\quad \cdot \frac{\left(\frac{\nu_i |x| + \mu_i}{n + \nu_i} + \left(1 + \frac{\nu_i}{n + \nu_i}\right) \frac{T}{n^{1-\alpha}}\right)^N}{N!}. \end{aligned} \tag{24}$$

Inequality (24) *implies the pointwise convergence of $H_n^*(f)(x)$ to $f(x)$ as $n \to \infty$ with speed $1/n^{1-\alpha}$.*

Proof. Let k be as in (5). We observe that

$$\begin{aligned} w_i f\left(\frac{k + \mu_i}{n + \nu_i}\right) &= \sum_{j=0}^{N} \frac{f^{(j)}(x)}{j!} w_i \left(\frac{k + \mu_i}{n + \nu_i} - x\right)^j \\ &+ w_i \int_x^{\frac{k+\mu_i}{n+\nu_i}} (f^{(N)}(t) - f^{(N)}(x)) \frac{\left(\frac{k+\mu_i}{n+\nu_i} - t\right)^{N-1}}{(N-1)!} \, dt, \qquad i = 1, \ldots, r. \end{aligned}$$

Set

$$V(x) = \sum_{k=\lceil nx - Tn^\alpha \rceil}^{[nx + Tn^\alpha]} b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right).$$

Hence

$$\frac{\left(\sum_{i=1}^{r} w_i f\left(\frac{k+\mu_i}{n+\nu_i}\right)\right) b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)}$$
$$= \sum_{j=0}^{N} \frac{f^{(j)}(x)}{j!} \left(\sum_{i=1}^{r} w_i \left(\frac{k+\mu_i}{n+\nu_i} - x \right)^j \right) \frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)}$$
$$+ \frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)} \left(\sum_{i=1}^{r} w_i \int_{x}^{\frac{k+\mu_i}{n+\nu_i}} \left(f^{(N)}(t) - f^{(N)}(x)\right) \frac{\left(\frac{k+\mu_i}{n+\nu_i} - t\right)^{N-1}}{(N-1)!}\, dt \right).$$

Therefore (see (9))

$$H_n^*(f)(x) - f(x) = \sum_{j=1}^{N} \frac{f^{(j)}(x)}{j!} \sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \sum_{i=1}^{r} w_i \left(\frac{k+\mu_i}{n+\nu_i} - x \right)^j \frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)} + R, \tag{25}$$

where

$$R = \sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)} \cdot \sum_{i=1}^{r} w_i \int_{x}^{\frac{k+\mu_i}{n+\nu_i}} \left(f^{(N)}(t) - f^{(N)}(x)\right) \frac{\left(\frac{k+\mu_i}{n+\nu_i} - t\right)^{N-1}}{(N-1)!}\, dt. \tag{26}$$

Hence

$$|H_n^*(f)(x) - f(x)|$$
$$\leq \sum_{j=1}^{N} \frac{|f^{(j)}(x)|}{j!} \sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \sum_{i=1}^{r} w_i \left| \frac{k+\mu_i}{n+\nu_i} - x \right|^j \frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)} + |R|$$
$$\leq \sum_{j=1}^{N} \frac{|f^{(j)}(x)|}{j!} \sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \sum_{i=1}^{r} w_i \left[\frac{\nu_i|x| + \mu_i}{n+\nu_i} + \left(1 + \frac{\nu_i}{n+\nu_i}\right) \frac{T}{n^{1-\alpha}} \right]^j \cdot \frac{b\left(n^{1-\alpha}\left(x-\frac{k}{n}\right)\right)}{V(x)} + |R|$$
$$= \sum_{j=0}^{N} \frac{|f^{(j)}(x)|}{j!} \sum_{i=1}^{r} w_i \left[\frac{\nu_i|x| + \mu_i}{n+\nu_i} + \left(1 + \frac{\nu_i}{n+\nu_i}\right) \frac{T}{n^{1-\alpha}} \right]^j + |R|.$$

Furthermore we see that

$$|R| \leq \sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{b\big(n^{1-\alpha}\big(x-\frac{k}{n}\big)\big)}{V(x)}$$

$$\cdot \sum_{i=1}^{r} w_i \left| \int_{x}^{\frac{k+\mu_i}{n+\nu_i}} (f^{(N)}(t)-f^{(N)}(x)) \frac{\big(\frac{k+\mu_i}{n+\nu_i}-t\big)^{N-1}}{(N-1)!}\, dt \right|$$

$$\leq \sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{b\big(n^{1-\alpha}\big(x-\frac{k}{n}\big)\big)}{V(x)} \gamma,$$

where

$$\gamma := \sum_{i=1}^{r} w_i \left| \int_{x}^{\frac{k+\mu_i}{n+\nu_i}} |f^{(N)}(t)-f^{(N)}(x)| \frac{\big|\frac{k+\mu_i}{n+\nu_i}-t\big|^{N-1}}{(N-1)!}\, dt \right|.$$

