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GENERAL METHOD OF REGULARIZATION.
II: RELAXATION PROPOSED BY SUQUET

Abstract. The aim of this paper is to prove that the relaxation of the elastic-perfectly plastic energy (of a solid made of a Hencky material) is the lower semicontinuous regularization of the plastic energy. We find the integral representation of a non-locally coercive functional. We show that the set of solutions of the relaxed problem is equal to the set of solutions of the relaxed problem proposed by Suquet. Moreover, we prove an existence theorem for the limit analysis problem.

1. Introduction. In the first part of this paper (see [2]) we investigate the convex functional

$$(1.1) \qquad BD \ni \mathbf{u} \mapsto \mathbb{B}(\boldsymbol{\varepsilon}(\mathbf{u})) = \int_{\overline{\Omega}} h(x, \boldsymbol{\varepsilon}(\mathbf{u}))$$

with constraints on the boundary of Ω, where $\boldsymbol{\varepsilon}(\mathbf{u})$ is the symmetrized gradient of $\mathbf{u}$ and $BD(\Omega)$ is the space of bounded deformations. Moreover, we assume that $\mathbb{B}(\boldsymbol{\varepsilon}(\mathbf{u})) = \infty$ if $\boldsymbol{\varepsilon}(\mathbf{u}) \notin L^1$. In [1] we find the *lower semicontinuous* (l.s.c.) relaxation of $\mathbb{B}$, and we show that the relaxation is a l.s.c. function (in the weak* BD topology), not greater than $\mathbb{B}$. Here we prove that the above mentioned relaxation (in the case of the relaxation proposed by Suquet) is the largest l.s.c. minorant less than $\mathbb{B}$, i.e. it is the l.s.c. regularization of $\mathbb{B}$. If the volume forces are 0, then we can omit the assumption of global coercivity of the functional considered. Moreover, we show that the set of solutions of the classical relaxed problem is equal to the set of solutions of the relaxed problem proposed by Suquet.

In [3], the global method for relaxation is applied for l.s.c. regularization of quasiconvex functionals with constraints (Dirichlet condition). These functionals are defined on $BV(\Omega)$. The constraints considered do not describe the relaxation proposed by Suquet (see [12]).

In [6] and [7] Christiansen finds the solution for the limit analysis problem, associated to the relaxed problem proposed by Suquet. But the limit analysis problem is not explicitly formulated in [6] and the relation between solutions of the relaxed problem and solutions of the relaxed problem proposed by Suquet is not considered.

The classical method of relaxation does not allow one to find a solution of the limit analysis problem $(P_0)_{AL}$ (see [13], [1]). Therefore we study the method of relaxation proposed by Suquet (cf. [12], [1]).

In Section 3, we obtain an existence theorem for the limit analysis problem, associated to the relaxed problem proposed by Suquet. In Corollary 10, we get a criterion of coercivity of the original problem $(P_{\lambda,j})$, or the relaxed problem $(RP_{\lambda,j}^{**})$ (see [2, (3.9)–(3.11), (5.1), (5.3), (5.4) and (5.8)]).

In the Appendix, we describe the scheme of duality in convex optimization in the case of Hencky plasticity.

We obtain the above mentioned results under the following assumptions.

ASSUMPTION 1. Ω and Ω_1 are bounded open connected sets of class C^1 in $\mathbb{R}^n$ such that $\Omega \subset\subset \Omega_1$. ∎

Let Γ_0 and Γ_1 $(= \overline{\Gamma}_1)$ be Borel subsets of the boundary Fr Ω of Ω such that $\Gamma_0 \cap \Gamma_1 = \emptyset$ and $ds(\text{Fr}\,\Omega - (\Gamma_0 \cup \Gamma_1)) = 0$. The Lebesgue and the Hausdorff measures on Ω and Fr Ω are denoted by dx and ds, respectively.

In this paper we consider the Banach space of measurable functions

$$(1.2) \qquad W^n(\Omega, \text{div}) \equiv \{\boldsymbol{\sigma} \in L^\infty(\Omega, \mathbf{E}_s^n) \mid \text{div}\,\boldsymbol{\sigma} \in L^n(\Omega, \mathbb{R}^n)\}$$

with the natural norm $\|\boldsymbol{\sigma}\|_{W^n(\Omega,\text{div})} = \|\boldsymbol{\sigma}\|_{L^\infty(\Omega,\mathbf{E}_s^n)} + \|\text{div}\,\boldsymbol{\sigma}\|_{L^n(\Omega,\mathbb{R}^n)}$ (where $\mathbf{E}_s^n$ is the space of symmetric real $n \times n$ matrices). Moreover, we consider the space

$$(1.3) \qquad C_{\text{div}}(\overline{\Omega}, \mathbf{E}_s^n) \equiv \{\boldsymbol{\sigma} \in C(\overline{\Omega}, \mathbf{E}_s^n) \mid \boldsymbol{\sigma}_{|\text{int}\,\Omega} \in W^n(\Omega, \text{div})\}.$$

Let $\mathcal{K} : \overline{\Omega} \to 2^{\mathbf{E}_s^n}$ be a multifunction.

ASSUMPTION 2 (see [11, p. 19, Lemma 1]). $\mathcal{K}(x)$ is a convex and closed subset of $\mathbf{E}_s^n$ for all $x \in \overline{\Omega}$, and there exists $\mathbf{z}_0 \in C^1(\overline{\Omega}, \mathbf{E}_s^n)$ such that

$$(1.4) \qquad\qquad \mathbf{z}_0(x) \in \mathcal{K}(x) \quad \text{for every } x \in \overline{\Omega},$$

and the following conditions hold:

   (i) if $\mathbf{z}(x) \in \mathcal{K}(x)$ for dx-a.e. $x \in \Omega$, where $\mathbf{z} \in C_{\text{div}}(\overline{\Omega}, \mathbf{E}_s^n)$, then $\mathbf{z}(y) \in \mathcal{K}(y)$ for every $y \in \overline{\Omega}$;

(ii) for every $y \in \overline{\Omega}$ and $\mathbf{w} \in \mathcal{K}(y)$ there exists $\mathbf{z} \in C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}_s^n)$ such that $\mathbf{z}(y) = \mathbf{w}$ and $\mathbf{z}(x) \in \mathcal{K}(x)$ for every $x \in \overline{\Omega}$. ∎

Conditions (i) and (ii) are equivalent to the condition that for every $y \in \overline{\Omega}$,

$$(1.5) \qquad \mathcal{K}(y) = \{\mathbf{z}(y) \mid \mathbf{z} \in C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}_s^n), \ \mathbf{z}(x) \in \mathcal{K}(x) \text{ for } dx\text{-a.e. } x \in \Omega\}.$$

DEFINITION 1. Let $j^* : \Omega \times \mathbf{E}_s^n \to \mathbb{R} \cup \{\infty\}$ be a *convex normal integrand*, i.e.

(a) the function $\mathbf{E}_s^n \ni \mathbf{w}^* \mapsto j^*(x, \mathbf{w}^*)$ is convex and l.s.c. for dx-a.e. $x \in \Omega$;

(b) there exists a Borel function $\widetilde{j^*} : \Omega \times \mathbf{E}_s^n \to \mathbb{R} \cup \{\infty\}$ such that $\widetilde{j^*}(x, \cdot) = j^*(x, \cdot)$ for dx-a.e. $x \in \Omega$.

Moreover, assume that

$$(1.6) \qquad \{\mathbf{w}^* \in \mathbf{E}_s^n \mid j^*(x, \mathbf{w}^*) < \infty\} = \mathcal{K}(x) \quad \text{for } dx\text{-a.e. } x \in \Omega.$$

ASSUMPTION 3. For every $\widehat{r} > 0$ there exists $c_{\widehat{r}}$ such that

$$(1.7) \qquad \sup \left\{ \int_{\Omega} j^*(x, \mathbf{z}^*)\, dx \ \middle| \ \mathbf{z}^* \in L^\infty(\Omega, \mathbf{E}_s^n), \ \|\mathbf{z}^*\|_{L^\infty} < \widehat{r} \right.$$
$$\left. \text{and } \mathbf{z}^*(x) \in \mathcal{K}(x) \text{ for } dx\text{-a.e. } x \in \Omega \right\} < c_{\widehat{r}} < \infty. \ ∎$$

ASSUMPTION 4. There exist $\mathbf{u}^e \in LD(\Omega)$ and $q \in L^1(\Omega, \mathbb{R})$ such that

$$(1.8) \qquad\qquad j^*(x, \mathbf{w}^*) \geq \boldsymbol{\varepsilon}(\mathbf{u}^e)(x) : \mathbf{w}^* + q(x)$$

for dx-a.e. $x \in \Omega$ and every $\mathbf{w}^* \in \mathbf{E}_s^n$, and $\boldsymbol{\gamma}_B(\mathbf{u}^e) = \mathbf{0}$ on $\mathrm{Fr}\,\Omega$. ∎

The set $\mathcal{K}(x)$ denotes the elasticity convex domain at the point x. Define

$$(1.9) \qquad j(x, \mathbf{w}) \equiv j^{**}(x, \mathbf{w}) \equiv \sup\{\mathbf{w} : \mathbf{w}^* - j^*(x, \mathbf{w}^*) \mid \mathbf{w}^* \in \mathbf{E}_s^n\}$$

for dx-a.e. $x \in \Omega$ and all $\mathbf{w} \in \mathbf{E}_s^n$. Then j is a convex normal integrand. Define $j_\infty : \overline{\Omega} \times \mathbf{E}_s^n \to \mathbb{R} \cup \{\infty\}$ by

$$(1.10) \qquad j_\infty(x, \mathbf{w}) \equiv \sup\{\mathbf{w} : \mathbf{w}^* - I_{\mathcal{K}(x)}(\mathbf{w}^*) \mid \mathbf{w}^* \in \mathbf{E}_s^n\}$$

for $x \in \overline{\Omega}$ and $\mathbf{w} \in \mathbf{E}_s^n$.

ASSUMPTION 5. There exists $\boldsymbol{\sigma}_0 \in C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}_s^n)$ such that $\boldsymbol{\beta}_B(\boldsymbol{\sigma}_0) = \lambda \mathbf{g}$ on Γ_1 and $\boldsymbol{\sigma}_0(x) \in \mathcal{K}(x)$ for dx-a.e. $x \in \Omega$, where $\mathbf{g}$ is a boundary force on Γ_1. ∎

ASSUMPTION 6. Let $\Gamma_1 = \mathrm{Fr}\,\Omega \cap \mathcal{C}$, where $\mathcal{C} = \mathrm{cl\,int}\,\mathcal{C} \subset \Omega_1$ is a closed Caccioppoli set and $ds(\mathrm{Fr}\,\Omega \cap \mathrm{Fr}\,\mathcal{C}) = 0$. ∎

2. Relaxed problem proposed by Suquet. The classical method of relaxation does not allow one to find a solution of the limit analysis problem $(P_0)_{AL}$ (see [4], [13], [1]). The problem has an interesting mechanical

interpretation as the *safety condition*. Therefore we turn to the following method of relaxation, proposed by Suquet (cf. [12], [1]).

