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SOLVABILITY OF THE HEAT EQUATION IN WEIGHTED SOBOLEV SPACES

Abstract. The existence of solutions to an initial-boundary value problem to the heat equation in a bounded domain in $\mathbb{R}^3$ is proved. The domain contains an axis and the existence is proved in weighted anisotropic Sobolev spaces with weight equal to a negative power of the distance to the axis. Therefore we prove the existence of solutions which vanish sufficiently fast when approaching the axis. We restrict our considerations to the Dirichlet problem, but the Neumann and the third boundary value problems can be treated in the same way. The proof of the existence is split into the following steps. First by an appropriate extension of initial data the initial-boundary value problem is reduced to an elliptic problem with a fixed $t \in \mathbb{R}$. Applying the regularizer technique it is considered locally. The most difficult part is to show the existence in weighted spaces near the axis, because the existence in neighbourhoods located at a positive distance from the axis is well known. In a neighbourhood of a point where the axis meets the boundary, the elliptic problem considered is transformed to a problem near an interior point of the axis by an appropriate reflection.

Using cutoff functions the problem near the axis is considered in $\mathbb{R}^3$ with sufficiently fast decreasing functions as $|x| \to \infty$. Then by applying the Fourier–Laplace transform we are able to show an appropriate estimate in weighted spaces and to prove local in space existence. The result of this paper is necessary to show the existence of global regular solutions to the Navier–Stokes equations which are close to axially symmetric solutions.

1. Introduction. The aim of this paper is to prove the existence of solutions to an initial-boundary value problem for the heat equation in weighted

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Sobolev spaces $H^{l,l/2}_{-\mu}(\Omega^T)$, $l\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$, $\mu\in\mathbb{R}_+\setminus\mathbb{Z}$, $\Omega^T=\Omega\times(0,T)$, where $\Omega\subset\mathbb{R}^3$ is a bounded domain which contains a distinguished axis L.

Definition 1.1. We denote by $H^{l,l/2}_{-\mu}(\Omega^T)$, $l\in\mathbb{N}_0$, $\mu\in\mathbb{R}_+$ the completion of the $C^\infty(\Omega\times(0,T))$ functions vanishing sufficiently fast on approaching L in the norm

$$\|u\|_{H^{l,l/2}_{-\mu}(\Omega^T)}=\Big(\sum_{|\alpha|+2a\le l}\int_{\Omega^T}|D^\alpha_x\partial^a_t u(x,t)|^2\varrho(x)^{2(-\mu+|\alpha|+2a-l)}\,dx\,dt\Big)^{1/2},$$

if l is even, where $|\alpha|=\alpha_1+\alpha_2+\alpha_3$, $D^\alpha_x=\partial^{\alpha_1}_{x_1}\partial^{\alpha_2}_{x_2}\partial^{\alpha_3}_{x_3}$, $\alpha_i,a\in\mathbb{N}_0$, $i=1,2,3$, $\varrho(x)=\operatorname{dist}(x,L)$, and

$$\|u\|_{H^{l,l/2}_{-\mu}(\Omega^T)}=\Big(\sum_{|\alpha|+2a\le l}\int_{\Omega^T}|D^\alpha_x\partial^a_t u|^2\varrho(x)^{2(-\mu+|\alpha|+2a-l)}\,dx\,dt + \int_\Omega\int_0^T\int_0^T\varrho(x)^{-2\mu}\frac{|\partial^{[l/2]}_{t'}u(x,t')-\partial^{[l/2]}_{t''}u(x,t'')|^2}{|t'-t''|^2}\,dx\,dt'\,dt''\Big)^{1/2}$$

for l odd.

We denote by $H^l_{-\mu}(\Omega)$, $l\in\mathbb{N}_0$, $\mu\in\mathbb{R}_+$, the space with the finite norm

$$\|u\|_{H^l_{-\mu}(\Omega)}=\Big(\sum_{|\alpha|\le l}\int_\Omega|D^\alpha_x u|^2\varrho(x)^{2(-\mu+|\alpha|-l)}\,dx\Big)^{1/2}.$$

We consider the Cauchy–Dirichlet problem

$$\begin{aligned} &u_t-\Delta u=f && \text{in } \Omega^T,\\ &u=0 && \text{on } S^T=S\times(0,T),\\ &u|_{t=0}=u_0 && \text{in } \Omega, \end{aligned}\tag{1.1}$$

where $S=\partial\Omega$. We assume that L meets S in two points: s_1 and s_2.

We shall restrict ourselves to problem (1.1) to simplify the presentation. The techniques of this paper can also be applied to the nonhomogeneous problem (1.1), the Neumann problem and the third boundary value problem. The tools of this paper are sufficient to show solvability of any initial-boundary value problem for a general linear parabolic system of the second order. Moreover, we can consider unbounded domains because the existence in a domain with boundary follows from the regularizer technique.

The main result of this paper is the following

Theorem 1.2. *Assume that* $f\in H^{l,l/2}_{-\mu}(\Omega^T)$, $u_0\in H^{l+1}_{-\mu}(\Omega)$, $l\in\mathbb{N}_0$, $\mu\in\mathbb{R}_+\setminus\mathbb{Z}$, $S\in C^{l+2}$. *Assume the compatibility condition*

$$u_0|_S=0 \bmod H^{l+1/2}_{-\mu}(S).\tag{1.2}$$

Then there exists a solution $u \in H^{l+2,l/2+1}_{-\mu}(\Omega^T)$ *to problem* (1.1) *such that*

$$\|u\|_{H^{l+2,l/2+1}_{-\mu}(\Omega^T)} \le c(\|f\|_{H^{l,l/2}_{-\mu}(\Omega^T)} + \|u_0\|_{H^{l+1}_{-\mu}(\Omega)}), \tag{1.3}$$

where c *does not depend on* u, f *and* u_0.

Let us consider the nonhomogeneous problem (1.1),

$$\begin{aligned} &u_t - \Delta u = f && \text{in } \Omega^T,\\ &u = g && \text{on } S^T,\\ &u|_{t=0} = u_0 && \text{in } \Omega. \end{aligned} \tag{1.4}$$

Similarly to Theorem 1.2 we prove

THEOREM 1.3. *Assume that* $f \in H^{l,l/2}_{-\mu}(\Omega^T)$, $g \in H^{l+3/2,l/2+3/4}_{-\mu}(S^T)$, $u_0 \in H^{l+1}_{-\mu}(\Omega)$, $l \in \mathbb{N}_0$, $\mu \in \mathbb{R}_+ \setminus \mathbb{Z}$, $S \in C^{l+2}$. *Assume the compatibility condition*

$$u_0|_S - g|_{t=0} = 0 \bmod H^{l+1}_{-\mu}(S). \tag{1.5}$$

Then there exists a solution $u \in H^{l+2,l/2+1}_{-\mu}(\Omega^T)$ *to problem* (1.4) *such that*

$$\begin{aligned} &\|u\|_{H^{l+2,l/2+1}_{-\mu}(\Omega^T)} \\ &\quad \le c(\|f\|_{H^{l,l/2}_{-\mu}(\Omega^T)} + \|g\|_{H^{l+3/2,l/2+3/4}_{-\mu}(S^T)} + \|u_0\|_{H^{l+1}_{-\mu}(\Omega)}), \end{aligned} \tag{1.6}$$

where c *does not depend on* u, f, g, u_0.

To omit some technical difficulties related to weighted spaces of traces on S we consider problem (1.1). The norms of weighted Sobolev spaces of traces can be found in [5, 8, 13].

Finally, we consider the Neumann problem

$$\begin{aligned} &u_t - \Delta u = f && \text{in } \Omega^T,\\ &\frac{\partial u}{\partial n} = g && \text{on } S^T,\\ &u|_{t=0} = u_0 && \text{in } \Omega, \end{aligned} \tag{1.7}$$

where $\frac{\partial}{\partial n}$ denotes the normal derivative on S. We have

THEOREM 1.4. *Assume that* $f \in H^{l,l/2}_{-\mu}(\Omega^T)$, $g \in H^{l+l/2,l/2+1/4}_{-\mu}(S^T)$, $u_0 \in H^{l+1}_{-\mu}(\Omega)$, $l \in \mathbb{N}_0$, $\mu \in \mathbb{R}_+ \setminus \mathbb{Z}$, $S \in C^{l+2}$. *Assume the compatibility condition*

$$\frac{\partial u_0}{\partial n}\bigg|_S - g|_{t=0} = 0 \bmod H^{l-1/2}_{-\mu}(S). \tag{1.8}$$

Then there exists a unique solution $u \in H^{l+2,l/2+1}_{-\mu}(\Omega^T)$ *to* (1.7) *such that*

$$\|u\|_{H^{l+2,l/2+1}_{-\mu}(\Omega^T)} \le c(\|f\|_{H^{l,l/2}_{-\mu}(\Omega^T)} + \|g\|_{H^{l+1/2,l/2+1/4}_{-\mu}(S^T)} + \|u_0\|_{H^{l+1}_{-\mu}(\Omega)}), \tag{1.9}$$

where c *does not depend on* u, f, g, u_0.

We prove Theorem 1.2 in the following steps. First we extend the initial data for $t > 0$ to a function $\tilde{u}_0$ such that

$$\tilde{u}_0|_{t=0} = u_0, \quad \tilde{u}_0|_S = 0 \quad \text{and} \quad \tilde{u}_0 \in H^{l+2,l/2+1}_{-\mu}(\Omega^T)$$

with

$$\|\tilde{u}_0\|_{H^{l+2,l/2+1}_{-\mu}(\Omega^T)} \le c\|u_0\|_{H^{l+1}_{-\mu}(\Omega)}. \tag{1.10}$$

Introducing the new function

$$v = u - \tilde{u}_0, \tag{1.11}$$

we see that v is a solution to the problem

$$\begin{aligned} &v_t - \Delta v = f - \tilde{u}_{0,t} + \Delta\tilde{u}_0 \equiv g && \text{in } \Omega \times (0,T),\\ &v|_S = 0 && \text{on } S \times (0,T),\\ &v|_{t=0} = 0 && \text{in } \Omega. \end{aligned} \tag{1.12}$$

Assuming the compatibility conditions

$$\partial_t^\sigma g|_{t=0} = 0, \quad \sigma \le [l/2] - 1, \tag{1.13}$$

we can extend g by 0 onto $\Omega \times (-\infty, 0]$.

Let g' be the extended function. Then

$$\|g'\|_{H^{l,l/2}_{-\mu}(\Omega\times(-\infty,T))} \le c\|g\|_{H^{l,l/2}_{-\mu}(\Omega\times(0,T))}. \tag{1.14}$$

The possibility of such extensions is described in Lemma 2.4.

By the Hestenes–Whitney method, g' can be extended onto $\Omega \times [T, \infty)$ in such a way that the extended function g'' is in $H^{l,l/2}_{-\mu}(\Omega \times \mathbb{R})$ and

$$\|g''\|_{H^{l,l/2}_{-\mu}(\Omega\times\mathbb{R})} \le c\|g'\|_{H^{l,l/2}_{-\mu}(\Omega\times(-\infty,T))}. \tag{1.15}$$

Then problem (1.12) will be considered in $\Omega \times \mathbb{R}$. Hence (1.12) takes the form

$$\begin{aligned} &v_t - \Delta v = g'' && \text{in } \Omega \times \mathbb{R},\\ &v|_S = 0 && \text{on } S \times \mathbb{R}. \end{aligned} \tag{1.16}$$

To apply the regularizer technique (see [4]) we examine problem (1.16) locally in space variables. We distinguish four cases:

1. in a neighbourhood of an interior point of L;
2. near a point where L meets S;

3. near an interior point of Ω located at a positive distance from L;
4. near a point of S located also at a positive distance from L.