Let first $x \leq \frac{k+\mu_i}{n+\nu_i}$. Then

$$\varepsilon_i := \int_{x}^{\frac{k+\mu_i}{n+\nu_i}} |f^{(N)}(t)-f^{(N)}(x)| \frac{\big|\frac{k+\mu_i}{n+\nu_i}-t\big|^{N-1}}{(N-1)!}\, dt \tag{27}$$

$$\leq \omega_1\left(f^{(N)}, \left|\frac{k+\mu_i}{n+\nu_i}-x\right|\right) \int_{x}^{\frac{k+\mu_i}{n+\nu_i}} \frac{\big(\frac{k+\mu_i}{n+\nu_i}-t\big)^{N-1}}{(N-1)!}\, dt$$

$$= \omega_1\left(f^{(N)}, \left|\frac{k+\mu_i}{n+\nu_i}-x\right|\right) \frac{\big|\frac{k+\mu_i}{n+\nu_i}-t\big|^{N}}{N!}$$

$$\overset{(12)}{\leq} \omega_1\left(f^{(N)}, \frac{\nu_i|x|+\mu_i}{n+\nu_i} + \left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right)$$

$$\cdot \frac{\big[\frac{\nu_i|x|+\mu_i}{n+\nu_i} + \big(1+\frac{\nu_i}{n+\nu_i}\big)\frac{T}{n^{1-\alpha}}\big]^{N}}{N!}.$$

Let now $x > \frac{k+\mu_i}{n+\nu_i}$. Then

$$\rho_i := \int_{\frac{k+\mu_i}{n+\nu_i}}^{x} |f^{(N)}(t)-f^{(N)}(x)| \frac{\big(t-\frac{k+\mu_i}{n+\nu_i}\big)^{N-1}}{(N-1)!}\, dt \tag{28}$$

$$\leq \omega_1\left(f^{(N)}, \left|\frac{k+\mu_i}{n+\nu_i}-x\right|\right) \frac{\big(x-\frac{k+\mu_i}{n+\nu_i}\big)^{N}}{N!}$$

$$= \omega_1\left(f^{(N)}, \left|\frac{k+\mu_i}{n+\nu_i}-x\right|\right) \frac{\big|\frac{k+\mu_i}{n+\nu_i}-x\big|^{N}}{N!}$$

$$\le \omega_1\left(f^{(N)}, \frac{\nu_i|x|+\mu_i}{n+\nu_i} + \left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right) \cdot \frac{\left(\frac{\nu_i|x|+\mu_i}{n+\nu_i} + \left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right)^N}{N!}.$$

Notice that in (27) and (28) we obtained the same upper bound. Hence

$$\gamma \le \sum_{i=1}^{r} w_i\omega_1\left(f^{(N)}, \frac{\nu_i|x|+\mu_i}{n+\nu_i} + \left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right) \cdot \frac{\left(\frac{\nu_i|x|+\mu_i}{n+\nu_i} + \left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right)^N}{N!} =: E. \tag{29}$$

Thus

$$|R| \le E, \tag{30}$$

proving the claim. ∎

Corollary 12. *Under the assumptions of Theorem* 11, *plus* $f^{(j)}(x) = 0$, $j = 1, \ldots, N$, *we have*

$$|H_n^*(f)(x) - f(x)| \le \sum_{i=1}^{r} w_i\omega_1\left(f^{(N)}, \frac{\nu_i|x|+\mu_i}{n+\nu_i} + \left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right) \cdot \frac{\left(\frac{\nu_i|x|+\mu_i}{n+\nu_i} + \left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right)^N}{N!}. \tag{31}$$

Proof. By (25), (26), (29) and (30). ∎

In (31) notice the extremely high speed of convergence, $1/n^{(1-\alpha)(N+1)}$.

Uniform convergence with rates follows from

Corollary 13. *Let* $x \in [-T^*, T^*]$, $T^* > 0$, $T > 0$ *and* $n \in \mathbb{N}$ *be such that* $n \ge \max(T + T^*, T^{-1/\alpha})$. *Let* $f \in C^N(\mathbb{R})$, $N \in \mathbb{N}$, *be such that* $f^{(N)}$ *is uniformly continuous or is continuous and bounded. Then*

$$\begin{aligned} \|H_n^*(f) - f\|_{\infty,[-T^*,T^*]} &\le \sum_{j=1}^{N} \frac{\|f^{(j)}\|_{\infty,[-T^*,T^*]}}{j!} \sum_{i=1}^{r} w_i\left[\frac{\nu_i T^*+\mu_i}{n+\nu_i} + \left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right]^j \\ &+ \sum_{i=1}^{r} w_i\omega_1\left(f^{(N)}, \frac{\nu_i T^*+\mu_i}{n+\nu_i} + \left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right) \cdot \frac{\left(\frac{\nu_i T^*+\mu_i}{n+\nu_i} + \left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right)^N}{N!}. \end{aligned} \tag{32}$$