We define the following Banach spaces:

$$(2.1) \quad LD(\Omega) \equiv \left\{ \mathbf{u} \in L^1(\Omega, \mathbb{R}^n) \, \middle| \right.$$

$$\left. \varepsilon_{ij}(\mathbf{u}) \equiv \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \in L^1(\Omega), \, i,j = 1, \dots, n \right\},$$

$$(2.2) \quad BD(\Omega) \equiv \{ \mathbf{u} \in L^1(\Omega, \mathbb{R}^n) \mid \varepsilon_{ij}(\mathbf{u}) \in \mathbb{M}_b(\Omega, \mathbb{R}), \, i,j = 1, \dots, n \},$$

with the natural norms

$$(2.3) \quad \|\mathbf{u}\|_{LD} = \|\mathbf{u}\|_{L^1} + \sum_{i,j}^{n} \|\varepsilon_{ij}(\mathbf{u})\|_{L^1},$$

$$\|\mathbf{u}\|_{BD} = \|\mathbf{u}\|_{L^1} + \sum_{i,j}^{n} \|\varepsilon_{ij}(\mathbf{u})\|_{\mathbb{M}_b},$$

where $\mathbb{M}_b(\Omega, \mathbb{R})$ is the space of $\mathbb{R}$-valued bounded measures defined on Ω. Moreover, we consider the space

$$(2.4) \quad \mathbf{Y}^1(\overline{\Omega}) \equiv \{ \mathbf{M} \in \mathbb{M}_b(\overline{\Omega}, \mathbf{E}_s^n) \mid \exists \mathbf{u}_1 \in BD(\Omega_1),$$

$$\varepsilon(\mathbf{u}_1)_{|\overline{\Omega}} = \mathbf{M}, \, \mathbf{u}_{1|\Omega_1 - \overline{\Omega}} = \mathbf{0} \}.$$

We now define the functional of the total elastic-perfectly plastic energy $\mathbb{H}_\lambda^j$, introduced in [1]. We then find its l.s.c. regularization in the topology $\sigma(\mathbf{Y}^1(\overline{\Omega}) \times \mathbf{M}^1(\Gamma_1), C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}_s^n) \times C(\Gamma_1, \mathbf{E}_s^n))$ (where the bilinear form is defined in (2.8) below). Let

$$(2.5) \quad \mathbf{M}^1(\Gamma_1) \equiv \{ \boldsymbol{\mu} \otimes_s \boldsymbol{\nu} \in \mathbb{M}_b(\Gamma_1, \mathbf{E}_s^n) \mid \boldsymbol{\mu} \in \mathbb{M}_b(\Gamma_1, \mathbb{R}^n) \},$$

where $\boldsymbol{\nu}$ is the outer unit vector, normal to $\mathrm{Fr}\,\Omega$. Consider the topological vector spaces

$$(2.6) \quad [C(\Gamma_1, \mathbf{E}_s^n), \sigma(C(\Gamma_1, \mathbf{E}_s^n), \mathbf{M}^1(\Gamma_1))], \, [\mathbf{M}^1(\Gamma_1), \sigma(\mathbf{M}^1(\Gamma_1), C(\Gamma_1, \mathbf{E}_s^n))]$$

(the latter is equivalent to $[\mathbf{M}^1(\Gamma_1), \text{weak}^* \, \mathbf{M}^1(\Gamma_1)]$).

We define the functional $\mathbb{H}_\lambda^j : \mathbf{Y}^1(\overline{\Omega}) \times \mathbf{M}^1(\Gamma_1) \to \mathbb{R} \cup \{\infty\}$ by

$$(2.7) \quad \mathbb{H}_\lambda^j(\varepsilon(\mathbf{u})_{|\overline{\Omega}}, \boldsymbol{\mu} \otimes_s \boldsymbol{\nu}) \equiv - \int_{\Gamma_1} \lambda \mathbf{g} \cdot \boldsymbol{\mu} + \int_{\Omega} j(x, \varepsilon(\mathbf{u})_a) \, dx$$

$$+ \int_{\Gamma_0} I_{\{\gamma_B^I(\mathbf{u}) \otimes_s \boldsymbol{\nu} = \mathbf{0}\}}(\gamma_B^I(\mathbf{u}) \otimes_s \boldsymbol{\nu}) \, ds$$

if $\mathbf{u}_{|\Omega} \in LD(\Omega)$ and $\boldsymbol{\mu} = \gamma_B^I(\mathbf{u}) \, ds$ for $|\boldsymbol{\mu} - \gamma_B^I(\mathbf{u}) \, ds|$-a.e. $x \in \Gamma_1$, and $\mathbb{H}_\lambda^j(\varepsilon(\mathbf{u})_{|\overline{\Omega}}, \boldsymbol{\mu} \otimes_s \boldsymbol{\nu}) \equiv \infty$ otherwise. Here $I_{\{\gamma_B^I(\mathbf{u}) \otimes_s \boldsymbol{\nu} = \mathbf{0}\}}(\gamma_B^I(\mathbf{u}) \otimes_s \boldsymbol{\nu})$ is the indicator function, which takes value 0 if $\gamma_B^I(\mathbf{u}) \otimes_s \boldsymbol{\nu} = \mathbf{0}$ and ∞ otherwise.

We assume that there exist $\widetilde{\mathbf{u}} \in BD(\Omega_1)$ and $\widetilde{\boldsymbol{\mu}} \in \mathbb{M}_b(\Gamma_1, \mathbb{R}^n)$ such that $\widetilde{\mathbf{u}}_{|\Omega} \in LD(\Omega)$ and $\mathbb{H}_\lambda^j(\boldsymbol{\varepsilon}(\widetilde{\mathbf{u}})_{|\overline{\Omega}}, \widetilde{\boldsymbol{\mu}} \otimes_s \boldsymbol{\nu}) < \infty$.

The bilinear form between $\mathbf{Y}^1(\overline{\Omega}) \times \mathbf{M}^1(\Gamma_1)$ and $C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}_s^n) \times C(\Gamma_1, \mathbf{E}_s^n)$ is given by

$$(2.8) \quad \langle (\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}}, \boldsymbol{\mu} \otimes_s \boldsymbol{\nu}); (\boldsymbol{\sigma}, \boldsymbol{\kappa}) \rangle_2 \equiv \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\mathbf{u})_{|\Omega}$$

$$+ \int_{\mathrm{Fr}\,\Omega} \boldsymbol{\sigma} : (-\boldsymbol{\gamma}_B^I(\mathbf{u}) \otimes_s \boldsymbol{\nu}) ds + \int_{\Gamma_1} \boldsymbol{\kappa} : [\boldsymbol{\mu} \otimes_s \boldsymbol{\nu}]$$

for $(\boldsymbol{\varepsilon}(\mathbf{u})_{|\Omega}, -\boldsymbol{\gamma}_B^I(\mathbf{u}) ds \otimes_s \boldsymbol{\nu}) \in \mathbf{Y}^1(\overline{\Omega})$, $\boldsymbol{\mu} \otimes_s \boldsymbol{\nu} \in \mathbf{M}^1(\Gamma_1)$, $\boldsymbol{\sigma} \in C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}_s^n)$ and $\boldsymbol{\kappa} \in C(\Gamma_1, \mathbf{E}_s^n)$. Because of the duality between $\mathbf{Y}^1(\overline{\Omega}) \times \mathbf{M}^1(\Gamma_1)$ and $C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}_s^n) \times C(\Gamma_1, \mathbf{E}_s^n)$, we define a functional $(\mathbb{H}_\lambda^j)^* : C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}_s^n) \times C(\Gamma_1, \mathbf{E}_s^n) \to \mathbb{R} \cup \{\infty\}$ by

$$(2.9) \quad (\mathbb{H}_\lambda^j)^*(\boldsymbol{\sigma}, \boldsymbol{\kappa}) = \sup\{\langle (\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}}, \boldsymbol{\mu} \otimes_s \boldsymbol{\nu}); (\boldsymbol{\sigma}, \boldsymbol{\kappa}) \rangle_2$$

$$- \mathbb{H}_\lambda^j(\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}}, \boldsymbol{\mu} \otimes_s \boldsymbol{\nu}) \mid \boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}} \in \mathbf{Y}^1(\overline{\Omega}), \boldsymbol{\mu} \otimes_s \boldsymbol{\nu} \in \mathbf{M}^1(\Gamma_1)\}.$$

The bidual functional $(\mathbb{H}_\lambda^j)^{**} : \mathbf{Y}^1(\overline{\Omega}) \times \mathbf{M}^1(\Gamma_1) \to \mathbb{R} \cup \{\infty\}$ is defined by

$$(2.10) \quad (\mathbb{H}_\lambda^j)^{**}(\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}}, \boldsymbol{\mu} \otimes_s \boldsymbol{\nu}) = \sup\{\langle (\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}}, \boldsymbol{\mu} \otimes_s \boldsymbol{\nu}); (\boldsymbol{\sigma}, \boldsymbol{\kappa}) \rangle_2$$

$$- (\mathbb{H}_\lambda^j)^*(\boldsymbol{\sigma}, \boldsymbol{\kappa}) \mid \boldsymbol{\sigma} \in C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}_s^n), \boldsymbol{\kappa} \in C(\Gamma_1, \mathbf{E}_s^n)\}.$$

The bilinear form between $\mathbb{M}_b(\Omega, \mathbf{E}_s^n) \times \mathbf{Y}^1(\overline{\Omega})_{|\mathrm{Fr}\,\Omega} \times \mathbf{M}^1(\Gamma_1)$ and $C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}_s^n) \times C(\Gamma_1, \mathbf{E}_s^n)$ is given by

$$(2.11) \quad \langle (\mathbf{w}, \boldsymbol{\gamma}_B^I(\mathbf{u}) \otimes_s \boldsymbol{\nu}, \boldsymbol{\mu} \otimes_s \boldsymbol{\nu}); (\boldsymbol{\sigma}, \boldsymbol{\kappa}) \rangle_3 = \int_\Omega \boldsymbol{\sigma} : \mathbf{w}$$

$$+ \int_{\mathrm{Fr}\,\Omega} \boldsymbol{\sigma} : (-\boldsymbol{\gamma}_B^I(\mathbf{u}) \otimes_s \boldsymbol{\nu}) \, ds + \int_{\Gamma_1} \boldsymbol{\kappa} : [\boldsymbol{\mu} \otimes_s \boldsymbol{\nu}]$$

for $\mathbf{w} \in \mathbb{M}_b(\Omega, \mathbf{E}_s^n)$, $\boldsymbol{\gamma}_B^I(\mathbf{u}) ds \otimes_s \boldsymbol{\nu} \in \mathbf{Y}^1(\overline{\Omega})_{|\mathrm{Fr}\,\Omega}$, $\boldsymbol{\mu} \otimes_s \boldsymbol{\nu} \in \mathbf{M}^1(\Gamma_1)$, $\boldsymbol{\sigma} \in C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}_s^n)$ and $\boldsymbol{\kappa} \in C(\Gamma_1, \mathbf{E}_s^n)$. The extension of $\mathbb{H}_\lambda^j$ onto the space $\mathbb{M}_b(\Omega, \mathbf{E}_s^n) \times \mathbf{Y}^1(\overline{\Omega})_{|\mathrm{Fr}\,\Omega} \times \mathbf{M}^1(\Gamma_1)$ (denoted by $\widetilde{\mathbb{H}}_\lambda^j$) is given by

