APPLICATIONES MATHEMATICAE
42,1 (2015), pp. 83–109

TADEUSZ RADZIK and PIOTR WIĘCEK (Wrocław)

SIMPLE EQUILIBRIA IN FINITE GAMES WITH CONVEXITY PROPERTIES

Abstract. This review paper gives a characterization of non-coalitional zero-sum and non-zero-sum games with finite strategy spaces and payoff functions having some concavity or convexity properties. The characterization is given in terms of the existence of two-point Nash equilibria, that is, equilibria consisting of mixed strategies with spectra consisting of at most two pure strategies. The structure of such simple equilibria is discussed in various cases. In particular, many of the results discussed can be seen as discrete counterparts of classical theorems about the existence of pure (or "almost pure") Nash equilibria in continuous concave (convex) games with compact convex spaces of pure strategies. The paper provides many examples illustrating the results presented and ends with four related open problems.

1. Introduction. Consider an n-person non-zero-sum game in normal form

$$G = \langle N, \{X_i\}_{i \in N}, \{F_i\}_{i \in N} \rangle,$$

where

(1) $N = \{1, \ldots, n\}$ is the set of players;
(2) for each $i \in N$, X_i is the space of the ith player's *pure strategies* x_i;
(3) for each vector $x = (x_1, \ldots, x_n)$ of the players' pure strategies and for each i, $F_i(x)$ is the *payoff function* of player i when the players use the strategies $x_1, \ldots, x_n$, respectively.

The classical concept of solution for such games is *pure Nash equilibrium.* It is defined as any *strategy profile* x^* consisting of the players' pure strate-

---

2010 *Mathematics Subject Classification*: 91A05, 91A06, 91A10, 91B52.
Key words and phrases: non-cooperative game, matrix game, bimatrix game, finite game, game on the unit square, saddle point, Nash equilibrium, convex payoff, pure strategy, two-point strategy.

DOI: 10.4064/am42-1-6 © Instytut Matematyczny PAN, 2015

gies, $x^* = (x_1^*, \ldots, x_n^*) \in \prod_{i \in N} X_i$, satisfying $F_i(x^*) \geq F_i(x_1^*, \ldots, x_{i-1}^*, x_i, x_{i+1}^*, \ldots, x_n^*)$ for each $i \in N$ and any $x_i \in X_i$. When all these inequalities hold up to an $\epsilon > 0$, we say that x^* is an *ϵ-pure Nash equilibrium.* As this kind of solution has a natural interpretation and is easy to implement, it is very desirable in practical applications. Thus, a lot of research has been devoted to the problem of existence of pure Nash equilibria. The most important result of this type was proved by Glicksberg (1952) and Debreu (1952). (Recall that by definition, a real-valued function f on a convex set X is *quasi-concave* when for each real c, the set $\{x : f(x) \geq c\}$ is convex. Of course, every concave function is quasi-concave. When the converse inequality holds, f is *quasi-convex.*)

THEOREM 1A. *Let $X_i \subset \mathbb{R}^k$ for $i \in N$ be non-empty, convex and compact. If all the functions F_i are continuous on $\prod_{i \in N} X_i$ and quasi-concave in x_i, then the n-person non-zero-sum game $G = \langle N, \{X_i\}_{i \in N}, \{F_i\}_{i \in N} \rangle$ has a pure Nash equilibrium.*

Some extensions of this result were given by Nikaido and Isoda (1955), Mertens (1986), Sion (1958) and Fan (1952), where the existence of pure Nash equilibria was shown under some weaker assumptions either about continuity or convexity. Unfortunately, in the case of finite games (games with finite strategy sets), the existence of a pure Nash equilibrium is not guaranteed. In fact, it is easy to find an example of a game with two players, each having two pure strategies, which has no pure Nash equilibrium (and also no ϵ-pure Nash equilibria for ϵ small enough).

The standard way to deal with the lack of pure Nash equilibria is to enrich the strategy sets by introducing so-called *mixed strategies.* A mixed strategy of player $i \in N$ in the game Γ is any probability distribution over the space X_i. We can then extend the definitions of the payoff functions F_i to this richer domain by defining them as expected values with respect to the product distribution induced by the strategies of all the players. Thus, we will treat the payoff functions F_i as defined also on the spaces of mixed strategy profiles. A Nash equilibrium in mixed strategies will then be called a *mixed Nash equilibrium.* It is known that any n-person game with all X_i finite always has a mixed Nash equilibrium (Nash, 1950).

One of the main points of criticism against the concept of mixed Nash equilibria has always been that such solutions are rather difficult to apply. If a game is only played once, any realization of a mixed Nash equilibrium is in fact a choice of pure strategies for the players, which may give payoffs far from the assumed optimum. Also if the game is repeated a finite number of times, the empirical distributions of pure strategies used by the players may fail to be close to the ones prescribed by the strategies. It is however important to notice that this problem becomes more visible as the supports of the equilibrium strategies become larger. Thus a number of papers consider the so-called *two-point Nash equilibria*, that is, Nash equilibria of the form $\mu^* = (\mu_1^*, \ldots, \mu_n^*)$, where for each i, μ_i^* is a probability distribution with spectrum consisting of at most two pure strategies of player i. The results found in the literature concern only two-person games on the unit square and two-person games with finite spaces of strategies. In the first group we can list the papers of Bohnenblust et al. (1984), Gómez (1988) and Radzik (1991b) for zero-sum games, and of Parthasarathy and Raghavan (1971) and Radzik (1993) for non-zero-sum games. As the last paper contains the most general results and is very closely related to the present one, we quote two of its theorems.

Let $G = \langle\{1,2\}, \{[0,1],[0,1]\}, \{F_1, F_2\}\rangle$ be a two-person non-zero-sum game on the unit square, where the payoff functions $F_1(x,y)$ and $F_2(x,y)$ for players 1 and 2 are assumed to be bounded and bounded from above on $[0,1] \times [0,1]$, respectively. Throughout, we denote by δ_t the degenerate probability distribution concentrated at t.

THEOREM 1B. *Let $F_1(x,y)$ be concave in x for each y. Then for any $\varepsilon > 0$ the game $G = \langle\{1,2\}, \{[0,1],[0,1]\}, \{F_1, F_2\}\rangle$ has an ε-Nash equilibrium of the form $(\mu_1^*, \mu_2^*) = (\alpha\delta_a + (1-\alpha)\delta_b, \beta\delta_c + (1-\beta)\delta_d)$ for some $0 \le \alpha, \beta, a, b, c, d \le 1$ with $|a-b| < \varepsilon$.*

THEOREM 1C. *Let $F_1(x,y)$ be convex in x for each y. Then for any $\varepsilon > 0$ the game $G = \langle\{1,2\}, \{[0,1],[0,1]\}, \{F_1, F_2\}\rangle$ has an ε-Nash equilibrium of the form $(\mu_1^*, \mu_2^*) = (\alpha\delta_0 + (1-\alpha)\delta_1, \beta\delta_c + (1-\beta)\delta_d)$ for some $0 \le \alpha, \beta, c, d \le 1$, where α is independent of ε.*

As far as two-person games with finite spaces of strategies are concerned, the most general results were obtained by Radzik (1991a, 2000) for the zero-sum case, and by Połowczuk (2003, 2006) for the non-zero sum case. The present paper is in a large part based on the results from these four articles.

More precisely, the above-mentioned papers, as well as the present one, are mainly devoted to the study of *finite concave/convex games Γ*, that is, n-person games G as defined above, with all the pure strategy spaces X_i finite, under the additional basic assumption that for each i the payoff function F_i is concave/convex in x_i. The new notion of concavity/convexity for functions with a finite domain is defined in a way very closely related to the standard definition of concavity/convexity of functions on intervals. It appears that such n-person finite games have very interesting properties. Namely,

(a) the games Γ well approximate n-person continuous games on the n-product of the unit interval $[0,1]$ with concave/convex payoff functions;

(b) there always exists a *two-point Nash equilibrium* in convex games Γ;
(c) there always exists a *two-adjoining Nash equilibrium* in concave games Γ, that is, an almost-pure Nash equilibrium where the strategy of player i, for any i, is a probability distribution with at most two-element spectrum, consisting of two adjoining pure strategies from X_i.

The exact form of these results will be given later. Right now we only note that (c) above can be seen as a discrete version of Glicksberg's theorem (Theorem 1A) and is very closely related to Theorem 1B. On the other hand, one can view (b) as a discrete version of Theorem 1C extended to n-person games. In this paper we not only discuss these problems, but in many cases also give procedures to find these equilibria. Moreover, we give many examples illustrating the results obtained and the properties discussed.

The organization of the paper is as follows: Some preliminaries are given in Section 2. The generalization of convexity to functions with finite domains is discussed in Section 3. Section 4 reviews the results about the existence of pure equilibria in matrix/bimatrix games. In Section 5 we formulate results on the existence of almost-pure (or two-point) Nash equilibria in matrix/bimatrix games. In Section 6 we give extensions of these results to n-person finite games. Section 7 contains the concluding remarks and four related open problems.

2. Preliminary definitions. We will be discussing *n-person non-zero-sum finite games Γ*, denoted by

$$\Gamma = \langle N, \{E_i\}_{i\in N}, \{H_i\}_{i\in N}\rangle\,, \tag{1}$$

where

(1) $N = \{1, \ldots, n\}$ is the set of players;
(2) for each $i \in N$, $E_i = \{1, \ldots, k_i\}$ is a finite space of the ith player's *pure strategies* e_i, with k_i a natural number;
(3) for each vector $e = (e_1, \ldots, e_n)$ of the players' pure strategies and for each i, $H_i(e)$ is the *payoff function* of player i when the strategies e are used.

For simplicity we set

$$E = \prod_{j=1}^{n} E_j, \quad E_{-i} = \prod_{j=1}^{i-1} E_j \times \prod_{j=i+1}^{n} E_j, \quad e_{-i} = (e_1, \ldots, e_{i-1}, e_{i+1}, \ldots, e_n),$$

and

$$(e_{-i}, t) = (e_1, \ldots, e_{i-1}, t, e_{i+1}, \ldots, e_n).$$

We will also say that the game Γ is of *size* $|E_1| \times \cdots \times |E_n|$, writing briefly $\Gamma = \Gamma_{k_1\times\cdots\times k_n}$.