Proof. By (24). ∎

COROLLARY 14. *Under the assumptions of Theorem* 11 *with* $N = 1$ *we have*

$$
\begin{aligned}
(33)\quad & |H_n^*(f)(x) - f(x)| \\
&\le |f'(x)| \sum_{i=1}^{r} w_i \left[\frac{\nu_i|x| + \mu_i}{n+\nu_i} + \left(1 + \frac{\nu_i}{n+\nu_i}\right) \frac{T}{n^{1-\alpha}} \right] \\
&\quad + \sum_{i=1}^{r} w_i \omega_1 \left(f', \frac{\nu_i|x| + \mu_i}{n+\nu_i} + \left(1 + \frac{\nu_i}{n+\nu_i}\right) \frac{T}{n^{1-\alpha}} \right) \\
&\quad \cdot \left(\frac{\nu_i|x| + \mu_i}{n+\nu_i} + \left(1 + \frac{\nu_i}{n+\nu_i}\right) \frac{T}{n^{1-\alpha}} \right).
\end{aligned}
$$

THEOREM 15. *Suppose the assumptions of Theorem* 11 *hold, but now with* $0 < \alpha < 1/2$. *Then*

$$
\begin{aligned}
(34)\quad & |K_n^*(f)(x) - f(x)| \le 2 \sum_{j=1}^{N} \frac{|f^{(j)}(x)|}{(j+1)!} \\
&\cdot \sum_{i=1}^{r} w_i (n+\nu_i) \left[\left(\frac{\nu_i|x| + \mu_i + 1}{n+\nu_i} + \left(1 + \frac{\nu_i}{n+\nu_i}\right) \frac{T}{n^{1-\alpha}} \right)^{j+1} \right] \\
&\quad + \sum_{i=1}^{r} w_i \omega_1 \left(f^{(N)}, \frac{\nu_i|x| + \mu_i + 1}{n+\nu_i} + \left(1 + \frac{\nu_i}{n+\nu_i}\right) \frac{T}{n^{1-\alpha}} \right) \\
&\quad \cdot \frac{\left(\frac{\nu_i|x|+\mu_i+1}{n+\nu_i} + \left(1 + \frac{\nu_i}{n+\nu_i}\right) \frac{T}{n^{1-\alpha}} \right)^N}{N!}.
\end{aligned}
$$

Inequality (34) *implies the pointwise convergence of* $K_n^*(f)(x)$ *to* $f(x)$ *as* $n \to \infty$ *with speed* $1/n^{1-2\alpha}$.

Proof. Let k be as in (5). We observe that

$$
\begin{aligned}
&\int_0^{\frac{1}{n+\nu_i}} f\left(t + \frac{k+\mu_i}{n+\nu_i}\right) dt \\
&= \sum_{j=0}^{N} \frac{f^{(j)}(x)}{j!} \int_0^{\frac{1}{n+\nu_i}} \left(t + \frac{k+\mu_i}{n+\nu_i} - x\right)^j dt \\
&\quad + \int_0^{\frac{1}{n+\nu_i}} \left(\int_x^{t+\frac{k+\mu_i}{n+\nu_i}} (f^{(N)}(z) - f^{(N)}(x)) \frac{\left(t + \frac{k+\mu_i}{n+\nu_i} - z\right)^{N-1}}{(N-1)!} dz \right) dt
\end{aligned}
$$

for $i = 1, \ldots, r$. That is,

$$\int_0^{\frac{1}{n+\nu_i}} f\left(t + \frac{k+\mu_i}{n+\nu_i}\right) dt$$
$$= \sum_{j=0}^{N} \frac{f^{(j)}(x)}{(j+1)!}\left[\left(\frac{k+\mu_i+1}{n+\nu_i} - x\right)^{j+1} - \left(\frac{k+\mu_i}{n+\nu_i} - x\right)^{j+1}\right]$$
$$+ \int_0^{\frac{1}{n+\nu_i}} \left(\int_x^{t+\frac{k+\mu_i}{n+\nu_i}} (f^{(N)}(z) - f^{(N)}(x)) \frac{\left(t + \frac{k+\mu_i}{n+\nu_i} - z\right)^{N-1}}{(N-1)!} dz\right) dt$$

for $i = 1, \ldots, r$. Furthermore we have

$$\sum_{i=1}^{r} w_i(n+\nu_i) \int_0^{\frac{1}{n+\nu_i}} f\left(t + \frac{k+\mu_i}{n+\nu_i}\right) dt$$
$$= \sum_{j=0}^{N} \frac{f^{(j)}(x)}{(j+1)!} \sum_{i=1}^{r} w_i(n+\nu_i)\left[\left(\frac{k+\mu_i+1}{n+\nu_i} - x\right)^{j+1} - \left(\frac{k+\mu_i}{n+\nu_i} - x\right)^{j+1}\right]$$
$$+ \sum_{i=1}^{r} w_i(n+\nu_i) \int_0^{\frac{1}{n+\nu_i}} \left(\int_x^{t+\frac{k+\mu_i}{n+\nu_i}} (f^{(N)}(z) - f^{(N)}(x)) \frac{\left(t + \frac{k+\mu_i}{n+\nu_i} - z\right)^{N-1}}{(N-1)!} dz\right) dt.$$

Set

$$V(x) = \sum_{k=\lceil nx - Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right).$$

