APPLICATIONES MATHEMATICAE
42,1 (2015), pp. 111–124

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ON FORMULAE FOR CENTRAL MOMENTS OF COUNTING DISTRIBUTIONS

Abstract. The aim of this article is to give new formulae for central moments of the binomial, negative binomial, Poisson and logarithmic distributions. We show that they can also be derived from the known recurrence formulae for those moments. Central moments for distributions of the Panjer class are also studied. We expect our formulae to be useful in many applications.

1. Introduction. There is an extensive literature about moments of discrete distributions. Here we are interested in the central moments μ_r of the binomial distribution $B(N,p)$,

$$P(X=x)=\binom{N}{x}p^x q^{N-x},\qquad x=0,1,\ldots,N, \tag{1.1}$$

the negative binomial distribution $NB(N,q)$,

$$P(X=x)=\binom{N+x-1}{N-1}p^N q^x,\qquad x=0,1,\ldots, \tag{1.2}$$

the Poisson distribution $P(\lambda)$,

$$P(X=x)=\frac{\lambda^x}{x!}e^{-\lambda},\quad x=0,1,2,\ldots, \tag{1.3}$$

and the logarithmic distribution $\mathrm{Log}(\theta)$,

$$P(X=x)=\frac{-1}{\ln(1-\theta)}\,\frac{\theta^x}{x},\quad x=1,2,\ldots, \tag{1.4}$$

where $\mu_r=E(X-EX)^r$, $r=0,1,2,\ldots.$ $N\in\mathbb{N}$, $p>0$, $p+q=1$, $\lambda>0$, $0<\theta<1$.

2010 *Mathematics Subject Classification*: Primary 60E99; Secondary 62E99.
Key words and phrases: moments, Stirling numbers, recursions, Panjer class, compound distributions.

DOI: 10.4064/am42-1-7

THEOREM 1.1. *The* $(r+1)$*th central moments of the binomial, negative binomial, Poisson and logarithmic distributions are given by the formulas*

$$\mu_{r+1} = q\sum_{i=1}^{r}\binom{r}{i}(-Np)^{r-i}\sum_{k=1}^{i} kS(i,k)N^{(k)}p^k, \tag{1.5}$$

$$\mu_{r+1} = (1/p)\sum_{i=1}^{r}\binom{r}{i}(-Nq/p)^{r-i}\sum_{k=1}^{i} kS(i,k)(N)_k(q/p)^k, \tag{1.6}$$

$$\mu_{r+1} = \sum_{i=1}^{r}\binom{r}{i}(-\lambda)^{r-i}\sum_{k=1}^{i} kS(i,k)\lambda^k, \tag{1.7}$$

$$\mu_{r+1} = \frac{d}{1-\theta}\sum_{i=1}^{r}\binom{r}{i}\left(-\frac{d\theta}{1-\theta}\right)^{r-i} \cdot \sum_{k=1}^{i}[k-d\theta]S(i,k)(k-1)!\theta^k(1-\theta)^{-k} \tag{1.8}$$

respectively, where $r \in \mathbb{N}$, $S(i,k)$ *is the Stirling number of the second kind,*

$$N^{(k)} = N\cdot(N-1)\cdot\ldots\cdot(N-k+1) = \frac{N!}{(N-k)!},$$

$$(N)_k = N\cdot(N+1)\cdot\ldots\cdot(N+k-1) = \frac{(N+k-1)!}{(N-1)!}$$

(*the Pochhammer symbol*) *and* $d := -1/\ln(1-\theta)$.

Proof. We use the formula

$$\mu_r = E(X-EX)^r = \sum_{i=0}^{r}\binom{r}{i}(-m_1)^{r-i}m_i, \tag{1.9}$$

where $m_1 = EX$ and $m_i = EX^i$ denotes the ith uncorrected moment (cf. Johnson et al. [JKK, p. 52]).

