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EXISTENCE FOR A CAUCHY–DIRICHLET PROBLEM FOR EVOLUTIONAL p-LAPLACIAN SYSTEMS

Abstract. We study the existence of a weak solution to a Cauchy–Dirichlet problem for evolutional p-Laplacian systems with constant coefficients and principal term only. The initial-boundary data is assumed to be a bounded weak solution of an evolutional p-Laplacian system with an L^1-function as external force. The key ingredient is the maximum principle for weak solutions.

1. Introduction. Let Ω be a bounded domain in $\mathbb{R}^m$, $m \geq 2$, with smooth boundary $\partial\Omega$, T' be a positive number and put $\Omega_{T'} = (0, T') \times \Omega$. Let $1 < p < \infty$. For a map $u : \Omega_{T'} \to \mathbb{R}^n$, $z = (t, x) = (t, x_1, \ldots, x_m)$, $u = u(z) = (u^1(z), \ldots, u^n(z))$, we consider the evolutional p-Laplacian system

$$\partial_t u^i - \sum_{\alpha,\beta=1}^{m} D_\alpha(|Du|_g^{p-2} g^{\alpha\beta} D_\beta u^i) = f^i, \quad i = 1, \ldots, n, \tag{1.1}$$

where $D_\alpha = \partial/\partial x^\alpha$, $\alpha = 1, \ldots, m$, Du is the spatial gradient of a map u, $Du = (D_\alpha u^i)$, $|Du|_g^2 = \sum_{i=1}^n \sum_{\alpha,\beta=1}^m g^{\alpha\beta} D_\alpha u^i D_\beta u^i$, f is an L^1-function defined on $\Omega_{T'}$ with values in $\mathbb{R}^n$ and $(g^{\alpha\beta})$ is a symmetric positive definite constant matrix. In particular, we assume that there exist positive constants γ, Γ such that

$$\gamma|\xi|^2 \leq g^{\alpha\beta}\xi_\alpha\xi_\beta \leq \Gamma|\xi|^2 \quad \text{for all } \xi = (\xi_\alpha) \in \mathbb{R}^m. \tag{1.2}$$

Here and in what follows, the notation $|\xi|^2 = \xi \cdot \xi = \xi_\alpha^i \xi_\alpha^i$ and $|\xi|_g^2 = g^{\alpha\beta}\xi_\alpha^i\xi_\beta^i$ is used for $\xi = (\xi_\alpha^i)$ and the summation convention over repeated indices is adopted.

As a typical example of an evolution system (1.1), we think of the negative gradient flow for the p-energy functional, defined for functions u belonging to the Sobolev space $W^{1,p}(\Omega, \mathbb{R}^n)$ by

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$$E_p(u) = \int_{\Omega} \frac{1}{p}(g^{\alpha\beta}(x,u) D_\alpha u \cdot D_\beta u)^{p/2}\, dx, \tag{1.3}$$

where $g = (g^{\alpha\beta}(x,u))$ is a symmetric matrix with C^1 entries $g^{\alpha\beta}(x,u)$ defined on $\Omega \times \mathbb{R}^n$ and satisfies the uniform ellipticity condition like (1.2) for every $x \in \Omega$ and all $u \in \mathbb{R}^n$. Then the equation which describes the negative gradient flow for (1.3) is (1.1) with constant coefficients replaced by the variable coefficients $g^{\alpha\beta}(x,u)$ and the lower order term

$$f(z) = -\frac{1}{2}\, |Du|_g^{p-2}\, \frac{dg^{\alpha\beta}}{du}(x,u) D_\alpha u \cdot D_\beta u. \tag{1.4}$$

Note that (1.4) is of pth power order growth in the gradient: for a positive constant a depending only on g,

$$|f| \le a|Du|^p. \tag{1.5}$$

It is also natural to study such systems in the class $L^\infty(\Omega_{T'}, \mathbb{R}^n) \cap L^p(0,T'; W^{1,p}(\Omega, \mathbb{R}^n))$ (see [8, Chapter II, Section 3, pp. 54–63]). For weak solutions in this class, the term (1.4) is exactly an L^1-function. We call such systems with lower order term as in (1.5) *evolutional p-Laplacian systems with critical* (or natural) *growth*.

Let $p \ge 2m/(m+2)$. Let $u \in L^\infty(\Omega_{T'}, \mathbb{R}^n)) \cap L^p(0,T'; W^{1,p}(\Omega, \mathbb{R}^n))$ be a bounded weak solution of (1.1) with $\sup_{\Omega_{T'}} |u| = M < \infty$. Let $B \subset \mathbb{R}^m$ be a domain compactly contained in Ω and $t_0 < t_0 + T < T'$ be positive numbers. Put $Q = (t_0, t_0+T) \times B$ and note that Q is compactly contained in $\Omega_{T'}$. For simplicity, we assume that $Q = (0,T) \times B$. When one studies the regularity of bounded weak solutions of evolutional p-Laplacian systems with critical growth, one needs to invoke a regularity estimate for a weak solution $v \in L^\infty(0,T; L^2(B, \mathbb{R}^n)) \cap L^p(0,T; W^{1,p}(B, \mathbb{R}^n))$ to the Cauchy–Dirichlet problem for the evolutional p-Laplacian system with constant coefficients and with principal term only,

$$\begin{aligned} \partial_t v &= D_\alpha(|Dv|_g^{p-2} g^{\alpha\beta} D_\beta v) \quad \text{in } Q, \\ v &= u \quad \text{on } \partial_p Q. \end{aligned} \tag{1.6}$$

Note that weak solutions of (1.6) satisfy interior Hölder and gradient Hölder estimates in Q. The proof of these estimates is based on Moser's and De Giorgi's iteration method and the so-called Campanato type estimates (we refer to [7, 5] and [2, 6, 8, 10] for $p = 2$); this argument is recognized to be fundamental in the regularity theory for evolutional p-Laplacian systems. On the other hand, Gehring's reverse Hölder inequality, which implies the higher integrability of the gradient, holds for "small" weak solutions of evolutional p-Laplacian systems with critical growth (see [12, 16]), where the smallness is defined, with a being the positive constant in (1.5), by

$$\sup_{\Omega_{T'}} |u| < \frac{\gamma^{p/2}}{2a}. \tag{1.7}$$

Combining Hölder regularity estimates for (1.6) with Gehring's reverse Hölder inequality, we can follow the scheme of the Campanato estimates (see [3, 15]) to establish a partial regularity of small weak solutions of evolutional p-Laplacian systems with critical growth ([16]). The equation (1.1) with (1.4) also concerns the negative gradient flow for p-harmonic maps between smooth, compact Riemannian manifolds (cf. [4, 17, 14] and, for $p = 2$, see [8, 10, 19]), and the smallness condition (1.7) implies a geometric relation between the curvature of the target manifold and the image of a solution (see [9]).

In a forthcoming paper, we will study the existence of a small weak solution to the Cauchy–Dirichlet problem for the negative gradient flow of p-harmonic maps with variational data of "small" image.