Consequently, we get

$$K_n^*(f)(x)$$
$$= \frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \left(\sum_{i=1}^{r} w_i(n+\nu_i) \int_0^{\frac{1}{n+\nu_i}} f\left(t + \frac{k+\mu_i}{n+\nu_i}\right) dt\right) b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}{V(x)}$$
$$= \sum_{j=0}^{N} \frac{f^{(j)}(x)}{(j+1)!} \sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}{V(x)}$$
$$\cdot \sum_{i=1}^{r} w_i(n+\nu_i)\left[\left(\frac{k+\mu_i+1}{n+\nu_i} - x\right)^{j+1} - \left(\frac{k+\mu_i}{n+\nu_i} - x\right)^{j+1}\right]$$

$$+\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\frac{b\big(n^{1-\alpha}\big(x-\frac{k}{n}\big)\big)}{V(x)}\sum_{i=1}^{r}w_i(n+\nu_i)$$

$$\cdot\int_0^{\frac{1}{n+\nu_i}}\left(\int_x^{t+\frac{k+\mu_i}{n+\nu_i}}(f^{(N)}(z)-f^{(N)}(x))\frac{\big(t+\frac{k+\mu_i}{n+\nu_i}-z\big)^{N-1}}{(N-1)!}\,dz\right)dt.$$

Therefore

$$K_n^*(f)(x)-f(x)=\sum_{j=0}^{N}\frac{f^{(j)}(x)}{(j+1)!}\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\frac{b\big(n^{1-\alpha}\big(x-\frac{k}{n}\big)\big)}{V(x)}$$
$$\cdot\sum_{i=1}^{r}w_i(n+\nu_i)\left[\left(\frac{k+\mu_i+1}{n+\nu_i}-x\right)^{j+1}-\left(\frac{k+\mu_i}{n+\nu_i}-x\right)^{j+1}\right]+R, \tag{35}$$

where

$$R=\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\frac{b\big(n^{1-\alpha}\big(x-\frac{k}{n}\big)\big)}{V(x)}\sum_{i=1}^{r}w_i(n+\nu_i)$$
$$\cdot\int_0^{\frac{1}{n+\nu_i}}\left(\int_x^{t+\frac{k+\mu_i}{n+\nu_i}}(f^{(N)}(z)-f^{(N)}(x))\frac{\big(t+\frac{k+\mu_i}{n+\nu_i}-z\big)^{N-1}}{(N-1)!}dz\right)dt. \tag{36}$$

We derive that

$$|K_n^*(f)(x)-f(x)|\le\sum_{j=1}^{N}\frac{f^{(j)}(x)}{(j+1)!}\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\frac{b\big(n^{1-\alpha}\big(x-\frac{k}{n}\big)\big)}{V(x)}$$
$$\cdot\sum_{i=1}^{r}w_i(n+\nu_i)\left[\left|\frac{k+\mu_i+1}{n+\nu_i}-x\right|^{j+1}-\left|\frac{k+\mu_i}{n+\nu_i}-x\right|^{j+1}\right]+|R|$$
$$\overset{(12)}{\le}\sum_{j=1}^{N}\frac{|f^{(j)}(x)|}{(j+1)!}\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\frac{b\big(n^{1-\alpha}\big(x-\frac{k}{n}\big)\big)}{V(x)}$$
$$\cdot\sum_{i=1}^{r}w_i(n+\nu_i)\left[\left(\frac{\nu_i|x|+\mu_i+1}{n+\nu_i}+\left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right)^{j+1}\right.$$
$$\left.+\left(\frac{\nu_i|x|+\mu_i}{n+\nu_i}+\left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right)^{j+1}\right]+|R|$$
$$=\sum_{j=1}^{N}\frac{|f^{(j)}(x)|}{(j+1)!}\sum_{i=1}^{r}w_i(n+\nu_i)\left[\left(\frac{\nu_i|x|+\mu_i+1}{n+\nu_i}+\left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right)^{j+1}\right.$$
$$\left.+\left(\frac{\nu_i|x|+\mu_i}{n+\nu_i}+\left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right)^{j+1}\right]+|R|.$$

Consequently,

$$
(37)\qquad |K_n^*(f)(x)-f(x)| \le 2\sum_{j=1}^{N}\frac{|f^{(j)}(x)|}{(j+1)!}
\cdot\sum_{i=1}^{r} w_i(n+\nu_i)\left[\frac{\nu_i|x|+\mu_i+1}{n+\nu_i}+\left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right]^{j+1}+|R|.
$$

Clearly the sum in (37) converges to zero with speed $1/n^{1-2\alpha}$ as $n\to\infty$ given that $0<\alpha<1/2$.

We notice that

$$
(38)\qquad |R| \overset{(36)}{\le} \sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]}\frac{b\big(n^{1-\alpha}\big(x-\frac{k}{n}\big)\big)}{V(x)}\sum_{i=1}^{r} w_i(n+\nu_i)
\cdot\int_0^{\frac{1}{n+\nu_i}}\left|\int_x^{t+\frac{k+\mu_i}{n+\nu_i}}|f^{(N)}(z)-f^{(N)}(x)|\frac{\left|t+\frac{k+\mu_i}{n+\nu_i}-z\right|^{N-1}}{(N-1)!}\,dz\right|dt =: (\xi).
$$