$$(2.12) \quad \widetilde{\mathbb{H}}_\lambda^j(\mathbf{w}, -\boldsymbol{\gamma}_B^I(\mathbf{u}) \otimes_s \boldsymbol{\nu}, \boldsymbol{\mu} \otimes_s \boldsymbol{\nu}) \equiv - \int_{\Gamma_1} \lambda \mathbf{g} \cdot \boldsymbol{\mu}$$

$$+ \int_\Omega j(x, \mathbf{w}) \, dx + \int_{\Gamma_0} I_{\{\boldsymbol{\gamma}_B^I(\mathbf{u}) \otimes_s \boldsymbol{\nu} = \mathbf{0}\}} (\boldsymbol{\gamma}_B^I(\mathbf{u}) \otimes_s \boldsymbol{\nu}) \, ds$$

if $\mathbf{w} \in L^1(\Omega, \mathbf{E}_s^n)$ and $\boldsymbol{\mu} = \boldsymbol{\gamma}_B^I(\mathbf{u}) ds$ for $|\boldsymbol{\mu} - \boldsymbol{\gamma}_B^I(\mathbf{u}) ds|$-a.e. $x \in \Gamma_1$, and $\widetilde{\mathbb{H}}_\lambda^j(\mathbf{w}, -\boldsymbol{\gamma}_B^I(\mathbf{u}) \otimes_s \boldsymbol{\nu}, \boldsymbol{\mu} \otimes_s \boldsymbol{\nu}) \equiv \infty$ otherwise.

By duality between $\mathbb{M}_b(\Omega, \mathbf{E}_s^n) \times \mathbf{Y}^1(\overline{\Omega})_{|\mathrm{Fr}\,\Omega} \times \mathbf{M}^1(\Gamma_1)$ and $C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}_s^n) \times C(\Gamma_1, \mathbf{E}_s^n)$, we define a functional $(\widetilde{\mathbb{H}}_\lambda^j)^* : C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}_s^n) \times C(\Gamma_1, \mathbf{E}_s^n) \to \mathbb{R} \cup \{\infty\}$

by

$$(2.13) \quad (\widetilde{\mathbb{H}}_\lambda^j)^*(\boldsymbol{\sigma}, \boldsymbol{\kappa}) = \sup\{\langle(\mathbf{w}, -\boldsymbol{\gamma}_B^I(\mathbf{u}) \otimes_s \boldsymbol{\nu}, \boldsymbol{\mu} \otimes_s \boldsymbol{\nu}); (\boldsymbol{\sigma}, \boldsymbol{\kappa})\rangle_3$$
$$- \widetilde{\mathbb{H}}_\lambda^j(\mathbf{w}, -\boldsymbol{\gamma}_B^I(\mathbf{u}) \otimes_s \boldsymbol{\nu}, \boldsymbol{\mu} \otimes_s \boldsymbol{\nu}) \mid \mathbf{w} \in L^1(\Omega, \mathbf{E}_s^n),$$
$$\mathbf{u} \in BD(\Omega) \text{ and } \boldsymbol{\mu} \otimes_s \boldsymbol{\nu} \in \mathbf{M}^1(\Gamma_1)\},$$

since $L^1 \subset \mathbb{M}_b$. The bidual functional $(\widetilde{\mathbb{H}}_\lambda^j)^{**} : \mathbf{Y}^1(\overline{\Omega}) \times \mathbf{M}^1(\Gamma_1) \to \mathbb{R} \cup \{\infty\}$ is defined by

$$(2.14) \quad (\widetilde{\mathbb{H}}_\lambda^j)^{**}(\mathbf{w}, -\boldsymbol{\gamma}_B^I(\mathbf{u}) \otimes_s \boldsymbol{\nu}, \boldsymbol{\mu} \otimes_s \boldsymbol{\nu}) = \sup\{\langle(\mathbf{w}, -\boldsymbol{\gamma}_B^I(\mathbf{u}) \otimes_s \boldsymbol{\nu};$$
$$(\boldsymbol{\sigma}, \boldsymbol{\kappa})\rangle_3 - (\widetilde{\mathbb{H}}_\lambda^j)^*(\boldsymbol{\sigma}, \boldsymbol{\kappa}) \mid \boldsymbol{\sigma} \in C_{\text{div}}(\overline{\Omega}, \mathbf{E}_s^n), \boldsymbol{\kappa} \in C(\Gamma_1, \mathbf{E}_s^n)\}$$

for $(\mathbf{w}, -\boldsymbol{\gamma}_B^I(\mathbf{u})ds \otimes_s \boldsymbol{\nu}) \in \mathbf{Y}^1(\overline{\Omega})$ and $\boldsymbol{\mu} \otimes_s \boldsymbol{\nu} \in \mathbf{M}^1(\Gamma_1)$.

LEMMA 1. *For every* $\mathbf{u} \in BD(\Omega_1)$ *and* $\boldsymbol{\mu} \in \mathbb{M}_b(\Gamma_1, \mathbb{R}^n)$ *such that* $\mathbf{u}_{|\Omega} \in LD(\Omega)$, $\mathbf{u}_{|\Omega_1 - \overline{\Omega}} = \mathbf{0}$, $\boldsymbol{\gamma}_B^I(\mathbf{u}) = \mathbf{0}$ *on* Γ_0 *and* $\boldsymbol{\gamma}_B^I(\mathbf{u})ds = \boldsymbol{\mu}$ *on* Γ_1, *we have*

$$(2.15) \quad (\mathbb{H}_\lambda^j)^{**}(\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}}, \boldsymbol{\mu} \otimes_s \boldsymbol{\nu}) = \mathbb{H}_\lambda^j(\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}}, \boldsymbol{\mu} \otimes_s \boldsymbol{\nu})$$
$$= (\widetilde{\mathbb{H}}_\lambda^j)^{**}(\boldsymbol{\varepsilon}(\mathbf{u})_{|\Omega}, -\boldsymbol{\gamma}_B^I(\mathbf{u}) \otimes_s \boldsymbol{\nu}, \boldsymbol{\mu} \otimes_s \boldsymbol{\nu}).$$

Proof. By [1, (4.80)], we have $(\widetilde{\mathbb{H}}_\lambda^j)^{**}(\boldsymbol{\varepsilon}(\mathbf{u})_{|\Omega}, -\boldsymbol{\gamma}_B^I(\mathbf{u})ds \otimes_s \boldsymbol{\nu}, \boldsymbol{\mu} \otimes_s \boldsymbol{\nu}) = \mathbb{H}_\lambda^j(\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}}, \boldsymbol{\mu} \otimes_s \boldsymbol{\nu})$ for every $\mathbf{u} \in BD(\Omega_1)$ and $\boldsymbol{\mu} \in \mathbb{M}_b(\Gamma_1, \mathbb{R}^n)$ such that $\mathbf{u}_{|\Omega} \in LD(\Omega)$, $\mathbf{u}_{|\Omega_1 - \overline{\Omega}} = \mathbf{0}$, $\boldsymbol{\gamma}_B^I(\mathbf{u}) = \mathbf{0}$ on Γ_0 and $\boldsymbol{\gamma}_B^I(\mathbf{u})ds = \boldsymbol{\mu}$ on Γ_1 (now $\mathbb{H}^{\#*}$ is denoted by $(\widetilde{\mathbb{H}}_\lambda^j)^{**}$). Similarly to the proofs of Lemmas 6 and 8 from [2] we obtain (2.15). ∎

LEMMA 2. *Let* $\boldsymbol{\kappa}_s \in C(\Gamma_1, \mathbf{E}_s^n)$, $\boldsymbol{\sigma}_s \in C_{\text{div}}(\overline{\Omega}, \mathbf{E}_s^n)$, *where* $\text{div}\,\boldsymbol{\sigma}_s = \mathbf{0}$ *on* Ω. *If for every* $\mathbf{t} \in L^1(\Gamma_1, \mathbb{R}^n)$, $\int_{\Gamma_1} \boldsymbol{\kappa}_s : [\mathbf{t} \otimes_s \boldsymbol{\nu}]ds = 0$, *then*

$$(\mathbb{H}_\lambda^j)^*(\boldsymbol{\sigma}, \boldsymbol{\kappa}) = (\mathbb{H}_\lambda^j)^*(\boldsymbol{\sigma} + \boldsymbol{\sigma}_s, \boldsymbol{\kappa} + \boldsymbol{\kappa}_s)$$

for every $\boldsymbol{\sigma} \in C_{\text{div}}(\overline{\Omega}, \mathbf{E}_s^n)$ *and* $\boldsymbol{\kappa} \in C(\Gamma_1, \mathbf{E}_s^n)$.

Proof. We prove this result similarly to [2, Lemma 9]. ∎

We say that a net $\{(\boldsymbol{\sigma}_t, \boldsymbol{\kappa}_t)\}_{t \in T} \subset C_{\text{div}}(\overline{\Omega}, \mathbf{E}_s^n) \times C(\Gamma_1, \mathbf{E}_s^n)$ converges to $(\widehat{\boldsymbol{\sigma}}, \widehat{\boldsymbol{\kappa}})$ in the topology

$$(2.16) \quad \sigma(C_{\text{div}}(\overline{\Omega}, \mathbf{E}_s^n) \times C(\Gamma_1, \mathbf{E}_s^n), L^1(\Omega, \mathbf{E}_s^n) \times$$
$$\{(\boldsymbol{\varphi}, \boldsymbol{\mu} \otimes_s \boldsymbol{\nu}) \in \mathbf{Y}^1(\overline{\Omega})_{|\text{Fr}\,\Omega} \times \mathbf{M}^1(\Gamma_1) \mid \boldsymbol{\varphi}_{|\Gamma_0} = \mathbf{0}, \boldsymbol{\varphi}_{|\Gamma_1}ds = \boldsymbol{\mu} \otimes_s \boldsymbol{\nu}\})$$

if $\langle(\mathbf{w}dx, \mathbf{p} \otimes_s \boldsymbol{\nu}, \mathbf{p}_{|\Gamma_1}ds \otimes_s \boldsymbol{\nu}); [(\boldsymbol{\sigma}_t, \boldsymbol{\kappa}_t) - (\widehat{\boldsymbol{\sigma}}, \widehat{\boldsymbol{\kappa}})]\rangle_3 \to 0$ for every $\mathbf{w} \in L^1(\Omega, \mathbf{E}_s^n)$ and $\mathbf{p} \in L^1(\text{Fr}\,\Omega, \mathbb{R}^n)$, where $\mathbf{p}_{|\Gamma_0} = \mathbf{0}$.