With the above motivation, and since the existence of a solution for (1.6) does not seem to be stated in the literature, we want to prove the existence of a weak solution $v \in L^\infty(0,T;L^2(B,\mathbb{R}^n)) \cap L^p(0,T;W^{1,p}(B,\mathbb{R}^n))$ to the Cauchy–Dirichlet problem (1.6). Our main theorem is the following:

THEOREM 1. *Let $u \in L^\infty(\Omega_{T'},\mathbb{R}^n) \cap L^p(0,T';W^{1,p}(\Omega,\mathbb{R}^n))$ be a bounded weak solution of (1.1) with $\sup_{\Omega_{T'}} |u| = M < \infty$. Then there exists a weak solution $v \in C([0,T];L^2(B,\mathbb{R}^n)) \cap L^p(0,T;W^{1,p}(B,\mathbb{R}^n))$ of (1.6) satisfying*

$$\sup_Q |v| \le \sup_Q |u| \tag{1.8}$$

and the energy inequality

$$\sup_{0\le t\le T} \int_{\{t\}\times B} |v|^2\,dx + \int_Q |Dv|^p\,dz \le C\Big(\sup_{0\le t\le T} \int_{\{t\}\times B} |u|^2\,dx + \int_Q (|f| + |Du|^p)\,dz\Big), \tag{1.9}$$

where the positive constant C depends only on m, p and the L^∞-norm of u.

REMARK. It can be shown that a bounded weak solution $u \in L^\infty(\Omega_{T'},\mathbb{R}^n) \cap L^p(0,T';W^{1,p}(\Omega,\mathbb{R}^n))$ of (1.1) belongs to $C([0,T];L^2(B,\mathbb{R}^n))$ (see Appendix).

First of all, note the following. Set $w = v - u$. Then we see from (1.1) that $w \in L^\infty(0,T;L^2(\Omega,\mathbb{R}^n)) \cap L^p(0,T;W^{1,p}(\Omega,\mathbb{R}^n))$ is a weak solution of the Cauchy–Dirichlet problem

$$\begin{aligned} \partial_t w &= D_\alpha(|Dw + Du|_g^{p-2} g^{\alpha\beta}(D_\beta w + D_\beta u)) \\ &\qquad - D_\alpha(|Du|_g^{p-2} g^{\alpha\beta} D_\beta u^i) - f^i \quad \text{in } Q, \\ w &= 0 \quad \text{on } \partial_p Q. \end{aligned} \tag{1.10}$$

By (1.1), the problem (1.10) is equivalent to (1.6).

2. Approximation. We consider an approximate problem for (1.6) by using the usual approximation of the initial-boundary data with the approximation parameter $\varepsilon > 0$ tending to 0. Let $\varrho = \varrho(z)$ be a smooth function in $\mathbb{R}^{m+1}$ such that $\operatorname{supp}\varrho \subset (-1,1)\times B_1(0)$ and $\int_{\mathbb{R}^{m+1}} \varrho(z)\,dz = 1$. Let $0 < \varepsilon < \min\{t_0, T' - T - t_0, \operatorname{dist}(B, \partial\Omega)\}$. Set, for any $\eta \in L^1(\Omega_{T'})$,

$$\begin{aligned}\varrho_\varepsilon * \eta(z) &= \int\limits_{\mathbb{R}^{m+1}} \varrho\biggl(\frac{z-z'}{\varepsilon}\biggr)\eta(z')\,dz' \\ &= \frac{1}{\varepsilon^{m+1}} \int\limits_{t-\varepsilon}^{t+\varepsilon} \int\limits_{B_\varepsilon(x)} \varrho\biggl(\frac{t-t'}{\varepsilon}, \frac{x-x'}{\varepsilon}\biggr)\eta(t',x')\,dx'\,dt', \qquad z \in Q.\end{aligned}$$

Since Q is compactly contained in $\Omega_{T'}$, we know that $\varrho_\varepsilon * \eta$ is smooth in $\overline{Q}$ and that $\varrho_\varepsilon * \eta$ strongly converges to η in $L^1(Q)$ as $\varepsilon \searrow 0$. Then we consider the problem

$$\begin{aligned}\partial_t v_\varepsilon &= D_\alpha(|Dv_\varepsilon|_g^{p-2} g^{\alpha\beta} D_\beta v_\varepsilon) \quad \text{in } Q, \\ v_\varepsilon &= \varrho_\varepsilon * u \quad \text{on } \partial_p Q,\end{aligned} \tag{2.1}$$

which we call the approximate problem for (1.6). Put $w_\varepsilon = v_\varepsilon - \varrho_\varepsilon * u$. Then (2.1) is seen to be equivalent to the problem

$$\begin{aligned}\partial_t w_\varepsilon &= D_\alpha(|Dw_\varepsilon + D\varrho_\varepsilon * u|_g^{p-2} g^{\alpha\beta}(D_\beta w_\varepsilon + D_\beta \varrho_\varepsilon * u)) \\ &\qquad - \partial_t \varrho_\varepsilon * u \quad \text{in } Q, \\ w_\varepsilon &= 0 \quad \text{on } \partial_p Q.\end{aligned} \tag{2.2}$$

By (1.1),

$$\partial_t \varrho_\varepsilon * u^i - D_\alpha \varrho_\varepsilon * (|Du|_g^{p-2} g^{\alpha\beta} D_\beta u^i) = \varrho_\varepsilon * f^i, \qquad i = 1, \ldots, n. \tag{2.3}$$

Substituting (2.3) into (2.2) shows that (2.2) is equivalent to

$$\begin{aligned}\partial_t w_\varepsilon &= D_\alpha(|Dw_\varepsilon + D\varrho_\varepsilon * u|_g^{p-2} g^{\alpha\beta}(D_\beta w_\varepsilon + D_\beta \varrho_\varepsilon * u)) \\ &\qquad - D_\alpha \varrho_\varepsilon * (|Du|_g^{p-2} g^{\alpha\beta} D_\beta u) - \varrho_\varepsilon * f \quad \text{in } Q, \\ w_\varepsilon &= 0 \quad \text{on } \partial_p Q,\end{aligned} \tag{2.4}$$

which we call the approximate problem for (1.10). We apply the Galerkin approximation to construct a solution of (2.2). We follow the argument in the proof of [13, Theorem 6.7, pp. 466–475] (see also [11, Sec. 2–4, pp. 499–505]). Let $\{\phi_i\}$, $i = 1, 2, \ldots$, be a system of smooth maps with compact supports in B, which is a fundamental system, dense in $W_0^{1,p}(B, \mathbb{R}^n)$. Since $p \geq 2m/(m+2)$, we may assume that $\{\phi_i\}$ is orthonormal in $L^2(B, \mathbb{R}^n)$. For any positive integer l, we seek the Galerkin approximation for (2.2),

$$w_{\varepsilon,l} = w_{\varepsilon,l}(t,x) = \sum_{i=1}^{l} a_{il}(t)\phi_i(x),$$

such that the unknown real-valued functions $a_{il} = a_{il}(t)$, $i = 1, \ldots, l$, are determined from a system of nonlinear ordinary differential equations