We distinguish two cases. If $t+\frac{k+\mu_i}{n+\nu_i}\ge x$, then

$$
\begin{aligned}
(39)\qquad \theta_i &:= \left|\int_x^{t+\frac{k+\mu_i}{n+\nu_i}}|f^{(N)}(t)-f^{(N)}(x)|\frac{\left|t+\frac{k+\mu_i}{n+\nu_i}-z\right|^{N-1}}{(N-1)!}\,dz\right| \\
&= \int_x^{t+\frac{k+\mu_i}{n+\nu_i}}|f^{(N)}(t)-f^{(N)}(x)|\frac{\left(t+\frac{k+\mu_i}{n+\nu_i}-z\right)^{N-1}}{(N-1)!}\,dz \\
&\le \omega_1\left(f^{(N)},|t|+\left|\frac{k+\mu_i}{n+\nu_i}-x\right|\right)\frac{\left(|t|+\left|\frac{k+\mu_i}{n+\nu_i}-x\right|\right)^{N}}{N!} \\
&\overset{(12)}{\le} \omega_1\left(f^{(N)},\frac{\nu_i|x|+\mu_i+1}{n+\nu_i}+\left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right) \\
&\quad\cdot\frac{\left(\frac{\nu_i|x|+\mu_i+1}{n+\nu_i}+\left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right)^{N}}{N!}.
\end{aligned}
$$

If $t+\frac{k+\mu_i}{n+\nu_i}<x$, then

$$
\begin{aligned}
\theta_i &:= \int_{t+\frac{k+\mu_i}{n+\nu_i}}^{x}|f^{(N)}(z)-f^{(N)}(x)|\frac{\left(z-\left(t+\frac{k+\mu_i}{n+\nu_i}\right)\right)^{N-1}}{(N-1)!}\,dz \\
&\le \omega_1\left(f^{(N)},|t|+\left|\frac{k+\mu_i}{n+\nu_i}-x\right|\right)\frac{\left(|t|+\left|\frac{k+\mu_i}{n+\nu_i}-x\right|\right)^{N}}{N!}
\end{aligned}
$$

$$\overset{(12)}{\leq} \omega_1\left(f^{(N)}, \frac{\nu_i|x|+\mu_i+1}{n+\nu_i} + \left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right) \cdot \frac{\left(\frac{\nu_i|x|+\mu_i+1}{n+\nu_i} + \left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right)^N}{N!},$$

the same estimate as in (39).

Therefore we derive (see (38))

$$(\xi) \leq \sum_{k=\lceil nx-Tn^\alpha\rceil}^{[nx+Tn^\alpha]} \frac{b(n^{1-\alpha}(x-\frac{k}{n}))}{V(x)} \cdot \sum_{i=1}^r w_i\omega_1\left(f^{(N)}, \frac{\nu_i|x|+\mu_i+1}{n+\nu_i} + \left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right) \cdot \frac{\left(\frac{\nu_i|x|+\mu_i+1}{n+\nu_i} + \left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right)^N}{N!}.$$

Finally, we have found the estimate

$$(40) \qquad |R| \leq \sum_{i=1}^r w_i\omega_1\left(f^{(N)}, \frac{\nu_i|x|+\mu_i+1}{n+\nu_i} + \left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right) \cdot \frac{\left(\frac{\nu_i|x|+\mu_i+1}{n+\nu_i} + \left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right)^N}{N!}.$$

Based on (37), (38) and (40) we derive (34). ■

COROLLARY 16. *Under the assumptions of Theorem* 15, *plus* $f^{(j)}(x)=0$, $j=1,\dots,N$, $0<\alpha<1$, *we have*

$$(41) \qquad |K_n^*(f)(x)-f(x)| \leq \sum_{i=1}^r w_i\omega_1\left(f^{(N)}, \frac{\nu_i|x|+\mu_i+1}{n+\nu_i} + \left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right) \cdot \frac{\left(\frac{\nu_i|x|+\mu_i+1}{n+\nu_i} + \left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right)^N}{N!}.$$

Proof. By (35), (36) and (40). ■

In (41) notice the extremely high speed of convergence, $1/n^{(1-\alpha)(N+1)}$. Uniform convergence with rates follows from

COROLLARY 17. *Let* $x \in [-T^*, T^*]$, $T^*>0$, $T>0$ *and* $n\in\mathbb{N}$ *with* $n \geq \max(T+T^*, T^{-1/\alpha})$, $0<\alpha<1/2$. *Let* $f\in C^N(\mathbb{R})$, $N\in\mathbb{N}$, *be such that* $f^{(N)}$ *is uniformly continuous or is continuous and bounded. Then*

$$(42)\qquad \|K_n^*(f)-f\|_{\infty,[-T^*,T^*]} \le 2\sum_{j=1}^{N}\frac{\|f^{(j)}\|_{\infty,[-T^*,T^*]}}{(j+1)!} \cdot \sum_{i=1}^{r} w_i(n+\nu_i)\left[\frac{\nu_i T^*+\mu_i+1}{n+\nu_i}+\left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right]^{j+1} + \sum_{i=1}^{r} w_i\omega_1\left(f^{(N)}, \frac{\nu_i T^*+\mu_i+1}{n+\nu_i}+\left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right) \cdot \frac{\left(\frac{\nu_i T^*+\mu_i+1}{n+\nu_i}+\left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right)^N}{N!}.$$