For $X \sim B(N,p)$ from (1.9) we get

$$\mu_r = \sum_{i=0}^{r}(-1)^{r-i}\binom{r}{i}(Np)^{r-i}\sum_{k=0}^{i} S(i,k)N^{(k)}p^k \tag{1.10}$$

since $m_1 = Np$ and $m_i = \sum_{k=0}^{i} S(i,k)m_{(k)}$ with

$$m_{(k)} = EX(X-1)(X-2)\cdot\ldots\cdot(X-k+1) = \frac{N!p^k}{(N-k)!} = N^{(k)}p^k$$

(cf. Johnson et al. [JKK, p. 109]). Taking $r+1$ in (1.10) we obtain

$$\mu_{r+1} = \sum_{i=0}^{r+1}(-1)^{r+1-i}\binom{r+1}{i}(Np)^{r+1-i}\sum_{k=0}^{i} S(i,k)N^{(k)}p^k.$$

The following calculations lead to a simpler formula for μ_{r+1}. Using the property

$$\binom{r+1}{i} = \binom{r}{i} + \binom{r}{i-1} \tag{1.11}$$

we have

$$\begin{aligned}\mu_{r+1} = &\sum_{i=0}^{r}(-1)^{r+1-i}\binom{r}{i}(Np)^{r+1-i}\sum_{k=0}^{i}S(i,k)N^{(k)}p^k \\ &+ \sum_{i=1}^{r+1}(-1)^{r+1-i}\binom{r}{i-1}(Np)^{r+1-i}\sum_{k=0}^{i}S(i,k)N^{(k)}p^k,\end{aligned}$$

which leads to

$$\begin{aligned}\mu_{r+1} = &\sum_{i=0}^{r}(-1)^{r+1-i}\binom{r}{i}(Np)^{r+1-i}\sum_{k=0}^{i}S(i,k)N^{(k)}p^k \\ &+ \sum_{i=0}^{r}(-1)^{r-i}\binom{r}{i}(Np)^{r-i}\sum_{k=0}^{i+1}S(i+1,k)N^{(k)}p^k.\end{aligned}$$

The recurrence relation for the Stirling numbers of the second kind,

$$S(n+1,k) = S(n,k-1) + kS(n,k), \tag{1.12}$$

allows us to write

$$\begin{aligned}\mu_{r+1} = &\sum_{i=0}^{r}(-1)^{r+1-i}\binom{r}{i}(Np)^{r+1-i}\sum_{k=0}^{i}S(i,k)N^{(k)}p^k \\ &+ \sum_{i=0}^{r}(-1)^{r-i}\binom{r}{i}(Np)^{r-i}\sum_{k=1}^{i+1}S(i,k-1)N^{(k)}p^k \\ &+ \sum_{i=1}^{r}(-1)^{r-i}\binom{r}{i}(Np)^{r-i}\sum_{k=1}^{i}kS(i,k)N^{(k)}p^k.\end{aligned}$$

Hence we get

$$\begin{aligned}\mu_{r+1} = &\sum_{i=0}^{r}(-1)^{r+1-i}\binom{r}{i}(Np)^{r+1-i}\sum_{k=0}^{i}S(i,k)N^{(k)}p^k \\ &+ \sum_{i=0}^{r}(-1)^{r-i}\binom{r}{i}(Np)^{r-i}\sum_{k=0}^{i}S(i,k)N^{(k+1)}p^{k+1} \\ &+ \sum_{i=1}^{r}(-1)^{r-i}\binom{r}{i}(Np)^{r-i}\sum_{k=1}^{i}kS(i,k)N^{(k)}p^k,\end{aligned}$$

which gives

$$\mu_{r+1} = \sum_{i=0}^{r}(-1)^{r+1-i}\binom{r}{i}(Np)^{r+1-i}\sum_{k=0}^{i}S(i,k)N^{(k)}p^k$$
$$+ \sum_{i=0}^{r}(-1)^{r-i}\binom{r}{i}(Np)^{r-i}\sum_{k=0}^{i}S(i,k)N^{(k)}(N-k)p^{k+1}$$
$$+ \sum_{i=1}^{r}(-1)^{r-i}\binom{r}{i}(Np)^{r-i}\sum_{k=1}^{i}kS(i,k)N^{(k)}p^k,$$

leading to

$$\mu_{r+1} = -\sum_{i=1}^{r}\binom{r}{i}(-Np)^{r-i}\sum_{k=1}^{i}kS(i,k)N^{(k)}p^{k+1}$$
$$+ \sum_{i=1}^{r}\binom{r}{i}(-Np)^{r-i}\sum_{k=1}^{i}kS(i,k)N^{(k)}p^k,$$

which ends the proof of (1.5).

In the case $X \sim NB(N,q)$ and $X \sim P(\lambda)$ the proofs of (1.6) and (1.7) are similar to the proof of (1.5).