$$
\begin{aligned}
\frac{da_{jl}}{dt} &= (\partial_t w_{\varepsilon,l}, \phi_j) \\
&= -\,(|Dw_{\varepsilon,l} + D\varrho_\varepsilon * u|_g^{p-2} g^{\alpha\beta}(D_\beta w_{\varepsilon,l} + D_\beta \varrho_\varepsilon * u), D_\alpha \phi_j)_{p^*,p} \\
&\quad - (\partial_t \varrho_\varepsilon * u, \phi_j), \\
a_{jl}(0) &= (w_{\varepsilon,l}(0), \phi_j) = 0, \qquad j = 1, \ldots, l,
\end{aligned}
\tag{2.5}
$$

where p^* is the dual exponent of p and $(\cdot,\cdot)_{p^*,p}$ denotes the dual product in $L^p(B,\mathbb{R}^n)$, which is abbreviated to $(\cdot,\cdot)$ if $p = 2$. Note that the right hand side in (2.5) is continuous with respect to the variable $Dw_{\varepsilon,l}$, and that $\varrho_\varepsilon * u$ is a smooth map in $\overline{Q}$. Thus the well known Peano theorem guarantees that there exists at least one time-local solution of (2.5). To extend it to a time-global one, we need an energy inequality for $w_{\varepsilon,l}$. Multiply the jth equation of (2.5) by $a_{jl}(t)$, sum the resulting equalities over j from 1 to l and then integrate the result over $(0,s)$ for any positive $s \le T$ to obtain

$$
\begin{aligned}
\int_0^s \frac{d}{dt} \sum_{j=1}^{l} \frac{1}{2} |a_{jl}(t)|^2 \, dt &= \frac{1}{2} \sum_{j=1}^{l} |a_{jl}(s)|^2 = \int_B \frac{1}{2} |w_{\varepsilon,l}(s)|^2 \, dx \\
= \int_{(0,s)\times B} &(-|Dw_{\varepsilon,l} + D\varrho_\varepsilon * u|_g^{p-2} g^{\alpha\beta}(D_\beta w_{\varepsilon,l} + D_\beta \varrho_\varepsilon * u) \cdot D_\alpha w_{\varepsilon,l} \\
&\qquad - \partial_t \varrho_\varepsilon * u \cdot w_{\varepsilon,l}) \, dz.
\end{aligned}
$$

Note that the external force $\partial_t \varrho_\varepsilon * u$ is a smooth map in $\overline{Q}$, and thus the usual estimation with Hölder's and Young's inequalities yields, for all $0 < s \le T$,

$$
\begin{aligned}
\sum_{j=1}^{l} |a_{jl}(s)|^2 &\le \int_B |w_{\varepsilon,l}(s)|^2 \, dx + \int_{(0,s)\times B} |Dw_{\varepsilon,l} + D\varrho_\varepsilon * u|^p \, dz \\
&\le C \int_{(0,s)\times B} (|\partial_t \varrho_\varepsilon * u|^{p/(p-1)} + |D\varrho_\varepsilon * u|^p) \, dz,
\end{aligned}
\tag{2.6}
$$

where to estimate the integral term containing $\partial_t \varrho_\varepsilon * u$ we use the Poincaré inequality for functions in $W_0^{1,p}(B,\mathbb{R}^n)$. By (2.6), we have the boundedness of $\sum_{j=1}^{l} |a_{jl}(T)|^2$. Then we can solve (2.5) with initial value $a_{jl}(T)$ to have a solution of (2.5) in $[0, T+\delta]$ for some positive number δ. Repeat the argument by the continuity method to get a time-global solution $w_{\varepsilon,l}$ of (2.5) (for the details, see [11, Sec. 3, 501–503]). From (2.6), we also obtain the boundedness of $\{w_{\varepsilon,l}\}$ in the space $L^\infty(0,T;L^2(B,\mathbb{R}^n)) \cap L^p(0,T;W^{1,p}(B,\mathbb{R}^n))$ and of $\{|Dw_{\varepsilon,l} + D\varrho_\varepsilon * u|_{\overline{g}}^{p-2} \overline{g}^{\alpha\beta}(D_\beta w_{\varepsilon,l} + D_\beta \varrho_\varepsilon * u)\}$ in $L^{p^*}(Q,\mathbb{R}^{mn})$. Thus, there is a subsequence $\{w_{\varepsilon,l}\}$, a limit map $w \in L^\infty(0,T;L^2(B,\mathbb{R}^n)) \cap L^p(0,T;W^{1,p}(B,\mathbb{R}^n))$ and $\sigma_\varepsilon = ((\sigma_\varepsilon)_\alpha^i) \in L^{p^*}(Q,\mathbb{R}^{mn})$

such that

$$w_{\varepsilon,l} \to w_\varepsilon \quad \text{weak}^* \text{ in } L^\infty(0,T;L^2(B,\mathbb{R}^n)) \text{ and weakly in } L^p(0,T;W^{1,p}(B,\mathbb{R}^n)), \tag{2.7}$$

$$|Dw_{\varepsilon,l} + D\varrho_\varepsilon * u|_g^{p-2} g^{\alpha\beta}(D_\beta w_{\varepsilon,l} + D_\beta \varrho_\varepsilon * u) \to \sigma_\varepsilon \quad \text{weakly in } L^{p^*}(Q,\mathbb{R}^{mn}). \tag{2.8}$$

Now we find that

$$\int_Q (w_\varepsilon \cdot \partial_t \phi + \sigma_\varepsilon \cdot D\phi + \partial_t \varrho_\varepsilon * u \cdot \phi)\, dz = 0 \tag{2.9}$$

for any smooth map ϕ with values in $\mathbb{R}^n$ and compact support in Q. In fact letting $l \nearrow \infty$ in (2.5) and using (2.8) yields (2.9). For the details, we refer to Appendix (see also [13, p. 470]). We note that $w_{\varepsilon,l}$ satisfies the zero initial and boundary conditions and use the strong convergence result in [1, Lemma 4.2, pp. 591–592] with [18, Corollary 4, Section 8, pp. 84–86] to apply the "monotonicity trick" of [4, Lemma 1.1, Corollary 1.3, pp. 27–28]. For the details, see the arguments below ([13, pp. 471–472]). Thus we obtain the existence result for (2.2).

Lemma 2. *For any sufficiently small positive number ε, there exists a weak solution $w_\varepsilon \in L^\infty(0,T;L^2(B,\mathbb{R}^n)) \cap L^p(0,T;W_0^{1,p}(B,\mathbb{R}^n))$ of* (2.2) *such that*

$$|w_\varepsilon(t)|_{L^2(B)} \to 0 \quad \text{as } t \searrow 0. \tag{2.10}$$

Let $v_\varepsilon = w_\varepsilon + \varrho_\varepsilon * u$. Then $v_\varepsilon \in L^\infty(0,T;L^2(B,\mathbb{R}^n)) \cap L^p(0,T;W^{1,p}(B,\mathbb{R}^n))$ is a weak solution of (2.1) *such that $v_\varepsilon(t) = \varrho_\varepsilon * u(t)$ on ∂B in the trace sense in $W^{1,p}(B,\mathbb{R}^n)$ for almost every $t \in (0,T)$ and $|v_\varepsilon(t) - \varrho_\varepsilon * u(t)|_{L^2(B)} \to 0$ as $t \searrow 0$.*

Now we state a maximum principle for weak solutions of (2.1), which plays a fundamental role in the limiting process.