In cases 1 and 3 problem (1.16) can be transformed to

$$v_t - \Delta v = \bar{g} \quad \text{in } \mathbb{R}^3 \times \mathbb{R}, \tag{1.17}$$

while in cases 2 and 4 it can be formulated as

$$\begin{aligned} v_t - \Delta v &= \bar{g} && \text{in } \mathbb{R}^3_+ \times \mathbb{R}, \\ v|_{x_3} &= 0 && \text{on } \mathbb{R}^2 \times \mathbb{R}, \end{aligned} \tag{1.18}$$

where $\bar{g}$ is g localized to the corresponding neighbourhood.

In cases 3 and 4 the weighted spaces $H^{l,l/2}_{-\mu}$ will not be used because the problems are considered in neighbourhoods located at a positive distance from L so the weighted and nonweighted spaces are equivalent.

We do not know how to examine problem (1.18) in case 2 as a boundary value problem. Therefore it is transformed to problem of the form (1.17) by reflection with respect to the plane $x_3 = 0$.

Summarizing, we need to examine the local problem (1.17) for case 1. This problem describes the behaviour of solutions near the distinguished axis L. Solvability and existence for other problems (cases 3 and 4) is well known (see [4]).

We show an appropriate estimate in weighted spaces for solutions to (1.17) in Section 3 but the existence is proved in Section 4. In Section 3 we perform all calculations on the Laplace transforms of solutions of (1.17), so we obtain the estimate in terms of Laplace transforms (see Definition 2.5).

The equivalence of norms introduced in Definitions 2.1 and 2.2 (in these spaces Theorem 1.2 is formulated) and in Definition 2.5 is shown in Lemma 2.6.

Having the existence for local problems, the existence in a bounded domain with vanishing initial data is proved by the regularizer technique in Section 5. Finally, in Section 6 the existence with nonvanishing initial data is proved.

The results of this paper are necessary for the proof of existence of global regular solutions to the Navier–Stokes equations which remain close to axially symmetric solutions (see [7, 11, 12, 13]).

2. Notation and auxiliary results. We consider a domain $\Omega \subset \mathbb{R}^3$ with a distinguished axis L. Assume that a global Cartesian system $x = (x_1, x_2, x_3)$ in Ω is such that L is the x_3-axis. 1 We denote by $H^{l,l/2}(\Omega \times (0,T))$, $l \in \mathbb{N}_0$, the classical Sobolev–Slobodetskiĭ spaces $W_2^{l,l/2}(\Omega \times (0,T))$

with the finite norm

$$\|u\|_{H^{l,l/2}(\Omega\times(0,T))}=\Big(\sum_{|\alpha|\le l}\int_0^T\int_\Omega|D_x^\alpha u|^2\,dx\,dt+\sum_{a\le[l/2]}\int_0^T\int_\Omega|\partial_t^a u|^2\,dx\,dt$$
$$+\int_\Omega\int_0^T\int_0^T\frac{|\partial_{t'}^{[l/2]}u(x,t')-\partial_{t''}^{[l/2]}u(x,t'')|^2}{|t'-t''|^2}\,dx\,dt'\,dt''\Big)^{1/2},$$

where the last integral disappears for l even. Then we define

$$L_2(\Omega\times(0,T))=H^{0,0}(\Omega\times(0,T)).$$

Definition 2.1. Let l be even, $\mu\in\mathbb{R}_+$. We denote by $H^{l,l/2}_{-\mu}(\Omega\times(0,T))$ the completion of $C^\infty(\Omega\times(0,T))$ functions vanishing sufficiently fast near L in the norm

$$\|u\|_{H^{l,l/2}_{-\mu}(\Omega\times(0,T))}=\Big(\sum_{|\alpha|+2a\le l}\int_0^T\int_\Omega|D_x^\alpha\partial_t^a u|^2|x'|^{2(-\mu-l+|\alpha|+2a)}\,dx\,dt\Big)^{1/2},$$

where $D_x^\alpha=\partial_{x_1}^{\alpha_1}\partial_{x_2}^{\alpha_2}\partial_{x_3}^{\alpha_3}$, $\alpha=(\alpha_1,\alpha_2,\alpha_3)$ is a multiindex, $\alpha_i\in\mathbb{N}_0$, $i=1,2,3$, $|\alpha|=\alpha_1+\alpha_2+\alpha_3$, $x'=(x_1,x_2)$, $|x'|=\sqrt{x_1^2+x_2^2}$.

Definition 2.2. Let l be odd, $\mu\in\mathbb{R}_+$. Then $H^{l,l/2}_{-\mu}(\Omega\times(0,T))$ is defined to be the closure of $C^\infty(\Omega\times(0,T))$ functions vanishing sufficiently fast near L in the norm

$$\|u\|_{H^{l,l/2}_{-\mu}(\Omega\times(0,T))}=\Big(\sum_{|\alpha|\le l}\int_0^T\int_\Omega|D_x^\alpha u|^2|x'|^{2(-\mu-l+|\alpha|)}\,dx\,dt$$
$$+\sum_{2a<l}\int_0^T\int_\Omega|\partial_t^a u|^2|x'|^{2(-\mu-l+2a)}\,dx\,dt$$
$$+\int_\Omega\int_0^T\int_0^T|x'|^{-2\mu}\frac{|\partial_{t'}^{[l/2]}u(x,t')-\partial_{t''}^{[l/2]}u(x,t'')|^2}{|t'-t''|^2}\,dx\,dt'\,dt''\Big)^{1/2},$$

where we have used the fact that $l/2-[l/2]=1/2$.

Definition 2.3 (see [1, 6]). Let $l\in\mathbb{N}_0$, $\mu\in\mathbb{R}_+$, $\gamma\ge0$. Then we introduce the space $H^{l,l/2}_{-\mu,\gamma}(\Omega\times(0,T))$ of functions $u\in H^{l,l/2}_{-\mu}(\Omega\times(0,T))$ such that $e^{-\gamma t}u\in H^{l,l/2}_{-\mu}(\Omega\times(0,T))$,

$$\partial_t^a u|_{t=0}=0\quad\text{for } a\le l/2-1 \text{ for even } l,\tag{2.1}$$

$$\partial_t^a u|_{t=0}=0\quad\text{for } a<l/2 \text{ for odd } l,\tag{2.2}$$

and

$$e^{-\gamma t}u\in H^{l,l/2}_{-\mu}(\Omega\times(-\infty,T)).$$

The space $H^{l,l/2}_{-\mu,\gamma}(\Omega\times(0,T))$ has the property that its elements can be extended by zero for $t<0$ and the extended function also belongs to $H^{l,l/2}_{-\mu,\gamma}(\Omega\times(-\infty,T))$.

In the nonweighted case the spaces $H^{l,l/2}_{\gamma}(\Omega\times(0,T))$ were introduced in [1, 6]. The time behaviour of elements of $H^{l,l/2}_{-\mu,\gamma}$ and $H^{l,l/2}_{\gamma}$ is the same.

We also write $L_{2,-\mu,\gamma}(\Omega\times(0,T))=H^{0,0}_{-\mu,\gamma}(\Omega\times(0,T))$ and $L_{2,-\mu}(\Omega\times(0,T))=H^{0,0}_{-\mu}(\Omega\times(0,T))$.

LEMMA 2.4 (see [6]). *Let u satisfy either* (2.1) *or* (2.2). *Let $T<\infty$. Then the norms*

$$\|u\|_{H^{l,l/2}_{-\mu}(\Omega\times(0,T))} \quad \textit{and} \quad \|u\|_{H^{l,l/2}_{-\mu,0}(\Omega\times(-\infty,T))}$$

are equivalent if and only if

$$\partial_t^a u|_{t=0}=0, \qquad a\le[l/2],$$
$$\int_\Omega dx\int_0^T |x'|^{-2\mu}t^{-2(l/2-[l/2])}|\partial_t^{[l/2]}u|^2\,dt<\infty.$$

Let us introduce the Laplace transform for a function $u\in H^{l,l/2}_{-\mu}(\Omega\times(0,T))$ satisfying either (2.1) or (2.2) by

$$\tilde u(x,s)=\int_0^\infty e^{-st}u(x,t)\,dt, \qquad s=\gamma+i\xi_0,\ \xi_0\in\mathbb{R},\ \gamma>0. \tag{2.3}$$

By the Paley–Wiener theorem the inverse Laplace transform of $\tilde u$ vanishes for $t<0$ and satisfies either (2.1) or (2.2).

DEFINITION 2.5. Let $\tilde H^{l,l/2}_{-\mu,\gamma}(\Omega\times\mathbb{R})$ be the space of functions u with the finite norm

$$\|u\|_{\tilde H^{l,l/2}_{-\mu,\gamma}(\Omega\times\mathbb{R})}=\Big(\sum_{j=0}^{l}\int_{-\infty}^{\infty}d\xi_0\,|s|^j\|\tilde u\|^2_{H^{l-j}_{-\mu}(\Omega)}\Big)^{1/2}, \tag{2.4}$$

where $\tilde u=\tilde u(x,s)$ is the Laplace transform of u defined by (2.3).

From [1] we have

LEMMA 2.6. *For any $\gamma\ge0$ there exist constants c_1 and c_2 independent of u and γ such that*

$$c_1\|u\|_{\tilde H^{l,l/2}_{-\mu,\gamma}(\Omega\times\mathbb{R})}\le\|u\|_{H^{l,l/2}_{-\mu,\gamma}(\Omega\times\mathbb{R})}\le c_2\|u\|_{\tilde H^{l,l/2}_{-\mu,\gamma}(\Omega\times\mathbb{R})}. \tag{2.5}$$

Recall that functions in $\tilde H^{l,l/2}_{-\mu,\gamma}(\Omega\times\mathbb{R})$ and $H^{l,l/2}_{-\mu,\gamma}(\Omega\times\mathbb{R})$ vanish for $t<0$.

Proof. To prove the lemma we use the Parseval identity

$$\int_{-\infty}^{\infty} d\xi_0 \int_{\mathbb{R}^n} |\tilde{f}(x,\gamma+i\xi_0)|^2\,dx = 2\pi \int_0^{\infty} e^{-2\gamma t}\,dt \int_{\mathbb{R}^n} |f(x,t)|^2\,dx. \tag{2.6}$$

From (2.6) we have

$$\int_0^{\infty} e^{-2\gamma t}\|u(t)\|^2_{H^l_{-\mu}(\Omega)}\,dt = (2\pi)^{-1}\int_{-\infty}^{\infty} d\xi_0\,\|\tilde{u}(\cdot,\gamma+i\xi_0)\|^2_{H^l_{-\mu}(\Omega)},$$

$$\int_0^{\infty} e^{-2\gamma t}\gamma^l\|u(t)\|^2_{L_{2,-\mu}(\Omega)}\,dt = (2\pi)^{-1}\int_{-\infty}^{\infty} d\xi_0\,\gamma^l\|\tilde{u}(\cdot,\gamma+i\xi_0)\|^2_{L_{2,-\mu}(\Omega)},$$

$$\int_0^{\infty} e^{-2\gamma t}\|\partial_t^{l/2}u\|^2_{L_{2,-\mu}(\Omega)}\,dt = (2\pi)^{-1}\int_{-\infty}^{\infty} d\xi_0\,|s|^l\|\tilde{u}(\cdot,\gamma+i\xi_0)\|^2_{L_{2,-\mu}(\Omega)},$$

for l even.

For l odd we consider

$$\int_0^{\infty} e^{-2\gamma t}dt \int_0^{\infty} \|\partial_t^k u(\cdot,t-\tau)-\partial_t^k u(\cdot,t)\|^2_{L_{2,-\mu}(\Omega)}\,\frac{d\tau}{\tau^{1+l-2k}}$$
$$= (2\pi)^{-1}\int_{-\infty}^{\infty} |s|^{2k}\|\tilde{u}(\cdot,s)\|^2_{L_{2,-\mu}(\Omega)}\,d\xi_0 \int_0^{\infty} |e^{-\tau s}-1|^2\,\frac{d\tau}{\tau^{1+l-2k}},$$

where $k=[l/2]$.