Now we prove (1.8). For $X \sim \mathrm{Log}(\theta)$ from (1.9) we get

$$\mu_r = \left(\frac{-d\theta}{1-\theta}\right)^r + \sum_{i=1}^{r}\binom{r}{i}\left(-\frac{d\theta}{1-\theta}\right)^{r-i}\sum_{k=1}^{i}S(i,k)(k-1)!d\theta^k(1-\theta)^{-k}. \tag{1.13}$$

Taking $r+1$ in (1.13) we obtain

$$\mu_{r+1} = \left(\frac{-d\theta}{1-\theta}\right)^{r+1}$$
$$+ \sum_{i=1}^{r+1}\binom{r+1}{i}\left(-\frac{d\theta}{1-\theta}\right)^{r+1-i}\sum_{k=1}^{i}S(i,k)(k-1)!d\theta^k(1-\theta)^{-k}.$$

Hence using (1.11), (1.12) and some calculations we get

$$\mu_{r+1} = \left(\frac{-d\theta}{1-\theta}\right)^{r+1}$$
$$+ \sum_{i=1}^{r}\binom{r}{i}\left(-\frac{d\theta}{1-\theta}\right)^{r+1-i}\sum_{k=1}^{i}S(i,k)(k-1)!d\theta^k(1-\theta)^{-k}$$
$$+ \sum_{i=0}^{r}\binom{r}{i}\left(-\frac{d\theta}{1-\theta}\right)^{r-i}\sum_{k=0}^{i}S(i,k)k!d\theta^{k+1}(1-\theta)^{-k-1}$$
$$+ \sum_{i=1}^{r}\binom{r}{i}\left(-\frac{d\theta}{1-\theta}\right)^{r-i}\sum_{k=1}^{i}kS(i,k)(k-1)!d\theta^k(1-\theta)^{-k},$$

which gives

$$\mu_{r+1} = -\frac{d\theta}{1-\theta}\sum_{i=1}^{r}\binom{r}{i}\left(-\frac{d\theta}{1-\theta}\right)^{r-i}\sum_{k=1}^{i} S(i,k)(k-1)!d\theta^k(1-\theta)^{-k}$$
$$+\frac{1}{1-\theta}\sum_{i=1}^{r}\binom{r}{i}\left(-\frac{d\theta}{1-\theta}\right)^{r-i}\sum_{k=1}^{i} kS(i,k)(k-1)!d\theta^k(1-\theta)^{-k},$$

and ends the proof of (1.8). ■

From (1.6) we have the following

COROLLARY 1.2. *Let $X \sim G(q)$, i.e. X has the geometric distribution with*

$$P(X = x) = q^x p, \qquad x = 0, 1, 2, \ldots.$$

Then

$$\mu_{r+1} = (1/p)\sum_{i=1}^{r}\binom{r}{i}(-q/p)^{r-i}\sum_{k=1}^{i} kS(i,k)k!(q/p)^k. \tag{1.14}$$

2. An alternative derivation of the formulae. We show now that the above formulae can be obtained via some recurrence relations for the central moments.

THEOREM 2.1. *The following statements hold true.*

(i) *Let $X \sim B(N,p)$. Formula* (1.5) *for the central moments of X follows from the recurrence relations*

$$\mu_{r+1} = pq\left(Nr\mu_{r-1} + \frac{d\mu_r}{dp}\right) \tag{2.1}$$

(*cf. Romanovsky* [R]*; Kendall and Stuart* [KS, p. 122]*; Johnson et al.* [JKK, p. 110]).

(ii) *Let $X \sim NB(N,q)$. Formula* (1.6) *follows from the recurrence relations*

$$\mu_{r+1} = q\left[(Nr/p^2)\mu_{r-1} + \frac{d\mu_r}{dq}\right]$$

(*cf. Johnson et al.* [JKK, p. 216]). *Formula* (1.14) *for the central moments of $X \sim G(q)$ follows from the recurrence relations*

$$\mu_{r+1} = q\left[(r/p^2)\mu_{r-1} + \frac{d\mu_r}{dq}\right]$$

(*cf. Johnson et al.* [JKK, p. 216]).

(iii) *Let $X \sim P(\lambda)$. Formula* (1.7) *follows from the recurrence relations*

$$\mu_{r+1} = \lambda\left(r\mu_{r-1} + \frac{d\mu_r}{d\lambda}\right)$$

(*cf. Craig* [C]*; Kendall and Stuart* [KS, p. 126]).

(iv) *Let* $X \sim \mathrm{Log}(\theta)$. *Formula* (1.8) *follows from the recurrence relations*

$$\mu_{r+1} = \theta \frac{d\mu_r}{d\theta} + r\mu_2\mu_{r-1} \tag{2.2}$$

(*cf. Johnson et al.* [JKK, p. 110]).