We now give the definitions of three special types of strategies and equilibria determined by them. They play a basic role in the present paper.

DEFINITION 1. A strategy μ^* of player i in a finite game Γ described by (1) is said to be a *two-point strategy* if it is of the form $\mu_i^* = \alpha\delta_a + (1-\alpha)\delta_b$ for some $0 \le \alpha \le 1$ and $a, b \in E_i$, $1 \le i \le n$. If additionally $b = a + 1$, then μ_i is called a *two-adjoining strategy.* If $a = 1$ and $b = k_i$, then μ_i is called a *two-marginal strategy.* Nash equilibria consisting only of strategies of the same type will be referred to by that type's name.

REMARK 1. In Sections 4–5 we will consider two-person finite games with strategy spaces $X_1 = \{1, \dots, m\}$ and $X_2 = \{1, \dots, n\}$ for two naturals m and n, and with payoff functions F_1 and F_2. Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$ be $m \times n$ matrices such that $a_{ij} = H_1(i, j)$ and $b_{ij} = H_2(i, j)$ for all i and j. We will call such games *bimatrix games* $\Gamma(A, B)$. When $B = -A$, a bimatrix game $\Gamma(A, B)$ becomes a two-person zero-sum game $\Gamma(A, -A)$ and will be briefly denoted by $\Gamma(A)$ and called a *matrix game.*

3. Convexity of functions defined over finite domains. In this section we introduce and thoroughly discuss discrete counterparts of quasi-concavity (or concavity) of functions. They play an essential role in our subsequent considerations. Throughout this section, E is a finite set of the form $E = \prod_{j=1}^{n} E_j$ with $E_i = \{1, \dots, k_i\}$.

DEFINITION 2. A function H on E is said to be *concave* (resp. *quasi-concave*) in the ith variable if for $j = 1, \dots, n$ there are strictly increasing sequences $x^j = (x_1^j, \dots, x_{k_j}^j)$ in $[0, 1]$ with $x_1^j = 0$ and $x_{k_j}^j = 1$, and there exists a continuous function $F(x_1, \dots, x_n)$ on $[0, 1]^n$, concave (resp. quasi-concave) in x_i, such that $F(x_{e_1}^1, \dots, x_{e_n}^n) = H(e_1, \dots, e_n)$ for all $(e_1, \dots, e_n) \in E$.

Convexity (quasi-convexity) is defined analogously.

REMARK 2. One could think that the above definition would be more natural if the set $[0, 1]^n$ was replaced by $\mathrm{conv}(E_1 \times \dots \times E_n)$ and the sequences x^j were taken constant of the form $x^j = (1, \dots, k_j)$ for $j = 1, \dots, n$. However this approach would be less general and lead to much smaller classes of concave functions (as shown in Example 1 below). For quasi-concave functions, both approaches lead to the same class.

EXAMPLE 1. Let $E_1 = \{1, 2, 3, 4\}$ and $E_2 = \{1, 2\}$. Define

$$H(k, l) = \frac{8l - 12}{k}.$$

It can be easily seen that if we take $x^1 = (0, 2/3, 8/9, 1)$, $x^2 = (0, 1)$ and $F(x_1, x_2) = (4 - 3x_1)(2x_2 - 1)$, then

$$H(k, l) = F(x_k^1, x_l^2) \quad \text{for } (k, l) \in E_1 \times E_2.$$

Thus, as F is linear in x_1, H is both convex and concave in its first variable according to Definition 2. If however $x^1 = (1,2,3,4)$ and $x^2 = (1,2)$, one can easily see that there is no convex or concave function $\overline{F}(x_1,x_2)$ defined on $[1,4]\times[1,2]$ and satisfying $H(k,l) = \overline{F}(x_k^1, x_l^2)$ for all $(k,l)\in E_1\times E_2$. Namely, by considering the values $H(k,1)$ and $H(k,2)$ for $k\in E_1$ we find that $\overline{F}(x_1,1)$ cannot be concave or convex in x_1.

Now we give two new theorems which allow one to verify easily whether a player's payoff function in an n-person non-zero-sum finite game is concave, convex, quasi-concave or quasi-convex.

Theorem 2A. *A function H on E is concave* (*resp. convex*) *in the ith variable if and only if for every natural u with $1 < u < k_i$ there exists a positive number C_u* (*resp. D_u*) *such that for each $e_{-i}\in E_{-i}$,*

$$H(e_{-i},u) - H(e_{-i},u-1) \ge C_u[H(e_{-i},u+1) - H(e_{-i},u)] \tag{2}$$

$$(\textit{resp. } H(e_{-i},u) - H(e_{-i},u-1) \le D_u[H(e_{-i},u+1) - H(e_{-i},u)]). \tag{3}$$

Theorem 2B. *A function H on E is quasi-concave in the ith variable if and only if for each $e_{-i}\in E_{-i}$ there exists a natural l with $1\le l\le k_i$ such that*

$$H(e_{-i},1) \le \cdots \le H(e_{-i},l) \ge H(e_{-i},l+1) \ge \cdots \ge H(e_{-i},k_i). \tag{4}$$

When all the inequalities in (4) *are reversed, H is quasi-convex in the ith variable.*

Proof of Theorem 2A. Fix $i\in N$. Obviously, it suffices to consider the case of H concave.

($\Leftarrow$) Assume that for each $1 < u < k_i$ the inequalities (2) hold, and define $C_1 := 1$. Now for $j = 1,\dots,n$ we define strictly increasing sequences $x^j = (x_1^j,\dots,x_{k_j}^j)$ in $[0,1]$ by

$$x_u^j = \frac{u-1}{k_j-1} \quad \text{for } j\ne i \text{ and } 1\le u\le k_j,$$

and

$$x_1^i = 0 \quad \text{and} \quad x_u^i = \sum_{k=1}^{u-1}\overline{C_k} \Big/ \sum_{k=1}^{k_i-1}\overline{C_k} \quad \text{for } u = 2,\dots,k_i,$$

where $\overline{C_1} = 1$ and $\overline{C_k} = 1/\prod_{u=2}^{k} C_u$ for $k\ge 2$. (Note that $x_1^j = 0$ and $x_{k_j}^j = 1$ for $1\le j\le n$.)

Define $X^j := \{x_1^j, x_2^j,\dots,x_{k_j}^j\}$ for $j = 1,\dots,n$, and

$$Q_i = X^1\times\cdots\times X^{i-1}\times X^{i+1}\times\cdots\times X^n.$$

Now, for every $q\in Q_i$, let f_q be the continuous function on $[0,1]$ defined by the two conditions:

(a) for $u = 1, \dots, k_i$,

$$f_q(x^i_u) = H(e_{-i}, u) \quad \text{if } q = (x^1_{e_1}, \dots, x^{i-1}_{e_{i-1}}, x^{i+1}_{e_{i+1}}, \dots, x^n_{e_n}),$$

(b) f_q is a linear on every interval $[x^i_u, x^i_{u+1}]$, $1 \le u \le k_i - 1$.

Hence, by (2), for every $q \in Q_i$ and $u = 2, \dots, k_i - 1$,

$$f_q(x^i_u) - f_q(x^i_{u-1}) \ge C_u[f_q(x^i_{u+1}) - f_q(x^i_u)].$$

But one can easily check that the last inequality is equivalent to

$$\text{(5)} \qquad \frac{f_q(x^i_u) - f_q(x^i_{u-1})}{x^i_u - x^i_{u-1}} \ge \frac{f_q(x^i_{u+1}) - f_q(x^i_u)}{x^i_{u+1} - x^i_u}.$$

Therefore, each f_q, $q \in Q_i$, is concave on every interval $[x^i_{u-1}, x^i_{u+1}]$, $u = 2, \dots, k_i - 1$. Consequently, it is continuous and concave on $[0, 1]$.

Now take any $x = (x_1, \dots, x_n) \in [0, 1]^n$. Since $x^j_1 = 0$ and $x^j_{k_j} = 1$ for all j, for every $1 \le j \le n$ there are $0 \le \alpha_j \le 1$ and $1 \le s_j \le k_j - 1$ such that $x_j = \alpha_j x^j_{s_j} + (1 - \alpha_j) x^j_{s_j+1}$. Hence, setting

$$K(x_{-i}) := \{y_{-i} \in [0, 1]^{n-1} \mid y_j = x^j_{s_j} \text{ or } y_j = x^j_{s_j+1} \text{ for } j = 1, \dots, n,\ j \ne i\},$$

one can easily conclude that x_{-i} can be written as a convex combination

$$x_{-i} = \sum_{w \in K(x_{-i})} \beta^{x_{-i}}_w w,$$

with the coefficients $\beta^{x_{-i}}_w$ depending on w and continuous in x_{-i}. (Here $y := (y_1, \dots, y_n)$.)

Now, for $x = (x_1, \dots, x_n) \in [0, 1]^n$ we define

$$F(x) = \sum_{w \in K(x_{-i})} \beta^{x_{-i}}_w f_w(x_i).$$

It is immediate that the function F is continuous on $[0, 1]^n$ and concave in x_i, and for all $(e_1, \dots, e_n) \in E$ we have $F(x^1_{e_1}, \dots, x^n_{e_n}) = H(e_1, \dots, e_n)$. This ends the proof of $(\Leftarrow)$.