From [6, Lemma 2.1] there exist constants c_3, c_4 such that

$$c_3|s|^{l-2k} \le \int_0^{\infty} |e^{-\tau s}-1|^2\,\frac{d\tau}{\tau^{1+l-2k}} \le c_4|s|^{l-2k}$$

which concludes the proof.

Lemma 2.7. *For $T<\infty$ the norms*

$$\|u\|_{H^{l,l/2}_{-\mu,\gamma}(\Omega\times(-\infty,T))} \quad \textit{and} \quad \|u\|_{H^{l,l/2}_{-\mu,0}(\Omega\times(-\infty,T))}$$

are equivalent.

Proof. The equivalence of the norms

$$\|u\|_{\tilde{H}^{l,l/2}_{-\mu,\gamma}(\Omega\times(0,T))} \quad \text{and} \quad \|u\|_{\tilde{H}^{l.l/2}_{-\mu,0}(\Omega\times(0,T))}$$

follows from $u|_{t<0}=0$, $T<\infty$ and $e^{-\gamma T}\le e^{-\gamma t}\le 1$ for $t\in[0,T]$. An application of the proof of Lemma 2.6 concludes the proof.

For the Laplace transform $\tilde{u}(x,s)$ of $u(x,t)$ and for a given s we introduce the space $\mathcal{E}^l_{-\mu}(\Omega)$ endowed with the norm

$$\|\tilde u(\cdot,s)\|^2_{\mathcal{E}^l_{-\mu}(\Omega)} = \sum_{j=0}^{l} |s|^j \|\tilde u(\cdot,s)\|^2_{H^{l-j}_{-\mu}(\Omega)}. \tag{2.7}$$

Applying the Laplace transform yields

$$\|u\|^2_{\tilde H^{l,l/2}_{-\mu,\gamma}(\Omega\times\mathbb{R})} = \int_{-\infty}^{\infty} d\xi_0\, \|\tilde u(\cdot,\gamma+i\xi_0)\|^2_{\mathcal{E}^l_{-\mu}(\Omega)}. \tag{2.8}$$

3. A priori estimates for solutions to problem (1.1) in $\mathbb{R}^3\times\mathbb{R}$. In this section we consider problem (1.1) in the form

$$u_t - \Delta u = f \quad \text{in } \mathbb{R}^3\times\mathbb{R}, \tag{3.1}$$

where functions u and f are extended by zero for $t<0$. Applying the Laplace transform

$$\tilde u(x,s) = \int_{\mathbb{R}_+} u(x,t)e^{-st}\,dt, \tag{3.2}$$

where $s=\gamma+i\xi_0$, $\gamma>0$, $\xi_0\in\mathbb{R}$, to problem (3.1), yields

$$s\tilde u - \Delta\tilde u = \tilde f \quad \text{in } \mathbb{R}^3. \tag{3.3}$$

Lemma 3.1. *Assume that $f\in L_{2,-\mu}(\mathbb{R}^3\times\mathbb{R})$, $\mu\in\mathbb{R}_+\setminus\mathbb{Z}$. Then solutions $u\in H^{2,1}_{-\mu}(\mathbb{R}^3\times\mathbb{R})$ of (3.1) satisfy*

$$\begin{aligned}\int_{\mathbb{R}} d\xi_0 \int_{\mathbb{R}^3} (|s|^2|\tilde u|^2 + |s|\,|\nabla\tilde u|^2)|x'|^{-2\mu}\,dx + \int_{\mathbb{R}} d\xi_0\, \|\tilde u\|^2_{H^2_{-\mu}(\mathbb{R}^3)} \\ \le c\int_{\mathbb{R}} d\xi_0\,|s| \int_{\mathbb{R}^3} |\tilde u|^2|x'|^{-2\mu-2}\,dx + c\int_{\mathbb{R}} d\xi_0 \int_{\mathbb{R}^3} |\tilde f|^2|x'|^{-2\mu}\,dx,\end{aligned} \tag{3.4}$$

where c does not depend on u and f.

Proof. Multiplying (3.3) by $\bar\varphi$, where $\varphi=(1+i\operatorname{sign}\operatorname{Ims})\tilde u|x'|^{-2\mu}$ and $\bar\varphi$ is the complex conjugate to φ, integrating the result over $\mathbb{R}^3$ and by parts, we obtain

$$\begin{aligned}\int_{\mathbb{R}^3} [s\tilde u(1-i\operatorname{sign}\operatorname{Im}s)\bar{\tilde u}|x'|^{-2\mu} + \nabla\tilde u\cdot\nabla((1-i\operatorname{sign}\operatorname{Im}s)\bar{\tilde u}|x'|^{-2\mu})]\,dx \\ = \int_{\mathbb{R}^3} \tilde f(1-i\operatorname{sign}\operatorname{Im}s)\bar{\tilde u}|x'|^{-2\mu}\,dx.\end{aligned} \tag{3.5}$$

Comparing the real parts and performing estimates for nonpositive terms yields

$$\begin{aligned}\int_{\mathbb{R}^3} (|s|\,|\tilde u|^2 + |\nabla\tilde u|^2)|x'|^{-2\mu}\,dx \\ \le 2\mu\int_{\mathbb{R}^3} |\nabla\tilde u|\,|\tilde u|\,|x'|^{-2\mu-1}\,dx + \int_{\mathbb{R}^3} |\tilde f|\,|\tilde u|\,|x'|^{-2\mu}\,dx.\end{aligned} \tag{3.6}$$

Applying the Hölder and the Young inequalities, the first term on the r.h.s. is estimated by

$$\frac{1}{2}\int_{\mathbb{R}^3}|\nabla\tilde{u}|^2|x'|^{-2\mu}\,dx+2\mu^2\int_{\mathbb{R}^3}|\tilde{u}|^2|x'|^{-2\mu-2}\,dx,$$

and the second by

$$\frac{1}{2}|s|\int_{\mathbb{R}^3}|\tilde{u}|^2|x'|^{-2\mu}\,dx+\frac{1}{2|s|}\int_{\mathbb{R}^3}|\tilde{f}|^2|x'|^{-2\mu}\,dx.$$

In view of the above estimates, (3.6) takes the form

$$\int_{\mathbb{R}^3}(|s|^2|\tilde{u}|^2+|s|\,|\nabla\tilde{u}|^2)|x'|^{-2\mu}\,dx \le 4\mu^2|s|\int_{\mathbb{R}^3}|\tilde{u}|^2|x'|^{-2\mu-2}\,dx+2\int_{\mathbb{R}^3}|\tilde{f}|^2|x'|^{-2\mu}\,dx. \tag{3.7}$$

From [14] we have

$$\|\tilde{u}\|_{H^2_{-\mu}(\mathbb{R}^3)}\le c\|\tilde{f}-s\tilde{u}\|^2_{L_{2,-\mu}(\mathbb{R}^3)}\le c\|\tilde{f}\|^2_{L_{2,-\mu}(\mathbb{R}^3)}+c|s|^2\|\tilde{u}\|^2_{L_{2,-\mu}(\mathbb{R}^3)}. \tag{3.8}$$

Inequalities (3.7) and (3.8) imply

$$\int_{\mathbb{R}^3}(|s|^2|\tilde{u}|^2+|s|\,|\nabla\tilde{u}|^2)|x'|^{-2\mu}\,dx+\|\tilde{u}\|^2_{H^2_{-\mu}(\mathbb{R}^3)} \le c|s|\int_{\mathbb{R}^3}|\tilde{u}|^2|x'|^{-2\mu-2}\,dx+c\int_{\mathbb{R}^3}|\tilde{f}|^2|x'|^{-2\mu}\,dx. \tag{3.9}$$

Integrating (3.9) with respect to ξ_0 yields (3.4). This concludes the proof.

To estimate the first term on the r.h.s. of (3.4) we need

LEMMA 3.2. *Let $f\in L_{2,-\mu}(\mathbb{R}^3\times\mathbb{R})$, $\mu\in\mathbb{R}_+$, and $u\in H^{2,1}_{-\mu}(\mathbb{R}^3\times\mathbb{R})$ be a solution to* (3.1). *Let $0<a_1<a_2$. Then*

$$\int_{\mathbb{R}}d\xi_0\,|s|\int_{\mathbb{R}^3}|\tilde{u}|^2|x'|^{-2\mu-2}\,dx\le 2a_1\int_{\mathbb{R}}d\xi_0\int_{\mathbb{R}^3}|\tilde{u}|^2|x'|^{-2\mu-4}\,dx + \frac{2}{a_2}\int_{\mathbb{R}}d\xi_0\int_{\mathbb{R}^3}|\tilde{u}|^2|s|^2|x'|^{-2\mu}\,dx. \tag{3.10}$$

Proof. We set

$$Q_1=\{(s,x'):|s|\,|x'|^2\le a_1\},$$
$$Q_2=\{(s,x'):|s|\,|x'|^2\ge a_2\},$$
$$Q_3=\{(s,x'):a_1\le|s|\,|x'|^2\le a_2\}.$$

For given s we introduce

$$
\begin{aligned}
d_1(s) &= \{x' \in \mathbb{R}^2 : |s|\,|x'|^2 \le a_1\},\\
d_2(s) &= \{x' \in \mathbb{R}^2 : |s|\,|x'|^2 \ge a_2\},\\
d_3(s) &= \{x' \in \mathbb{R}^2 : a_1 \le |s|\,|x'|^2 \le a_2\}.
\end{aligned}
$$

For $\lambda > 0$ we define

$$\Omega^\lambda = \{(s,x') : \lambda|s|\,|x'|^2 \le 1\}, \qquad w^\lambda(s) = \{x' \in \mathbb{R}^2 : \lambda|s|\,|x'|^2 \le 1\}.$$

Clearly, we have

$$Q_3 \subset \Omega^\lambda \quad \text{for} \quad \lambda \in (0, a_2^{-1}]. \tag{3.11}$$

We express the first term on the r.h.s. of (3.4) in the form

$$
\begin{aligned}
&\int_{\mathbb{R}} |s| \int_{\mathbb{R}^3} |\tilde u|^2 |x'|^{-2\mu-2}\,dx\,d\xi_0\\
&\qquad = \sum_{i=1}^{3} \int_{\mathbb{R}} dx_3 \int_{Q_i} d\xi_0\,dx'\,|s|\,|\tilde u|^2|x'|^{-2\mu-2} \equiv \sum_{i=1}^{3} I_i.
\end{aligned}
\tag{3.12}
$$

From the properties of the sets Q_i, $i = 1,2,3$, we have

$$
\begin{aligned}
I_1 &\le a_1 \int_{\mathbb{R}} d\xi_0 \int_{\mathbb{R}^3} |\tilde u|^2 |x'|^{-2\mu-4} dx,\\
I_2 &\le \frac{1}{a_2} \int_{\mathbb{R}} d\xi_0 \int_{\mathbb{R}^3} |\tilde u|^2 |s|^2 |x'|^{-2\mu}\,dx,\\
I_3 &\le \frac{1}{a_1^{1+\mu}} \int_{\mathbb{R}} d\xi_0 \int_{\mathbb{R}^3} |s|^{2+\mu} |\tilde u|^2\,dx \equiv I.
\end{aligned}
\tag{3.13}
$$

Let us introduce a smooth function $\chi = \chi(t)$ such that $\chi(t) = 1$ for $t \le 1$ and $\chi(t) = 0$ for $t \ge 2$, $0 \le \chi(t) \le 1$, $|\chi'(t)| \le 2$. Set $\chi_\lambda(x',s) = \chi(\lambda|s|\,|x'|^2)$. Then $\nabla\chi_\lambda(x',s) \ne 0$ for $\lambda^{-1} \le |s|\,|x'|^2 \le 2\lambda^{-1}$.