LEMMA 3. *Let*

$$(2.17) \qquad \widetilde{f} : C_{\text{div}}(\overline{\Omega}, \mathbf{E}_s^n) \times C(\Gamma_1, \mathbf{E}_s^n) \to \mathbb{R}$$

be a linear functional, continuous in the topology (2.16), *such that there exist* $\boldsymbol{\kappa}_0 \in C(\Gamma_1, \mathbf{E}_s^n)$ *and* $\widetilde{\boldsymbol{\sigma}}_1 \in C_{\text{div}}(\overline{\Omega}, \mathbf{E}_s^n)$ *such that for every* $\boldsymbol{\sigma}_s \in C(\overline{\Omega}, \mathbf{E}_s^n)$

with div $\boldsymbol{\sigma}_s = \mathbf{0}$, we have

(2.18) $$\widetilde{f}(\widetilde{\boldsymbol{\sigma}}_1 + \boldsymbol{\sigma}_s, \boldsymbol{\kappa}_0) = \widetilde{f}(\widetilde{\boldsymbol{\sigma}}_1, \boldsymbol{\kappa}_0).$$

Moreover, assume that for every $\boldsymbol{\kappa}_s \in C(\Gamma_1, \mathbf{E}_s^n)$ *with* $\int_{\Gamma_1} \boldsymbol{\kappa}_s : [\mathbf{t} \otimes_s \boldsymbol{\nu}]\, ds = 0$ *for every* $\mathbf{t} \in L^1(\Gamma_1, \mathbb{R}^n)$, *we have*

(2.19) $\quad \forall \boldsymbol{\sigma} \in C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}_s^n),\ \forall \boldsymbol{\kappa} \in C(\Gamma_1, \mathbf{E}_s^n), \quad \widetilde{f}(\boldsymbol{\sigma}, \boldsymbol{\kappa}) = \widetilde{f}(\boldsymbol{\sigma}, \boldsymbol{\kappa} + \boldsymbol{\kappa}_s).$

Then there exists $\widetilde{\mathbf{u}}_1 \in LD(\Omega)$ *such that* $\boldsymbol{\gamma}_B(\widetilde{\mathbf{u}}_1) = \mathbf{0}$ *on* Γ_0 *and for every* $(\boldsymbol{\sigma}, \boldsymbol{\kappa}) \in C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}_s^n) \times C(\Gamma_1, \mathbf{E}_s^n)$ *we get*

(2.20) $$\widetilde{f}(\boldsymbol{\sigma}, \boldsymbol{\kappa}) = \int_{\Omega} \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\widetilde{\mathbf{u}}_1)\, dx - \int_{\mathrm{Fr}\,\Omega} \boldsymbol{\sigma} : (\boldsymbol{\gamma}_B(\widetilde{\mathbf{u}}_1) \otimes_s \boldsymbol{\nu})\, ds$$
$$+ \int_{\Gamma_1} \boldsymbol{\kappa} : (\boldsymbol{\gamma}_B(\widetilde{\mathbf{u}}_1) \otimes_s \boldsymbol{\nu})\, ds.$$

Proof. Since $\widetilde{f}$ is continuous in the topology (2.16), by Theorem V.3.9 of [8] there exist $\mathbf{m} \in L^1(\Omega, \mathbf{E}_s^n)$ and $\widehat{\mathbf{u}} \in BD(\Omega)$ with $\boldsymbol{\gamma}_B(\widehat{\mathbf{u}}) = \mathbf{0}$ on Γ_0 such that, for every $\boldsymbol{\sigma} \in C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}_s^n)$ and $\boldsymbol{\kappa} \in C(\Gamma_1, \mathbf{E}_s^n)$,

(2.21) $$\widetilde{f}(\boldsymbol{\sigma}, \boldsymbol{\kappa}) = \int_{\Omega} \boldsymbol{\sigma} : \mathbf{m}\, dx - \int_{\mathrm{Fr}\,\Omega} \boldsymbol{\sigma} : (\boldsymbol{\gamma}_B(\widehat{\mathbf{u}}) \otimes_s \boldsymbol{\nu})\, ds$$
$$+ \int_{\Gamma_1} \boldsymbol{\kappa} : (\boldsymbol{\gamma}_B(\widehat{\mathbf{u}}) \otimes_s \boldsymbol{\nu})\, ds.$$

Indeed, in (2.17) we can replace $C(\Gamma_1, \mathbf{E}_s^n)$ by

(2.22) $C(\Gamma_1, \mathbf{E}_s^n)/\Big\{ \boldsymbol{\kappa} \in C(\Gamma_1, \mathbf{E}_s^n) \,\Big|\, \forall \mathbf{t} \in L^1(\Gamma_1, \mathbb{R}^n),\ \int_{\Gamma_1} \boldsymbol{\kappa} : [\mathbf{t} \otimes_s \boldsymbol{\nu}]\, ds = 0 \Big\}.$

By the linearity of $\widetilde{f}$, we get $\widetilde{f}(\boldsymbol{\sigma} + \boldsymbol{\sigma}_s, \boldsymbol{\kappa}) = \widetilde{f}(\boldsymbol{\sigma}, \boldsymbol{\kappa})$ for every $\boldsymbol{\kappa} \in C(\Gamma_1, \mathbf{E}_s^n)$, $\boldsymbol{\sigma} \in C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}_s^n)$ and $\boldsymbol{\sigma}_s \in C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}_s^n)$ such that div $\boldsymbol{\sigma}_s = \mathbf{0}$. Indeed, by (2.18) and (2.21) we have

(2.23) $$\widetilde{f}(\boldsymbol{\sigma} + \boldsymbol{\sigma}_s, \boldsymbol{\kappa}_0) = \widetilde{f}(\widetilde{\boldsymbol{\sigma}}_1 + \boldsymbol{\sigma}_s, \boldsymbol{\kappa}_0) + \widetilde{f}(\boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}}_1, \mathbf{0})$$
$$= \widetilde{f}(\widetilde{\boldsymbol{\sigma}}_1, \boldsymbol{\kappa}_0) + \widetilde{f}(\boldsymbol{\sigma} - \widetilde{\boldsymbol{\sigma}}_1, \mathbf{0}) = \widetilde{f}(\boldsymbol{\sigma}, \boldsymbol{\kappa}_0),$$

and by (2.21) we get

(2.24) $$\widetilde{f}(\boldsymbol{\sigma} + \boldsymbol{\sigma}_s, \boldsymbol{\kappa}) = \widetilde{f}(\boldsymbol{\sigma} + \boldsymbol{\sigma}_s, \boldsymbol{\kappa}_0) + \widetilde{f}(\mathbf{0}, \boldsymbol{\kappa} - \boldsymbol{\kappa}_0)$$
$$= \widetilde{f}(\boldsymbol{\sigma}, \boldsymbol{\kappa}_0) + \widetilde{f}(\mathbf{0}, \boldsymbol{\kappa} - \boldsymbol{\kappa}_0) = \widetilde{f}(\boldsymbol{\sigma}, \boldsymbol{\kappa}).$$

Thus, $\widetilde{f}(\boldsymbol{\sigma}_s, \mathbf{0}) = 0$ for every $\boldsymbol{\sigma}_s \in C(\overline{\Omega}, \mathbf{E}_s^n)$ with div $\boldsymbol{\sigma}_s = \mathbf{0}$. Taking $\boldsymbol{\kappa} = \mathbf{0}$ in (2.21), we may proceed as in [2, Lemma 10]. ∎

Let $Q_1 : C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}_s^n) \times C(\Gamma_1, \mathbf{E}_s^n) \to \mathbb{R} \cup \{\infty\}$ be defined by

(2.25) $Q_1(\boldsymbol{\sigma}, \boldsymbol{\kappa})$

$$= \inf_{\boldsymbol{\sigma}_s} \{ (\widetilde{\mathbb{H}}_\lambda^j)^*(\boldsymbol{\sigma} + \boldsymbol{\sigma}_s, \boldsymbol{\kappa}) \mid \boldsymbol{\sigma}_s \in C(\overline{\Omega}, \mathbf{E}_s^n) \text{ with } \operatorname{div} \boldsymbol{\sigma}_s = \mathbf{0} \}$$

for $\boldsymbol{\sigma} \in C_{\operatorname{div}}(\overline{\Omega}, \mathbf{E}_s^n)$ and $\boldsymbol{\kappa} \in C(\Gamma_1, \mathbf{E}_s^n)$.

PROPOSITION 4. *For every* $\boldsymbol{\sigma} \in C_{\operatorname{div}}(\overline{\Omega}, \mathbf{E}_s^n)$ *and* $\boldsymbol{\kappa} \in C(\Gamma_1, \mathbf{E}_s^n)$ *we have*

(2.26) $$(\mathbb{H}_\lambda^j)^*(\boldsymbol{\sigma}, \boldsymbol{\kappa}) = \operatorname{cl}_{(2.16)} Q_1(\boldsymbol{\sigma}, \boldsymbol{\kappa}),$$

where $\operatorname{cl}_{(2.16)} Q_1$ *denotes the largest minorant which is less than* Q_1 *and is l.s.c. in the topology* (2.16) (*i.e.* $\operatorname{cl}_{(2.16)} Q_1$ *is the l.s.c. regularization of* Q_1 *in* (2.16)).

Proof. Step 1. First notice that

(2.27) $$Q_1(\boldsymbol{\sigma}, \boldsymbol{\kappa}) = \inf_{\boldsymbol{\sigma}_s, \boldsymbol{\kappa}_s} \left\{ (\widetilde{\mathbb{H}}_\lambda^j)^*(\boldsymbol{\sigma} + \boldsymbol{\sigma}_s, \boldsymbol{\kappa} + \boldsymbol{\kappa}_s) \,\Big|\, \boldsymbol{\sigma}_s \in C(\overline{\Omega}, \mathbf{E}_s^n), \right.$$

$$\left. \boldsymbol{\kappa}_s \in C(\Gamma_1, \mathbf{E}_s^n), \operatorname{div} \boldsymbol{\sigma}_s = \mathbf{0} \text{ and } \forall \mathbf{t} \in L^1(\Gamma_1, \mathbb{R}^n), \int_{\Gamma_1} \boldsymbol{\kappa}_s : [\mathbf{t} \otimes_s \boldsymbol{\nu}] \, ds = 0 \right\}$$

(see (2.11)–(2.13)). Next, assume the existence of a couple $(\widehat{\boldsymbol{\sigma}}_1, \boldsymbol{\kappa}_0) \in C_{\operatorname{div}}(\overline{\Omega}, \mathbf{E}_s^n) \times C(\Gamma_1, \mathbf{E}_s^n)$ and a constant $\delta_0 > 0$ such that $(\mathbb{H}_\lambda^j)^*(\widehat{\boldsymbol{\sigma}}_1, \boldsymbol{\kappa}_0) + \delta_0 < \operatorname{cl}_{(2.16)} Q_1(\widehat{\boldsymbol{\sigma}}_1, \boldsymbol{\kappa}_0)$. On account of Lemma 2, it suffices to show that this assumption leads to a contradiction, since $(\mathbb{H}_\lambda^j)^*(\boldsymbol{\sigma}, \boldsymbol{\kappa}) \leq (\widetilde{\mathbb{H}}_\lambda^j)^*(\boldsymbol{\sigma}, \boldsymbol{\kappa})$ for every $(\boldsymbol{\sigma}, \boldsymbol{\kappa}) \in C_{\operatorname{div}}(\overline{\Omega}, \mathbf{E}_s^n) \times C(\Gamma_1, \mathbf{E}_s^n)$ (cf. (2.7)–(2.9), (2.11)–(2.13)).