Lemma 3. *Let $v_\varepsilon \in L^\infty(0,T;L^2(B,\mathbb{R}^n)) \cap L^p(0,T;W^{1,p}(B,\mathbb{R}^n))$ be a weak solution of* (2.1) *as in Lemma* 2. *Then*

$$\sup_Q |v_\varepsilon| \le \sup_Q |\varrho_\varepsilon * u| \le \sup_Q |u|, \tag{2.11}$$

and so for $w_\varepsilon = v_\varepsilon - \varrho_\varepsilon * u$ we have

$$\sup_Q |w_\varepsilon| \le 2 \sup_Q |u|. \tag{2.12}$$

Proof. Let $k_0 = \sup_Q |\varrho_\varepsilon * u|^2$ and $v = v_\varepsilon$, for brevity. Fix $t \in (0,T]$ and let $0 < h < \min\{2^{-1}t, T-t\}$. Let σ_h be a real-valued Lipschitz function defined on $(-\infty,\infty)$ such that $\sigma_h = 1$ in $[2h,t]$ and $\operatorname{supp}\sigma_h = [h, t+h]$. Then the usual regularization argument (using the Steklov averages on the time variable; for the details, see the proof of Lemma 2.15 below) shows that

$\sigma_h v((|v|^2)_L - k_0)^+$ is an admissible test function in (1.6), where $L > k_0$, $(f)_L = \min\{f, L\}$ and $(f)^+ = \max\{f, 0\}$. Use this test function in (2.1) to obtain

$$\int_Q \{ - ((|v|^2)_L - k_0)^+ (2\left(|v|^2 - k_0\right)^+ - ((|v|^2)_L - k_0)^+)\partial_t \sigma_h \\ + \sigma_h |Dv|_g^{p-2}(4((|v|^2)_L - k_0)^+ g^{\alpha\beta} D_\beta v \cdot D_\alpha v \\ + 2g^{\alpha\beta} D_\beta ((|v|^2)_L - k_0)^+ D_\alpha ((|v|^2)_L - k_0)^+)\}\, dz = 0. \tag{2.13}$$

Let $h \searrow 0$ and $L \nearrow \infty$ in (2.13) to get

$$\int_{\{t\}\times B_R} ((|v|^2 - k_0)^+)^2\, dx = 0 \quad \text{for almost all } t \in (0, T). \tag{2.14}$$

The assertion (2.11) follows from (2.14) immediately.

We state an energy inequality for weak solutions w_ε, which also plays a crucial role in the limiting process.

LEMMA 4. *Let $w_\varepsilon \in L^\infty(0, T; L^2(B, \mathbb{R}^n)) \cap L^p(0, T; W_0^{1,p}(B, \mathbb{R}^n))$ be a weak solution of* (2.4) *satisfying* (2.12). *Then there exists a positive constant C depending only on m, p and the L^∞-norm of u such that*

$$\int_{\{s\}\times B} |w_\varepsilon|^2\, dx + \int_Q |Dw_\varepsilon + D\varrho_\varepsilon * u|^p\, dz \\ \le C \int_{(0,s)\times B} (|\varrho_\varepsilon * f| + |D\varrho_\varepsilon * u|^p)\, dz \le C \int_{\Omega_{T'}} (|f| + |Du|^p)\, dz \tag{2.15}$$

for almost every $s \in (0, T)$.

Proof. For an integrable function η in $(0, T)$, denote its Steklov average by

$$(\eta)_\delta(s) = \frac{1}{\delta} \int_s^{s+\delta} \eta(t)\, dt \quad \text{for } s \in (0, T) \text{ and } 0 < \delta < \min\{s, T - s\}.$$

Let $s', s \in [0, T]$, $s' < s$. Let $0 < h < 2^{-1} \min\{s', T - s\}$. Let σ_h be a real-valued Lipschitz function defined on $(-\infty, \infty)$ such that $\sigma_h = 1$ in $[s', s]$ and $\operatorname{supp} \sigma_h = [s' - h, s + h]$. Also, let $0 < \delta < \min\{h, T - s - h\}$. For brevity, we put $w = w_\varepsilon$. Note that $\sup_Q |w| \le 2 \sup_Q |u|$, and use the test function $(\sigma_h (w)_\delta)_{-\delta}$ in (2.4) to obtain

$$\frac{1}{h}\Big(\int_s^{s+h} - \int_{s'}^{s'+h} \Big) \int_B |(w)_\delta(t)|^2\, dx\, dt \\ + \int_Q 2\sigma_h ((|Dw + D\varrho_\varepsilon * u|_g^{p-2} g^{\alpha\beta}(D_\beta w + D_\beta \varrho_\varepsilon * u))_\delta \cdot D_\alpha (w)_\delta \\ - (\varrho_\varepsilon * (|Du|_g^{p-2} g^{\alpha\beta} D_\beta u))_\delta \cdot D_\alpha (w)_\delta + (\varrho_\varepsilon * f)_\delta \cdot (w)_\delta)\, dz = 0; \tag{2.16}$$

note that $\operatorname{supp}(\sigma_h(w)_\delta)_{-\delta} \subset (0,T)$. First let $\delta \searrow 0$ in (2.16). Use the strong convergence of $(w)_\delta$ and of its derivatives in $L^p_{\mathrm{loc}}(0,T;L^p(B,\mathbb{R}^n))$, and again the boundedness (2.12), to obtain

$$\begin{aligned}(2.17)\quad & \frac{1}{h}\Big(\int_s^{s+h} - \int_{s'}^{s'+h}\Big)\int_B |w(t)|^2\,dx\,dt \\ &+ 2\int_Q \sigma_h(|Dw + D\varrho_\varepsilon * u|_g^{p-2} g^{\alpha\beta}(D_\beta w + D_\beta\varrho_\varepsilon * u)\cdot D_\alpha w \\ &\qquad - \varrho_\varepsilon * (|Du|_g^{p-2} g^{\alpha\beta} D_\beta u)\cdot D_\alpha w + \varrho_\varepsilon * f\cdot w)\,dz = 0.\end{aligned}$$

Using the pointwise convergence of the Steklov averages in t almost everywhere in $(0,T)$ in the first term of (2.17), let $h \searrow 0$ in (2.17) to obtain

$$\begin{aligned}(2.18)\quad & \int_B |w(s)|^2\,dx - \int_B |w(s')|^2\,dx \\ &+ 2\int_{(s',s)\times B} (|Dw + D\varrho_\varepsilon * u|_{\overline{g}}^{p-2} \overline{g}^{\alpha\beta}(D_\beta w + D_\beta\varrho_\varepsilon * u)\cdot D_\alpha w \\ &\qquad - \varrho_\varepsilon * (|Du|_g^{p-2} g^{\alpha\beta} D_\beta u)\cdot D_\alpha w + \varrho_\varepsilon * f\cdot w)\,dz = 0.\end{aligned}$$

Applying (2.10), let $s' \searrow 0$ in (2.18). Use (2.12) and make a routine estimation with Hölder's and Young's inequalities to arrive at (2.15).