$(\Rightarrow)$ Fix $i \in N$ and $e_i \in E_i$, and let $x' = (x^1_{e_1}, \dots, x^n_{e_n})$. Then, by the concavity of F in the ith variable, for $1 < u < k_i$ we have

$$\frac{F(x'_{-i}, x^i_u) - F(x'_{-i}, x^i_{u-1})}{x^i_u - x^i_{u-1}} \ge \frac{F(x'_{-i}, x^i_{u+1}) - F(x'_{-i}, x^i_u)}{x^i_{u+1} - x^i_u}.$$

But this is equivalent to

$$H(e_{-i}, u) - H(e_{-i}, u - 1) \ge \frac{x^i_u - x^i_{u-1}}{x^i_{u+1} - x^i_u}[H(e_{-i}, u + 1) - H(e_{-i}, u)].$$

Therefore (2) holds with

$$C_u = \frac{x^i_u - x^i_{u-1}}{x^i_{u+1} - x^i_u},$$

completing the proof. ∎

Proof of Theorem 2B. Without loss of generality we can assume that $i = 1$. We will show that there is a function F defined on $[0,1]^n$ with the properties required by Definition 2 for $i = 1$.

($\Leftarrow$) Assume that for each $e_{-1} \in E_{-1}$ there is $1 \le l \le k_1$ such that the inequalities (4) with $i = 1$ hold. Now for $j = 1, \ldots, n$ we define strictly increasing sequences $x^j = (x_1^j, x_2^j, \ldots, x_{k_j}^j)$ in $[0,1]$ by

$$x_u^j = \frac{u-1}{k_j - 1} \quad \text{for } 1 \le u \le k_j.$$

Let $X^j := \{x_1^j, \ldots, x_{k_j}^j\}$, $j = 1, \ldots, n$, and $K_s := X^s \times X^{s+1} \times \cdots \times X^n$. For every $q^2 \in K_2$, let $f_{q^2}^1$ be the continuous function on $[0,1]$ defined by the two conditions:

(a) $f_{q^2}^1(x_{e_1}^1) = H(e_{-1}, e_1)$ for $e_1 \in E_1$ if $q^2 = (x_{e_2}^2, x_{e_3}^3, \ldots, x_{e_n}^n)$,

(b) $f_{q^2}^1$ is linear on every interval $[x_u^1, x_{u+1}^1]$, $1 \le u \le k_1 - 1$.

Hence, because of (4) with $i = 1$, for each $q^2 \in K_2$ the function $f_{q^2}^1$ is continuous and quasi-concave on $[0,1]$. Therefore the function G_1 defined by $G_1(x_1, q^2) = f_{q^2}^1(x_1)$ is continuous on $[0,1] \times K_2$, quasi-concave in x_1 for each $q^2 \in K_2$, and satisfies $G_1(x_{e_1}^1, \ldots, x_{e_n}^n) = H(e)$ for every $e = (e_1, \ldots, e_n) \in E$.

Assume now that for some $1 \le s < n$ there exists a continuous function $G_s(x_1, \ldots, x_s, q^{s+1})$ on $[0,1]^s \times K_{s+1}$ which is quasi-concave in x_1 for each $(x_2, \ldots, x_s, q^{s+1}) \in [0,1]^{s-1} \times K_{s+1}$, and satisfies

$$G_s(x_{e_1}^1, x_{e_1}^1, \ldots, x_{e_n}^n) = H(e_1, \ldots, e_n) \quad \text{for every } (e_1, \ldots, e_n) \in E. \tag{6}$$

We will construct a function $G_{s+1}(x_1, \ldots, x_{s+1}, q^{s+2})$ on $[0,1]^{s+1} \times K_{s+2}$. Define $y^{s+1} = (y_1^{s+1}, \ldots, y_{k_{s+1}-1}^{s+1})$, where

$$y_u^{s+1} = x_u^{s+1} + \frac{1}{2(k_{s+1} - 1)} \quad \text{for } u = 1, \ldots, k_{s+1} - 1.$$

Notice that

$$0 = x_1^{s+1} < y_1^{s+1} < x_2^{s+1} < y_2^{s+1} < \cdots < x_{k_{s+1}-1}^{s+1} < y_{k_{s+1}-1}^{s+1} < x_{k_{s+1}}^{s+1} = 1. \tag{7}$$

Let $(x_1, \ldots, x_s, x_{s+1}, q^{s+2}) \in [0,1]^{s+1} \times K_{s+2}$. Since $x_{s+1} \in [0,1]$, (7) implies that $x_{s+1} \in [x_u^{s+1}, y_u^{s+1}]$ or $x_{s+1} \in [y_u^{s+1}, x_{u+1}^{s+1}]$ for some $1 \le u \le k_{s+1} - 1$. In the former case, $x_{s+1} = \alpha_{x_{s+1}} x_u^{s+1} + (1 - \alpha_{x_{s+1}}) y_u^{s+1}$ for some $0 \le \alpha_{x_{s+1}} \le 1$, and we define

$$G_{s+1}(x_1, \ldots, x_s, x_{s+1}, q^{s+2}) = \alpha_{x_{s+1}} G_s(x_1, \ldots, x_s, x_u^{s+1}, q^{s+2}).$$

In the latter case, $x_{s+1} = \beta_{x_{s+1}} y_u^{s+1} + (1 - \beta_{x_{s+1}}) x_{u+1}^{s+1}$ for some $0 \le \beta_{x_{s+1}} \le 1$, and we define

$$G_{s+1}(x_1, \ldots, x_s, x_{s+1}, q^{s+2}) = (1 - \beta_{x_{s+1}}) G_s(x_1, \ldots, x_s, x_u^{s+1}, q^{s+2}).$$

In this way the function $G_{s+1}(x_1,\dots,x_s,x_{s+1},q^{s+2})$ has been defined on $[0,1]^{s+1}\times K_{s+2}$, and one can easily see that it is continuous, quasi-concave in x_1 for each $(x_2,\dots,x_{s+1},q^{s+2})\in[0,1]^s\times K_{s+2}$, and

$$(8)\qquad G_{s+1}(x^1_{e_1},\dots,x^n_{e_n})=H(e_1,\dots,e_n)\quad\text{for every }(e_1,\dots,e_n)\in E.$$

Therefore, by induction, the function F defined by $F(x_1,\dots,x_n)=G_n(x_1,\dots,x_n)$ on $[0,1]^n$ has the required properties, which completes the proof of ($\Leftarrow$).

($\Rightarrow$) Let $e_{-1}\in E_{-1}$. Then inequalities (4) for $i=1$ immediately follow from the quasi-concavity of $F(t,x^2_{e_2},\dots,x^n_{e_n})$ in $t\in[0,1]$ and its relation to the function $H(s,e_{-1})$ of $s\in\{1,\dots,k_1\}$. This completes the proof of the theorem.

3.1. Convexity properties of matrices. In Definitions 3–6 below, we introduce several easily verifiable properties of matrices, which are basic for our considerations in the next sections. Since every matrix can be seen as a function of two variables defined on the product of two finite sets, Theorems 2A and 2B (for $n=2$) can be used to define the convexity properties of matrices in terms of their rows and columns. We do this in the first two definitions and later justify it in Proposition 1.

Definition 3. A matrix $T=[t_{ij}]_{m\times n}$ is *row-concave* (resp. *column-concave*) if for each $1<j<n$ (resp. $1<i<m$) there is a positive number C_j (resp. D_i) such that

$$(9)\qquad t_{ij}-t_{i,j-1}\ge C_j[t_{i,j+1}-t_{ij}]\quad\text{for all } i.$$

$$(10)\qquad (\text{resp. } t_{ij}-t_{i-1,j}\ge D_i[t_{i+1,j}-t_{ij}]\quad\text{for all } j.)$$

When the inequalities in (9) and (10) are reversed, the matrix T is *row-convex* or *column-convex*, respectively.

Definition 4. A matrix $T=[t_{ij}]_{m\times n}$ is *row-quasi-concave* (resp. *column-quasi-concave*) if for each $1\le i\le m$ (resp. $1\le j\le n$), there exist $1\le k\le n$ (resp. $1\le l\le m$) such that

$$(11)\qquad t_{i1}\le t_{i2}\le\dots\le t_{ik}\ge t_{i,k+1}\ge\dots\ge t_{in},$$

$$(12)\qquad (\text{resp. } t_{1j}\le t_{2j}\le\dots\le t_{lj}\ge t_{l+1,j}\ge\dots\ge t_{mj}).$$

When the inequalities in (11) and (12) are reversed, the matrix T is *row-quasi-convex* or *column-quasi-convex*, respectively.

For $T=[t_{ij}]_{m\times n}$, let H_T be the function defined on $K:=\{1,\dots,m\}\times\{1,\dots,n\}$ by $H_T(i,j)=t_{ij}$ for $(i,j)\in K$.

Remark 3. The properties of the matrix T described in Definitions 3 and 4 are closely related to the concavity and quasi-concavity of the function H_T defined on the finite set K. We state this in Proposition 1 below. It suffices to formulate it only for row-(quasi-)concave matrices because of the following obvious implications:

- a matrix W is row-(quasi-)convex iff $-W$ is row-(quasi-)concave;
- a matrix W is column-(quasi-)concave iff W^T is row-(quasi-)concave;
- a matrix W is column-(quasi-)convex iff $-W^T$ is row-(quasi-)concave.

Theorems 2A and 2B immediately imply

Proposition 1.

(i) *A matrix T is row-concave* (*resp. column-concave*) *if and only if the function $H_T(i,j)$ on K is concave in the variable j* (*resp. in the variable i*),

(ii) *A matrix T is row-quasi-concave* (*resp. column-quasi-concave*) *if and only if the function $H_T(i,j)$ on K is quasi-concave in j* (*resp. in i*).

Remark 4. The verification of row-quasi-concavity of a matrix is trivial. One can also easily check whether a matrix is concave: a simple algorithm is given in Proposition 1 of Połowczuk et al. (2012).

The next two definitions describe some natural stronger versions of the properties of matrices given in Definition 4.

Definition 5. A matrix $T = [t_{ij}]_{m\times n}$ is *strongly row-quasi-concave* (resp. *strongly column-quasi-concave*) if for each $1 \le i \le m$ (resp. $1 \le j \le n$), there exist $1 \le k \le s \le n$ (resp. $1 \le l \le u \le m$) such that

$$t_{i1} < t_{i2} < \cdots < t_{ik} = t_{i,k+1} = \cdots = t_{is} > \cdots > t_{in} \tag{13}$$

$$(\text{resp. } t_{1j} < t_{2j} < \cdots < t_{lj} = t_{l+1,j} = \cdots = t_{lu} > \cdots > t_{mj}). \tag{14}$$

When the inequalities in (13) and (14) are reversed, the matrix T is *strongly row-quasi-convex* or *strongly column-quasi-convex*, respectively.