Multiplying (3.3) by $\overline{(1 + i\,\mathrm{sign}\,\mathrm{Im}\,s)\tilde u \chi_\lambda^2}$ and integrating over $\mathbb{R}^3$ we obtain

$$\int_{\mathbb{R}^3} (s\tilde u - \Delta\tilde u)(1 - i\,\mathrm{sign}\,\mathrm{Im}\,s)\bar{\tilde u}\chi_\lambda^2\,dx = \int_{\mathbb{R}^3} \tilde f(1 - i\,\mathrm{sign}\,\mathrm{Im}\,s)\bar{\tilde u}\chi_\lambda^2\,dx.$$

Integrating by parts, taking the real parts and estimating we get

$$\int_{\mathbb{R}^3} (|s|\,|\tilde u|^2 + |\nabla\tilde u|^2)\chi_\lambda^2\,dx \le \int_{\mathbb{R}^3} |\nabla\tilde u\bar{\tilde u}| 2\chi_\lambda\nabla\chi_\lambda\,dx + \int_{\mathbb{R}^3} |\tilde f\bar{\tilde u}|\chi_\lambda^2\,dx. \tag{3.14}$$

The first term on the r.h.s. of (3.14) is estimated by

$$\frac{1}{2}\int_{\mathbb{R}^3} |\nabla\tilde u|^2\chi_\lambda^2\,dx + 2\int_{\mathbb{R}^3} |\tilde u|^2|\nabla\chi_\lambda|^2\,dx$$

and the second by

$$\frac{\varepsilon}{2}\int_{\mathbb{R}}|\tilde u|^2|s|^{1+\mu}|x'|^{2\mu}\chi_\lambda^2dx+\frac{1}{2\varepsilon}\,\frac{1}{|s|^{1+\mu}}\int_{\mathbb{R}^3}|\tilde f|^2|x'|^{-2\mu}\chi_\lambda^2\,dx\equiv J.$$

On $\operatorname{supp}\chi_\lambda$ we have $|s|^\mu|x'|^{2\mu}\le\left(\frac{2}{\lambda}\right)^\mu$, so the first term in J is bounded by

$$\frac{\varepsilon}{2}\left(\frac{2}{\lambda}\right)^\mu\int_{\mathbb{R}^3}|s|\,|\tilde u|^2\chi_\lambda^2\,dx.$$

Assuming that $\varepsilon\left(\frac{2}{\lambda}\right)^\mu=1$ we obtain from (3.14) the inequality

$$\int_{\mathbb{R}^3}(|s|\,|\tilde u|^2+|\nabla\tilde u|^2)\chi_\lambda^2\,dx \le 4\int_{\mathbb{R}^3}|\tilde u|^2|\nabla\chi_\lambda|^2\,dx+\left(\frac{2}{\lambda}\right)^\mu\frac{1}{|s|^{1+\mu}}\int_{\mathbb{R}^3}|\tilde f|^2|x'|^{-2\mu}\chi_\lambda^2\,dx.$$

Multiplying the above inequality by $|s|^{1+\mu}$ and integrating the result with respect to ξ_0 yields

$$\int_{\mathbb{R}}d\xi_0\,|s|^{1+\mu}\int_{\mathbb{R}^3}(|s|\,|\tilde u|^2+|\nabla\tilde u|^2)\chi_\lambda^2\,dx \le 4\int_{\mathbb{R}}d\xi_0\,|s|^{1+\mu}\int_{\mathbb{R}^3}|\tilde u|^2|\nabla'\chi_\lambda|^2\,dx+\left(\frac{2}{\lambda}\right)^\mu\int_{\mathbb{R}}d\xi_0\int_{\mathbb{R}^3}|\tilde f|^2|x'|^{-2\mu}\chi_\lambda^2\,dx, \tag{3.15}$$

where ∇' denotes $(\partial_{x_1},\partial_{x_2})$. Using $|\nabla'\chi_\lambda|\le 4\lambda|s|\,|x'|$ in (3.15) implies the inequality

$$\int_{\mathbb{R}}d\xi_0\,|s|^{1+\mu}\int_{\mathbb{R}^3}(|s|\,|\tilde u|^2+|\nabla\tilde u|^2)\chi_\lambda^2\,dx \le 2^6\lambda\int_{\mathbb{R}}d\xi_0\int_{\mathbb{R}^3\cap\operatorname{supp}\nabla\chi_\lambda}|s|^{2+\mu}|\tilde u|^2\,dx+\left(\frac{2}{\lambda}\right)^\mu\int_{\mathbb{R}}d\xi_0\int_{\mathbb{R}^3}|\tilde f|^2|x'|^{-2\mu}\,dx. \tag{3.16}$$

On $\operatorname{supp}\nabla\chi_\lambda$ we have $|s|\,|x'|^2\le\frac{2}{\lambda}$. Then (3.16) takes the form

$$\int_{\mathbb{R}}d\xi_0\,|s|^{1+\mu}\int_{\mathbb{R}^3}(|s|\,|\tilde u|^2+|\nabla\tilde u|^2)\chi_\lambda^2\,dx \le 2^6\lambda\int_{\mathbb{R}}d\xi_0\int_{\mathbb{R}^3\cap\operatorname{supp}\nabla\chi_\lambda}|s|^{2+\mu}|\tilde u|^2\,dx+\left(\frac{2}{\lambda}\right)^{\mu+1}\int_{\mathbb{R}}d\xi_0\int_{\mathbb{R}^3}|\tilde f|^2|x'|^{-2\mu}\,dx, \tag{3.17}$$

where $\lambda<2$ was used. Since $\nabla'\chi_\lambda\ne0$ for $\lambda^{-1}\le|s|\,|x'|^2\le(\lambda/2)^{-1}$, we have $\operatorname{supp}\nabla'\chi_\lambda\subset w^{\lambda/2}(s)\setminus w^\lambda(s)$ for any s.

Multiplying (3.17) by $(\lambda/2)^{\mu+1}$ yields

$$\left(\frac{\lambda}{2}\right)^{\mu+1}\int_{\mathbb{R}} d\xi_0\,|s|^{1+\mu}\int_{\mathbb{R}^3}(|s|\,|\tilde u|^2+|\nabla\tilde u|^2)\chi_\lambda^2 dx \tag{3.18}$$
$$\le 2^7 2^\mu\lambda\left(\frac{\lambda/2}{2}\right)^{\mu+1}\int_{\mathbb{R}} d\xi_0\,|s|^{2+\mu}\int_{w^{\lambda/2}(s)\setminus w^\lambda(s)}|\tilde u|^2\,dx+\int_{\mathbb{R}} d\xi_0\int_{\mathbb{R}^3}|\tilde f|^2|x'|^{-2\mu}\,dx.$$

Let λ be so small that $2^7 2^\mu\lambda\le\frac12$. Then iterating (3.18) k times we obtain

$$\left(\frac{\lambda}{2}\right)^{\mu+1}\int_{\mathbb{R}} d\xi_0\,|s|^{2+\mu}\int_{w^\lambda(s)}|\tilde u|^2\,dx \tag{3.19}$$
$$\le\frac{1}{2^k}\left(\frac{\lambda/2^k}{2}\right)^{\mu+1}\int_{\mathbb{R}} d\xi_0\,|s|^{2+\mu}\int_{w^{\lambda/2^{k+1}}(s)\setminus w^{\lambda/2^k}(s)}|\tilde u|^2\,dx+2\int_{\mathbb{R}} d\xi_0\int_{\mathbb{R}^3}|\tilde f|^2|x'|^{-2\mu}\,dx,$$

where

$$w^{\lambda/2^{k+1}}(s)\setminus w^{\lambda/2^k}(s)=\left\{x'\in\mathbb{R}^2:\frac{2}{\lambda/2^k}\le|s|\,|x'|^2\le\frac{2}{\lambda/2^{k+1}}\right\} \tag{3.20}$$
$$=\left\{x'\in\mathbb{R}^2:\frac{2^{k+1}}{\lambda}\le|s|\,|x'|^2\le\frac{2^{k+2}}{\lambda}\right\}.$$

On the set (3.20) we have

$$|s|\le 2\frac{2^{k+1}}{\lambda}|x'|^{-2},$$

so the first term on the r.h.s. of (3.19) is estimated by

$$\frac{1}{2^k}\left(\frac{\lambda}{2^{k+1}}\right)^{\mu+1}\left(\frac{2^{k+2}}{\lambda}\right)^{\mu+1}\int_{\mathbb{R}} d\xi_0\,|s|\int_{w^{\lambda/2^{k+1}}(s)\setminus w^{\lambda/2^k}(s)}|\tilde u|^2|x'|^{-2\mu-2}\,dx \tag{3.21}$$
$$=\frac{1}{2^k}\int_{\mathbb{R}} d\xi_0\,|s|\int_{w^{\lambda/2^{k+1}}(s)\setminus w^{\lambda/2^k}(s)}|\tilde u|^2|x'|^{-2\mu-2}\,dx.$$

From (3.12), (3.13), (3.19) and (3.21) we obtain

$$\int_{\mathbb{R}} d\xi_0\,|s|\int_{\mathbb{R}^3}|\tilde u|^2|x'|^{-2\mu-2}\,dx \tag{3.22}$$
$$\le a_1\int_{\mathbb{R}} d\xi_0\int_{\mathbb{R}^3}|\tilde u|^2|x'|^{-2\mu-4}\,dx+\frac{1}{a_2}\int_{\mathbb{R}} d\xi_0\int_{\mathbb{R}^3}|\tilde u|^2|s|^2|x'|^{-2\mu}\,dx$$
$$+\left(\frac{2}{a_1}\right)^{1+\mu}\frac{1}{2^k}\int_{\mathbb{R}} d\xi_0|s|\int_{\mathbb{R}^3}|\tilde u|^2|x'|^{-2\mu-2}\,dx.$$

For k so large that $(2/a_1)^{1+\mu}2^{-k} \le 1/2$ we obtain from (3.22) inequality (3.10). This concludes the proof.

From (3.4), (3.10) and for sufficiently small a_1 and sufficiently large a_2 we obtain

$$
\int\limits_{\mathbb{R}^3} (|s|^2|\tilde{u}|^2 + |s|\,|\nabla\tilde{u}|^2)|x'|^{-2\mu}\,dx + \int\limits_{\mathbb{R}^3} |s|\,|\tilde{u}|^2|x'|^{-2\mu-2}\,dx + \|\tilde{u}\|^2_{H^2_{-\mu}(\mathbb{R}^3)} \le c\int\limits_{\mathbb{R}^3} |\tilde{f}|^2|x'|^{-2\mu}dx. \tag{3.23}
$$

In view of notation (2.7) we can express (3.23) in the form

$$
\|\tilde{u}\|^2_{\mathcal{E}^2_{-\mu}(\Omega)} \le c\|\tilde{f}\|^2_{L_{2,-\mu}(\Omega)}. \tag{3.24}
$$

By (2.8) we have

Lemma 3.3. *Assume that* $f \in L_{2,-\mu}(\mathbb{R}^3\times\mathbb{R})$, $\mu \in \mathbb{R}_+ \setminus \mathbb{Z}$, *and* $u \in \tilde{H}^{2,1}_{-\mu,\gamma}(\mathbb{R}^3\times\mathbb{R})$ *is a solution to* (3.1). *Then*

$$
\|u\|_{\tilde{H}^{2,1}_{-\mu,\gamma}(\mathbb{R}^3\times\mathbb{R})} \le c\|f\|_{L_{2,-\mu}(\mathbb{R}^3\times\mathbb{R})}, \tag{3.25}
$$

where c does not depend on u and f.