3. Passage to the limit. In this section, we study the convergence of the solutions w_ε of (2.4) to construct a weak solution of (1.10). The energy inequality (2.15) gives the boundedness of $\{w_\varepsilon\}$ in $L^\infty(Q)\cap L^p(0,T;W^{1,p}(B,\mathbb{R}^n))$ and of $\{|Dw_\varepsilon + D\varrho_\varepsilon * u|_g^{p-2} g^{\alpha\beta}(D_\beta w_\varepsilon + D_\beta\varrho_\varepsilon * u)\}$ in $L^{p^*}(Q,\mathbb{R}^{mn})$. Thus, there are a subsequence $\{w_\varepsilon\}$ and limit maps $w \in L^\infty(0,T;L^2(B,\mathbb{R}^n))\cap L^p(0,T;W^{1,p}(B,\mathbb{R}^n))$ and $\chi = (\chi^i_\alpha) \in L^{p^*}(Q,\mathbb{R}^{mn})$ such that, as $\varepsilon \searrow 0$,

$$\begin{aligned}(3.1)\quad & w_\varepsilon \to w \quad \text{weak}^* \text{ in } L^\infty(0,T;L^2(B,\mathbb{R}^n)) \\ &\qquad\qquad \text{and weakly in } L^p(0,T;W^{1,p}(B,\mathbb{R}^n)),\end{aligned}$$

$$(3.2)\quad |Dw_\varepsilon + D\varrho_\varepsilon * u|_g^{p-2} g^{\alpha\beta}(D_\beta w_\varepsilon + D_\beta\varrho_\varepsilon * u) \to \chi \text{ weakly in } L^{p^*}(Q,\mathbb{R}^{mn}).$$

For any $\phi \in L^p(0,T;W^{1,p}(B,\mathbb{R}^n))$, define $\mathcal{A}\phi \in L^{p^*}(0,T;W^{-1,p^*}(B,\mathbb{R}^n))$ by

$$(3.3)\qquad \mathcal{A}\phi = -D_\alpha(|D\phi|_g^{p-2} g^{\alpha\beta} D_\beta\phi),$$

$$(3.4)\qquad \langle \mathcal{A}\phi,\eta\rangle = \int_Q |D\phi|_g^{p-2} g^{\alpha\beta} D_\beta\phi\cdot D_\alpha\eta\,dz \quad \text{for all } \eta \in L^p(0,T;W^{1,p}(B,\mathbb{R}^n)).$$

For any $\mathcal{T} \in L^{p^*}(0,T;W^{-1,p^*}(B,\mathbb{R}^n))$, write $\langle \mathcal{T},\eta\rangle$ for all $\eta \in L^p(0,T;W^{1,p}(B,\mathbb{R}^n))$.

Now we find that

$$\int_Q (-w \cdot \partial_t \phi + \chi \cdot D\phi - |Du|_g^{p-2} g^{\alpha\beta} D_\beta u \cdot D_\alpha \phi + f \cdot \phi)\, dz = 0 \tag{3.5}$$

for any smooth map ϕ with values in $\mathbb{R}^n$ and compact support in Q. In fact, we use a test function ϕ in the weak form of (2.2) and the weak convergence in (3.1) and (3.2) in the resulting equality to deduce (3.5).

To pass to the limit in the weak form of the p-Laplace term in (2.2), we use the monotonicity trick of [4, Lemma 1.1, Corollary 1.3, pp. 27–28]. For that purpose, we need more compactness of weak solutions $\{w_\varepsilon\}$ to (2.2).

LEMMA 5. *There exists a subsequence $\{w_\varepsilon\}$ such that, as $\varepsilon \searrow 0$,*

$$w_\varepsilon \to w \quad \text{strongly in } L^q(Q, \mathbb{R}^n) \text{ for any } q \geq 1, \tag{3.6}$$

$$w_\varepsilon \to w \quad \text{almost everywhere in } Q, \tag{3.7}$$

$$\sup_Q |w| \leq 2 \sup_Q |u|. \tag{3.8}$$

Proof. Since $f \in L^1(\Omega_{T'})$ and $\overline{Q} \subset \Omega_{T'}$, the family $\{\varrho_\varepsilon * f\}$ is a bounded sequence in $L^1(Q)$, and thus bounded in the space of Radon measures, which is the dual to the space of continuous functions with compact support in Q. Here we use the strong convergence result of [1, Lemma 4.2, pp. 591–592], which is an extension of Rellich's theorem to the evolution case. Now note that $w_\varepsilon = 0$ on $\partial_p Q$. In the proof of [1, Lemma 4.2, pp. 591–592], without the cut-off functions $\psi = \psi(x)$ and $\eta = \eta(t)$, replace the approximation parameters n, v_n, α_n and β_n by ε, w_ε, the family of the p-Laplace operators in (2.2) and the maps $\varrho_\varepsilon * f$, respectively, and argue similarly to [1, Lemma 4.2, pp. 591–592]. Note the compact embedding of $W_0^{1,p}(B, \mathbb{R}^n)$ into $L^p(B, \mathbb{R}^n)$ to apply the compactness result of [18, Corollary 4, Section 8, pp. 84–86]. Thus, we have (3.6). Then, in the usual way, we have a subsequence satisfying (3.7), which satisfies (3.8) by (2.11).

Now we can state the following convergence:

LEMMA 6.

$$\mathcal{A}(w_\varepsilon + \varrho_\varepsilon * u) \to -D \cdot \chi = \mathcal{A}(w + u) \quad \textit{weakly in } L^{p^*}(0, T; W^{-1,p^*}(B, \mathbb{R}^n)). \tag{3.9}$$

Proof. Recall that the following algebraic inequalities hold for any $P = (P_\alpha^i)$, $Q = (Q_\alpha^i) \in \mathbb{R}^{mn}$ with a positive constant C depending only on p, γ and Γ (see [17, 14]): for $p \geq 2$,

$$g^{\alpha\beta}(|P|_g^{p-2} P_\alpha - |Q|_g^{p-2} Q_\alpha) \cdot (P_\beta - Q_\beta) \geq C|P - Q|^p, \tag{3.10}$$

$$|(|P|_g^{p-2} g^{\alpha\beta} P_\beta - |Q|_g^{p-2} g^{\alpha\beta} Q_\beta)| \leq C|P - Q|^2(|P| + |Q|)^{p-2}, \tag{3.11}$$

and for $1 < p < 2$,

$$g^{\alpha\beta}(|P|_g^{p-2}P_\alpha - |Q|_g^{p-2}Q_\alpha) \cdot (P_\beta - Q_\beta) \geq C|P-Q|^2(|P|+|Q|)^{p-2}, \tag{3.12}$$

$$|(|P|_g^{p-2}g^{\alpha\beta}P_\beta - |Q|_g^{p-2}g^{\alpha\beta}Q_\beta)| \leq C|P-Q|^{p-1}. \tag{3.13}$$