Definition 6. A matrix $T = [t_{ij}]_{m\times n}$ is *strictly row-quasi-concave* (resp. *strictly column-quasi-concave*) if for each $1 \le i \le m$ (resp. $1 \le j \le n$), there exist $1 \le k \le n$ (resp. $1 \le l \le m$) such that

$$t_{i1} < t_{i2} < \cdots < t_{ik} > \cdots > t_{in} \tag{15}$$

$$(\text{resp. } t_{1j} < t_{2j} < \cdots < t_{lj} > \cdots > t_{mj}). \tag{16}$$

When the inequalities in (15) and (16) are reversed, the matrix T is *strictly row-quasi-convex* or *strictly column-quasi-convex*, respectively.

The properties of matrices described in Definitions 3 and 5 are also related to each other, as shown in the next proposition.

Proposition 2. *If a matrix T is row-concave* (*resp. column-concave*) *then it is also strongly row-quasi-concave* (*resp. strongly column-quasi-concave*). *An analogous statement holds for convexity.*

Proof. It suffices to show that (9) implies (13). But this can be easily derived by considering separately the sides of the inequalities in (9) with positive, negative and zero values. The details are omitted. ■

We end this section by quoting a theorem that gives a correspondence between the "quasi-convexity-concavity" of a matrix and of a function of two variables on the unit square with an analogous property. This result comes from Radzik (2000) and is implied by Theorems 3.1 and 3.2, Definition 3.1 and Remark 3.2 given there. It completes the results of Theorems 2A and 2B in the case of two-person zero-sum finite games.

THEOREM 3. *A matrix $T = [t_{ij}]_{m\times n}$ is row-(quasi-)convex and column-(quasi-)concave if and only if there exists a continuous function $F(x,y)$ on the unit square, (quasi-)convex in y for each x and (quasi-)concave in x for each y, and if there are strictly increasing sequences $\{x_i\}_{i=1}^{m}$ and $\{y_j\}_{j=1}^{n}$ in $[0,1]$ with $x_1 = y_1 = 0$ and $x_m = y_n = 1$ such that $F(x_i, y_j) = t_{ij}$ for all i and j.*

4. Pure equilibria in two-person games

4.1. Saddle points in matrix games. Below we recall six results on the existence of Nash equilibria in pure strategies for two-person zero-sum matrix games, together with several examples illustrating them. Four of these results concern matrix games with matrices having convexity properties described in the previous section. They provide necessary and sufficient conditions for the existence of pure equilibria in such games.

We start by quoting two classical results on the existence of a saddle point of payoff matrices. They belong to Shapley (1964) (see also Parthasarathy and Raghavan (1971, Theorems 3.2.1 and 3.2.3)). For a payoff matrix $A = [a_{ij}]$, a *saddle point* of A is defined as any pair (i,j) such that a_{ij} is minimal in the ith row and maximal in the jth column of A. If we denote by $\Gamma(A)$ the zero-sum two-person matrix game $\Gamma(A,-A)$, we immediately see that a pair (i,j) is a saddle point of A if and only if it is a pure Nash equilibrium in $\Gamma(A)$.

THEOREM 4A. *Assume that every 2×2 submatrix of A has a saddle point. Then A also has a saddle point.*

THEOREM 4B. *Assume that $2 \le p \le m$ and $2 \le q \le n$. Let A be an $m\times n$ matrix none of whose rows and none of whose columns contains two equal elements. If every $p\times q$ submatrix of A has a saddle point, then A also has a saddle point.*

Now we quote four theorems of Radzik (1991a) on the existence of saddle points in matrix games with convexity properties. To formulate them, some notation is needed.

For $W = [w_{rs}]_{m\times n}$, let W^{ij}_{kl}, $1 \le i \le k \le m$, $1 \le j \le l \le n$, be the submatrix

$$W^{ij}_{kl} := \begin{bmatrix} w_{ij} & w_{i,j+1} & \dots & w_{i,l} \\ w_{i+1,j} & w_{i+1,j+1} & \dots & w_{i+1,l} \\ \vdots & \vdots & & \vdots \\ w_{kj} & w_{k,j+1} & \dots & w_{kl} \end{bmatrix}. \tag{17}$$

Theorem 5A. *Let A be a row-quasi-convex and column-quasi-concave $m \times n$ matrix. Then the game $\Gamma(A)$ has a saddle point if and only if every submatrix A^{ij}_{kl}, $1 \le i < k \le m$, $1 \le j < l \le n$, has a saddle point.*

Theorem 5B. *Let A be a strongly row-quasi-convex and strongly column-quasi-concave $m \times n$ matrix. Then the game $\Gamma(A)$ has a saddle point if and only if all the 2×2 submatrices $A^{ij}_{i+1,j+1}$, 3×2 submatrices $A^{ij}_{i+2,j+1}$, and 2×3 submatrices $A^{ij}_{i+1,j+2}$ have saddle points.*

Theorem 5C. *Let A be a strongly row-quasi-convex and strongly column-quasi-concave $m \times n$ matrix. Assume that $3 \le p \le m$ and $3 \le q \le n$. Then the game $\Gamma(A)$ has a saddle point if and only if every $p \times q$ submatrix $A^{ij}_{i+p-1,j+q-1}$ has a saddle point.*

Theorem 5D. *Let A be a strictly row-quasi-convex and strictly column-quasi-concave matrix. Then the game $\Gamma(A)$ has a saddle point if and only if every 2×2 submatrix $A^{ij}_{i+1,j+1}$ has a saddle point.*

Remark 5. Theorems 5A–5C remain true under the assumption that A is only row-convex and column-concave. This immediately follows from Proposition 2 and Definitions 3–5.

Remark 6. In Theorem 5A, "every submatrix A^{ij}_{kl}" cannot be weakened to "every proper submatrix A^{ij}_{kl}". Similarly, strong quasi-concavity in Theorem 5B cannot be replaced by quasi-concavity. Finally, strict quasi-concavity in Theorem 5D cannot be replaced by strong quasi-concavity. All this is discussed in Example 2.

Also as far as Theorem 5C is concerned, its assumption cannot be weakened by removing "strongly". To see this, it suffices to consider the matrix B of Example 2 below which is row-quasi-convex and column-quasi-concave. All its relevant 3×3 submatrices have saddle points, but the entire matrix C does not.

Example 2. Consider the following three matrices:

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 3 & 1 & 1 \\ 3 & 1 & 1 & 3 \\ 3 & 1 & 1 & 3 \\ 1 & 1 & 3 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} -2 & -2 & -2 & 3 \\ -1 & -1 & 1 & 2 \\ 3 & 2 & 1 & 0 \end{bmatrix}.$$

It is easily seen that A is row-quasi-convex and column-quasi-concave, and all its proper submatrices A^{ij}_{kl} mentioned in Theorem 5B have saddle points. However, there is no saddle point in A.

Further, B is row-quasi-convex and column-quasi-concave, but not strongly row-quasi-convex or strongly column-quasi-concave. Moreover, all of its 2×2, 3×2 and 2×3 submatrices mentioned in Theorem 5B have saddle points. But B has no saddle points.

Similarly, C is strongly row-quasi-convex and strongly column-quasi-concave but not strictly row-quasi-convex or strictly column-quasi-concave. Moreover, one can easily check that all its 2×2 submatrices mentioned in Theorem 5B have saddle points. But C has no saddle points.

Now we give several counterexamples showing that no direct generalization of Theorems 4A–4B and 5A–5D to non-zero-sum two-person games is possible.

Let $\Gamma(A, B)$ be a bimatrix game, where $A = [a_{rs}]_{m\times n}$ and $B = [b_{rs}]_{m\times n}$. By the *subgame* Γ^{ij}_{kl} of $\Gamma(A, B)$, where $1 \le i < k \le m$ and $1 \le j < l \le n$, we mean the game $\Gamma(A^{ij}_{kl}, B^{ij}_{kl})$.

Example 3. Assume that

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 3 & 2 \\ 2 & 1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix}.$$

All the 2×2 subgames of $\Gamma(A, B)$ have pure Nash equilibria here, but the entire game $\Gamma(A, B)$ does not. So Theorem 4A is not true in the non-zero-sum case. However, after strengthening the assumptions it appears to be true (see Theorem 6A in the next section).

The next example shows that Theorem 4B cannot be extended to the non-zero-sum case either.

Example 4. Let

$$A = \begin{bmatrix} 4 & 6 & 10 & 13 \\ 2 & 7 & 11 & 14 \\ 1 & 8 & 12 & 15 \\ 3 & 5 & 9 & 16 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 3 & 4 & 1 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 16 & 14 & 13 & 15 \end{bmatrix}.$$

Let $p = 2$ and $q = 4$. All the $p \times q$ subgames of $\Gamma(A, B)$ have pure Nash equilibria, but the entire game does not.

EXAMPLE 5. Consider the strictly quasi-concave game $\Gamma(A, B)$ with

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 2 & 2 \\ 1 & 1 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}.$$

Here $(1, 1)$ is a pure Nash equilibrium for $\Gamma(A, B)$, but the 2×2 subgame Γ_{33}^{22} and the 2×3 subgame Γ_{33}^{21} do not have pure Nash equilibria. Hence, the $\Rightarrow$ part in Theorems 5A–5D cannot be generalized to non-zero-sum bimatrix games.

EXAMPLE 6. In the case of zero-sum games $\Gamma(A)$ considered in Theorem 5B, it is enough to assume that all 2×2, 2×3 and 3×2 subgames of $\Gamma(A)$ have pure Nash equilibria (see $\Leftarrow$ of Theorem 5B), and then the entire game has a solution of the same type. But consider the two-person non-zero-sum game $\Gamma(A, B)$ with A strongly column-quasi-concave and B strongly row-quasi-concave, where

$$A = \begin{bmatrix} 4 & 1 & 1 & 1 \\ 3 & 1 & 1 & 2 \\ 2 & 1 & 1 & 3 \\ 1 & 1 & 1 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix}.$$

Note that all the 2×2, 3×2 and 2×3 subgames have pure Nash equilibria but the entire game does not. So the $\Leftarrow$ part of Theorem 5B does not hold in the non-zero-sum case. This means that some more conditions are needed to get a non-zero-sum counterpart of Theorem 5B. For this, see Theorem 6B in the next section.