Proof. By (34). ∎

Corollary 18. *Under the assumptions of Theorem* 15 *with* $N=1$, *we have*

$$(43)\qquad |K_n^*(f)(x)-f(x)| \le |f'(x)|\sum_{i=1}^{r} w_i(n+\nu_i)\left[\frac{\nu_i|x|+\mu_i+1}{n+\nu_i}+\left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right]^2 + \sum_{i=1}^{r} w_i\omega_1\left(f', \frac{\nu_i|x|+\mu_i+1}{n+\nu_i}+\left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right) \cdot \left(\frac{\nu_i|x|+\mu_i+1}{n+\nu_i}+\left(1+\frac{\nu_i}{n+\nu_i}\right)\frac{T}{n^{1-\alpha}}\right).$$

Proof. By (34). ∎

Theorem 19. *Suppose the assumptions of Theorem* 11 *hold. Then*

$$(44)\qquad |M_n^*(f)(x)-f(x)| \le \sum_{j=1}^{N}\frac{|f^{(j)}(x)|}{j!}\left[\frac{T}{n^{1-\alpha}}+\frac{1}{n}\right]^j + \omega_1\left(f^{(N)}, \frac{T}{n^{1-\alpha}}+\frac{1}{n}\right)\frac{\left(\frac{T}{n^{1-\alpha}}+\frac{1}{n}\right)^N}{N!}.$$

Inequality (44) *implies the pointwise convergence of* $M_n^*(f)(x)$ *to* $f(x)$, *as* $n\to\infty$, *with speed* $1/n^{1-\alpha}$.

Proof. Let k be as in (5). Again by Taylor's formula we have

$$\sum_{i=1}^{r} w_i f\left(\frac{k}{n}+\frac{i}{nr}\right) = \sum_{j=0}^{N}\frac{f^{(j)}(x)}{j!}\sum_{i=1}^{r} w_i\left(\frac{k}{n}+\frac{i}{nr}-x\right)^j + \sum_{i=1}^{r} w_i \int_x^{\frac{k}{n}+\frac{i}{nr}} (f^{(N)}(t)-f^{(N)}(x))\frac{\left(\frac{k}{n}+\frac{i}{nr}-t\right)^{N-1}}{(N-1)!}\,dt.$$

Set

$$V(x) = \sum_{k=\lceil nx - Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right).$$

Then

$$M_n^*(f)(x) = \frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \left(\sum_{i=1}^{r} w_i f\left(\frac{k}{n} + \frac{i}{nr}\right)\right) b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}{V(x)}$$

$$= \sum_{j=0}^{N} \frac{f^{(j)}(x)}{j!} \sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}{V(x)}$$

$$\cdot \sum_{i=1}^{r} w_i \left(\frac{k}{n} + \frac{i}{nr} - x\right)^j + \sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}{V(x)}$$

$$\cdot \sum_{i=1}^{r} w_i \int_{x}^{\frac{k}{n}+\frac{i}{nr}} \left(f^{(N)}(t) - f^{(N)}(x)\right) \frac{\left(\frac{k}{n} + \frac{i}{nr} - t\right)^{N-1}}{(N-1)!}\, dt.$$

Therefore we get

$$M_n^*(f)(x) - f(x) = \sum_{j=1}^{N} \frac{f^{(j)}(x)}{j!} \sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}{V(x)}$$

$$\cdot \sum_{i=1}^{r} w_i \left(\frac{k}{n} + \frac{i}{nr} - x\right)^j + R,$$

where

$$R = \sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}{V(x)}$$

$$\cdot \sum_{i=1}^{r} w_i \int_{x}^{\frac{k}{n}+\frac{i}{nr}} \left(f^{(N)}(t) - f^{(N)}(x)\right) \frac{\left(\frac{k}{n} + \frac{i}{nr} - t\right)^{N-1}}{(N-1)!}\, dt.$$

Hence

$$|M_n^*(f)(x) - f(x)| \leq \sum_{j=1}^{N} \frac{f^{(j)}(x)}{j!} \sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{b\left(n^{1-\alpha}\left(x - \frac{k}{n}\right)\right)}{V(x)}$$

$$\cdot \sum_{i=1}^{r} w_i \left[\left|\frac{k}{n} - x\right| + \frac{i}{nr}\right]^j + |R|$$

$$\leq \sum_{j=1}^{N} \frac{|f^{(j)}(x)|}{j!} \frac{\sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} b\big(n^{1-\alpha}\big(x-\frac{k}{n}\big)\big)}{V(x)} \sum_{i=1}^{r} w_i \left[\frac{T}{n^{1-\alpha}} + \frac{1}{n}\right]^j + |R|$$
$$= \sum_{j=1}^{N} \frac{|f^{(j)}(x)|}{j!} \left[\frac{T}{n^{1-\alpha}} + \frac{1}{n}\right]^j + |R|.$$

Next we observe

$$|R| \leq \sum_{k=\lceil nx-Tn^{\alpha}\rceil}^{[nx+Tn^{\alpha}]} \frac{b\big(n^{1-\alpha}\big(x-\frac{k}{n}\big)\big)}{V(x)} \cdot \sum_{i=1}^{r} w_i \left| \int_{x}^{\frac{k}{n}+\frac{i}{nr}} |f^{(N)}(t) - f^{(N)}(x)| \frac{\left|\frac{k}{n}+\frac{i}{nr}-t\right|^{N-1}}{(N-1)!}\, dt \right|. \tag{45}$$