Step 2. The linear space

(2.28) $$\mathcal{M}_1 \equiv \{ \boldsymbol{\sigma}_s \in C_{\operatorname{div}}(\overline{\Omega}, \mathbf{E}_s^n) \mid \operatorname{div} \boldsymbol{\sigma}_s = \mathbf{0} \}$$

$$\times \left\{ \boldsymbol{\kappa} \in C(\Gamma_1, \mathbf{E}_s^n) \,\Big|\, \forall \mathbf{t} \in L^1(\Gamma_1, \mathbb{R}^n), \int_{\Gamma_1} \boldsymbol{\kappa} : [\mathbf{t} \otimes_s \boldsymbol{\nu}] \, ds = 0 \right\}$$

is a closed subspace of $C_{\operatorname{div}}(\overline{\Omega}, \mathbf{E}_s^n) \times C(\Gamma_1, \mathbf{E}_s^n)$ endowed with the topology (2.16). Indeed, by Green's formula (see [2, (2.11) and (4.23)]) we have

(2.29) $$\bigcap_{\mathbf{u} \in LD(\Omega), \boldsymbol{\gamma}_B(\mathbf{u}) = \mathbf{0} \text{ on } \Gamma_0} \left\{ (\boldsymbol{\sigma}, \boldsymbol{\kappa}) \in C_{\operatorname{div}}(\overline{\Omega}, \mathbf{E}_s^n) \times C(\Gamma_1, \mathbf{E}_s^n) \,\Big|\, \right.$$

$$\left. \int_\Omega (\operatorname{div} \boldsymbol{\sigma}) \cdot \mathbf{u} \, dx = \int_{\Gamma_1} \boldsymbol{\kappa} : (\boldsymbol{\gamma}_B(\mathbf{u}) \otimes_s \boldsymbol{\nu}) \, ds \right\}$$

$$= \bigcap_{\mathbf{u} \in LD(\Omega), \boldsymbol{\gamma}_B(\mathbf{u}) = \mathbf{0} \text{ on } \Gamma_0} \left\{ (\boldsymbol{\sigma}, \boldsymbol{\kappa}) \in C_{\operatorname{div}}(\overline{\Omega}, \mathbf{E}_s^n) \times C(\Gamma_1, \mathbf{E}_s^n) \,\Big|\, \right.$$

$$\left. \operatorname{div} \boldsymbol{\sigma} = \mathbf{0} \text{ in } \Omega, \int_{\Gamma_1} \boldsymbol{\kappa} : (\boldsymbol{\gamma}_B(\mathbf{u}) \otimes_s \boldsymbol{\nu}) \, ds = 0 \right\}$$

$$= \left\{ (\boldsymbol{\sigma}, \boldsymbol{\kappa}) \,\Big|\, \operatorname{div} \boldsymbol{\sigma} = \mathbf{0}, \int_{\Gamma_1} \boldsymbol{\kappa} : [\mathbf{t} \otimes_s \boldsymbol{\nu}] \, ds = 0, \forall \mathbf{t} \in L^1(\Gamma_1, \mathbb{R}^n) \right\} = \mathcal{M}_1,$$

since the trace $\boldsymbol{\gamma}_B$ is a surjection onto $L^1(\operatorname{Fr} \Omega, \mathbb{R}^n)$.

Step 3. Let

$$(2.30) \quad \Phi_1 : C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}_s^n) \times C(\Gamma_1, \mathbf{E}_s^n) \to [C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}_s^n) \times C(\Gamma_1, \mathbf{E}_s^n)]/\mathcal{M}_1$$

be the linear function (canonical homomorphism) such that $\mathcal{M}_1 = \ker \Phi_1$. Moreover, let $[C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}_s^n) \times C(\Gamma_1, \mathbf{E}_s^n)]/\mathcal{M}_1$ be endowed with the strongest topology for which Φ_1 is continuous, where $C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}_s^n) \times C(\Gamma_1, \mathbf{E}_s^n)$ is endowed with the topology (2.16). Then

$$(2.31) \quad \widehat{A}_1 \equiv \{((\widetilde{\boldsymbol{\sigma}}, \widetilde{\boldsymbol{\kappa}}), a) \in [(C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}_s^n) \times C(\Gamma_1, \mathbf{E}_s^n))/\mathcal{M}_1] \times \mathbb{R} \mid$$
$$\exists (\boldsymbol{\sigma}, \boldsymbol{\kappa}) \text{ such that } \mathrm{cl}_{(2.16)} Q_1(\boldsymbol{\sigma}, \boldsymbol{\kappa}) \leq a \text{ and } \Phi_1(\boldsymbol{\sigma}, \boldsymbol{\kappa}) = (\widetilde{\boldsymbol{\sigma}}, \widetilde{\boldsymbol{\kappa}})\}$$

is a convex closed set. By the Hahn–Banach theorem, there exists a closed affine hyperplane $\mathcal{H}_1$ which strictly separates the set $\widehat{A}_1$ and $(\Phi_1(\widehat{\boldsymbol{\sigma}}_1, \boldsymbol{\kappa}_0),$ $(\mathbb{H}_\lambda^j)^*(\widehat{\boldsymbol{\sigma}}_1, \boldsymbol{\kappa}_0) + \delta_0)$. From [1, (4.79)] and Assumption 4, we deduce that $\inf\{(\widetilde{\mathbb{H}}_\lambda^j)^*(\boldsymbol{\sigma}, \boldsymbol{\kappa}) - \int_\Omega \boldsymbol{\varepsilon}(\mathbf{u}^e) : \boldsymbol{\sigma}\, dx \mid \boldsymbol{\sigma} \in C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}_s^n),\ \boldsymbol{\kappa} \in C(\Gamma_1, \mathbf{E}_s^n)\}$ is finite (here $\mathbb{H}^\#$ is denoted by $(\widetilde{\mathbb{H}}_\lambda^j)^*$). Then, similarly to Step 4 of the proof of Proposition 11 from [2], there exists a continuous linear functional $\widetilde{f}_3 :$ $[C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}_s^n) \times C(\Gamma_1, \mathbf{E}_s^n)]/\mathcal{M}_1 \to \mathbb{R}$ and $\widetilde{c}_3 \in \mathbb{R}$ such that

$$(2.32) \quad \widetilde{f}_3(\Phi_1(\widehat{\boldsymbol{\sigma}}_1, \boldsymbol{\kappa}_0)) + \widetilde{c}_3 > (\mathbb{H}_\lambda^j)^*(\widehat{\boldsymbol{\sigma}}_1, \boldsymbol{\kappa}_0) + \delta_0, \qquad f_3(\widetilde{\boldsymbol{\sigma}}, \widetilde{\boldsymbol{\kappa}}) + c_3 < a$$

for every $((\widetilde{\boldsymbol{\sigma}}, \widetilde{\boldsymbol{\kappa}}), a) \in \widehat{A}_1$.

Step 4. Therefore the linear functional $\widetilde{f}_4$ defined by $\widetilde{f}_4 = \widetilde{f}_3 \circ \Phi_1$ strictly separates $\mathrm{epi}\, \mathrm{cl}_{(2.16)} Q_1$ and

$$(2.33) \quad \{(\boldsymbol{\sigma}, \boldsymbol{\kappa}, a) \in C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}_s^n) \times C(\Gamma_1, \mathbf{E}_s^n) \times \mathbb{R} \mid$$
$$(\boldsymbol{\sigma}, \boldsymbol{\kappa}) \in \mathcal{M}_1 + \{(\widehat{\boldsymbol{\sigma}}_1, \boldsymbol{\kappa}_0)\},\ a = (\mathbb{H}_\lambda^j)^*(\widehat{\boldsymbol{\sigma}}_1, \boldsymbol{\kappa}_0) + \delta_0\}.$$

Moreover $\mathcal{M}_1 \subset \ker \widetilde{f}_4$ and $\widetilde{f}_4$ is continuous in the topology (2.16), since $\widetilde{f}_4 = \widetilde{f}_3 \circ \Phi_1$.

Step 5. By Lemma 3, there exists $\widetilde{\mathbf{u}}_1 \in LD(\Omega)$ such that $\boldsymbol{\gamma}_B(\widetilde{\mathbf{u}}_1) = \mathbf{0}$ on Γ_0 and for every $(\boldsymbol{\sigma}, \boldsymbol{\kappa}) \in C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}_s^n) \times C(\Gamma_1, \mathbf{E}_s^n)$,

$$(2.34) \quad \widetilde{f}_4(\boldsymbol{\sigma}, \boldsymbol{\kappa}) = \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\widetilde{\mathbf{u}}_1)\, dx - \int_{\mathrm{Fr}\, \Omega} \boldsymbol{\sigma} : (\boldsymbol{\gamma}_B(\widetilde{\mathbf{u}}_1) \otimes_s \boldsymbol{\nu})\, ds$$
$$+ \int_{\Gamma_1} \boldsymbol{\kappa} : (\boldsymbol{\gamma}_B(\widetilde{\mathbf{u}}_1) \otimes_s \boldsymbol{\nu})\, ds.$$

Step 6. We say that a net $\{(\boldsymbol{\sigma}_t, \boldsymbol{\kappa}_t)\}_{t \in T} \subset C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}_s^n) \times C(\Gamma_1, \mathbf{E}_s^n)$ converges to $(\widehat{\boldsymbol{\sigma}}, \widehat{\boldsymbol{\kappa}})$ in the topology

$$(2.35) \quad \sigma(C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}_s^n) \times C(\Gamma_1, \mathbf{E}_s^n); \{(\boldsymbol{\varphi}, \boldsymbol{\mu} \otimes_s \boldsymbol{\nu}) \in \mathbf{Y}^1(\overline{\Omega}) \times \mathbf{M}^1(\Gamma_1) \mid$$
$$\exists \mathbf{u} \in BD(\Omega_1),\ \boldsymbol{\varepsilon}(\mathbf{u}) = \boldsymbol{\varphi},\ \mathbf{u}_{|\Omega} \in LD(\Omega),\ \mathbf{u}_{|\Omega_1 - \overline{\Omega}} = \mathbf{0},$$
$$\boldsymbol{\gamma}_B^I(\mathbf{u}) = \mathbf{0} \text{ on } \Gamma_0 \text{ and } \boldsymbol{\gamma}_B^I(\mathbf{u})\, ds = \boldsymbol{\mu} \text{ on } \Gamma_1\})$$

if

$$(2.36) \qquad \langle (\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}}, \boldsymbol{\gamma}_B^I(\mathbf{u})\, ds \otimes_s \boldsymbol{\nu}); (\boldsymbol{\sigma}_t, \boldsymbol{\kappa}_t) - (\widehat{\boldsymbol{\sigma}}, \widehat{\boldsymbol{\kappa}}) \rangle_2 \to 0$$