It follows from (3.10) and (3.12) that $\mathcal{A}$ is a monotone operator, that is,

$$\langle \mathcal{A}\eta - \mathcal{A}\zeta, \eta - \zeta\rangle \geq 0 \quad \text{for all } \eta, \zeta \in L^p(0,T; W^{1,p}(B,\mathbb{R}^n)).$$

By (3.10) and (3.12), we find that the operator $\mathcal{A}$ is hemicontinuous, that is, the function $s \mapsto \langle \mathcal{A}(\eta + s\zeta), \phi\rangle$ is continuous on $s \in [0,T]$ for all $\eta, \zeta, \phi \in L^p(0,T; W^{1,p}(B,\mathbb{R}^n))$. Now we prove that

$$\limsup_{\varepsilon \searrow 0} \langle \mathcal{A}(w_\varepsilon + \varrho_\varepsilon * u), w_\varepsilon + \varrho_\varepsilon * u\rangle \leq \langle -D\cdot\chi, w + u\rangle. \tag{3.14}$$

For this purpose, we first claim that, as $\varepsilon \searrow 0$,

$$w_\varepsilon(T) \to w(T) \quad \text{weakly in } L^2(B,\mathbb{R}^n). \tag{3.15}$$

For the proof of (3.15), see Appendix. Use (2.4), (3.5), (3.15), (2.10) and (2.12) to find subsequences $\{w_\varepsilon\}$, $\{\varrho_\varepsilon * u\}$ and $\{\varrho_\varepsilon * f\}$ such that

$$\begin{aligned}
&\limsup_{\varepsilon \searrow 0} \langle \mathcal{A}(w_\varepsilon + \varrho_\varepsilon * u), w_\varepsilon\rangle \\
&= \limsup_{\varepsilon \searrow 0} \int_Q |Dw_\varepsilon + D\varrho_\varepsilon * u|_g^{p-2} g^{\alpha\beta}(D_\beta w_\varepsilon + D_\beta \varrho_\varepsilon * u)\cdot D_\alpha w_\varepsilon \, dz \\
&= \limsup_{\varepsilon \searrow 0} \Big(\int_B (-|w_\varepsilon(T)|^2 + |w_\varepsilon(0)|^2)\, dx \\
&\quad + \int_Q (\varrho_\varepsilon * (|Du|_g^{p-2} g^{\alpha\beta} D_\beta u^i)\cdot D_\alpha w_\varepsilon - \varrho_\varepsilon * f \cdot w_\varepsilon)\, dz \Big) \\
&\leq -\liminf_{\varepsilon \searrow 0} \int_B |w_\varepsilon(T)|^2\, dx \\
&\quad + \lim_{\varepsilon \searrow 0} \int_Q (\varrho_\varepsilon * (|Du|_g^{p-2} g^{\alpha\beta} D_\beta u^i)\cdot D_\alpha w_\varepsilon - \varrho_\varepsilon * f \cdot w_\varepsilon)\, dz \\
&\leq -\int_B |w(T)|^2\, dx + \int_Q (|Du|_g^{p-2} g^{\alpha\beta} D_\beta u^i \cdot D_\alpha w - f\cdot w)\, dz \\
&= \langle -D\cdot\chi, w\rangle.
\end{aligned} \tag{3.16}$$

Note the following fact: If $\zeta_\varepsilon \to \zeta$ strongly in $L^p(Q,\mathbb{R}^n)$ and $\eta_\varepsilon \to \eta$ weakly in $L^{p^*}(Q,\mathbb{R}^n)$ as $\varepsilon \searrow 0$, then

$$\int_Q \eta_\varepsilon \cdot \zeta_\varepsilon \, dz \to \int_Q \eta\cdot\zeta\, dz \quad \text{as } \varepsilon \searrow 0. \tag{3.17}$$

Also, we use (2.12), (3.7) and the strong convergence of $\varrho_\varepsilon * f$ to f in $L^1(Q,\mathbb{R}^n)$ as $\varepsilon \searrow 0$ and then apply the Lebesgue convergence theorem to choose a subsequence $\{w_\varepsilon\}$ such that

$$\int_Q f\cdot w_\varepsilon\,dz \to \int_Q f\cdot w\,dz \quad \text{as } \varepsilon\searrow 0. \tag{3.18}$$

Then we have, by (3.6) and (3.18),

$$\begin{aligned}\int_Q \varrho_\varepsilon * f\cdot w_\varepsilon\,dz &= \int_Q (\varrho_\varepsilon * f - f)\cdot w_\varepsilon\,dz + \int_Q f\cdot w_\varepsilon\,dz \\ &\to \int_Q f\cdot w\,dz \quad \text{as } \varepsilon\searrow 0.\end{aligned} \tag{3.19}$$

Again, from (3.17) with (3.2), we obtain

$$\begin{aligned}\lim_{\varepsilon\searrow 0}\int_Q |Dw_\varepsilon + D\varrho_\varepsilon * u|_g^{p-2} g^{\alpha\beta}(D_\beta w_\varepsilon + D_\beta\varrho_\varepsilon * u)\cdot D_\alpha\varrho_\varepsilon * u\,dz \\ = \int_Q \chi\cdot D_\alpha u\,dz.\end{aligned} \tag{3.20}$$

Combine (3.20) with (3.16) to obtain

$$\begin{aligned}&\limsup_{\varepsilon\searrow 0}\,\langle\mathcal{A}(w_\varepsilon+\varrho_\varepsilon * u),\, w_\varepsilon+\varrho_\varepsilon * u\rangle \\ &\quad\le \limsup_{\varepsilon\searrow 0}\,\langle\mathcal{A}(w_\varepsilon+\varrho_\varepsilon * u), w_\varepsilon\rangle + \limsup_{\varepsilon\searrow 0}\,\langle\mathcal{A}(w_\varepsilon+\varrho_\varepsilon * u), \varrho_\varepsilon * u\rangle \\ &\quad\le \langle -D\cdot\chi, w+u\rangle,\end{aligned} \tag{3.21}$$

which actually gives (3.14).

From the monotonicity and hemicontinuity of the operator $\mathcal{A}$ and (3.14), we can apply the monotonicity trick of [4, Lemma 1.1, Corollary 1.3, pp. 27–28] to arrive at the assertion (3.9).