Proof. (i) Taking $r-1$ in (1.10) (cf. Johnson et al. [JKK, pp. 52–53], yields

$$\mu_{r-1} = \sum_{i=0}^{r-1} (-1)^{r-1-i} \binom{r-1}{i} (Np)^{r-1-i} \sum_{k=0}^{i} S(i,k) N^{(k)} p^k. \tag{2.3}$$

Now from (1.10) we get

$$\begin{aligned} \frac{d\mu_r}{dp} &= \sum_{i=0}^{r} (-1)^{r-i} \binom{r}{i} (r-i) N (Np)^{r-i-1} \sum_{k=0}^{i} S(i,k) N^{(k)} p^k \\ &\quad + \sum_{i=1}^{r} (-1)^{r-i} \binom{r}{i} (Np)^{r-i} \sum_{k=1}^{i} S(i,k) N^{(k)} k p^{k-1}. \end{aligned}$$

Hence using the property

$$\binom{r}{i}(r-i) = r\binom{r-1}{i} \tag{2.4}$$

we obtain

$$\begin{aligned} \frac{d\mu_r}{dp} &= Nr \sum_{i=0}^{r} (-1)^{r-i} \binom{r-1}{i} (Np)^{r-i-1} \sum_{k=0}^{i} S(i,k) N^{(k)} p^k \\ &\quad + \sum_{i=1}^{r} (-1)^{r-i} \binom{r}{i} (Np)^{r-i} \sum_{k=1}^{i} k S(i,k) N^{(k)} p^{k-1}, \end{aligned}$$

which by (2.3) gives

$$\frac{d\mu_r}{dp} = -Nr\mu_{r-1} + \sum_{i=1}^{r} (-1)^{r-i} \binom{r}{i} (Np)^{r-i} \sum_{k=1}^{i} k S(i,k) N^{(k)} p^{k-1}. \tag{2.5}$$

Putting (2.5) in (2.1) we obtain (1.5).

The proof of (ii) and (iii) is similar to the proof of (i).

Now we prove (iv). From (1.13) we have

$$\mu_2 = d\theta(1-d\theta)(1-\theta)^{-2}, \tag{2.6}$$

$$\begin{aligned} \mu_{r-1} &= \left(\frac{-d\theta}{1-\theta}\right)^{r-1} \\ &\quad + \sum_{i=1}^{r-1} \binom{r-1}{i} \left(-\frac{d\theta}{1-\theta}\right)^{r-1-i} \sum_{k=1}^{i} S(i,k)(k-1)!\,d\theta^k (1-\theta)^{-k}, \end{aligned} \tag{2.7}$$

and

$$\frac{d\mu_r}{d\theta} = \frac{dr(\theta d-1)}{(1-\theta)^2}\left(\frac{-d\theta}{1-\theta}\right)^{r-1}$$
$$+ \frac{d(\theta d-1)}{(1-\theta)^2}\sum_{i=1}^{r}(r-i)\binom{r}{i}$$
$$\cdot \left(-\frac{d\theta}{1-\theta}\right)^{r-1-i}\sum_{k=1}^{i} S(i,k)(k-1)!d\theta^k(1-\theta)^{-k}$$
$$+ \sum_{i=1}^{r}\binom{r}{i}\left(-\frac{d\theta}{1-\theta}\right)^{r-i}$$
$$\cdot \sum_{k=1}^{i} S(i,k)(k-1)!\left[\frac{-d^2}{1-\theta}\left(\frac{\theta}{1-\theta}\right)^{k} + \frac{d}{(1-\theta)^2}k\left(\frac{\theta}{1-\theta}\right)^{k-1}\right],$$

respectively. Hence, using (2.4) and (2.7) we get

$$\frac{d\mu_r}{d\theta} = \frac{dr(\theta d-1)}{(1-\theta)^2}\mu_{r-1} \tag{2.8}$$
$$+ \frac{d}{1-\theta}\sum_{i=1}^{r}\binom{r}{i}\left(-\frac{d\theta}{1-\theta}\right)^{r-i}\sum_{k=1}^{i} S(i,k)(k-1)!\theta^{k-1}(1-\theta)^{-k}[k-d\theta].$$

Finally putting (2.6) and (2.8) in (2.2) we get (1.8). ∎

REMARK 2.2. Some of the above statements were applied in Steliga and Szynal [SS] where the central moments for the α-modified binomial and α-modified Poisson distributions were studied.