for every $\mathbf{u} \in BD(\Omega_1)$, where $\mathbf{u}_{|\Omega} \in LD(\Omega)$, $\mathbf{u}_{|\Omega_1 - \overline{\Omega}} = \mathbf{0}$ and $\boldsymbol{\gamma}_B^I(\mathbf{u}) = \mathbf{0}$ on Γ_0. The l.s.c. regularization of $(\widetilde{\mathbb{H}}_\lambda^j)^*$ in the topology (2.35) (denoted by $\mathrm{cl}_{(2.35)}(\widetilde{\mathbb{H}}_\lambda^j)^*)$ is given by

$$(2.37) \qquad \mathrm{cl}_{(2.35)}(\widetilde{\mathbb{H}}_\lambda^j)^*(\boldsymbol{\sigma}, \boldsymbol{\kappa}) = \sup\{\langle (\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}}, \boldsymbol{\gamma}_B^I(\mathbf{u})\, ds \otimes_s \boldsymbol{\nu}); (\boldsymbol{\sigma}, \boldsymbol{\kappa}) \rangle_2$$
$$- (\widetilde{\mathbb{H}}_\lambda^j)^{**}(\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}}, \boldsymbol{\gamma}_B^I(\mathbf{u})\, ds \otimes_s \boldsymbol{\nu}) \mid \mathbf{u} \in BD(\Omega_1),\ \mathbf{u}_{|\Omega} \in LD(\Omega),$$
$$\mathbf{u}_{|\Omega_1 - \overline{\Omega}} = \mathbf{0} \text{ and } \boldsymbol{\gamma}_B^I(\mathbf{u}) = \mathbf{0} \text{ on } \Gamma_0\} = \sup\{\langle \boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}}, (\boldsymbol{\gamma}_B^I(\mathbf{u})\, ds \otimes_s \boldsymbol{\nu});$$
$$(\boldsymbol{\sigma}, \boldsymbol{\kappa}) \rangle_2 - \mathbb{H}_\lambda^j(\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}}, \boldsymbol{\gamma}_B^I(\mathbf{u})\, ds) \mid \mathbf{u} \in BD(\Omega_1),$$
$$\mathbf{u}_{|\Omega} \in LD(\Omega),\ \mathbf{u}_{|\Omega_1 - \overline{\Omega}} = \mathbf{0} \text{ and } \boldsymbol{\gamma}_B^I(\mathbf{u}) = \mathbf{0} \text{ on } \Gamma_0\} = (\mathbb{H}_\lambda^j)^*(\boldsymbol{\sigma}, \boldsymbol{\kappa}),$$

for every $\boldsymbol{\sigma} \in C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}_s^n)$ and $\boldsymbol{\kappa} \in C(\Gamma_1, \mathbf{E}_s^n)$ (see Lemma 1).

By (2.32), (2.34) and (2.37) we obtain a contradiction, because $\widetilde{f}_4$ is continuous in (2.35). ∎

We say that a net $\{(\boldsymbol{\sigma}_t, \boldsymbol{\kappa}_t)\}_{t \in T} \subset C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}_s^n) \times C(\Gamma_1, \mathbf{E}_s^n)$ converges to $(\overline{\boldsymbol{\sigma}}, \overline{\boldsymbol{\kappa}})$ in the topology

$$(2.38) \qquad \sigma(C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}_s^n) \times C(\Gamma_1, \mathbf{E}_s^n), \mathbf{Y}^1(\overline{\Omega}) \times L^1(\Gamma_1, \mathbb{R}^n) \otimes_s \boldsymbol{\nu})$$

if

$$(2.39) \qquad \langle (\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}}, \mathbf{z}\, ds \otimes_s \boldsymbol{\nu}); (\boldsymbol{\sigma}_t, \boldsymbol{\kappa}_t) - (\overline{\boldsymbol{\sigma}}, \overline{\boldsymbol{\kappa}}) \rangle_2 \to 0$$

for every $\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}} \in \mathbf{Y}^1(\overline{\Omega})$ and $\mathbf{z} \in L^1(\Gamma_1, \mathbb{R}^n)$.

LEMMA 5. *For every $k > 0$, the topology (2.16) is stronger than (2.38) over the set $\overline{A}_k \equiv \{(\boldsymbol{\sigma}, \boldsymbol{\kappa}) \in C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}_s^n) \times C(\Gamma_1, \mathbf{E}_s^n) \mid \|\mathrm{div}\,\boldsymbol{\sigma}\|_{L^n} \leq k, \|\boldsymbol{\kappa}\|_{C(\Gamma_1, \mathbf{E}_s^n)} \leq k\}$.*

Proof. Suppose a net $\{(\boldsymbol{\sigma}_t, \boldsymbol{\kappa}_t)\}_{t \in T} \subset \overline{A}_k$ converges to $(\overline{\boldsymbol{\sigma}}, \overline{\boldsymbol{\kappa}})$ in the topology (2.16). Then by the Green formula (see [2]) we obtain $\int_{\Gamma_1} (\boldsymbol{\kappa}_t - \overline{\boldsymbol{\kappa}}) : (\boldsymbol{\gamma}_B(\mathbf{u}) \otimes_s \boldsymbol{\nu})\, ds - \int_\Omega \mathrm{div}(\boldsymbol{\sigma}_t - \overline{\boldsymbol{\sigma}}) \cdot \mathbf{u}\, dx \to 0$ for every $\mathbf{u} \in LD(\Omega)$ such that $\boldsymbol{\gamma}_B(\mathbf{u}) = \mathbf{0}$ on Γ_0. Since $C_c^1(\Omega, \mathbb{R}^n)$ is dense in $L^{n/(n-1)}(\Omega, \mathbb{R}^n)$, so is $C_c^1(\Omega, \mathbb{R}^n) + \{\mathbf{u}\}$ for every $\mathbf{u} \in LD(\Omega)$. Therefore the set

$$(2.40) \qquad \{(\mathbf{w}, \mathbf{z} \otimes_s \boldsymbol{\nu}) \in L^{n/(n-1)}(\Omega, \mathbb{R}^n) \times (L^1(\Gamma_1, \mathbb{R}^n) \otimes_s \boldsymbol{\nu}) \mid$$
$$\exists \widehat{\mathbf{u}}_1 \in C_c^1(\Omega, \mathbb{R}^n), \exists \widehat{\mathbf{u}}_2 \in LD(\Omega),\ \mathbf{w} = \widehat{\mathbf{u}}_1 + \widehat{\mathbf{u}}_2,\ \boldsymbol{\gamma}_B(\widehat{\mathbf{u}}_2) = \mathbf{z} \text{ on } \Gamma_1\}$$

is dense in $L^{n/(n-1)}(\Omega, \mathbb{R}^n) \times (L^1(\Gamma_1, \mathbb{R}^n) \otimes_s \boldsymbol{\nu})$. By [8, Theorem II.1.18],

$$(2.41) \qquad \int_{\Gamma_1} (\boldsymbol{\kappa}_t - \overline{\boldsymbol{\kappa}}) : (\mathbf{z} \otimes_s \boldsymbol{\nu})\, ds - \int_\Omega \mathrm{div}(\boldsymbol{\sigma}_t - \overline{\boldsymbol{\sigma}}) \cdot \mathbf{u}\, dx \to 0$$

for all $\mathbf{u} \in BD(\Omega)$ and $\mathbf{z} \in L^1(\Gamma_1, \mathbb{R}^n)$. By the Green formula, $\{(\boldsymbol{\sigma}_t, \boldsymbol{\kappa}_t)\}_{t \in T}$ converges to $(\overline{\boldsymbol{\sigma}}, \overline{\boldsymbol{\kappa}})$ in the topology (2.38). ∎

PROPOSITION 6. *Let $\overline{A}_k$ be as in Lemma 5. For every $\widehat{\boldsymbol{\sigma}} \in C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}_s^n)$ and $\widehat{\boldsymbol{\kappa}} \in C(\Gamma_1, \mathbf{E}_s^n)$ there exists $k_{(\widehat{\boldsymbol{\sigma}}, \widehat{\boldsymbol{\kappa}})} > 0$ such that for every $k \geq k_{(\widehat{\boldsymbol{\sigma}}, \widehat{\boldsymbol{\kappa}})}$,*

$$(2.42) \qquad (\mathbb{H}_\lambda^j)^*(\widehat{\boldsymbol{\sigma}}, \widehat{\boldsymbol{\kappa}}) = \mathrm{cl}_{\overline{A}_k} Q_1(\widehat{\boldsymbol{\sigma}}, \widehat{\boldsymbol{\kappa}}),$$

where $\mathrm{cl}_{\overline{A}_k} Q_1(\cdot, \cdot)$ is the l.s.c. regularization of $(\boldsymbol{\sigma}, \boldsymbol{\kappa}) \mapsto Q_1(\boldsymbol{\sigma}, \boldsymbol{\kappa}) + I_{\overline{A}_k}(\boldsymbol{\sigma}, \boldsymbol{\kappa})$ in the topology (2.16) and $I_{\overline{A}_k}(\cdot)$ is the indicator function of $\overline{A}_k$.

Proof. We prove this result similarly to [2, Proposition 13]. ∎

THEOREM 7. *For every $\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}} \in \mathbf{Y}^1(\overline{\Omega})$ and $\boldsymbol{\mu} \in \mathbb{M}_b(\Gamma_1, \mathbb{R}^n)$ we have*

$$(2.43) \qquad (\widetilde{\mathbb{H}}_\lambda^j)^{**}(\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}}, \boldsymbol{\mu} \otimes_s \boldsymbol{\nu}) = (\mathbb{H}_\lambda^j)^{**}(\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}}, \boldsymbol{\mu} \otimes_s \boldsymbol{\nu}).$$

Proof. Step 1. By Proposition 4, Lemma 5 and Proposition 6 (similarly to the proof of Theorem 14 from [2]) we get $(\widetilde{\mathbb{H}}_\lambda^j)^{**}(\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}}, \mathbf{z} \otimes_s \boldsymbol{\nu}) = (\mathbb{H}_\lambda^j)^{**}(\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}}, \mathbf{z} \otimes_s \boldsymbol{\nu})$ for every $\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}} \in \mathbf{Y}^1(\overline{\Omega})$ and $\mathbf{z} \in L^1(\Gamma_1, \mathbb{R}^n)$.