Set

$$\varepsilon_i := \left| \int_{x}^{\frac{k}{n}+\frac{i}{nr}} |f^{(N)}(t) - f^{(N)}(x)| \frac{\left|\frac{k}{n}+\frac{i}{nr}-t\right|^{N-1}}{(N-1)!}\, dt \right|.$$

We distinguish two cases. If $\frac{k}{n}+\frac{i}{nr} \geq x$, then

$$\varepsilon_i := \int_{x}^{\frac{k}{n}+\frac{i}{nr}} |f^{(N)}(t) - f^{(N)}(x)| \frac{\left(\frac{k}{n}+\frac{i}{nr}-t\right)^{N-1}}{(N-1)!}\, dt \tag{46}$$
$$\leq \omega_1\left(f^{(N)}, \left|\frac{k}{n}-x\right| + \frac{1}{n}\right) \frac{\left(\frac{k}{n}+\frac{i}{nr}-x\right)^{N}}{N!}$$
$$\leq \omega_1\left(f^{(N)}, \frac{T}{n^{1-\alpha}} + \frac{1}{n}\right) \frac{\left(\frac{T}{n^{1-\alpha}}+\frac{1}{n}\right)^{N}}{N!}.$$

If $\frac{k}{n}+\frac{i}{nr} < x$, then

$$\varepsilon_i := \int_{\frac{k}{n}+\frac{i}{nr}}^{x} |f^{(N)}(t) - f^{(N)}(x)| \frac{\left(t-\left(\frac{k}{n}+\frac{i}{nr}\right)\right)^{N-1}}{(N-1)!}\, dt$$
$$\leq \omega_1\left(f^{(N)}, x-\left(\frac{k}{n}+\frac{i}{nr}\right)\right) \frac{\left(x-\left(\frac{k}{n}+\frac{i}{nr}\right)\right)^{N}}{N!}$$
$$\leq \omega_1\left(f^{(N)}, \frac{T}{n^{1-\alpha}} + \frac{1}{n}\right) \frac{\left(\frac{T}{n^{1-\alpha}}+\frac{1}{n}\right)^{N}}{N!}.$$

So we obtain (46) again.

Clearly now by (45) we derive that

$$|R| \leq \omega_1\left(f^{(N)}, \frac{T}{n^{1-\alpha}} + \frac{1}{n}\right) \frac{\left(\frac{T}{n^{1-\alpha}}+\frac{1}{n}\right)^{N}}{N!},$$

proving the claim. ∎

COROLLARY 20. *Under the assumptions of Theorem* 19, *plus* $f^{(j)}(x) = 0$, $j = 1, \dots, N$, *we have*

$$(47)\qquad |M_n^*(f)(x) - f(x)| \le \omega_1\left(f^{(N)}, \frac{T}{n^{1-\alpha}} + \frac{1}{n}\right)\frac{\left(\frac{T}{n^{1-\alpha}} + \frac{1}{n}\right)^N}{N!}.$$

Proof. By (44). ∎

In (47) notice the extremely high speed of convergence, $1/n^{(1-\alpha)(N+1)}$. The uniform convergence estimate follows:

COROLLARY 21. *Under the assumptions of Corollary* 13,

$$(48)\qquad \|M_n^*(f) - f\|_{\infty,[-T^*,T^*]} \le \sum_{j=1}^{N} \frac{\|f^{(j)}\|_{\infty,[-T^*,T^*]}}{j!}\left(\frac{T}{n^{1-\alpha}} + \frac{1}{n}\right)^j + \omega_1\left(f^{(N)}, \frac{T}{n^{1-\alpha}} + \frac{1}{n}\right)\frac{\left(\frac{T}{n^{1-\alpha}} + \frac{1}{n}\right)^N}{N!}.$$

Proof. By (44). ∎

COROLLARY 22. *Under the assumptions of Theorem* 19 *with* $N = 1$, *we have*

$$(49)\qquad |M_n^*(f)(x) - f(x)| \le \left[|f'(x)| + \omega_1\left(f', \frac{T}{n^{1-\alpha}} + \frac{1}{n}\right)\right]\left(\frac{T}{n^{1-\alpha}} + \frac{1}{n}\right).$$

Proof. By (44). ∎

NOTE 23. We also observe that all the right-hand sides of convergence inequalities (24), (31)–(34), (41)–(44), (47–(49) are independent of b.

NOTE 24. We observe that

$$H_n^*(1) = K_n^*(1) = M_n^*(1) = 1,$$

thus they are unitary operators.

Also, given that f is bounded, we get

$$(50)\qquad \|H_n^*(f)\|_{\infty,\mathbb{R}} \le \|f\|_{\infty,\mathbb{R}},$$

$$(51)\qquad \|K_n^*(f)\|_{\infty,\mathbb{R}} \le \|f\|_{\infty,\mathbb{R}},$$

$$(52)\qquad \|M_n^*(f)\|_{\infty,\mathbb{R}} \le \|f\|_{\infty,\mathbb{R}}.$$

The operators H_n^*, K_n^*, M_n^* are positive linear operators, and of course bounded, by (50)–(52).