3. Generalizations. It is known that the binomial, Poisson and negative binomial distributions belong to the Panjer class which has applications in insurance (cf. Klugman et al. [KPW, p. 221]; Sundt and Vernic [SV, p. 38]). The evaluations of the previous sections allow us to give the following general formula for the central moments of distributions belonging to the Panjer class.

DEFINITION 3.1. Let p_k be the probability function of a discrete random variable. The class of counting distributions satisfying the recursion

$$\frac{p_k}{p_{k-1}} = a + \frac{b}{k}, \qquad k = 1,2,\ldots, \tag{3.1}$$

is called the *Panjer class*. We write $X \sim \mathscr{P}(a,b)$ if X has the probability function given by (3.1).

THEOREM 3.2. *Let X have a distribution satisfying* (3.1). *The* $(r+1)$*th central moment of X is*

$$\mu_{r+1} = \frac{1}{1-a}\sum_{i=1}^{r}\binom{r}{i}\left(-\frac{a+b}{1-a}\right)^{r-i}\sum_{k=1}^{i} kS(i,k) \cdot\left[\left(\frac{a+b}{1-a}\right)^{k} + \sum_{j=0}^{k-1} s(k,j)\left(\frac{a+b}{1-a}\right)^{j}\left(\frac{a}{1-a}\right)^{k-j}\right] \tag{3.2}$$

where $s(k,j)$ is the signless Stirling number of the first kind.

Proof. From (1.9) with

$$m_1 = \frac{a+b}{1-a}, \qquad m_i = \sum_{k=0}^{i} S(i,k)m_{(k)},$$

$$m_{(k)} = \left(\frac{a+b}{1-a}\right)^{k} + \sum_{j=0}^{k-1} s(k,j)\left(\frac{a+b}{1-a}\right)^{j}\left(\frac{a}{1-a}\right)^{k-j},$$

we get

$$\mu_{r} = \sum_{i=0}^{r}\binom{r}{i}\left(-\frac{a+b}{1-a}\right)^{r-i}\sum_{k=0}^{i} S(i,k) \cdot\left[\left(\frac{a+b}{1-a}\right)^{k} + \sum_{j=0}^{k-1} s(k,j)\left(\frac{a+b}{1-a}\right)^{j}\left(\frac{a}{1-a}\right)^{k-j}\right]. \tag{3.3}$$

Taking $r+1$ in (3.3) we obtain

$$\mu_{r+1} = \sum_{i=0}^{r+1}\binom{r+1}{i}\left(-\frac{a+b}{1-a}\right)^{r+1-i}\sum_{k=0}^{i} S(i,k) \cdot\left[\left(\frac{a+b}{1-a}\right)^{k} + \sum_{j=0}^{k-1} s(k,j)\left(\frac{a+b}{1-a}\right)^{j}\left(\frac{a}{1-a}\right)^{k-j}\right].$$

Hence using (1.11), (1.12) and some calculations we get

$$\mu_{r+1} = \sum_{i=0}^{r}\binom{r}{i}(-1)^{r+1-i}\left(\frac{a+b}{1-a}\right)^{r+1-i}\sum_{k=0}^{i} S(i,k) \cdot\left[\left(\frac{a+b}{1-a}\right)^{k} + \sum_{j=0}^{k-1} s(k,j)\left(\frac{a+b}{1-a}\right)^{j}\left(\frac{a}{1-a}\right)^{k-j}\right]$$
$$+ \sum_{i=0}^{r}\binom{r}{i}(-1)^{r-i}\left(\frac{a+b}{1-a}\right)^{r-i}\sum_{k=0}^{i} S(i,k)$$

$$\cdot\left[\left(\frac{a+b}{1-a}\right)^{k+1}+\sum_{j=0}^{k}s(k+1,j)\left(\frac{a+b}{1-a}\right)^{j}\left(\frac{a}{1-a}\right)^{k+1-j}\right]$$
$$+\sum_{i=0}^{r}\binom{r}{i}(-1)^{r-i}\left(\frac{a+b}{1-a}\right)^{r-i}\sum_{k=0}^{i}kS(i,k)$$
$$\cdot\left[\left(\frac{a+b}{1-a}\right)^{k}+\sum_{j=0}^{k-1}s(k,j)\left(\frac{a+b}{1-a}\right)^{j}\left(\frac{a}{1-a}\right)^{k-j}\right].$$