Lemma 3.4. *Assume that* $f \in \tilde{H}^{l,l/2}_{-\mu,\gamma}(\mathbb{R}^3\times\mathbb{R})$, $l \in \mathbb{N}$, $\mu \in \mathbb{R}_+ \setminus \mathbb{Z}$, *and* $u \in \tilde{H}^{l+2,l/2+1}_{-\mu,\gamma}(\mathbb{R}^3\times\mathbb{R})$ *is a solution to* (3.1). *Then*

$$
\|u\|_{\tilde{H}^{l+2,l/2+1}_{-\mu,\gamma}(\mathbb{R}^3\times\mathbb{R})} \le c\|f\|_{\tilde{H}^{l,l/2}_{-\mu,\gamma}(\mathbb{R}^3\times\mathbb{R})}, \tag{3.26}
$$

where c does not depend on u and f.

Proof. First we prove the assertion for $l = 1$. Let $\tilde{f} \in \mathcal{E}^1_{-\mu}(\mathbb{R}^3)$. Then

$$
\|\tilde{f}\|^2_{\mathcal{E}^1_{-\mu}(\mathbb{R}^3)} = |s|\,\|\tilde{f}\|^2_{L_{2,-\mu}(\mathbb{R}^3)} + \|\tilde{f}\|^2_{H^1_{-\mu}(\mathbb{R}^3)} < \infty.
$$

From the elliptic equation

$$
\Delta\tilde{u} = \tilde{f} - s\tilde{u} \tag{3.27}
$$

we have

$$
\|\tilde{u}\|^2_{H^3_{-\mu}(\mathbb{R}^3)} \le c\|\tilde{f}\|^2_{H^1_{-\mu}(\mathbb{R}^3)} + c|s|^2\|\tilde{u}\|^2_{H^1_{-\mu}(\mathbb{R}^3)}. \tag{3.28}
$$

Multiplying (3.23) by $|s|$ we get

$$
|s|^3\|\tilde{u}\|^2_{L_{2,-\mu}(\mathbb{R}^3)} + |s|^2\|\tilde{u}\|^2_{H^1_{-\mu}(\mathbb{R}^3)} + |s|\,\|\tilde{u}\|^2_{H^2_{-\mu}(\mathbb{R}^3)} \le c|s|\,\|\tilde{f}\|^2_{L_{2,-\mu}(\mathbb{R}^3)} \le c\|\tilde{f}\|^2_{\mathcal{E}^1_{-\mu}(\mathbb{R}^3)}. \tag{3.29}
$$

From (3.28) and (3.29) we have

$$
\|\tilde{u}\|^2_{\mathcal{E}^3_{-\mu}(\mathbb{R}^3)} \le c\|\tilde{f}\|^2_{\mathcal{E}^1_{-\mu}(\mathbb{R}^3)}. \tag{3.30}
$$

Using (2.8) proves the assertion for $l = 1$.

Assume that the assertion is proved for $l=\sigma$. Then we have

$$\|u\|^2_{\tilde H^{\sigma,\sigma/2}_{-\mu,\gamma}(\mathbb{R}^3\times\mathbb{R})} \le c\|f\|^2_{\tilde H^{\sigma-2,\sigma/2-1}_{-\mu,\gamma}(\mathbb{R}^3\times\mathbb{R})}. \tag{3.31}$$

Then for the Laplace transforms we obtain

$$\|\tilde u\|^2_{\mathcal{E}^{\sigma}_{-\mu}(\mathbb{R}^3)} \le c\|\tilde f\|^2_{\mathcal{E}^{\sigma-2}_{-\mu}(\mathbb{R}^3)}. \tag{3.32}$$

Let us assume that $\tilde f\in\mathcal{E}^{\sigma-1}_{-\mu}(\mathbb{R}^3)$, so

$$\|\tilde f\|^2_{H^{\sigma-1}_{-\mu}(\mathbb{R}^3)} + |s|\,\|\tilde f\|^2_{\mathcal{E}^{\sigma-2}_{-\mu}(\mathbb{R}^3)} < \infty. \tag{3.33}$$

Let us consider (3.27). Then it follows that

$$\|\tilde u\|^2_{H^{\sigma+1}_{-\mu}(\mathbb{R}^3)} \le c\|\tilde f\|^2_{H^{\sigma-1}_{-\mu}(\mathbb{R}^3)} + c|s|^2\|\tilde u\|^2_{H^{\sigma-1}_{-\mu}(\mathbb{R}^3)}. \tag{3.34}$$

From (3.31) we get

$$|s|^2\|\tilde u\|^2_{H^{\sigma-1}_{-\mu}(\mathbb{R}^3)} \le |s|\,\|\tilde u\|^2_{\mathcal{E}^{\sigma}_{-\mu}(\mathbb{R}^3)} \le c|s|\,\|\tilde f\|^2_{\mathcal{E}^{\sigma-2}_{-\mu}(\mathbb{R}^3)} \le c\|\tilde f\|^2_{\mathcal{E}^{\sigma-1}_{-\mu}(\mathbb{R}^3)}. \tag{3.35}$$

Hence (3.34) and (3.35) imply that

$$\|\tilde u\|^2_{\mathcal{E}^{\sigma+1}_{-\mu}(\mathbb{R}^3)} \le c\|\tilde f\|^2_{\mathcal{E}^{\sigma-1}_{-\mu}(\mathbb{R}^3)}. \tag{3.36}$$

Applying (2.8) shows that (3.36) proves the assertion for $l=\sigma+1$.

Hence by a recurrence argument the lemma is proved.

Some ideas used in this section were borrowed from [2, 3].

4. Existence in $\mathbb{R}^3\times\mathbb{R}_+$. Since we are going to prove the existence of solutions to problem (1.1) in a bounded domain and in weighted Sobolev spaces, we have to consider it locally. Local considerations require examining problem (1.1) with vanishing initial data and either in $\mathbb{R}^3$ or in $\mathbb{R}^3_+$. But we are interested in proving the existence of solutions to problem (1.1) in weighted Sobolev spaces so local solutions in neighbourhoods of L must be treated in a special way. We shall restrict ourselves to such neighbourhoods only because properties of local solutions in neighbourhoods located at a positive distance from L are well known. Natural neighbourhoods near L are cylinders with axis of symmetry equal to L. Now we examine local solutions to (1.1) in such cylinders. First, we shall restrict our considerations to the case $l=0$ (see Theorem 1.2).

Let $A>0$ be given. We denote by C_A the cylinder

$$C_A=\{x\in\mathbb{R}^3 : |x'|<A,\ x_3\in\mathbb{R}\},$$

where $x'=(x_1,x_2)$, $|x'|=\sqrt{x_1^2+x_2^2}$, $x=(x_1,x_2,x_3)$ is the Cartesian system in $\mathbb{R}^3$ with L as the x_3-axis.

Let $R > 0$ be given. Then we consider the problem

$$
\begin{aligned}
& u_t - \Delta u = f && \text{in } C_R \times \mathbb{R},\\
& u = 0 && \text{on } \partial C_R \times \mathbb{R}, \qquad (4.1)\\
& u \to 0 && \text{as } |x_3| \to \infty,
\end{aligned}
$$

where $\partial C_R = \{x \in \mathbb{R}^3 : \ |x'| = R,\ x_3 \in \mathbb{R}\}$.

To prove the existence of solutions to problem (4.1) in weighted Sobolev spaces we introduce an approximate solution. Let $\delta \in (0, R)$ be given. Let $C_{R,\delta} = C_R \setminus \bar{C}_\delta$, where $\bar{C}_\delta$ is the closure of C_δ. Then we consider the problem

$$
\begin{aligned}
& u_{\delta,t} - \Delta u_\delta = f_\delta && \text{in } C_{R,\delta} \times \mathbb{R}_+,\\
& u_\delta = 0 && \text{on } \partial C_{R,\delta} \times \mathbb{R}_+, \qquad (4.2)\\
& u_\delta \to 0 && \text{as } |x_3| \to \infty,
\end{aligned}
$$

We have

LEMMA 4.1. *Assume that $f_\delta \in L_{2,\gamma}(C_{R,\delta} \times \mathbb{R})$. Then there exists a solution $u_\delta \in H^{2,1}_\gamma(C_{R,\delta} \times \mathbb{R})$ to problem* (4.2) *such that*

$$
\|u_\delta\|_{H^{2,1}_\gamma(C_{R,\delta}\times\mathbb{R})} \le c\|f_\delta\|_{L_{2,\gamma}(C_{R,\delta}\times\mathbb{R})}, \qquad (4.3)
$$

where c does not depend on u_δ and f_δ.

Let us introduce the cylindrical coordinates (r, φ, z) by the relations $x_1 = r\cos\varphi$, $x_2 = r\sin\varphi$, $x_3 = z$.

Now we extend solutions of (4.2) for $|x'| > R$ by zero because we are not interested in their behaviour as $|x'| \to \infty$. In reality we could consider problem (4.2) with $R = \infty$. Then the behaviour of solutions as $|x'| \to \infty$ would be determined by the solution space $H^{2,1}(\mathbb{R}^3 \setminus \bar{C}_\delta \times \mathbb{R})$. In Section 5 we prove the existence of solutions to problem (1.1) by the regularizer technique. Therefore examining solutions vanishing outside $\bar{C}_R$ is close to the considerations from Section 5.

Let $\zeta_R(x')$ be a smooth function such that $\zeta_R(x') = 1$ for $|x'| \le \frac{3}{4}R$ and $\zeta_R(x') = 0$ for $|x'| \ge R$. Let $u'_\delta = u_\delta\zeta_R$, $f'_\delta = f_\delta\zeta_R$. Then problem (4.2) takes the form

$$
\begin{aligned}
& u'_{\delta,t} - \Delta u'_\delta = f'_\delta - 2\zeta_{R,x'}u_{\delta,x'} - \zeta_{R,x'x'}u_\delta \equiv f''_\delta && \text{in } \mathbb{R}^3 \setminus \bar{C}_\delta \times \mathbb{R},\\
& u'_\delta = 0 && \text{on } \partial C_\delta \times \mathbb{R}, \qquad (4.4)\\
& u'_\delta \to 0 && \text{as } |x_3| \to \infty.
\end{aligned}
$$

In view of Lemma 4.1 we have $f''_\delta \in H^{2,1}_\gamma(\mathbb{R}^3 \setminus \bar{C}_\delta \times \mathbb{R})$ and there exists a solution $u'_\delta \in H^{2,1}_\gamma(\mathbb{R}^3 \setminus \bar{C}_\delta \times \mathbb{R})$ to problem (4.4) such that

$$
\|u'_\delta\|_{H^{2,1}_\gamma(\mathbb{R}^3\setminus\bar{C}_\delta\times\mathbb{R})} \le c\|f''_\delta\|_{L_{2,\gamma}(\mathbb{R}^3\setminus\bar{C}_\delta\times\mathbb{R})} \le c\|f_\delta\|_{L_{2,\gamma}(C_{R,\delta}\times\mathbb{R})}. \qquad (4.5)
$$

Now we follow some ideas from [10]. To prove the existence of solutions to problem (1.1) in the weighted Sobolev spaces $H^{l,l/2}_{-\mu,\gamma}$ introduced in Definition 1.1 we extend f''_δ and u'_δ by zero for $r<\delta$. Let us denote the extended functions by $\bar u_\delta$ and $\bar f_\delta$. Additionally, we assume that $f'_\delta|_{r=\delta}=0$. Then $(4.4)_1$ implies that $(u'_{\delta,t}-\Delta u'_\delta)|_{r=\delta}=0$, where we assume that $\delta<\frac34 R$. Let

$$\begin{aligned} v_\delta &= u'_\delta|_{x=x(r,\varphi,z)}, & h_\delta &= f''_\delta|_{x=x(r,\varphi,z)},\\ \bar v_\delta &= \bar u_\delta|_{x=x(r,\varphi,z)}, & \bar h_\delta &= \bar f_\delta|_{x=x(r,\varphi,z)}. \end{aligned}$$