EXAMPLE 7. Consider the game $\Gamma(A, B)$ with the following strictly column-quasi-concave matrix A and strictly row-quasi-concave matrix B:

$$A = \begin{bmatrix} 1 & 4 & 1 & 4 \\ 2 & 3 & 2 & 3 \\ 3 & 2 & 3 & 2 \\ 4 & 1 & 4 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \\ 1 & 2 & 3 & 4 \end{bmatrix}.$$

In this game, all the 3×3 subgames $\Gamma_{i+2,j+2}^{ij}$ have pure Nash equilibria, but the entire game does not. Moreover, all the $3 \times k$ and $k \times 3$ subgames have pure Nash equilibria, for $k \geq 3$. Therefore, also the $\Leftarrow$ part of Theorem 5C cannot be directly generalized to the non-zero-sum case, even if we strengthen the assumptions to strict quasi-concavity. The case of such games $\Gamma(A, B)$ is discussed in Theorem 6C and in Corollaries 1 and 2 in the next section.

4.2. Pure Nash equilibria in bimatrix games. Below we give five results of Połowczuk (2003) about the existence of pure Nash equilibria in bimatrix games with payoff matrices having convexity properties defined in Section 3.1. Such bimatrix games will be considered in the form $\Gamma := \Gamma(A, B)$, with payoff matrices A and B of size $m \times n$, $m, n \geq 2$. These results complete the results of Theorems 4A–5D to the non-zero-sum case. Our first result is related to Theorem 4A.

THEOREM 6A. *Let A and B be strongly column-quasi-concave and strongly row-quasi-concave $m \times n$ matrices, respectively. Assume that every 2×2 subgame of the game Γ (obtained by removing $m - 2$ rows and $n - 2$ columns from A and B, the same for A and B) has a pure Nash equilibrium. Then Γ has a pure Nash equilibrium as well.*

Theorem 5B has the following counterpart in terms of bimatrix games.

THEOREM 6B. *Let A and B be strongly column-quasi-concave and strongly row-quasi-concave $m \times n$ matrices, respectively. If all the $2 \times l$ subgames $\Gamma^{ij}_{i+1,j+l-1}$ and all the $k \times 2$ subgames $\Gamma^{ij}_{i+k-1,j+1}$ $(1 \leq i < m$, $1 \leq j < n$, $k = 2, \ldots, m$, $l = 2, \ldots, n)$ have pure Nash equilibria, then Γ also has a pure Nash equilibrium.*

The next theorem deals with strictly quasi-concave bimatrix games. It says that if such a game can be divided into two subgames having pure Nash equilibria, then it also has a pure Nash equilibrium.

THEOREM 6C. *Let A and B be strictly column-quasi-concave and strictly row-quasi-concave $m \times n$ matrices, respectively. Assume that one of the following conditions holds:*

(i) *there exists k, $1 < k < m$, such that both subgames $\Gamma_1 = \Gamma^{11}_{kn}$ and $\Gamma_2 = \Gamma^{k1}_{mn}$ have pure Nash equilibria;*
(ii) *there exists l, $1 < l < n$, such that both subgames $\Gamma_1 = \Gamma^{11}_{ml}$ and $\Gamma_2 = \Gamma^{1l}_{mn}$ have pure Nash equilibria.*

Then Γ also has a pure Nash equilibrium.

Theorem 6C has two interesting corollaries. The first one can be seen as a counterpart of Theorem 5D for two-person non-zero-sum games.

COROLLARY 1. *Let A and B be strictly column-quasi-concave and strictly row-quasi-concave $m \times n$ matrices, respectively. If all the 2×2 subgames of the form $\Gamma^{ij}_{i+1,j+1}$ have pure Nash equilibria, then Γ also has a pure Nash equilibrium.*

Corollary 2. *Let A and B be strictly column-quasi-concave and strictly row-quasi-concave $m \times n$ matrices, respectively. If all the $m \times 2$ subgames $\Gamma^{1j}_{m,j+1}$ or all the $2 \times n$ subgames $\Gamma^{i1}_{i+1,n}$ have pure Nash equilibria, then Γ also has a pure Nash equilibrium.*

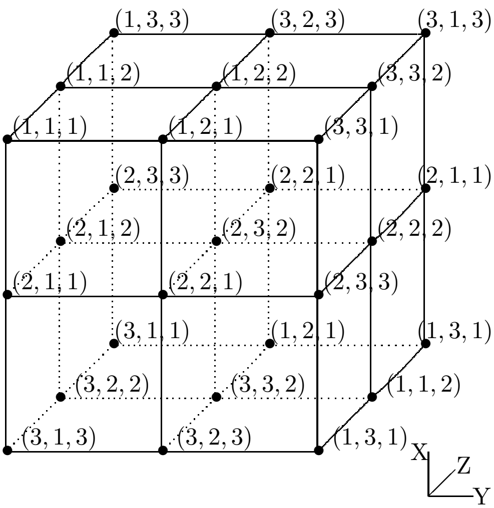
Now we give two examples describing other consequences of Theorem 6C.

Example 8. Let A and B be strictly column-quasi-concave and strictly row-quasi-concave matrices, respectively, both of size $(2k+1)\times n$ (the number of pure strategies of player 1 is odd). From Theorem 6C one can easily conclude that if all the $3 \times n$ subgames $\Gamma^{2i+1,1}_{2i+3,n}$ for $i = 0, 1, \dots, k-1$ have pure Nash equilibria, then also Γ has a pure Nash equilibrium. Namely, by Theorem 6C, the game Γ^{11}_{5n} has such a solution, because it occurs in the games Γ^{11}_{3n} and Γ^{31}_{5n}. Then, by repeating this argument, we conclude that all the games $\Gamma^{11}_{7n}, \dots, \Gamma^{11}_{2k+1,n} = \Gamma$ have pure Nash equilibria as well.

Example 9. Let A and B be strictly column-quasi-concave and strictly row-quasi-concave matrices, respectively, both of size $(2k+1) \times (2l+1)$. If all the 3×3 subgames $\Gamma^{2i+1,2j+1}_{2i+3,2j+3}$ for $i = 0, 1, \dots, k-1$ and $j = 0, 1, \dots, l-1$ have pure Nash equilibria, then Γ also has a pure Nash equilibrium. This can be easily justified with the help of Theorem 6C, as in the previous example.

In the next example we show that the results of this section, concerning bimatrix games (Theorems 6A–6C), cannot be extended to n-person games with $n > 2$.

Example 10. Consider the three-person non-zero-sum game Γ with three pure strategies $X = \{0, 1, 2\}$ for each player and with the payoff functions $K_i(x_1, x_2, x_3)$, $i = 1, 2, 3$, defined on X^3 as indicated in the figure:



Player 1 chooses one of his three pure strategies $x_1 \in X$ along line X, players 2 and 3 choose $x_2 \in X$ and $x_3 \in X$ along lines Y and Z, respectively. The triples represent the payoffs for each pure strategy combination. It is easily verifiable that the function $K_i(x_1, x_2, x_3)$ is strictly concave in x_i ($i = 1, 2, 3$) and all the $2 \times 2 \times 2$ subgames have pure Nash equilibria. Similarly, there are pure Nash equilibria in all the subgames of size $2\times2\times3$ or $2\times3\times2$ or $3 \times 2 \times 2$. But the entire game does not have such a solution.

5. Two-point equilibria in two-person games

5.1. Two-point Nash equilibria in matrix games. We present here three results (Theorems 7A–7C) of Radzik (2000) that complete Theorems 5A–5D. They show the existence of Nash equilibria in two-point strategies in two-person zero-sum games with payoff matrices having convexity properties. The first theorem, concerning matrix games with row-convex and column-concave matrices, provides a full characterization of optimal strategies and a simple search procedure. Also a discussion of various relationships between the results is given.

THEOREM 7A. *Let* $A = [a_{ij}]_{m\times n}$ *be row-convex and column-concave. Then for the matrix game* $\Gamma = \Gamma(A)$ *and its value* $v^* = \mathrm{val}(\Gamma(A))$, *one of the following four cases must occur:*

CASE 1: *There exists a saddle point* (s, r) *in* A. *Then* $(\mu^*, \nu^*) = (\delta_s, \delta_r)$ *is a pair of respective pure optimal strategies for players* 1 *and* 2 *in* Γ.

CASE 2: *There exists a* 2×2 *submatrix* $A^{sr}_{s+1,r+1}$ *without saddle points. Then* $v^* = \mathrm{val}(\Gamma(A^{sr}_{s+1,r+1}))$ *and the optimal strategies for players* 1 *and* 2 *in* $\Gamma(A^{sr}_{s+1,r+1})$ *are also optimal in* $\Gamma(A)$, *and are of the form* $\mu^* = \lambda\delta_s + (1-\lambda)\delta_{s+1}$ *and* $\nu^* = \gamma\delta_r + (1-\gamma)\delta_{r+1}$ *for some* $0 < \lambda, \gamma < 1$.

CASE 3: *Cases* 1 *and* 2 *do not hold and there is a* 2×3 *submatrix* $A^{sr}_{s+1,r+2}$ *without saddle points. Then* $v^* = \mathrm{val}(\Gamma(A^{sr}_{s+1,r+2}))$,

$$A^{sr}_{s+1,r+2} = \begin{bmatrix} a_{sr} & v^* & a_{s,r+2} \\ a_{s+1,r} & v^* & a_{s+1,r+2} \end{bmatrix}, \tag{18}$$

and the optimal strategies for players 1 *and* 2 *in* $\Gamma(A^{sr}_{s+1,r+2})$ *are* $\mu^* = \lambda\delta_s + (1-\lambda)\delta_{s+1}$ *and* $\nu^* = \delta_{r+1}$ *for some* $0 < \lambda < 1$, *and they remain optimal in* $\Gamma(A)$.