From the recurrence relation for the signless Stirling numbers of the first kind,

$$s(k+1,j)=ks(k,j)+s(k,j-1),$$

we obtain

$$\mu_{r+1}=\sum_{i=0}^{r}\binom{r}{i}(-1)^{r+1-i}\left(\frac{a+b}{1-a}\right)^{r+1-i}\sum_{k=0}^{i}S(i,k)$$
$$\cdot\left[\left(\frac{a+b}{1-a}\right)^{k}+\sum_{j=0}^{k-1}s(k,j)\left(\frac{a+b}{1-a}\right)^{j}\left(\frac{a}{1-a}\right)^{k-j}\right]$$
$$+\sum_{i=0}^{r}\binom{r}{i}(-1)^{r-i}\left(\frac{a+b}{1-a}\right)^{r-i}\sum_{k=0}^{i}S(i,k)$$
$$\cdot\left[\left(\frac{a+b}{1-a}\right)^{k+1}+\sum_{j=1}^{k}s(k,j-1)\left(\frac{a+b}{1-a}\right)^{j}\left(\frac{a}{1-a}\right)^{k+1-j}\right]$$
$$+\sum_{i=0}^{r}\binom{r}{i}(-1)^{r-i}\left(\frac{a+b}{1-a}\right)^{r-i}\sum_{k=0}^{i}S(i,k)$$
$$\cdot\left[\sum_{j=0}^{k}ks(k,j)\left(\frac{a+b}{1-a}\right)^{j}\left(\frac{a}{1-a}\right)^{k+1-j}\right]$$
$$+\sum_{i=0}^{r}\binom{r}{i}(-1)^{r-i}\left(\frac{a+b}{1-a}\right)^{r-i}\sum_{k=0}^{i}kS(i,k)$$
$$\cdot\left[\left(\frac{a+b}{1-a}\right)^{k}+\sum_{j=0}^{k-1}s(k,j)\left(\frac{a+b}{1-a}\right)^{j}\left(\frac{a}{1-a}\right)^{k-j}\right].$$

Hence

$$\mu_{r+1}=\left(1+\frac{a}{1-a}\right)\sum_{i=1}^{r}\binom{r}{i}(-1)^{r-i}\left(\frac{a+b}{1-a}\right)^{r-i}\sum_{k=1}^{i}kS(i,k)$$
$$\cdot\left[\left(\frac{a+b}{1-a}\right)^{k}+\sum_{j=0}^{k-1}s(k,j)\left(\frac{a+b}{1-a}\right)^{j}\left(\frac{a}{1-a}\right)^{k-j}\right].\ \blacksquare$$

One can verify that for $a = -p/q$, $b = (N+1)p/q$ (for the binomial distribution $B(N,p)$) from (3.2) we get (1.5). For $a = q$, $b = (N-1)q$ (for the negative binomial distribution $NB(N,q)$) from (3.2) we get (1.6). For $a = q$, $b = 0$ (for the geometric distribution $G(q)$) from (3.2) we get (1.14). For $a = 0$, $b = \lambda$ (for the Poisson distribution $P(\lambda)$) from (3.2) we get (1.7).

Recurrence formulae for moments of Panjer distributions can be found in Sundt and Vernic [SV], Murat and Szynal [MS], [MMS] and Szynal and Teugels [ST].

4. Compound distributions. In this section we consider the central moments of the random sum

$$S_N = X_1 + \cdots + X_N, \tag{4.1}$$

where $\{X_i,\, i \ge 1\}$ is a sequence of independent identically distributed random variables, and N is a random variable independent of $\{X_i,\ i \ge 1\}$. It is known that for compound distributions the following formulae hold true:

$$\begin{aligned} &E(S_N - ES_N) = 0,\\ &\sigma^2 S_N = E(S_N - ES_N)^2 = EN\sigma^2 X + \sigma^2 N E^2 X,\\ &E(S_N - ES_N)^3 = ENE(X - EX)^3 + 3\sigma^2 NEX\sigma^2 X\\ &\qquad\qquad + E(N - EN)^3 E^3 X \end{aligned} \tag{4.2}$$

(Klugman et al. [KPW, p. 298]). One can show that

$$\begin{aligned} E(S_N - ES_N)^4 = {} & ENE(X - EX)^4 + E(N - EN)^4 E^4 X\\ & + 6E(N - EN)^3 E^2 X \sigma^2 X + 6\sigma^2 NENE^2 X\sigma^2 X\\ & + 4\sigma^2 NEXE(X - EX)^3 + 3EN(N-1)(\sigma^2 X)^2, \end{aligned} \tag{4.3}$$

under the assumption that the moments exist.