Then problem (4.4) reads

$$\begin{aligned} &\bar v_{\delta,t}-\left(\frac1r(r\bar v_{\delta,r})_{,r}+\frac{1}{r^2}\bar v_{\delta,\varphi\varphi}+\bar v_{\delta,zz}\right)=\bar h_\delta && \text{in } \mathbb{R}^3\times\mathbb{R},\\ &\bar v_\delta|_{r\le\delta}=0,\ \ \bar h_\delta|_{r\le\delta}=0, && \\ &\bar v_\delta\to 0 && \text{as } |z|\to\infty,\ r>R. \end{aligned} \tag{4.6}$$

From (4.6) it follows that $\bar v_{\delta,t}=0$, $\bar v_{\delta,\varphi\varphi}=0$, $\bar v_{\delta,zz}=0$ for $r\le\delta$. Moreover, we have $\frac1r(r\bar v_{\delta,r})_{,r}|_{r=\delta}=0$. Lemma 4.1 implies that $\frac1r(r\bar v_{\delta,r})_{,r}\in L_{2,\gamma}(\mathbb{R}^3\times\mathbb{R})$ so $(4.4)_1$ yields that $\frac1r(r\bar v_{\delta,r})_{,r},\bar v_{\delta,t}\in L_{2,-\mu,\gamma}(\mathbb{R}^3\times\mathbb{R})$. By the Hardy inequality it follows that $\bar v_\delta\in H^{2,1}_{-\mu,\gamma}(\mathbb{R}^3\times\mathbb{R})$. Then Lemma 3.3 implies the estimate

$$\|\bar v_\delta\|_{H^{2,1}_{-\mu,\gamma}(\mathbb{R}^3\times\mathbb{R})}\le c\|\bar f_\delta\|_{L_{2,-\mu,\gamma}(\mathbb{R}^3\times\mathbb{R})},$$

where c does not depend on δ.

Letting $\delta\to 0$ we obtain

Lemma 4.2. *Assume that $f\in L_{2,-\mu,\gamma}(\mathbb{R}^3\times\mathbb{R})$, $\mu\in\mathbb{R}_+\setminus\mathbb{Z}$. Then there exists a solution $u\in H^{2,1}_{-\mu,\gamma}(\mathbb{R}^3\times\mathbb{R})$ to the problem*

$$\begin{aligned} &u_t-\Delta u=f && \text{in } \mathbb{R}^3\times\mathbb{R},\\ &u\to 0 && \text{as } |x|\to\infty, \end{aligned}$$

such that

$$\|u\|_{H^{2,1}_{-\mu,\gamma}(\mathbb{R}^3\times\mathbb{R})}\le c\|f\|_{L_{2,-\mu,\gamma}\mathbb{R}^3\times\mathbb{R})}. \tag{4.7}$$

Let us consider the case $l>0$. We have

Lemma 4.3. *Assume that $f\in H^{l,l/2}_\gamma(C_{R,\delta}\times\mathbb{R})$, $l\in\mathbb{N}$. Assume the compatibility conditions*

$$\partial_t^i u_\delta|_{t=0}=0,\quad i\le[l/2],\qquad \partial_t^i f_\delta|_{t=0}=0,\quad i\le[l/2]-1.$$

Then there exists a solution $u_\delta\in H^{l+2,l/2+1}_\gamma(C_{R,\delta}\times\mathbb{R})$ to problem (4.2) such that

$$\|u_\delta\|_{H^{l+2,l/2+1}_\gamma(C_{R,\delta}\times\mathbb{R})}\le c\|f_\delta\|_{H^{l,l/2}_\gamma(C_{R,\delta}\times\mathbb{R})}, \tag{4.8}$$

where c does not depend on u_δ and f_δ.

Assuming that $f\in H^{l,l/2}_{-\mu,\gamma}(\mathbb{R}^3\times\mathbb{R})$ we prove a lemma similar to Lemma 4.2. Let us consider problem (4.4). In view of Lemma 4.3, $f''_\delta\in H^{l,l/2}_\gamma(\mathbb{R}^3\backslash\bar C_\delta)$

so there exists a solution $u'_\delta \in H^{l+2,l/2+1}_\gamma(\mathbb{R}^3\setminus\bar{C}_\delta)$ to problem (4.4) such that and

$$\|u'_\delta\|_{H^{l+2,l/2+1}_\gamma(\mathbb{R}^3\setminus\bar{C}_\delta\times\mathbb{R})} \le c\|f''_\delta\|_{H^{l,l/2}_\gamma(\mathbb{R}^3\setminus\bar{C}_\delta\times\mathbb{R})}. \tag{4.9}$$

Let us recall the extensions denoted by $\bar{v}_\delta$ and $\bar{h}_\delta$. In view of (4.9) we have $\bar{h}_\delta \in H^{l,l/2}_{-\mu,\gamma}(\mathbb{R}^3\times\mathbb{R})$ and

$$\begin{aligned}\|\bar{v}_t\|_{H^{l,l/2}_{\mu,\gamma}(\mathbb{R}^3\times\mathbb{R})} &+ \left\|\frac{1}{r}(r,\bar{v}_{\delta,r})_{,r}\right\|_{H^{l,l/2}_{-\mu,\gamma}(\mathbb{R}^3\times\mathbb{R})}\\ &+ \left\|\frac{1}{r^2}\bar{v}_{\delta,\varphi\varphi}\right\|_{H^{l,l/2}_{-\mu,\gamma}(\mathbb{R}^3\times\mathbb{R})} + \|\bar{v}_{\delta,zz}\|_{H^{l,l/2}_{-\mu,\gamma}(\mathbb{R}^3\times\mathbb{R}))}\\ &\le c\|\bar{f}_\delta\|_{H^{l,l/2}_{-\mu,\gamma}(\mathbb{R}^3\times\mathbb{R})}.\end{aligned} \tag{4.10}$$

By the Hardy inequality it follows that $\bar{v}_\delta \in H^{l+2,l/2+1}_{-\mu,\gamma}(\mathbb{R}^3\times\mathbb{R})$ (see the proofs of Lemmas 2.1, 2.3 from [10]).

Remark. We need to choose a sequence of $\bar{f}_\delta$ that converges to $f \in H^{l,l/2}_{-\mu}$ as $\delta\to 0$. For $\mu\in(0,1)$ it is meaningful to assume that $\partial_r^i\bar{f}_\delta|_{r=\delta}=0$ for $i\le l-1$ but for $\mu>1$, $\partial_r^i\bar{f}_\delta|_{r=\delta}=0$ for $i\le l$ should be imposed. In view of these assumptions we obtain the corresponding restrictions on $\bar{v}_\delta$: $\partial_r^i\big(\frac{1}{r}(r\bar{v}_{\delta,r})_{,r}\big)\big|_{r=\delta}=0$ for $i\le l-1$ if $\mu\in(0,1)$, and for $i\le l$ if $\mu>1$.

Moreover, repeating the considerations from [10] we can show that

$$\partial_r^i\bar{v}_\delta|_{r=\delta}=0 \quad \text{for } i\le l-1.$$

Then the Hardy inequality works and $\bar{v}_\delta\in H^{l+2,l/2+1}_{-\mu,\gamma}(\mathbb{R}^3\times\mathbb{R})$.

We have to add that the existence in $H^{l+2,l/2+1}_{-\mu,\gamma}$ is shown step by step starting from $l=0$. Then Lemma 3.4 yields

$$\|\bar{v}_\delta\|_{H^{l+2,l/2+1}_{-\mu,\gamma}(\mathbb{R}^3\times\mathbb{R})} \le c\|\bar{f}_\delta\|_{H^{l,l/2}_{-\mu,\gamma}(\mathbb{R}^3\times\mathbb{R})}, \tag{4.11}$$

where c does not depend on δ. Letting $\delta\to 0$ we obtain

Lemma 4.4. *Assume that $f\in H^{l,l/2}_{-\mu,\gamma}(\mathbb{R}^3\times\mathbb{R})$, $\mu\in\mathbb{R}_+\setminus\mathbb{Z}$, $l\in\mathbb{N}_0$. Assume the compatibility conditions from Lemma 4.3. Then there exists a solution $u\in H^{l+2,l/2+1}_{-\mu,\gamma}(\mathbb{R}^3\times\mathbb{R})$ to the problem*

$$\begin{aligned} &u_t-\Delta u=f && \text{in } \mathbb{R}^3\times\mathbb{R},\\ &u\to 0 && \text{as } |x|\to\infty,\end{aligned} \tag{4.12}$$

such that

$$\|u\|_{H^{l+2,l/2+1}_{-\mu,\gamma}(\mathbb{R}^3\times\mathbb{R})} \le c\|f\|_{H^{l,l/2}_{-\mu,\gamma}(\mathbb{R}^3\times\mathbb{R})}, \tag{4.13}$$

where c does not depend on u and f.

Finally, we have to recall that the considerations from [9, 14] imply that there is no existence result for solutions to problem (1.1) in $H^{l+2,l/2+1}_{-\mu}$ for $\mu \in \mathbb{Z}$.

5. Existence in a bounded domain. The aim of this section is to prove Theorem 1.2 for solutions to (1.1) with vanishing initial data. For this purpose we use the regularizer technique so we need a partition of unity. Let us define two collections of open subsets $\{\omega^{(k)}\}$ and $\{\Omega^{(k)}\}$, $k \in \bigcup_{i=1}^4 \mathcal{M}_i$, such that $\overline{\omega^{(k)}} \subset \Omega^{(k)}$, $\bigcup_k \omega^{(k)} = \bigcup_k \Omega^{(k)} = \Omega$, $\Omega^{(k)} \cap S = \emptyset$ for $k \in \mathcal{M}_1 \cup \mathcal{M}_3$ and $\Omega^{(k)} \cap S \neq \emptyset$ for $k \in \mathcal{M}_2 \cup \mathcal{M}_4$. Here $\Omega^{(k)}$, $k \in \mathcal{M}_1$, is a neighbourhood of an interior point of $L \cap \Omega$; $\Omega^{(k)}$, $k \in \mathcal{M}_2$, is a neighbourhood of a point where L meets S; $\Omega^{(k)}$, $k \in \mathcal{M}_3$, is a neighbourhood of an interior point of Ω, located at a positive distance from L; $\Omega^{(k)}$, $k \in \mathcal{M}_4$, is a neighbourhood of a point of S, located at a positive distance from L. We assume that at most N_0 of the $\Omega^{(k)}$ have nonempty intersection, and $\sup_k \operatorname{diam} \Omega^{(k)} \le 2\lambda$ for some $\lambda > 0$.

Let $\zeta^{(k)}(x)$ be a smooth function such that $0 \le \zeta^{(k)}(x) \le 1$, $\zeta^{(k)}(x) = 1$ for $x \in \omega^{(k)}$, $\operatorname{supp} \zeta^{(k)} \subset \Omega^{(k)}$ and $|D_x^\nu \zeta^{(k)}(x)| \le c/|\lambda|^\nu$. Then $1 \le \sum_k (\zeta^{(k)}(x))^2 \le N_0$. Introducing the function $\eta^{(k)}(x) = \zeta^{(k)}(x)/\sum_l (\zeta^{(l)}(x))^2$ we have $\operatorname{supp} \eta^{(k)} \subset \Omega^{(k)}$, $\sum_k \eta^{(k)}(x)\zeta^{(k)}(x) = 1$, $|D_x^\nu \eta^{(k)}| \le c/|\lambda|^\nu$. By $\xi^{(k)}$ we denote a fixed interior point of $\omega^{(k)}$ and $\Omega^{(k)}$ for $k \in \mathcal{M}_1 \cup \mathcal{M}_3$, and a point of $\overline{\omega^{(k)}} \cap S$ and $\overline{\Omega^{(k)}} \cap S$ for $k \in \mathcal{M}_2 \cup \mathcal{M}_4$.