Case 4: *Cases* 1 *and* 2 *do not hold and there is a* 3×2 *submatrix* $A^{sr}_{s+2,r+1}$ *of* A *without saddle points. Then* $v^* = \mathrm{val}(\Gamma(A^{sr}_{s+2,r+1}))$,

$$A^{sr}_{s+2,r+1} = \begin{bmatrix} a_{sr} & a_{s,r+1} \\ v^* & v^* \\ v_{s+2,r} & v_{s+2,r+1} \end{bmatrix}, \tag{19}$$

and the optimal strategies for players 1 *and* 2 *in* $\Gamma(A^{sr}_{s+2,r+1})$ *are* $\mu^* = \delta_{s+1}$ *and* $\nu^* = \gamma\delta_r + (1-\gamma)\delta_{r+1}$ *for some* $0 < \gamma < 1$, *and they remain optimal in* $\Gamma(A)$.

COROLLARY 3. *Let* $A = [a_{ij}]_{m\times n}$ *be a row-convex and column-concave matrix such that all the* 2×2 *submatrices* $A^{ij}_{i+1,j+1}$ *have saddle points. Then one of the players has a pure optimal strategy in* $\Gamma(A)$. *If additionally there is exactly one minimal element in each row of* A, *and exactly one maximal element in each column of* A, *then both players have pure optimal strategies in* $\Gamma(A)$.

Proof. The first part is an immediate consequence of Theorem 7A since its Case 2 cannot happen. On the other hand, if there were a submatrix $A^{sr}_{s+1,r+2}$ of the form (18), then we could use column-concavity of A, Proposition 2 and (14) to show that v^* is the maximal element in its column. But this would contradict the assumption that there is exactly one maximal element in each column. So Case 3, and analogously Case 4, of Theorem 7A cannot hold, and the corollary follows. ∎

REMARK 7. In view of Theorem 3, Theorem 7A can be viewed as a discrete zero-sum counterpart of the stronger version (without "quasi") of the result of Debreu and Glicksberg (a pure Nash equilibrium in Theorem 1A corresponds to a Nash equilibrium in two-adjoining strategies for a matrix game in Theorem 7A). Hence, in view of the assumptions of Theorem 1A, one could expect that Theorem 7A should also hold for all matrices that are column-quasi-concave and row-quasi-convex. However, a counterexample is given in Example 11 below, where the support of the unique optimal strategy for a player may consist of two non-adjoining points. Moreover, in Example 12 we give another example of a matrix game with the same property that has unique optimal strategies with supports consisting of three neighboring points each. In view of Theorem 1A, these facts seem rather surprising.

EXAMPLE 11. Consider the two-person zero-sum matrix game with payoff matrix

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}.$$

Obviously, this matrix is row-quasi-convex and column-quasi-concave (but not row-convex or column-concave). One can easily check that the strategies $\mu^* = (1/2)\delta_1 + (1/2)\delta_2$ and $\nu^* = (1/2)\delta_1 + (1/2)\delta_4$ are the unique optimal strategies in this game. Hence, the optimal strategy of player 2 is not two-adjoining, in contrast to Theorem 7A.

EXAMPLE 12. Consider the two-person zero-sum matrix game with payoff matrix

$$A = \begin{bmatrix} 4 & 3 & 1 \\ 1 & 2 & 3 \\ 0 & 1 & 4 \end{bmatrix},$$

which is row-quasi-convex and column-quasi-concave (but not row-convex or column-concave). It is not difficult to check that the strategies $\mu^* = (1/2)\delta_1+(1/4)\delta_2+(1/4)\delta_3$ and $\nu^* = (1/4)\delta_1+(1/4)\delta_2+(1/2)\delta_3$ are the unique optimal strategies for the players in this game. So, the only equilibrium consists of the two players' optimal strategies in the form of a three-point strategy each.

Now we present two theorems that can be viewed as discrete counterparts of Theorems 1B and 1C. They show that the structure of ϵ-optimal strategies for infinite games is inherited (at least in the zero-sum case) by all their finite subgames. The first theorem corresponds to Theorem 1B.

We shall need the following notation: for $A = [a_{ij}]$, we put

$$\widehat{A}^{ij}_{kl} = \begin{bmatrix} a_{ij} & a_{il} \\ a_{kj} & a_{kl} \end{bmatrix}.$$

THEOREM 7B. *Let $A = [a_{ij}]_{m\times n}$ be a column-concave matrix, and let v^*, μ^* and ν^* be the value of the matrix game $\Gamma = \Gamma(A)$ and the optimal strategies for players* 1 *and* 2, *respectively. Then one of the following three cases must occur:*

CASE 1*: There exists a saddle point (s, r) in A. Then the pair $(\mu^*, \nu^*) = (\delta_s, \delta_r)$ of strategies is optimal in $\Gamma(A)$ and $v^* = a_{sr}$.*

CASE 2*: For some $1 \le i < m$, there exists a $2 \times n$ submatrix $A^{i1}_{i+1,n}$ without saddle points. Then $v^* = \mathrm{val}(A^{i1}_{i+1,n})$, and there are optimal strategies (μ^*, ν^*) in $\Gamma(A^{i1}_{i+1,n})$ of the form $\mu^* = \lambda\delta_i + (1-\lambda)\delta_{i+1}$ and $\nu^* = \gamma\delta_s + (1-\gamma)\delta_r$ for some $0 < \lambda < 1$, $0 \le \gamma \le 1$ and $1 \le s < r \le n$, and they are also optimal in $\Gamma(A)$.*

CASE 3*: Cases* 1 *and* 2 *do not hold. Then there exists a submatrix $\widehat{A}^{i-1,s}_{i+1,r}$ without saddle points, with $a_{is} = a_{ir} = v$, where $v = \min_{1\le j\le n} a_{ij}$. Moreover, $v^* = a_{is}$, $\mu^* = \delta_i$ and $\nu^* = \gamma\delta_s + (1-\gamma)\delta_r$ for some $0 < \gamma < 1$, where ν^* is also an optimal strategy for player* 2 *in $\Gamma(\widehat{A}^{i-1,s}_{i+1,r})$.*

The last Theorem 7C is a discrete zero-sum counterpart of Theorem 1C. To formulate it we need additional notation (also used in the next section).

For $C = [c_{ij}]_{m\times n}$, we write

$$C_m^1 = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{m1} & c_{m2} & \cdots & c_{mn} \end{bmatrix}.$$

THEOREM 7C. *Let $A = [a_{ij}]_{m\times n}$ be column-convex. Then one of the following two cases must occur:*

CASE 1*: There exists a saddle point (s, r) in the submatrix A_m^1. Then the pair of strategies $(\mu^*, \nu^*) = (\delta_s, \delta_r)$ is optimal in $\Gamma(A)$ and $v^* = a_{sr}$.*

CASE 2*: A_m^1 does not have a saddle point. Then $v^* = \mathrm{val}(\Gamma(A_m^1))$ is the value of $\Gamma(A)$, there are optimal strategies (μ^*, ν^*) in $\Gamma(A_m^1)$ of the form $\mu^* = \lambda\delta_1 + (1-\lambda)\delta_m$ and $\nu^* = \gamma\delta_s + (1-\gamma)\delta_r$ for some $0 < \lambda < 1$, $0 \le \gamma \le 1$ and $1 \le s < r \le n$, and they are also optimal in $\Gamma(A)$.*

Theorems 1B and 1C suggest that the last two theorems can be directly generalized to bimatrix games. This is discussed below.

5.2. Two-point Nash equilibria in bimatrix games. In this subsection we formulate four theorems (Theorems 8A–8C) about the existence of Nash equilibria in two-point strategies in bimatrix games, proved in Połowczuk (2006). They provide a full characterization of such equilibria and and a simple search procedure. Moreover, they complete Theorems 5A–7C.

The first result generalizes Theorem 7A to non-zero-sum games. It may also be seen as a discrete counterpart of Theorem 1A for the two-person case (considered with the concavity condition on payoff functions). To formulate it, we need to introduce some notation.

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be fixed matrices of the same size $m \times n$, $m, n \ge 2$.

The game $\Gamma_1(A_1, B_1)$ is said to be a *subgame* of $\Gamma(A, B)$ if the matrices A_1 and B_1 can be obtained by removing some rows and (or) columns from A and B (the same for A and B).

Now let $\Gamma_{kl}^{ij} = \Gamma(A_{kl}^{ij}, B_{kl}^{ij})$, $1 \le i \le k \le m$, $1 \le j \le l \le n$, where A_{kl}^{ij} and B_{kl}^{ij} are matrices of the form (17), corresponding to A and B, respectively. Obviously, each Γ_{kl}^{ij} is a subgame of $\Gamma(A, B)$.

Further, define

$$\underline{\lambda_{kl}^{ij}} = \min(b_{kl}^{ij}, b_{k,l+1}^{i,j+1}), \qquad \overline{\lambda_{kl}^{ij}} = \max(b_{kl}^{ij}, b_{k,l+1}^{i,j+1}), \tag{20}$$

$$\underline{\gamma_{kl}^{ij}} = \min(a_{kl}^{ij}, a_{k+1,l}^{i+1,j}), \qquad \overline{\gamma_{kl}^{ij}} = \max(a_{kl}^{ij}, a_{k+1,l}^{i+1,j}), \tag{21}$$

where

$$b_{kl}^{ij} = \frac{b_{kl} - b_{kj}}{b_{kl} - b_{kj} + b_{ij} - b_{il}}, \tag{22}$$

$$a_{kl}^{ij} = \frac{a_{il} - a_{kl}}{a_{il} - a_{kl} + a_{kj} - a_{ij}}. \tag{23}$$

Now we are ready to formulate the first theorem.