Step 2. In [1, (4.80)] we obtain the explicit representation of $(\widetilde{\mathbb{H}}_\lambda^j)^{**}$:

$$(2.44) \qquad (\widetilde{\mathbb{H}}_\lambda^j)^{**}(\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}}, \boldsymbol{\mu} \otimes_s \boldsymbol{\nu}) = -\int_{\Gamma_1} \lambda \mathbf{g} \boldsymbol{\mu} + \int_\Omega j(x, \boldsymbol{\varepsilon}(\mathbf{u})_a)\, dx$$

$$+ \int_{\Gamma_1} j_\infty \left(x, \frac{d((\boldsymbol{\mu} - \gamma_B^I(\mathbf{u})\, ds) \otimes_s \boldsymbol{\nu})}{d|(\boldsymbol{\mu} - \gamma_B^I(\mathbf{u})\, ds) \otimes_s \boldsymbol{\nu}|} \right) d|(\boldsymbol{\mu} - \gamma_B^I(\mathbf{u})\, ds) \otimes_s \boldsymbol{\nu}|$$

$$+ \int_\Omega j_\infty \left(x, \frac{d\boldsymbol{\varepsilon}(\mathbf{u})_s}{d|\boldsymbol{\varepsilon}(\mathbf{u})_s|} \right) d|\boldsymbol{\varepsilon}(\mathbf{u})_s| + \int_{\Gamma_0} j_\infty(x, (-\boldsymbol{\gamma}_B^I(\mathbf{u})) \otimes_s \boldsymbol{\nu})\, ds$$

for every $\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}} \in \mathbf{Y}^1(\overline{\Omega})$ and $\boldsymbol{\mu} \in \mathbb{M}_b(\Gamma_1, \mathbb{R}^n)$ (now $\mathbb{H}^{\#*}$ is denoted by $(\widetilde{\mathbb{H}}_\lambda^j)^{**}$). Consider the function $Q_{\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}}} : \mathbf{M}^1(\Gamma_1) \to \mathbb{R} \cup \{\infty\}$ for $\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}} \in \mathbf{Y}^1(\overline{\Omega})$, defined by

$$(2.45) \qquad \boldsymbol{\mu} \otimes_s \boldsymbol{\nu} \mapsto Q_{\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}}}(\boldsymbol{\mu} \otimes_s \boldsymbol{\nu}) = \begin{cases} (\widetilde{\mathbb{H}}_\lambda^j)^{**}(\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}}, \boldsymbol{\mu} \otimes_s \boldsymbol{\nu}) \\ \qquad \text{if } \boldsymbol{\mu} \in L^1(\Gamma_1, \mathbb{R}^n), \\ \infty \quad \text{otherwise.} \end{cases}$$

By (2.44),

$$(2.46) \qquad Q_{\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}}}(\boldsymbol{\mu} \otimes_s \boldsymbol{\nu}) = -\int_{\Gamma_1} \lambda \mathbf{g} \boldsymbol{\mu} + c_{\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}}}$$

$$+ \int_{\Gamma_1} j_\infty \left(x, \frac{d((\boldsymbol{\mu} - \gamma_B^I(\mathbf{u})\, ds) \otimes_s \boldsymbol{\nu})}{d|(\boldsymbol{\mu} - \gamma_B^I(\mathbf{u})\, ds) \otimes_s \boldsymbol{\nu}|} \right) d|(\boldsymbol{\mu} - \boldsymbol{\gamma}_B^I(\mathbf{u})\, ds) \otimes_s \boldsymbol{\nu}|$$

for every $\boldsymbol{\mu} \in L^1(\Gamma_1, \mathbb{R}^n)$, where $c_{\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}}}$ depends only on $\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}} \in \mathbf{Y}^1(\overline{\Omega})$. We now prove that the l.s.c. regularization of $Q_{\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}}}$ in the topology $\sigma(\mathbf{M}^1(\Gamma_1), C(\Gamma_1, \mathbf{E}_s^n))$ (denoted by $\operatorname{cl} Q_{\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}}}$) is given by

$$(2.47) \qquad \operatorname{cl} Q_{\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}}}(\boldsymbol{\mu} \otimes_s \boldsymbol{\nu}) = (\widetilde{\mathbb{H}}_\lambda^j)^{**}(\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}}, \boldsymbol{\mu} \otimes_s \boldsymbol{\nu})$$

for $\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}} \in \mathbf{Y}^1(\overline{\Omega})$ and $\boldsymbol{\mu} \in \mathbb{M}_b(\Gamma_1, \mathbb{R}^n)$.

Indeed, by [5], we obtain

$$(2.48) \qquad Q_{\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}}}^*(\boldsymbol{\kappa}) = \sup \Big\{ \int_{\Gamma_1} (\boldsymbol{\kappa} \cdot \boldsymbol{\nu} + \lambda \mathbf{g}) \cdot \boldsymbol{\mu} \, ds$$

$$- \int_{\Gamma_1} j_\infty(x, (\boldsymbol{\mu} - \boldsymbol{\gamma}_B^I(\mathbf{u})) \otimes_s \boldsymbol{\nu}) \, ds - c_{\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}}} \, \Big| \, \boldsymbol{\mu} \in L^1(\Gamma_1, \mathbb{R}^n) \Big\}$$

$$= \sup \Big\{ \int_{\Gamma_1} (\boldsymbol{\kappa} \cdot \boldsymbol{\nu} + \lambda \mathbf{g}) \cdot (\boldsymbol{\mu} - \boldsymbol{\gamma}_B^I(\mathbf{u})) \, ds - \int_{\Gamma_1} j_\infty(x, (\boldsymbol{\mu} - \boldsymbol{\gamma}_B^I(\mathbf{u})) \otimes_s \boldsymbol{\nu}) \, ds$$

$$- c_{\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}}} \, \Big| \, \boldsymbol{\mu} \in L^1(\Gamma_1, \mathbb{R}^n) \Big\} + \int_{\Gamma_1} (\boldsymbol{\kappa} \cdot \boldsymbol{\nu} + \lambda \mathbf{g}) \cdot \boldsymbol{\gamma}_B^I(\mathbf{u}) \, ds$$

$$= \int_{\Gamma_1} (\boldsymbol{\kappa} \cdot \boldsymbol{\nu} + \lambda \mathbf{g}) \cdot \boldsymbol{\gamma}_B^I(\mathbf{u}) \, ds + \int_{\Gamma_1} I_{\{\boldsymbol{\kappa} \cdot \boldsymbol{\nu} + \lambda \mathbf{g} \in \mathcal{K}(x) \cdot \boldsymbol{\nu}(x)\}}(\boldsymbol{\kappa} \cdot \boldsymbol{\nu} + \lambda \mathbf{g}) \, ds - c_{\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}}}$$

for every $\boldsymbol{\kappa} \in C(\Gamma_1, \mathbf{E}_s^n)$, and

$$(2.49) \qquad Q_{\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}}}^{**}(\boldsymbol{\mu} \otimes_s \boldsymbol{\nu}) = \sup \Big\{ \int_{\Gamma_1} \boldsymbol{\kappa} : (\boldsymbol{\mu} \otimes_s \boldsymbol{\nu}) - \int_{\Gamma_1} (\boldsymbol{\kappa} \cdot \boldsymbol{\nu} + \lambda \mathbf{g}) \cdot \boldsymbol{\gamma}_B^I(\mathbf{u}) \, ds$$

$$- \int_{\Gamma_1} I_{\{\boldsymbol{\kappa} \cdot \boldsymbol{\nu} + \lambda \mathbf{g} \in \mathcal{K}(x) \cdot \boldsymbol{\nu}(x)\}}(\boldsymbol{\kappa} \cdot \boldsymbol{\nu} + \lambda \mathbf{g}) \, ds + c_{\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}}} \, \Big| \, \boldsymbol{\kappa} \in C(\Gamma_1, \mathbf{E}_s^n) \Big\}$$

$$= \sup \Big\{ \int_{\Gamma_1} (\boldsymbol{\kappa} \cdot \boldsymbol{\nu} + \lambda \mathbf{g}) \cdot (\boldsymbol{\mu} - \boldsymbol{\gamma}_B^I(\mathbf{u}) \, ds) - \int_{\Gamma_1} I_{\{\boldsymbol{\kappa} \cdot \boldsymbol{\nu} + \lambda \mathbf{g} \in \mathcal{K}(x) \cdot \boldsymbol{\nu}(x)\}}(\boldsymbol{\kappa} \cdot \boldsymbol{\nu} + \lambda \mathbf{g}) \, ds \, \Big|$$

$$\boldsymbol{\kappa} \in C(\Gamma_1, \mathbf{E}_s^n) \Big\} - \int_{\Gamma_1} \lambda \mathbf{g} \cdot \boldsymbol{\mu} + c_{\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}}}$$

$$= \int_{\Gamma_1} j_\infty \Big(x, \frac{d((\boldsymbol{\mu} - \boldsymbol{\gamma}_B^I(\mathbf{u}) \, ds) \otimes_s \boldsymbol{\nu})}{d|(\boldsymbol{\mu} - \boldsymbol{\gamma}_B^I(\mathbf{u}) \, ds) \otimes_s \boldsymbol{\nu}|} \Big) \, d|(\boldsymbol{\mu} - \boldsymbol{\gamma}_B^I(\mathbf{u}) \, ds) \otimes_s \boldsymbol{\nu}|$$

$$- \int_{\Gamma_1} \lambda \mathbf{g} \cdot \boldsymbol{\mu} + c_{\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}}}$$

for all $\boldsymbol{\mu} \in \mathbb{M}_b(\Gamma_1, \mathbb{R}^n)$. Thus we get (2.47). Since $(\widetilde{\mathbb{H}}_\lambda^j)^{**}(\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}}, \boldsymbol{\mu} \otimes_s \boldsymbol{\nu})$ $= (\mathbb{H}_\lambda^j)^{**}(\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}}, \boldsymbol{\mu} \otimes_s \boldsymbol{\nu})$ for every $(\boldsymbol{\varepsilon}(\mathbf{u})_{|\overline{\Omega}}, \boldsymbol{\mu}) \in \mathbf{Y}^1(\overline{\Omega}) \times L^1(\Gamma_1, \mathbb{R}^n)$, and $(\mathbb{H}_\lambda^j)^{**}$ is the l.s.c. regularization of $\mathbb{H}_\lambda^j$, we obtain (2.43). ∎

Now we turn to mechanical conclusions.

The following is known as the *Suquet problem* (see [12]):

(2.50) $(SQ_{\lambda,j})$ Find $\inf\{\lambda F_{SQ}(\mathbf{u}, \boldsymbol{\mu}) + G_j(\boldsymbol{\varepsilon}(\mathbf{u})) \mid$
$$\mathbf{u} \in LD(\Omega), \boldsymbol{\mu} \in \mathbb{M}_b(\Gamma_1, \mathbb{R}^n)\},$$

where the functional of the total loading has the form

(2.51) $\lambda F_{SQ}(\mathbf{u}, \boldsymbol{\mu}) \equiv -\lambda\Big(\int\limits_{\Omega} \mathbf{f} \cdot \mathbf{u}\, dx + \int\limits_{\Gamma_1} \mathbf{g} \cdot \boldsymbol{\mu} \Big)$

$$+ \int\limits_{\Gamma_0} I_{\{\boldsymbol{\gamma}_B(\mathbf{u})(x)=\mathbf{0}\}}(-\boldsymbol{\gamma}_B(\mathbf{u}) \otimes_s \boldsymbol{\nu})\, ds$$

if $\boldsymbol{\mu} = \boldsymbol{\gamma}_B^I(\mathbf{u})\, ds$ for $|\boldsymbol{\mu} - \boldsymbol{\gamma}_B^I(\mathbf{u})\, ds|$-a.e. $x \in \Gamma_1$, and $\lambda F_{SQ}(\mathbf{u}, \boldsymbol{\mu}) \equiv \infty$ otherwise. The elastic-plastic potential G_j is defined by [2, (3.11)].