Since we consider a problem invariant with respect to translations and rotations we can introduce a local coordinate system $y = (y_1, y_2, y_3)$ with centre at $\xi^{(k)}$ such that for $k \in \mathcal{M}_2 \cup \mathcal{M}_4$ the part $\tilde{S}^{(k)} = S \cap \overline{\Omega^{(k)}}$ of the boundary is described by $y_3 = F(y_1, y_2)$. We assume that a point with coordinates (y_1, y_2, y_3), $y_3 > 0$, belongs to Ω. Then we introduce new coordinates by

$$z_i = y_i, \quad i = 1, 2, \quad z_3 = y_3 - F(y_1, y_2). \tag{5.1}$$

We denote by Ψ_k the transformation $\Omega^{(k)} \ni y \mapsto \Psi_k(y) = z \in \hat{\Omega}^{(k)}$, described by (5.1), such that $\omega^{(k)} \ni y \mapsto \Psi_k(y) = z \in \hat{\omega}^{(k)}$. We assume that the sets $\hat{\omega}^{(k)}$, $\hat{\Omega}^{(k)}$ are described in local coordinates at $\xi^{(k)}$ by the inequalities

$$\begin{aligned} &|y_i| < \lambda, && i = 1, 2, && 0 < y_3 - F(y_1, y_2) < \lambda, \\ &|y_i| < 2\lambda, && i = 1, 2, && 0 < y_3 - F(y_1, y_2) < 2\lambda, \end{aligned} \tag{5.2}$$

respectively.

Let $y = Y_k(x)$ be a transformation from the x coordinates to local coordinates with origin at $\xi^{(k)}$ which is a composition of a translation and a rotation.

We denote $\Phi_k = \Psi_k \circ Y_k$ and

$$\hat{u}^{(k)}(z) = u(\Phi_k^{-1}(z)), \quad \tilde{u}^{(k)}(z) = \hat{u}^{(k)}(z)\hat{\zeta}^{(k)}(z). \tag{5.3}$$

First, we prove

Lemma 5.1. *Assume that $f \in L_{2,-\mu,\gamma}(\Omega\times\mathbb{R}_+)$, $\mu \in (0,1)$, $u_0 = 0$. Then there exists a solution $u \in H^{2,1}_{-\mu,\gamma}(\Omega\times\mathbb{R}_+)$ to problem* (1.1) *such that*

$$\|u\|_{H^{2,1}_{-\mu,\gamma}(\Omega\times\mathbb{R}_+)} \le c\|f\|_{L_{2,-\mu,\gamma}(\Omega\times\mathbb{R}_+)}. \tag{5.4}$$

Proof. Since $f \in L_{2,-\mu,\gamma}(\Omega\times\mathbb{R}_+)$, $\mu > 0$, we have $f \in L_{2,\gamma}(\Omega\times\mathbb{R}_+)$ and

$$\|f\|_{L_{2,\gamma}(\Omega\times\mathbb{R}_+)} \le c\|f\|_{L_{2,-\mu,\gamma}(\Omega\times\mathbb{R}_+)}, \tag{5.5}$$

because Ω is bounded. Thus there exists a solution $u \in H^{2,1}_\gamma(\Omega\times\mathbb{R}_+)$ to problem (1.1) such that

$$\|u\|_{H^{2,1}_\gamma(\Omega\times\mathbb{R}_+)} \le c\|f\|_{L_{2,\gamma}(\Omega\times\mathbb{R}_+)}. \tag{5.6}$$

Now we consider problem (1.1) locally.

Let $\xi^{(k)} \in L\cap\Omega$, $k \in \mathcal{M}_1$. Let us introduce a local Cartesian system $y = (y_1, y_2, y_3)$ with origin at $\xi^{(k)}$ such that L is the y_3-axis. Let R and a be given positive numbers and let Q be a cylinder of the form

$$Q = \{y \in \mathbb{R}^3 : |y'| < R,\ |y_3| < a\},$$

where $y' = (y_1, y_2)$. We assume that $Q \cap S = \emptyset$ and $\Omega^{(k)} \subset Q$, $k \in \mathcal{M}_1$. Let $\zeta = \zeta^{(k)}(y)$, $k \in \mathcal{M}_1$, be a smooth function from the partition of unity such that $\operatorname{supp}\zeta \subset Q$. Let $\tilde{u} = u\zeta$, $\tilde{f} = f\zeta$. Then problem (1.1) with $u_0 = 0$ takes the form

$$\begin{aligned} &\tilde{u}_t - \Delta\tilde{u} = \tilde{f} - 2\nabla\zeta\nabla u - \Delta\zeta u \equiv \tilde{f}_0,\\ &\tilde{u}|_{\partial Q} = 0. \end{aligned} \tag{5.7}$$

To apply Lemma 4.2 we have to know that $\tilde{f}_0 \in L_{2,-\mu,\gamma}(\mathbb{R}^3\times\mathbb{R}_+)$. Since $\mu \in (0,1)$ and by (5.6), $\tilde{u} \in H^{2,1}_\gamma(\mathbb{R}^3\times\mathbb{R}_+)$, the Hardy inequality shows that $\nabla u \in L_{2,-\mu,\gamma}(\mathbb{R}^3\times\mathbb{R}_+)$ and $u \in L_{2,-\mu,\gamma}(\mathbb{R}^3\times\mathbb{R}_+)$. Then Lemma 4.2 implies the existence of a solution $\tilde{u} \in H^{2,1}_{-\mu,\gamma}(\mathbb{R}^3\times\mathbb{R}_+)$ to problem (5.7) such that

$$\|\tilde{u}\|_{H^{2,1}_{-\mu,\gamma}(\mathbb{R}^3\times\mathbb{R}_+)} \le c\|f\|_{L_{2,-\mu,\gamma}(\Omega\times\mathbb{R}_+)}, \tag{5.8}$$

because

$$\|\nabla u\|_{L_{2,-\mu,\gamma}(\mathbb{R}^3\cap Q\times\mathbb{R}_+)} + \|u\|_{L_{2,-\mu,\gamma}(\mathbb{R}^3\cap Q\times\mathbb{R}_+)} \le c\|f\|_{L_{2,\gamma}(\Omega\times\mathbb{R}_+)}.$$

Let $\xi^{(k)}$, $k \in \mathcal{M}_2$, be a point where L meets S. Let us introduce a local system of coordinates $y = (y_1, y_2, y_3)$ with origin in $\xi^{(k)}$ such that in the subdomain $\Omega^{(k)}$, $k \in \mathcal{M}_2$, the part of the boundary $\tilde{S}^{(k)} = S \cap \overline{\Omega^{(k)}}$ is described by

$$y_3 = F(y_1, y_2),$$

where $F(0,0)=0$ and the point (y_1,y_2,y_3) with $y_3>0$ belongs to Ω. Let $\hat\nabla_i=\frac{\partial z_k}{\partial x_i}\frac{\partial}{\partial z_k}$, where the summation convention over the repeated indices is assumed, and $z=\Phi_k(x)$, where $k\in\mathcal{M}_2$. Then locally problem (1.1) takes the form

$$
\begin{aligned}
&\hat u^{(k)}_{,t}-\hat\nabla_i^2\hat u^{(k)}=\hat f^{(k)} && \text{in } \Omega^{(k)}\times\mathbb{R}_+,\\
&\hat u^{(k)}|_{z_3}=0 && \text{on } S^{(k)}\times\mathbb{R}_+,\\
&\hat u^{(k)}|_{t=0}=0 && \text{in } \Omega^{(k)},
\end{aligned}
\tag{5.9}
$$

where $k\in\mathcal{M}_2$.

Let us extend problem (5.9) to $z_3<0$ by reflection. We denote the extended functions by $\hat u^{(k)\prime}$, $\hat f^{(k)\prime}$. Let $\hat\zeta=\hat\zeta^{(k)}$, $k\in\mathcal{M}_2$, be a function from the partition of unity and let $\hat\zeta'$ be its extension to $z_3<0$. We assume that $\operatorname{supp}\hat\zeta'\subset Q'$, where

$$Q'=\{z\in\mathbb{R}^3:|z'|<R,\ |z_3|<a\}.$$

Set $\tilde u=\hat u^{(k)\prime}\hat\zeta'$, $\tilde f=\hat f^{(k)\prime}\hat\zeta'$. Then problem (5.9) assumes the form

$$
\begin{aligned}
&\tilde u_{,t}-\hat\nabla_i^2\tilde u=\tilde f-2\hat\nabla\hat u^{(k)\prime}\nabla\hat\zeta'-\hat\Delta\hat\zeta'\hat u^{(k)\prime}\equiv\tilde f_0' && \text{in } Q'\times\mathbb{R}_+,\\
&\tilde u|_{\partial Q'}=0 && \text{on } \partial Q'\times\mathbb{R}_+,\\
&\tilde u|_{t=0}=0 && \text{in } Q'.
\end{aligned}
\tag{5.10}
$$

In view of (5.6) and the Hardy inequality we have $\tilde f_0'\in L_{2,-\mu,\gamma}(Q'\times\mathbb{R}_+)$, $\mu\in(0,1)$. Then Lemma 4.2 implies the existence of $\tilde u\in H^{2,1}_{-\mu,\gamma}(Q'\times\mathbb{R}_+)$. From (5.6) it follows that $\tilde u\in H^{2,1}_{-\mu,\gamma}(\Omega^{(k)}\times\mathbb{R}_+)$ in neighbourhoods $\Omega^{(k)}$, $k\in\mathcal{M}_3\cup\mathcal{M}_4$, located at a positive distance from L.

To introduce the regularizer we define all local problems in a uniform way.

Let $k\in\mathcal{M}_1$. Then problem (5.7) is expressed as

$$
\begin{aligned}
&\tilde u^{(k)}_t-\nabla_x^2\tilde u^{(k)}=\tilde f^{(k)}_* && \text{in } \mathbb{R}^3\times\mathbb{R}_+,\\
&\tilde u^{(k)}|_{t=0}=0 && \text{in } \mathbb{R}^3.
\end{aligned}
\tag{5.11}
$$

For $k\in\mathcal{M}_2$ problem (5.10) takes the form

$$
\begin{aligned}
&\tilde u^{(k)}_t-\nabla_z^2\tilde u^{(k)}=\tilde f^{(k)}_* && \text{in } \mathbb{R}^3\times\mathbb{R}_+,\\
&\tilde u^{(k)}|_{t=0}=0 && \text{in } \mathbb{R}^3.
\end{aligned}
\tag{5.12}
$$

For $k\in\mathcal{M}_3$ we consider the problem

$$
\begin{aligned}
&\tilde u^{(k)}_t-\nabla_x^2\tilde u^{(k)}=\tilde f^{(k)}_* && \text{in } \mathbb{R}^3\times\mathbb{R}_+,\\
&\tilde u^{(k)}|_{t=0}=0 && \text{in } \mathbb{R}^3.
\end{aligned}
\tag{5.13}
$$

For $k \in \mathcal{M}_4$ we have

$$
\begin{aligned}
&\tilde u_t^{(k)} - \nabla_z^2 \tilde u^{(k)} = \tilde f_*^{(k)} && \text{in } \mathbb{R}_+^3 \times \mathbb{R}_+,\\
&\tilde u^{(k)}|_{z_3=0} = 0 && \text{on } \mathbb{R}^2 \times \mathbb{R}_+,\\
&\tilde u^{(k)}|_{t=0} = 0 && \text{in } \mathbb{R}_+^3.
\end{aligned}
\tag{5.14}
$$

Lemma 4.2 implies the existence of solutions to problems (5.11) and (5.12). The neighbourhoods $\Omega^{(k)}$, $k \in \mathcal{M}_3 \cup \mathcal{M}_4$, are at a positive distance from L, so the existence of solutions to problems (5.13) and (5.14) in H_γ^{l+2} is equivalent to the existence in $H^{l+2}_{-\mu,\gamma}$.