THEOREM 8A. *Let A and B be $m \times n$ matrices such that A is column-concave and B is row-concave. Then for the game $\Gamma = \Gamma(A, B)$ one of the following four cases must occur:*

CASE 1*: There exists a pure Nash equilibrium (s, r) in Γ.*

CASE 2*: There exists a 2×2 subgame $\Gamma_{s+1,r+1}^{sr}$ without pure Nash equilibria. Then there is a Nash equilibrium in $\Gamma_{s+1,r+1}^{sr}$ of the form $\mu^* = \lambda\delta_s + (1-\lambda)\delta_{s+1}$ and $\nu^* = \gamma\delta_r + (1-\gamma)\delta_{r+1}$, with $\lambda = b_{s+1,r+1}^{sr}$ and $\gamma = a_{s+1,r+1}^{sr}$, which is also a Nash equilibrium in Γ.*

CASE 3*: For some $k \geq 3$ there is a $k \times 2$ subgame $\Gamma_{s+k-1,r+1}^{sr}$ without pure Nash equilibrium, which satisfies*

$$b_{lr} = b_{l,r+1} \quad \text{whenever } s < l < s+k-1. \tag{24}$$

Then if $s < l < s+k-1$ and $\underline{\gamma_{l,r+1}^{l-1,r}} \leq \gamma \leq \overline{\gamma_{l,r+1}^{l-1,r}}$, there is a Nash equilibrium in $\Gamma_{s+k-1,r+1}^{sr}$ of the form $\mu^* = \delta_l$ and $\nu^* = \gamma\delta_r + (1-\gamma)\delta_{r+1}$, which is also a Nash equilibrium in Γ.

CASE 4*: For some $k \geq 3$ there is a $2 \times k$ subgame $\Gamma_{s+1,r+k-1}^{sr}$ without pure Nash equilibrium, for which*

$$a_{sl} = a_{s+1,l} \quad \text{whenever } r < l < r+k-1. \tag{25}$$

Then if $r < l < r+k-1$ and $\underline{\lambda_{s+1,l}^{s,l-1}} \leq \lambda \leq \overline{\lambda_{s+1,l}^{s,l-1}}$, there is a Nash equilibrium in $\Gamma_{s+1,r+k-1}^{sr}$ of the form $\mu^* = \lambda\delta_s + (1-\lambda)\delta_{s+1}$ and $\nu^* = \delta_l$, which is also a Nash equilibrium in Γ.

REMARK 8. A zero-sum version of Theorem 8A was proved in Radzik (2000, Theorem 4.3). However, for zero-sum games it is enough to consider only 2×3 and 3×2 subgames in Cases 3 and 4.

The second theorem generalizes Theorem 7B to non-zero-sum finite games. It can also be seen as a discrete counterpart of Theorem 1B given in the previous section.

THEOREM 8B. *Let A and B be $m \times n$ matrices, and let A be column-concave. Then for the game $\Gamma = \Gamma(A, B)$ one of the following three cases must occur:*

CASE 1: *There exists a pure Nash equilibrium (s, r) in Γ.*

CASE 2: *For some* $1 \le s < m$ *with there exists a* $2 \times n$ *subgame* $\Gamma^{s1}_{s+1,n}$ *without pure Nash equilibria. Then there is a Nash equilibrium in* $\Gamma^{s1}_{s+1,n}$ *of the form* $\mu^* = \lambda\delta_s + (1-\lambda)\delta_{s+1}$ *and* $\nu^* = \gamma\delta_r + (1-\gamma)\delta_u$ *for some* $0 < \lambda < 1$, $0 \le \gamma \le 1$ *and* $1 \le r < u \le n$, *which is also a Nash equilibrium in* Γ.

CASE 3: *For some* $1 < l < m$ *and* $1 \le r < u \le n$ *there exists a* 3×2 *subgame of* Γ *of the form*

$$\Gamma' = \Gamma\left(\begin{bmatrix} a_{l-1,r} & a_{l-1,u} \\ a_{lr} & a_{lu} \\ a_{l+1,r} & a_{l+1,u} \end{bmatrix} \begin{bmatrix} b_{l-1,r} & b_{l-1,u} \\ b_{lr} & b_{lu} \\ b_{l+1,r} & b_{l+1,u} \end{bmatrix}\right)$$

satisfying

$$b_{lr} = b_{lu} \ge b_{lj} \quad \text{for all } 1 \le j \le n \tag{26}$$

and

$$\text{(a)} \begin{cases} a_{l-1,r} < a_{lr} < a_{l+1,r} \\ a_{l-1,u} > a_{lu} > a_{l+1,u} \end{cases} \quad \text{or} \quad \text{(b)} \begin{cases} a_{l-1,r} > a_{lr} > a_{l+1,r} \\ a_{l-1,u} < a_{lu} < a_{l+1,u}. \end{cases} \tag{27}$$

Then, for $\underline{\gamma^{l-1,r}_{lu}} \le \gamma \le \overline{\gamma^{l-1,r}_{lu}}$, *the game* Γ' *has a mixed Nash equilibrium* (μ^*, ν^*) *of the form* $\mu^* = \delta_l$ *and* $\nu^* = \gamma\delta_r + (1-\gamma)\delta_u$, *which is also a Nash equilibrium in* Γ.

The next theorem generalizes Theorem 7C to finite non-zero-sum games. It can also be seen as a discrete counterpart of Theorem 1C. Here, for a game $\Gamma(A,B)_{m\times n}$, we define $\Gamma^1_m = \Gamma(A^1_m, B^1_m)$ (see the notation introduced before Theorem 7C).

THEOREM 8C. *Let* A *and* B *be two* $m \times n$ *matrices, with* A *column-convex. Then for the game* $\Gamma = \Gamma(A,B)$ *one of the following three cases must occur:*

CASE 1: *There exists a pure Nash equilibrium* (s,r) *in the subgame* Γ^1_m. *Then the pair* $(\mu^*, \nu^*) = (\delta_s, \delta_r)$ *of strategies is also a pure Nash equilibrium in* Γ.

CASE 2: *The game* Γ^1_m *does not have a pure Nash equilibrium. Then there is a Nash equilibrium in* Γ^1_m *of the form* $\mu^* = \lambda\delta_1 + (1-\lambda)\delta_m$ *and* $\nu^* = \gamma\delta_s + (1-\gamma)\delta_r$ *for some* $0 < \lambda < 1$, $0 \le \gamma \le 1$ *and* $1 \le s < r \le n$, *which is also a Nash equilibrium in* Γ.

Our last result is a modification of Theorem 8A where "concavity" is replaced by "convexity".

THEOREM 8D. *Let A and B be two $m\times n$ matrices such that A is column-concave and B is row-concave, and let*

$$\Gamma'' = \Gamma\left(\begin{bmatrix} a_{11} & a_{1n} \\ a_{m1} & a_{mn}\end{bmatrix}\begin{bmatrix} b_{11} & b_{1n} \\ b_{m1} & b_{mn}\end{bmatrix}\right).$$

Then for the game $\Gamma = \Gamma(A, B)$ one of the following two cases must occur:

CASE 1: *There exists a pure Nash equilibrium (s, r) in the subgame Γ''. Then the pair $(\mu^*, \nu^*) = (\delta_s, \delta_r)$ of strategies is also a pure Nash equilibrium in Γ.*

CASE 2: *The game Γ'' does not have a pure Nash equilibrium. Then there is a Nash equilibrium in Γ'' of the form (μ^*, ν^*), where $\mu^* = \lambda\delta_1 + (1-\lambda)\delta_m$ and $\nu^* = \gamma\delta_1 + (1-\gamma)\delta_n$, with $\lambda = b^{11}_{mn}$ and $\gamma = a^{11}_{mn}$, which is also a Nash equilibrium in Γ.*

REMARK 9. Bimatrix games $\Gamma(A, B)$ with payoff matrices A and B of size $m \times \infty$ are discussed in Połowczuk et al. (2012). The assumptions on the payoff matrices made there are the same as in Theorems 8A–8D, and the results obtained are similar.

6. Two-point Nash equilibria in n-person finite games. In this section we quote five results, Theorems 9A–9C and 10A and 10B (proved in Połowczuk et al. (2007)), about the structure of Nash equilibria in finite n-person games, with the additional basic assumption that the payoff functions of the players are concave or convex in their variables. We show here the existence of two-point Nash equilibria and give their characterizations.

The Glicksberg theorem (Theorem 1A) says that continuous quasi-concave games always have pure Nash equilibria. This strongly suggests that any n-person finite game with payoff functions quasi-concave in each variable should have two-adjoining Nash equilibria. However, this turns out to be false.

EXAMPLE 13. Consider a two-person non-zero-sum game with the payoff functions H_1 and H_2 described by the $n \times n$ matrices ($n \geq 3$)

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & & 0 & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ 0 & 0 & 0 & & 1 & 0 \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & & 0 & 0 & 0 \\ \vdots & & \ddots & & \vdots & \vdots \\ 0 & 0 & & 1 & 0 & 0 \\ 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 \end{bmatrix}.$$

Obviously H_1 and H_2 are quasi-concave (but not concave) in their variables. However, the only Nash equilibrium in this game consists of the following mixed strategies:

$$\mu^* = \frac{1}{n}\delta_1 + \cdots + \frac{1}{n}\delta_n \quad \text{and} \quad \nu^* = \frac{1}{n}\delta_1 + \cdots + \frac{1}{n}\delta_n.$$

Sketch of the proof. By the Nash Theorem there are $\mu_1^* = \alpha_1\delta_1 + \cdots + \alpha_n\delta_n$ and $\mu_2^* = \beta_1\delta_1 + \cdots + \beta_n\delta_n$ which form an equilibrium in the above game. We will prove that $\alpha_1 = \cdots = \alpha_n = \beta_1 = \cdots = \beta_n = 1/n$. The proof is in two steps.

STEP 1. Suppose $\mathrm{supp}(\mu_1^*) = \mathrm{supp}(\mu_2^*) = \{1, \ldots, n\}$. Then it is well known that $F_1(1, \mu_2^*) = F_1(2, \mu_2^*) = \cdots = F_1(n, \mu_2^*)$ and $F_2(\mu_1^*, 1) = F_2(\mu_1^*, 2) = \cdots = F_2(\mu_1^*, n)$, which is equivalent to $\beta_1 = \cdots = \beta_n = 1/n$.