Finally, we show that $w \in L^\infty(Q,\mathbb{R}^n)\cap L^p(0,T;W^{1,p}(B,\mathbb{R}^n))$ is actually a weak solution of (1.10). Let ϕ be a smooth map defined on B with values in $\mathbb{R}^n$ and compact support in Q. Use a test function ϕ in (2.4) to obtain

$$\begin{aligned}\int_Q (w_\varepsilon\cdot\partial_t\phi + |Dw_\varepsilon + D\varrho_\varepsilon * u|_g^{p-2} g^{\alpha\beta}(D_\beta w_\varepsilon + D_\beta\varrho_\varepsilon * u)\cdot D_\alpha\phi)\,dz \\ = \int_Q (\varrho_\varepsilon * (|Du|_g^{p-2} g^{\alpha\beta}D_\beta u)\cdot D_\alpha\phi - \varrho_\varepsilon * f\cdot\phi)\,dz.\end{aligned}$$

Now we use the convergence results (3.1) and (3.9) and let $\varepsilon\searrow 0$ to conclude that

$$\begin{aligned}\int_Q (w\cdot\partial_t\phi + |Dw + Du|_g^{p-2} g^{\alpha\beta}(D_\beta w + D_\beta u)\cdot D_\alpha\phi)\,dz \\ = \int_Q (|Du|_g^{p-2} g^{\alpha\beta}D_\beta u\cdot D_\alpha\phi - f\cdot\phi)\,dz.\end{aligned} \tag{3.22}$$

Appendix. Here we prove (2.10) and (3.15). Fix $s \in (0, T]$. Since $\{w_{\varepsilon,l}\}$ is bounded in $L^\infty(0, T; L^2(B, \mathbb{R}^n))$ by the energy inequality (2.6), there are a subsequence $\{w_{\varepsilon,l}\}$ and a limit function $\omega \in L^2(B, \mathbb{R}^n)$ such that

$$w_{\varepsilon,l}(s) \to \omega \quad \text{weakly in } L^2(B, \mathbb{R}^n) \text{ as } l \nearrow \infty. \tag{3.23}$$

We have to show that $\omega = w_\varepsilon(T)$ almost everywhere in B.

First, we prove that

$$\int_B \omega \cdot \phi(s)\, dx = \int_{(0,s)\times B} (w_\varepsilon \cdot \partial_t \phi - \chi \cdot D\phi - \partial_t \varrho_\varepsilon * u \cdot \phi)\, dz \tag{3.24}$$

for any $\phi \in C^\infty([0, s]; C_0^\infty(B, \mathbb{R}^n))$. Consider the expansion

$$\phi(t, x) = \sum_{j=1}^{\infty} c_j(t)\phi_j(x), \qquad c_j(t) = (\phi(t), \phi_j)_{L^2(B,\mathbb{R}^n)},$$

strongly convergent in $L^p(0, T; W_0^{1,p}(B, \mathbb{R}^n))$, where $\{\phi_j\}$ is a fundamental system dense in $L^p(0, T; W_0^{1,p}(B, \mathbb{R}^n))$ (see [13, Lemma 4.12, p. 89; p. 156; p. 470]). Denote the partial sum up to k by $\psi_k = \sum_{j=1}^k c_j\phi_j$. Let $k \le l$ be a positive integer. Now, multiply (2.5) by c_j, $j = 1, 2, \ldots$, sum resulting equalities over j from 1 to k and integrate in t over $(0, s)$ to obtain, for $l \ge k$,

$$\begin{aligned}&\int_B \omega \cdot \psi_k(s)\, dx - \int_B w_{\varepsilon,l}(0) \cdot \psi_k(0)\, dx \\ &= \int_{(0,s)\times B} \big(w_{\varepsilon,l} \cdot \partial_t \psi_k - |Dw_{\varepsilon,l} + D\varrho_\varepsilon * u|_{\overline{g}}^{p-2}\, \overline{g}^{\alpha\beta}(D_\beta w_{\varepsilon,l} + D_\beta \varrho_\varepsilon * u) \cdot D_\beta \psi_k \\ &\qquad\qquad - \partial_t \varrho_\varepsilon * u \cdot \psi_k\big)\, dz.\end{aligned} \tag{3.25}$$

Note that $w_{\varepsilon,l} = 0$ on $\partial_p Q$ and use (3.23) and (2.7) to let $l \nearrow \infty$ and then $k \nearrow \infty$ in (3.25) to arrive at the assertion (3.24). In particular, by choosing $s = T$ and $\phi(T) = 0$ in (3.24), we obtain (2.9).

Next, we claim that

$$\begin{aligned}&\int_B w_\varepsilon(s) \cdot \phi(s)\, dx - \int_B w_\varepsilon(0) \cdot \phi(0)\, dx \\ &\qquad = \int_{(0,s)\times B} (w_\varepsilon \cdot \partial_t \phi - \sigma_\varepsilon \cdot D\phi - \partial_t \varrho_\varepsilon * u \cdot \phi)\, dz\end{aligned} \tag{3.26}$$

for any $\phi \in C^\infty([0, T]; C_0^\infty(B, \mathbb{R}^n))$. Let $0 < h < 4^{-1}\min\{s, T - s\}$. Let σ_h be a real-valued Lipschitz function defined on $(-\infty, \infty)$ such that $\sigma_h = 1$ in $[2h, s]$ and $\operatorname{supp} \sigma_h = [h, s + h]$. Use the test function $\sigma_h\, \phi$ in (2.9) to obtain

$$\int_{(0,s)\times B} \{-w_\varepsilon \cdot \phi \partial_t \sigma_h + \sigma_h(w_\varepsilon \cdot \partial_t \phi + \sigma_\varepsilon \cdot D\phi + \partial_t \varrho_\varepsilon * u \cdot \phi)\}\, dz = 0. \tag{3.27}$$

Similarly to the proof of Lemma 4, let $h \searrow 0$ in (3.27) to arrive at (3.26).

Combine (3.26) with (3.24) to find that

$$\int_B w_\varepsilon(s)\cdot\phi(s)\,dx - \int_B w_\varepsilon(0)\cdot\phi(0)\,dx = \int_B \omega\cdot\phi(s)\,dx \tag{3.28}$$

for any $\phi \in C^\infty([0,T];C_0^\infty(B,\mathbb{R}^n))$. For any $\eta=\eta(t)\in C^\infty([0,T])$ and $\psi=\psi(x)\in C_0^\infty(B,\mathbb{R}^n)$, choose $\phi=\eta(t)\psi(x)$ in (3.28). Set $\eta(0)=0$ to obtain

$$w_{\varepsilon,l}(s)\to w_\varepsilon(s) \quad \text{weakly in } L^2(B,\mathbb{R}^n) \text{ as } l\nearrow\infty. \tag{3.29}$$

Put $\eta=0$ in $[s,T]$ to conclude that $w_\varepsilon(0)=0$ almost everywhere in B.

Choose $\phi=\psi(x)\in C_0^\infty(B,\mathbb{R}^n)$ in (3.26) with $w_\varepsilon(0)=0$ to get

$$\Big|\int_B w_\varepsilon(s)\cdot\psi\,dx\Big| = \int_{(0,s)\times B} |\sigma_\varepsilon\cdot D\psi - \partial_t\varrho_\varepsilon * u\cdot\psi|\,dz. \tag{3.30}$$

Taking into account (2.7), (2.8), let $s\searrow 0$ in (3.30) to find that

$$\lim_{s\searrow 0}(w_\varepsilon(s),\phi)_{L^2(B,\mathbb{R}^n)}\to 0 \quad \text{as } s\searrow 0. \tag{3.31}$$

On the other hand, use (3.29) and the weak lower semicontinuity in $L^2(B,\mathbb{R}^n)$ to let $l\nearrow\infty$ in (2.6) and conclude that, for almost every $s\in(0,T)$,

$$\int_B |w_\varepsilon(s)|^2\,dx \le C\int_{(0,s)\times B}(|\partial_t\varrho_\varepsilon * u|^{p/(p-1)} + |D\varrho_\varepsilon * u|^p)\,dz. \tag{3.32}$$

This yields

$$\lim_{s\searrow 0}\int_B |w_\varepsilon(s)|^2\,dx \le 0, \tag{3.33}$$

which is exactly (2.10).