Let $R^{(k)}$ be the operator which solves the kth problem. Then we define (see [4, Ch. 4])

$$
Rf = \sum_{k\in\mathcal{M}} \eta^{(k)}(x) u^{(k)}(x,t),
$$

where

$$
u^{(k)}(x,t) = \begin{cases} R^{(k)}\zeta^{(k)} f & \text{for } k \in \mathcal{M}_1 \cup \mathcal{M}_3,\\ \Phi_k^{-1} R^{(k)}(\Phi_k \zeta^{(k)} f) & \text{for } k \in \mathcal{M}_2 \cup \mathcal{M}_4. \end{cases}
$$

Let us introduce the spaces $H = L_{2,-\mu,\gamma}$ and $V = H^{2,1}_{-\mu,\gamma}$, $\mu \in (0,1)$, endowed with the norms

$$
\begin{aligned}
\|f\|_H &= \sum_{k\in\mathcal{M}} \|f^{(k)}\|_{L_{2,-\mu,\gamma}(\mathbb{R}^{(k)}\times\mathbb{R}_+)},\\
\|u\|_V &= \sum_{k\in\mathcal{M}} \|u^{(k)}\|_{H^{2,1}_{-\mu,\gamma}(\mathbb{R}^{(k)}\times\mathbb{R}_+)},
\end{aligned}
$$

where

$$
\mathbb{R}^{(k)} = \begin{cases} \mathbb{R}^3 & \text{for } k \in \mathcal{M}_1 \cup \mathcal{M}_2 \cup \mathcal{M}_3,\\ \mathbb{R}^3_+ & \text{for } k \in \mathcal{M}_4. \end{cases}
$$

Since the solvability of problems (5.11)–(5.14) is known we find that $R : H \to V$ is a bounded operator.

Let $\mathcal{Z} = \partial_t - \Delta$. It can be shown that for $f \in H$ there exists an operator T such that

$$
\mathcal{Z} R f = (I + T) f, \tag{5.15}
$$

where $T : H \to H$, $\|T\| < 1$, and I is the identity operator. Moreover, for $v \in V$ we obtain

$$
R \mathcal{Z} v = (I + W) v, \tag{5.16}
$$

where $W : V \to V$ and $\|W\| < 1$.

Relations (5.15) and (5.16) imply the existence of a solution $v \in V(\Omega)$ to problem (1.12). We stress that the considerations leading to (5.15) and (5.16) are done for $f \in H(\Omega)$ and $v \in V(\Omega)$. Therefore, choosing the spaces $H(\Omega)$ and $V(\Omega)$ we have been able to prove the existence of solutions to problem (1.12) in $V(\Omega)$. If either $\|T\| > 1$ or $\|W\| > 1$ the existence cannot be proved in $V(\Omega)$ that way. This concludes the proof.

LEMMA 5.2. *Assume that $f \in L_{2,-\mu,\gamma}(\Omega\times\mathbb{R}_+)$, $\mu \in (1,2)$, $S \in C^2$. Then there exists a solution $u \in H^{2,1}_{-\mu,\gamma}(\Omega \times \mathbb{R}_+)$ to problem* (1.12) *such that*

$$\|u\|_{H^{2,1}_{-\mu,\gamma}(\Omega\times\mathbb{R}_+)} \le c\|f\|_{L_{2,-\mu,\gamma}(\Omega\times\mathbb{R}_+)}, \quad \mu \in (1,2). \tag{5.17}$$

Proof. Since $f \in L_{2,-\mu,\gamma}(\Omega \times \mathbb{R}_+)$, $\mu \in (1,2)$ and Ω is bounded, we have $f \in L_{2,-\mu,\gamma}(\Omega \times \mathbb{R}_+)$, $\mu \in (0,1)$. Hence Lemma 5.1 can be applied. By the Hardy inequality we get $\tilde{f}^{(k)}_* \in L_{2,-\mu,\gamma}(\mathbb{R}^3\times\mathbb{R}_+)$ for $\mu \in (1,2)$, $k \in \mathcal{M}_1\cup\mathcal{M}_2$. Then repeating the considerations from the proof of Lemma 5.1 in the case $H(\Omega) = L_{2,-\mu,\gamma}$, $V(\Omega) = H^2_{-\mu,\gamma}$, $\mu \in (1,2)$, we conclude the proof.

Continuing the considerations we obtain

LEMMA 5.3. *Assume that $f \in L_{2,-\mu}(\Omega \times \mathbb{R}_+)$, $\mu \in (k,k+1)$, $k \in \mathbb{N}_0$, $u_0 = 0$, $S \in C^2$. Then there exists a solution $u \in H^{2,1}_{-\mu,\gamma}(\Omega \times \mathbb{R}_+)$ to problem* (1.1) *such that*

$$\|u\|_{H^{2,1}_{-\mu,\gamma}(\Omega\times\mathbb{R}_+)} \le c\|f\|_{L_{2,-\mu,\gamma}(\Omega\times\mathbb{R}_+)}, \quad \mu \in (k,k+1). \tag{5.18}$$

Finally, we prove

LEMMA 5.4. *Assume that $f \in H^{l,l/2}_{-\mu}(\Omega \times \mathbb{R}_+)$, $\mu \in \mathbb{R}_+ \setminus \mathbb{Z}$, $l \in \mathbb{N}_0$, $u_0 = 0$, $S \in C^{l+2}$. Then there exists a solution $u \in H^{l+2,l/2+1}_{-\mu,\gamma}(\Omega \times \mathbb{R}_+)$ to problem* (1.1) *such that*

$$\|u\|_{H^{l+2,l/2+1}_{-\mu,\gamma}(\Omega\times_+)} \le c\|f\|_{H^{l,l/2}_{-\mu,\gamma}(\Omega\times\mathbb{R}_+)}. \tag{5.19}$$

Proof. We prove the lemma recurrently. Take $l = 1$. From Lemma 5.3 we see that the r.h.s. of (5.11) and (5.12) belong to $H^{1,1/2}_{-\mu,\gamma}(\Omega \times \mathbb{R}_+)$. Then $u \in H^{3,3/2}_{-\mu,\gamma}(\Omega \times \mathbb{R}_+)$ and (5.19) holds for $l = 1$. To prove the existence we apply the regularizer technique with $H(\Omega) = H^{1,1/2}_{-\mu,\gamma}(\Omega \times \mathbb{R}_+)$ and $V(\Omega) = H^{3,3/2}_{-\mu,\gamma}(\Omega \times \mathbb{R}_+)$. This yields the existence of solutions in $H^{3,3/2}_{-\mu,\gamma}(\Omega \times \mathbb{R}_+)$ to problem (1.1) with $u_0 = 0$.

Take $l = 2$. Then $u \in H^{3,3/2}_{-\mu,\gamma}(\Omega \times \mathbb{R}_+)$ implies that the r.h.s. of (5.11) and (5.12) belong to $H^{2,1}_{-\mu,\gamma}(\Omega \times \mathbb{R}_+)$. Hence we can repeat the above considerations with $H(\Omega) = H^{2,1}_{-\mu,\gamma}(\Omega \times \mathbb{R}_+)$ and $V(\Omega) = H^{4,2}_{-\mu,\gamma}(\Omega \times \mathbb{R}_+)$.

Continuing the considerations we prove the lemma. This concludes the proof.

6. Existence of solutions to problem (1.1). In this section we prove the existence of solutions to problem (1.1). Our aim is to prove Theorem 1.2. We use [6].

Assume that $f \in H^{l,l/2}_{-\mu}(\Omega\times(0,T))$, $\mu \in \mathbb{R}_+\setminus\mathbb{Z}$. Moreover, $u_0 \in H^{l+1}_{-\mu}(\Omega)$. Then there exists a function $\tilde u_0 \in H^{l+2,l/2+1}_{-\mu}(\Omega\times(0,T))$ such that $\tilde u_0|_S = 0$ and

$$\|\tilde u_0\|_{H^{l+2,l/2+1}_{-\mu}(\Omega\times(0,T))} \le c\|u_0\|_{H^{l+1}_{-\mu}(\Omega)} \tag{6.1}$$

Then problem (1.1) is transformed into (1.12), where $g \in H^{l,l/2}_{-\mu}(\Omega\times(0,T))$ and $v = u - \tilde u_0$. Since the compatibility conditions (1.13) are satisfied we have $g \in H^{l,l/2}_{-\mu,0}(\Omega\times(0,T))$ and

$$\|g\|_{H^{l,l/2}_{-\mu,0}(\Omega\times(0,T))} \le c\|g\|_{H^{l,l/2}_{-\mu}(\Omega\times(0,T))}. \tag{6.2}$$

For T finite the norms of $H^{l,l/2}_{-\mu,\gamma}(\Omega\times(0,T))$ and $H^{l,l/2}_{-\mu,0}(\Omega\times(0,T))$ are equivalent and

$$\|g\|_{H^{l,l/2}_{-\mu,\gamma}(\Omega\times(0,T))} \le c\|g\|_{H^{l,l/2}_{-\mu,0}(\Omega\times(0,T))}. \tag{6.3}$$

Extending g for $t > T$ and by zero for $t < 0$ we obtain

$$\|g\|_{H^{l,l/2}_{-\mu,\gamma}(\Omega\times\mathbb{R})} \le c\|g\|_{H^{l,l/2}_{-\mu,\gamma}(\Omega\times(0,T))}, \tag{6.4}$$

where the extended function is also denoted by g.

Lemma 5.4 yields the existence of a solution $v \in H^{l+2,l/2+1}_{-\mu,\gamma}(\Omega\times\mathbb{R})$ to problem (1.12) satisfying the estimate

$$\begin{aligned}\|v\|_{H^{l+2,l/2+1}_{-\mu,\gamma}(\Omega\times\mathbb{R})} &\le c\|g\|_{H^{l,l/2}_{-\mu,\gamma}(\Omega\times\mathbb{R})} \le c\|g\|_{H^{l,l/2}_{-\mu}(\Omega\times(0,T))}\\ &\le c(\|f\|_{H^{l,l/2}_{-\mu}(\Omega\times(0,T))} + \|u_0\|_{H^{l+1}_{-\mu}(\Omega)}).\end{aligned} \tag{6.5}$$

Using the fact that

$$\begin{aligned}\|v\|_{H^{l+2,l/2+1}_{-\mu,\gamma}(\Omega\times\mathbb{R})} &\ge c\|v\|_{H^{l+2,l/2+1}_{-\mu,\gamma}(\Omega,(-\infty,T))} \ge c\|v\|_{H^{l+2,l/2+1}_{-\mu,0}(\Omega\times(0,T))}\\ &\ge c\|v\|_{H^{l+2,l/2+1}_{-\mu}(\Omega\times(0,T))}\end{aligned}$$

and $u = v + \tilde u_0$, we obtain from (6.5) the estimate

$$\|u\|_{H^{l+2,l/2+1}_{-\mu}(\Omega\times(0,T))} \le c(\|f\|_{H^{l,l/2}_{-\mu}(\Omega\times(0,T))} + \|u_0\|_{H^{l+1}_{-\mu}(\Omega)}). \tag{6.6}$$

Since we also have the existence, we conclude the proof of Theorem 1.2.

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