Assume now that for a random variable N the recursion (3.1) holds. Then from (3.2), (4.2) and (4.3) we obtain formulae for central moments of compound distributions where the random variable N belongs to the Panjer class $\mathscr{P}(a,b)$:

$$\sigma^2 S_N = \frac{a+b}{1-a}\left[\sigma^2 X + \frac{1}{1-a}E^2 X\right], \tag{4.4}$$

$$E(S_N - ES_N)^3 = \frac{a+b}{1-a}\left[E(X - EX)^3 + \frac{3}{1-a}EX\sigma^2 X + \frac{1+a}{(1-a)^2}E^3 X\right],$$

(4.4) [cont.]
$$E(S_N-ES_N)^4=\frac{a+b}{1-a}\Big[E(X-EX)^4+\frac{4}{1-a}EXE(X-EX)^3 + \frac{3(2a+b)}{1-a}(\sigma^2X)^2+\frac{6(2a+b+1)}{(1-a)^2}E^2X\sigma^2X + \frac{3(a+b)+a^2+4a+1}{(1-a)^3}E^4X\Big].$$

In particular for the compound Poisson distribution, i.e. $N\sim P(\lambda)$, we have:

(i) for $N\sim P(\lambda)$, $X\sim P(\beta)$,

$$\sigma^2S_N=\lambda\beta(1+\beta),\quad E(S_N-ES_N)^3=\lambda\beta(1+3\beta+\beta^2),$$
$$E(S_N-ES_N)^4=\lambda\beta(1+7\beta+6\beta^2+\beta^3)+3\lambda^2(\beta+\beta^2)^2,$$

(ii) for $N\sim P(\lambda)$, $X\sim B(t,p)$,

$$\sigma^2S_N=\lambda(tp+t(t-1)p^2),$$
$$E(S_N-ES_N)^3=\lambda(tp+3t(t-1)p^2+t(t-1)(t-2)p^3),$$
$$E(S_N-ES_N)^4=\lambda\big(tp+7t(t-1)p^2+6t(t-1)(t-2)p^3 + t(t-1)(t-2)(t-3)p^4\big)+3\lambda^2(tp+t(t-1)p^2)^2;$$

(iii) for $N\sim P(\lambda)$, $X\sim NB(\alpha,q)$,

$$\sigma^2S_N=\lambda(\alpha q/p+\alpha(\alpha+1)q^2/p^2),$$
$$E(S_N-ES_N)^3=\lambda\big(\alpha q/p+3\alpha(\alpha+1)q^2/p^2+\alpha(\alpha+1)(\alpha+2)q^3/p^3\big),$$
$$E(S_N-ES_N)^4=\lambda\big(\alpha q/p+7\alpha(\alpha+1)q^2/p^2+6\alpha(\alpha+1)(\alpha+2)q^3/p^3 + \alpha(\alpha+1)(\alpha+2)(\alpha+3)q^4/p^4\big) + 3\lambda^2\big(\alpha q/p+\alpha(\alpha+1)q^2/p^2\big)^2;$$

(iv) for $N\sim P(\lambda)$, $X\sim G(q)$,

$$\sigma^2S_N=\lambda(q/p+2q^2/p^2),$$
$$E(S_N-ES_N)^3=\lambda(q/p+6q^2/p^2+6q^3/p^3),$$
$$E(S_N-ES_N)^4=\lambda(q/p+14q^2/p^2+36q^3/p^3+24q^4/p^4) + 3\lambda^2(q/p+2q^2/p^2)^2.$$

From our calculations one can get the following result.

THEOREM 4.1. *The* $(r+1)$*th central moment of a Poisson–Panjer distribution* ($N\sim P(\lambda)$, $X\sim \mathscr{P}(a,b)$), *i.e. Poisson-binomial* ($N\sim P(\lambda)$, $X\sim B(t,p)$), *Poisson-negative binomial* ($N\sim P(\lambda)$, $X\sim NB(\alpha,q)$), *Poisson-geometric* ($N\sim P(\lambda)$, $X\sim G(q)$) *and Poisson-Poisson* ($N\sim P(\lambda)$,