Now we verify (3.15): We can choose a subsequence $\{w_\varepsilon\}$ such that

$$w_\varepsilon(s)\to w(s) \quad \text{weakly in } L^2(B,\mathbb{R}^n) \text{ as } \varepsilon\searrow 0 \tag{3.34}$$

for almost every $s\in(0,T)$. By the energy inequality (2.15) and the boundedness of $\{w_\varepsilon\}$ in $L^\infty(0,T;L^2(B,\mathbb{R}^n))$, there are a subsequence $\{w_\varepsilon\}$ and a limit function $\omega\in L^2(B,\mathbb{R}^n)$ such that

$$w_\varepsilon(s)\to\omega \quad \text{weakly in } L^2(B,\mathbb{R}^n) \text{ as } \varepsilon\searrow 0. \tag{3.35}$$

Using the same cut-off function σ_h as above and the test function $\sigma_h\phi$ in (2.4), we have

$$\begin{aligned}\int_{(0,s)\times B}\{&-w_\varepsilon\cdot\phi\partial_t\sigma_h \\ &+\sigma_h(w_\varepsilon\cdot\partial_t\phi + |Dw_\varepsilon + D\varrho_\varepsilon * u|_{\overline{g}}^{p-2}\overline{g}^{\alpha\beta}(D_\beta w_\varepsilon + D_\beta\varrho_\varepsilon * u)\cdot D\phi \\ &-\varrho_\varepsilon * (|Du|_g^{p-2}g^{\alpha\beta}D_\beta u)\cdot D_\alpha\phi + \varrho_\varepsilon * f\cdot\phi)\}\,dz = 0.\end{aligned} \tag{3.36}$$

From (3.33), (3.35) and (3.1), we let $h \searrow 0$ and $\varepsilon \searrow 0$ in (3.36) to obtain

$$\begin{aligned}(3.37)\qquad &\int_B \omega \cdot \phi(s)\,dx \\ &= \int_{(0,s)\times B} (w\cdot\partial_t\phi - \chi\cdot D\phi + |Du|_g^{p-2} g^{\alpha\beta} D_\beta u \cdot D_\alpha\phi - f\cdot\phi)\,dz\end{aligned}$$

for any $\phi \in C^\infty([0,T]; C_0^\infty(B,\mathbb{R}^n))$.

We also prove that

$$\begin{aligned}(3.38)\qquad &\int_B w(s)\cdot\phi(s)\,dx - \int_B w(0)\cdot\phi(0)\,dx \\ &= \int_{(0,s)\times B} (w\cdot\partial_t\phi - \chi\cdot D\phi + |Du|_g^{p-2} g^{\alpha\beta} D_\beta u \cdot D_\alpha\phi - f\cdot\phi)\,dz\end{aligned}$$

for any $\phi \in C^\infty([0,T]; C_0^\infty(B,\mathbb{R}^n))$. As in (3.36), use a test function $\sigma_h\phi$ in (3.5) to deduce that

$$\begin{aligned}(3.39)\qquad &\int_{(0,s)\times B} \{-w\cdot\phi\partial_t\sigma_h \\ &+ \sigma_h(w\cdot\partial_t\phi + \chi\cdot D\phi - |Du|_g^{p-2} g^{\alpha\beta} D_\beta u\cdot D_\alpha\phi + f\cdot\phi)\}\,dz = 0.\end{aligned}$$

Apply (3.35) and let $h \searrow 0$ in (3.39) to arrive at (3.38).

Combine (3.38) with (3.37) to find that

$$(3.40)\qquad \int_B w(s)\cdot\phi(s)\,dx - \int_B w(0)\cdot\phi(0)\,dx = \int_B \omega\cdot\phi(s)\,dx$$

for any $\phi \in C^\infty([0,T]; C_0^\infty(B,\mathbb{R}^n))$. As in (3.29)–(3.31), we use (3.36), (3.39) and (3.40) to find that

$$(3.41)\qquad \lim_{s\searrow 0} (w(s),\,\phi)_{L^2(B,\mathbb{R}^n)} \to 0 \quad \text{as } s\searrow 0.$$

Taking into account (3.41), let $\varepsilon \searrow 0$ in (2.15) to verify that

$$(3.42)\qquad \int_B |w(s)|^2\,dx + \int_{(0,s)\times B} |Dw + Du|^p\,dz \le C \int_{(0,s)\times B} (|f| + |Du|^p)\,dz$$

for almost every $s \in (0,T)$. From (3.42), we immediately obtain (1.9). By letting $s \searrow 0$ in (3.42), we also have

$$(3.43)\qquad \lim_{s\searrow 0} \int_B |w(s)|^2\,dx = 0.$$

Let $s', s \in [0,T]$, $s' < s$. Let σ_h be the same cut-off function as in the proof of Lemma 4. Use a test function $\sigma_h\,\phi$ in (3.5) as in that proof to show that

$$\int_B w(s)\cdot\phi(s)\,dx - \int_B w(s')\cdot\phi(0)\,dx = \int_{(s',s)\times B} (w\cdot\partial_t\phi - \chi\cdot D\phi + |Du|_g^{p-2}g^{\alpha\beta}D_\beta u\cdot D_\alpha\phi - f\cdot\phi)\,dz \tag{3.44}$$

for any $\phi \in C^\infty([0,T];C_0^\infty(B,\mathbb{R}^n))$. From (3.44) and the density of $C_0^\infty(B,\mathbb{R}^n))$ in $L^2(B,\mathbb{R}^n)$, we find that $w(s)$ is a weakly continuous function of $s\in(0,T)$ with values in $L^2(B,\mathbb{R}^n)$.

By substitution of the test function $(\sigma_h(w)_\delta)_{-\delta}$ into (3.5), we make the same estimation as in the proof of Lemma 4 to obtain

$$\begin{aligned}\int_B |w(s)|^2\,dx - \int_B |w(s')|^2\,dx + 2\int_{(s',s)\times B} (&|Dw+Du|_{\overline{g}}^{p-2}\overline{g}^{\alpha\beta}(D_\beta w + D_\beta u)\cdot D_\alpha w \\ &- |Du|_g^{p-2}g^{\alpha\beta}D_\beta u\cdot D_\alpha w + f\cdot w)\,dz = 0.\end{aligned} \tag{3.45}$$

Use (2.12) and make a routine estimation with Hölder's and Young's inequalities in (3.45) to find that $|w(s)|_{L^2(B,\mathbb{R}^n)}$ is continuous in $s\in[0,T]$, in view of (3.43). Finally, from (3.44) and (3.45), we see that $w \in C([0,T];L^2(B,\mathbb{R}^n))$.

Remark. The argument of the proof that $w\in C([0,T];L^2(B,\mathbb{R}^n))$ can be applied to find that the approximations $w_{\varepsilon,l}$, w_ε and a weak solution u of (1.1) also belong to $C([0,T];L^2(B,\mathbb{R}^n))$.

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