References

[1] G. A. Anastassiou, *Rate of convergence of some neural network operators to the unit-univariate case*, J. Math. Anal. Appl. 212 (1997), 237–262.

[2] G. A. Anastassiou, *Rate of convergence of some multivariate neural network operators to the unit*, J. Comput. Math. Appl. 40 (2000), 1–19.

[3] G. A. Anastassiou, *Quantitative Approximations*, Chapman&Hall/CRC, Boca Raton, FL, 2001.

[4] G. A. Anastassiou, *Rate of convergence of some neural network operators to the unit-univariate case, revisited*, Mat. Vesnik 65 (2013), 511–518.

[5] G. A. Anastassiou, *Rate of convergence of some multivariate neural network operators to the unit, revisited*, J. Comput. Anal. Appl. 15 (2013), 1300–1309.

[6] A. R. Barron, *Universal approximation bounds for superpositions of a sigmoidal function*, IEEE Trans. Inform. Theory 39 (1993), 930–945.

[7] F. L. Cao, T. F. Xie and Z. B. Xu, *The estimate for approximation error of neural networks: a constructive approach*, Neurocomputing 71 (2008), 626–630.

[8] P. Cardaliaguet and G. Euvrard, *Approximation of a function and its derivative with a neural network*, Neural Networks 5 (1992), 207–220.

[9] Z. Chen and F. Cao, *The approximation operators with sigmoidal functions*, Comput. Math. Appl. 58 (2009), 758–765.

[10] T. P. Chen and H. Chen, *Universal approximation to nonlinear operators by neural networks with arbitrary activation functions and its applications to a dynamic system*, IEEE Trans. Neural Networks 6 (1995), 911–917.

[11] C. K. Chui and X. Li, *Approximation by ridge functions and neural networks with one hidden layer*, J. Approx. Theory 70 (1992), 131–141.

[12] D. Costarelli and R. Spigler, *Approximation results for neural network operators activated by sigmoidal functions*, Neural Networks 44 (2013), 101–106.

[13] D. Costarelli and R. Spigler, *Multivariate neural network operators with sigmoidal activation functions*, Neural Networks 48 (2013), 72–77.

[14] G. Cybenko, *Approximation by superpositions of a sigmoidal function*, Math. Control Signals Systems 2 (1989), 303–314.

[15] S. Ferrari and R. F. Stengel, *Smooth function approximation using neural networks*, IEEE Trans. Neural Networks 16 (2005), 24–38.

[16] K. I. Funahashi, *On the approximate realization of continuous mappings by neural networks*, Neural Networks 2 (1989), 183–192.

[17] N. Hahm and B. I. Hong, *An approximation by neural networks with a fixed weight*, Computers Math. Appl. 47 (2004), 1897–1903.

[18] S. Haykin, *Neural Networks: A Comprehensive Foundation*, 2nd ed., Prentice-Hall, New York, 1998.

[19] K. Hornik, M. Stinchombe and H. White, *Multilayer feedforward networks are universal approximations*, Neural Networks 2 (1989), 359–366.

[20] K. Hornik, M. Stinchombe and H. White, *Universal approximation of an unknown mapping and its derivatives using multilayer feedforward networks*, Neural Networks 3 (1990), 551–560.

[21] M. Leshno, V. Y. Lin, A. Pinks and S. Schocken, *Multilayer feedforward networks with a nonpolynomial activation function can approximate any function*, Neural Networks 6 (1993), 861–867.

[22] V. Maiorov and R. S. Meir, *Approximation bounds for smooth functions in* $C(R^d)$ *by neural and mixture networks*, IEEE Trans. Neural Networks 9 (1998), 969–978.

[23] Y. Makovoz, *Uniform approximation by neural networks*, J. Approx. Theory 95 (1998), 215–228.

[24] W. McCulloch and W. Pitts, *A logical calculus of the ideas immanent in nervous activity*, Bull. Math. Biophys. 7 (1943), 115–133.

[25] H. N. Mhaskar and C. A. Micchelli, *Approximation by superposition of sigmoidal and radial basis functions*, Adv. Appl. Math. 13 (1992), 350–373.

[26] H. N. Mhaskar and C. A. Micchelli, *Degree of approximation by neural networks with a single hidden layer*, Adv. Appl. Math. 16 (1995), 151–183.

[27] T. M. Mitchell, *Machine Learning*, WCB-McGraw-Hill, New York, 1997.

[28] D. D. Stancu, *On a generalization of the Bernstein polynomials*, Studia Univ. Babeş-Bolyai Ser. Math.-Phys. 14 (1969), 31–45.

[29] S. Suzuki, *Constructive function approximation by three-layer artificial neural networks*, Neural Networks 11 (1998), 1049–1058.

[30] Z. B. Xu and F. L. Cao, *The essential order of approximation for neural networks*, Sci. in China (Ser. F) 47 (2004), 97–112.

George A. Anastassiou
Department of Mathematical Sciences
University of Memphis
Memphis, TN 38152, U.S.A.
E-mail: ganastss@memphis.edu

Received on 18.5.2014 (2216)