$X \sim P(b)$) is given by

(4.5) $$E(S_N - ES_N)^{r+1}$$

$$= \begin{cases} \sum_{i=1}^{r} \binom{r}{i} \left(-\lambda \frac{a+b}{1-a}\right)^{r-i} \sum_{n=1}^{i} S(i,n) \left(\frac{a}{1-a}\right)^{n} \\ \cdot n! \Big\{ n \sum_{k=1}^{n} \frac{\lambda^k}{k!} \sum_{j=0}^{k-1} \binom{k}{j} (-1)^j \binom{(k-j)(1+\frac{b}{a})+n-1}{n} \\ + \lambda \frac{a+b}{1-a} \Big[\binom{\frac{b}{a}+n+1}{n} \sum_{k=0}^{n} \frac{(-\lambda)^k}{k!} \\ + \sum_{k=1}^{n} \frac{\lambda^k}{k!} \sum_{j=0}^{k-1} \binom{k}{j} (-1)^j \sum_{m=0}^{n-1} \binom{(k-j)(1+\frac{b}{a})+n-1}{m} \binom{2+\frac{b}{a}}{n-m} \Big] \Big\} \\ \qquad\qquad \textit{if } a \neq 0, \\ \sum_{i=1}^{r} \binom{r}{i} (-\lambda b)^{r-i} \sum_{n=1}^{i} S(i,n) b^n \sum_{k=1}^{n} \lambda^k S(n,k)(n+bk) \quad \textit{if } a = 0. \end{cases}$$

Proof. Note that for $X \sim \mathscr{P}(a,b)$,

$$G_X(s) = \begin{cases} \left(\frac{1-a}{1-sa}\right)^{1+b/a} & \text{if } a \neq 0, \\ \exp\{-b(1-s)\} & \text{if } a = 0. \end{cases}$$

Moreover, it is known that

$$G_{S_N}(s) = G_N(G_X(s))$$

(cf. Klugman et al. [KPW, p. 237]). Hence we can show that the factorial moments

$$m_{(r)}(S_N) := ES_N \cdot (S_N - 1) \cdot \ldots \cdot (S_N - r + 1)$$

are given by

$$m_{(r)}(S_N) = \begin{cases} \left(\frac{a}{1-a}\right)^r \sum_{k=1}^{r} \frac{\lambda^k}{k!} \sum_{j=0}^{k-1} \binom{k}{j} (-1)^j \left((k-j)\left(1+\frac{b}{a}\right)\right)_r & \text{if } a \neq 0, \\ b^r \sum_{k=1}^{r} S(r,k) \lambda^k & \text{if } a = 0, \end{cases}$$

leading to the moments ES_N^r (cf. Johnson et al. [JKK, p. 53]) and the central moments $E(S_N - ES_N)^r$ (cf. Johnson et al. [JKK, p. 52]). Now using, among other things, the evaluations for the central moments μ_r and μ_{r+1} of Section 1 we obtain formulae (4.5). ∎

Remark 4.2. From (4.5) we get the central moments $E(S_N - ES_N)^{r+1}$ when

$$\begin{aligned} &N \sim P(\lambda),\ X \sim NB(\alpha, q), && \text{i.e. } a = q,\ b = (\alpha-1)q,\\ &N \sim P(\lambda),\ X \sim G(q), && \text{i.e. } a = q,\ b = 0,\\ &N \sim P(\lambda),\ X \sim P(b), && \text{i.e. } a = 0,\ b > 0. \end{aligned}$$

In the case $N \sim P(\lambda)$ and $X \sim B(t,p)$, i.e. $a = -p/q$, $b = (t+1)p/q$, we have (after some calculations) the formula

$$\begin{aligned} E(S_N - ES_N)^{r+1} = {}& \sum_{i=1}^{r} \binom{r}{i} (-tp\lambda)^{r-i} \sum_{n=1}^{i} S(i,n) p^n \\ & \cdot n! \Big\{ n \sum_{k=1}^{n} \frac{\lambda^k}{k!} \sum_{j=0}^{k-1} \binom{k}{j} (-1)^j \binom{t(k-j)}{n} \\ & + tp\lambda \Big[\binom{t-1}{n} \sum_{k=0}^{n} \frac{(-\lambda)^k}{k!} \\ & + \sum_{k=1}^{n} \frac{\lambda^k}{k!} \sum_{j=0}^{k-1} \binom{k}{j} (-1)^j \sum_{m=0}^{n-1} \binom{t(k-j)}{m} \binom{t-1}{n-m} \Big] \Big\}. \end{aligned}$$

Moreover, from Theorem 4.1 with $r = 1, 2, 3$ one can get formulae (i)–(iv) of Section 4.

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Received on 2.7.2014;
revised version on 21.11.2014 (2237)