STEP 2. Suppose that there exists i such that $\alpha_i = 0$. Then it is easy to deduce that $\beta_i = 0$, since otherwise μ_2^* would not maximize the utility of player 2. Similarly we can show that $\beta_i = 0$ implies $\alpha_{i-1} = 0$ or $\alpha_n = 0$ if $i > 1$ or $i = 1$, respectively. Repeating this procedure, we finally get $\alpha_1 = \cdots = \alpha_n = \beta_1 = \cdots = \beta_n = 0$, which is a contradiction, since μ_1^* and μ_2^* are probability distributions. When we start with some $\beta_i = 0$, the result will be the same.

The example discussed above shows that the direct discrete counterpart of the Glicksberg theorem (Theorem 1A) would be false. However, the situation changes when "quasi-concavity" is replaced by "concavity". We study this case in some of the theorems below.

Let Γ be a finite n-person game of the form (1). We begin with two theorems about games with properties of "partial" concavity or convexity, respectively.

THEOREM 9A. *Let $1 \le s \le n$. If for $i = 1, \ldots, s$ the payoff function H_i is concave in the variable e_i then there exists a mixed Nash equilibrium $(\mu_1^*, \ldots, \mu_n^*)$ in the game Γ such that $\mu_1^*, \ldots, \mu_s^*$ are two-adjoining strategies.*

THEOREM 9B. *Let $1 \le s \le n$. If for $i = 1, \ldots, s$ the payoff function H_i is convex in e_i then there exists a mixed Nash equilibrium $(\mu_1^*, \ldots, \mu_n^*)$ in the game Γ such that $\mu_1^*, \ldots, \mu_s^*$ are two-marginal strategies.*

The next theorem combines in some sense the results of two previous ones.

THEOREM 9C. *Let $1 \le s \le n$. Assume that for $i = 1, \ldots, s$ the payoff function H_i is concave in e_i and for $j = s+1, \ldots, n$ the payoff function H_j is convex in e_j. Then there exists a two-point Nash equilibrium $(\mu_1^*, \ldots, \mu_n^*)$ in the game Γ such that $\mu_1^*, \ldots, \mu_s^*$ are two-adjoining strategies and $\mu_{s+1}^*, \ldots, \mu_n^*$ are two-marginal strategies.*

We end this section with two immediate but important corollaries of Theorems 9A and 9B. The first of them can be seen as a discrete counterpart of the stronger version of Glicksberg's Theorem (with "concavity" instead of "quasi-concavity" in Theorem 1A). It also generalizes the results of Theorems 7A and 8A. The second of them generalizes Theorem 8D.

THEOREM 10A. *If for $i = 1, \dots, n$ the payoff function H_i is concave in the variable e_i then there exists a two-adjoining Nash equilibrium in the game Γ.*

THEOREM 10B. *If for $i = 1, \dots, n$ the payoff function H_i is convex in e_i then there exists a two-marginal Nash equilibrium in the game Γ.*

REMARK 10. An interesting open problem is to find a simple procedure giving the Nash equilibrium described in Theorem 10A or in Theorem 9C, and similar to the ones described in Theorems 8A–8D in Połowczuk (2003).

REMARK 11. Another question is whether Theorems 10A and 9C remain true when convexity of payoff functions is replaced by quasi-convexity. It appears that this is not true, as shown in Example 14 below. Example 13 given earlier shows that a similar weakening of the assumptions of Theorems 10A and 9C, from concavity to quasi-concavity, is not possible.

EXAMPLE 14. Consider a two-person non-zero-sum game with the payoff functions F_1 and F_2 described by

$$A = \begin{bmatrix} 1 & 1 & 7 & 2 \\ 1 & 4 & 6 & 1 \\ 1 & 6 & 4 & 1 \\ 2 & 7 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 7 & 6 & 4 & 1 \\ 7 & 6 & 4 & 1 \\ 1 & 4 & 6 & 7 \\ 1 & 4 & 6 & 7 \end{bmatrix}.$$

One can easily see that this game is quasi-convex. However, the only Nash equilibrium in this game consists of the mixed strategies: $\mu^* = \frac{1}{2}\delta_2 + \frac{1}{2}\delta_3$ and $\nu^* = \frac{1}{2}\delta_2 + \frac{1}{2}\delta_3$. Justification of this fact is left to the reader.

7. Conclusions and open problems. We have presented a thorough review of the known results about the existence and form of Nash equilibria in n-person finite games with payoff functions having some concavity/convexity properties. It turns out that the most important results about the existence of pure Nash equilibria in games with compact convex strategy sets and concave/convex payoff functions have their counterparts in this case, yet the assumptions often have to be strengthened. In these theorems, instead of the existence of pure Nash equilibria, the existence of equilibria in mixed strategies where each of the players uses only two of his pure strategies is proved. We have also shown that in case of 2-person finite games, the two-point equilibria existing according to those results can be found using some simple search procedures. However, there are essential questions which still remain open.

Problem 1. Let $n \geq 3$. Are there “simple” algorithms allowing one to find two-point Nash equilibria in n-person finite games determined in Theorems 9A–9C?

Problem 2. Are there intermediate properties of payoff functions, weaker than concavity and stronger than quasi-concavity, that guarantee the existence of two-point equilibria in finite games with such rewards?

Problem 3. For a natural s, let $\Gamma(s)$ be an n-person non-zero-sum finite game of the form $G = \langle N, \{E_i\}_{i\in N}, \{H_i\}_{i\in N}\rangle$, where for $i \in N$ the strategy spaces E_i are $\{1,\dots,l_i\}^s$ and the payoff function H_i is concave in its ith variable. (Here the concavity is defined by the clear generalization of Definition 2.) The results of the previous section show that in the case $s = 1$, the game $\Gamma(1)$ always has a mixed Nash equilibrium in two-adjoining strategies. The question is whether also in the game $\Gamma(s)$ with $s > 1$, there is a mixed Nash equilibrium with supports of the players’ strategies, consisting only of “small” sets of adjoining pure strategies. Note that $\Gamma(s)$ can be seen as a discrete counterpart of the game considered in Theorem 1A where quasi-concavity is replaced by concavity.

Problem 4. Are there any essential generalizations of Theorems 1B and 1C with the strategy spaces $[0,1]$ replaced by $[0,1]^k$, where $k > 1$? No results are known here, even under continuity of the payoff functions.

Acknowledgements. The authors thank an anonymous referee for his useful comments, questions and suggestions, which have enabled the authors to improve the paper.

This research was supported by GRANT 530103/I-18 of Wrocław University of Technology.

References

H. F. Bohnenblust, S. Karlin and L. S. Shapley (1950), *Games with continuous, convex pay-off*, in: H. W. Kuhn and A. W. Tucker (eds.), Contributions to the Theory of Games, I, Ann. of Math. Stud. 24, Princeton Univ. Press, Princeton, 1950, 181–192.

G. Debreu (1952), *A social equilibrium existence theorem*, Proc. Nat. Acad. Sci. USA 38, 886–893.

K. Fan (1952), *Fixed point and minimax theorems in locally convex topological linear spaces*, Proc. Nat. Acad. Sci. USA 38, 121–126.

I. Glicksberg (1952), *A further generalization of the Kakutani fixed point theorem, with application to Nash equilibrium points*, Proc. Amer. Math. Soc. 3, 170–174.

E. Gómez (1988), *Games with convex payoff functions in the first variable*, Int. J. Game Theory 17, 201–204.

J.-F. Mertens (1986), *The minmax theorem for u.s.c.-l.s.c. payoff functions*, Int. J. Game Theory 15, 237–250.

J. Nash (1950), *Equilibrium points in n-person games*, Proc. Nat. Acad. Sci. U.S.A. 36, 48–49.

H. Nikaido and K. Isoda (1955), *Note on noncooperative convex games*, Pacific J. Math. 5, 807–815.
T. Parthasarathy and T. E. S. Raghavan (1971), *Some Topics in Two-Person Games*, American Elsevier, New York.
W. Połowczuk (2003), *Pure Nash equilibria in finite two-person non-zero-sum games*, Int. J. Game Theory 32, 229–240.
W. Połowczuk (2006), *On almost-pure-strategy Nash equilibria in bimatrix games with convexity properties*, Appl. Math. (Warsaw) 33, 71-84.
W. Połowczuk, P. Więcek and T. Radzik (2007), *On the existence of almost-pure-strategy Nash equilibrium in n-person finite games*, Math. Methods Oper. Research 65, 141–152.
W. Połowczuk, T. Radzik and P. Więcek (2012), *Simple equilibria in semi-infinite games*, Int. Game Theory Rev. 14, 1250014, 19 pp.
T. Radzik and G. Ravindran (1989), *On two counter-examples of non-cooperative games without Nash equilibrium*, Sankyā Ser. A 51, 236–240.
T. Radzik (1991a), *Saddle points theorems*, Int. J. Game Theory 20, 23–32.
T. Radzik (1991b), *Pure-strategy ε-Nash equilibrium in two-person non-zero-sum games*, Games Economic Behavior 3, 356–371.
T. Radzik (1993), *Nash equilibria of discontinuous non-zero-sum two-person games*, Int. J. Game Theory 21, 429–437.
T. Radzik (2000), *Characterization of optimal strategies in matrix games with convexity properties*, Int. J. Game Theory 29, 211–227.
L. Shapley (1964), *Some topics in two-person games*, in: Advances in Game Theory, Ann. of Math. Stud. 52, Princeton Univ. Press, 1–28.
M. Sion (1958), *On general minimax theorems*, Pacific J. Math. 8, 171–176.

Tadeusz Radzik, Piotr Więcek
Institute of Mathematics
Wrocław University of Technology
Wybrzeże Wyspiańskiego 27
50-370 Wrocław, Poland
E-mail: tadeusz.radzik@pwr.wroc.pl

Received on 25.2.2013;
revised version on 11.8.2014 (2173)