The following bidual relaxed *Suquet problem* is studied in [1]:

(2.52) $(SP_{\lambda,j}^{**})$ Find $\inf\{(\lambda F_H)^{**}(\mathbf{u}, \boldsymbol{\mu}) + G_j^{**}(\boldsymbol{\varepsilon}(\mathbf{u})) \mid$
$$\mathbf{u} \in BD(\Omega), \boldsymbol{\mu} \in \mathbb{M}_b(\Gamma_1, \mathbb{R}^n)\},$$

where for every $\mathbf{u} \in BD(\Omega)$ and $\boldsymbol{\mu} \in \mathbb{M}_b(\Gamma_1, \mathbb{R}^n)$,

(2.53) $(\lambda F_H)^{**}(\mathbf{u}, \boldsymbol{\mu}) \equiv -\lambda\Big[\int\limits_{\Omega} \mathbf{f} \cdot \mathbf{u}\, dx + \int\limits_{\Gamma_1} \mathbf{g} \cdot \boldsymbol{\mu} \Big]$

$$+ \int\limits_{\Gamma_0} j_\infty(x, (-\boldsymbol{\gamma}_B(\mathbf{u}) \otimes_s \boldsymbol{\nu}))\, ds$$

$$+ \int\limits_{\Gamma_1} j_\infty\left(x, \frac{d((\boldsymbol{\mu} - \boldsymbol{\gamma}_B(\mathbf{u})\, ds) \otimes_s \boldsymbol{\nu})}{d|(\boldsymbol{\mu} - \boldsymbol{\gamma}_B(\mathbf{u})\, ds) \otimes_s \boldsymbol{\nu}|} \right) d|(\boldsymbol{\mu} - \boldsymbol{\gamma}_B(\mathbf{u})\, ds) \otimes_s \boldsymbol{\nu}|$$

and $G_j^{**}(\boldsymbol{\varepsilon}(\cdot))$ is defined in [2, (5.4)].

ASSUMPTION 7. There exist $k_b > 0$ and $r_1 > 0$ such that $j^*(x, \mathbf{w}^*) \le k_b$ for every $\mathbf{w}^* \in B_{\mathbf{E}_s^n}(0, r_1)$ and for dx-a.e. $x \in \Omega$, where $B_{\mathbf{E}_s^n}(0, r_1)$ is the closed ball in $\mathbf{E}_s^n$ with center 0 and radius r_1. ∎

Let the functional

(2.54) $(\mathbf{u}, \boldsymbol{\mu}) \mapsto \lceil SQ_{\lambda,j} \rceil(\mathbf{u}, \boldsymbol{\mu}) \equiv \lambda F_{SQ}(\mathbf{u}, \boldsymbol{\mu}) + G_j(\boldsymbol{\varepsilon}(\mathbf{u}))$

be coercive over $BD(\Omega) \times \mathbb{M}_b(\Gamma_1, \mathbb{R}^n)$, i.e.,

(2.55)   if $\|\mathbf{u}_m\|_{BD} + \|\boldsymbol{\mu}_m\|_{\mathbb{M}_b} \to \infty$ then $\lceil SQ_{\lambda,j} \rceil(\mathbf{u}_m, \boldsymbol{\mu}_m) \to \infty$

for any sequences $\{\mathbf{u}_m\}_{m\in\mathbb{N}} \subset BD(\Omega)$ and $\{\boldsymbol{\mu}_m\}_{m\in\mathbb{N}} \subset \mathbb{M}_b(\Gamma_1, \mathbb{R}^n)$. Moreover, let $0 \le \lambda_1 < \lambda$. Then the functional $\lceil SQ_{\lambda_1,j} \rceil$ is coercive on $BD(\Omega) \times \mathbb{M}_b(\Gamma_1, \mathbb{R}^n)$. Similarly, if the functional

(2.56) $(\mathbf{u}, \boldsymbol{\mu}) \mapsto \lceil SP_{\lambda,j}^{**} \rceil(\mathbf{u}, \boldsymbol{\mu}) \equiv (\lambda F_H)^{**}(\mathbf{u}, \boldsymbol{\mu}) + G_j^{**}(\boldsymbol{\varepsilon}(\mathbf{u}))$

is coercive and $0 \le \lambda_1 \le \lambda$, then $\lceil SP_{\lambda_1,j}^{**} \rceil$ is coercive over $BD(\Omega) \times \mathbb{M}_b(\Gamma_1, \mathbb{R}^n)$. Moreover, we obtain $\lceil SQ_{\lambda,j} \rceil \ge \lceil SP_{\lambda,j}^{**} \rceil$.

We say that a net $\{(\mathbf{u}_t, \boldsymbol{\mu}_t)\}_{t \in T} \subset BD(\Omega) \times \mathbb{M}_b(\Gamma_1, \mathbb{R}^n)$ converges to $(\mathbf{u}_0, \boldsymbol{\mu}_0)$ in the weak* $BD(\Omega) \times \mathbb{M}_b(\Gamma_1, \mathbb{R}^n)$ topology if $\mathbf{u}_t \rightharpoonup \mathbf{u}_0$ in the weak* $BD(\Omega)$ topology and $\boldsymbol{\mu}_t \rightharpoonup \boldsymbol{\mu}_0$ in the weak* $\mathbb{M}_b(\Gamma_1, \mathbb{R}^n)$ topology.

THEOREM 8. *Let* $ds(\Gamma_0) \neq 0$ *and* $\mathbf{f} \in L^{n+\delta}(\Omega, \mathbb{R}^n)$, *where* $\delta \geq 0$. *If the functional* $\lceil SQ_{\lambda,j} \rceil$ *is coercive over* $BD(\Omega) \times \mathbb{M}_b(\Gamma_1, \mathbb{R}^n)$, *then the l.s.c. regularization of* $\lceil SQ_{\lambda,j} \rceil$ *in the weak* $BD(\Omega) \times \mathbb{M}_b(\Gamma_1, \mathbb{R}^n)$ *topology is the functional* $\lceil SP_{\lambda,j}^{**} \rceil$ (cf. (2.50), (2.52), (2.53)).

Proof. Step 1. Let $(\widetilde{\mathbf{u}}, \widetilde{\boldsymbol{\mu}}) \in BD(\Omega) \times \mathbb{M}_b(\Gamma_1, \mathbb{R}^n)$ and $\{(\mathbf{u}_p, \boldsymbol{\mu}_p)\}_{p \in P}$ be a net such that $(\mathbf{u}_p, \boldsymbol{\mu}_p) \rightharpoonup (\widetilde{\mathbf{u}}, \widetilde{\boldsymbol{\mu}})$ in the weak* $BD(\Omega) \times \mathbb{M}_b(\Gamma_1, \mathbb{R}^n)$ topology. By Steps 1 and 2 of [2, proof of Theorem 18], $\liminf_{p \in P} \lceil SQ_{\lambda,j} \rceil (\mathbf{u}_p, \boldsymbol{\mu}_p) \geq \lceil SP_{\lambda,j}^{**} \rceil (\widetilde{\mathbf{u}}, \widetilde{\boldsymbol{\mu}})$, since $\lceil SQ_{\lambda,j} \rceil$ is equal to the function defined in [2, (3.8)] and $\lceil RP_{\lambda,j}^{**} \rceil$ is equal to $\lceil SP_{\lambda,j}^{**} \rceil$ (cf. [2, (5.6)]) if $\boldsymbol{\mu} = \boldsymbol{\gamma}_B(\mathbf{u}) \, ds$ on Γ_1.

Step 2. Let $(\widetilde{\mathbb{H}}_\lambda^j)^{**}(\boldsymbol{\varepsilon}(\widetilde{\mathbf{u}})_{|\overline{\Omega}}, \widetilde{\boldsymbol{\mu}} \otimes_s \boldsymbol{\nu}) < \infty$. By Theorem 7, there is a net $\{(\mathbf{u}_t, \boldsymbol{\mu}_t)\}_{t \in \Sigma} \subset BD(\Omega) \times \mathbb{M}_b(\Gamma_1, \mathbb{R}^n)$ such that $(\mathbf{u}_t, \boldsymbol{\mu}_t \otimes_s \boldsymbol{\nu}) \rightharpoonup (\widetilde{\mathbf{u}}, \widetilde{\boldsymbol{\mu}} \otimes_s \boldsymbol{\nu})$ in the topology $\sigma(\mathbf{Y}^1(\overline{\Omega}) \times \mathbf{M}^1(\Gamma_1), C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}_s^n) \times C(\Gamma_1, \mathbf{E}_s^n))$, where the bilinear form $\langle \cdot; \cdot \rangle_2$ between $\mathbf{Y}^1(\overline{\Omega}) \times \mathbf{M}^1(\Gamma_1)$ and $C_{\mathrm{div}}(\overline{\Omega}, \mathbf{E}_s^n) \times C(\Gamma_1, \mathbf{E}_s^n)$ is given by (2.8). Moreover,

$$(2.57) \quad \lim_{t \in \Sigma} \left(\lceil SQ_{\lambda,j} \rceil (\mathbf{u}_t, \boldsymbol{\mu}_t) + \lambda \int_\Omega \mathbf{f} \cdot \mathbf{u}_t \, dx \right) = (\widetilde{\mathbb{H}}_\lambda^j)^{**}(\boldsymbol{\varepsilon}(\widetilde{\mathbf{u}})_{|\overline{\Omega}}, \widetilde{\boldsymbol{\mu}} \otimes_s \boldsymbol{\nu}).$$

Theorem 7 holds in the special case when $\mathbf{g} = \mathbf{0}$ on Γ_1. Then

$$(2.58) \quad \lim_{t \in \Sigma} \left[\lceil SQ_{\lambda,j} \rceil (\mathbf{u}_t, \boldsymbol{\mu}_t) + \lambda \left(\int_\Omega \mathbf{f} \cdot \mathbf{u}_t \, dx + \int_{\Gamma_1} \mathbf{g} \cdot \boldsymbol{\mu}_t \right) \right]$$
$$= (\widetilde{\mathbb{H}}_\lambda^j)^{**}(\boldsymbol{\varepsilon}(\widetilde{\mathbf{u}})_{|\overline{\Omega}}, \widetilde{\boldsymbol{\mu}} \otimes_s \boldsymbol{\nu}) + \lambda \int_{\Gamma_1} \mathbf{g} \cdot \widetilde{\boldsymbol{\mu}} \in \mathbb{R},$$

because $|\lambda \int_{\Gamma_1} \mathbf{g} \cdot \widetilde{\boldsymbol{\mu}}| < \infty$. By Assumption 7, [2, Lemma 16] and (2.58), the net $\{(\mathbf{u}_t, \boldsymbol{\mu}_t)\}_{t \in \Sigma}$ is bounded and $(\mathbf{u}_t, \boldsymbol{\mu}_t) \rightharpoonup (\widetilde{\mathbf{u}}, \widetilde{\boldsymbol{\mu}})$ in the weak* $BD(\Omega) \times \mathbb{M}_b(\Gamma_1, \mathbb{R}^n)$ topology. Therefore, by Theorem 7 and [2, Lemma 15], we get

$$(2.59) \quad \lim_{t \in \Sigma} \lceil SQ_{\lambda,j} \rceil (\mathbf{u}_t, \boldsymbol{\mu}_t) = \lceil SP_{\lambda,j}^{**} \rceil (\widetilde{\mathbf{u}}, \widetilde{\boldsymbol{\mu}}).$$

Step 3. Let $(\widetilde{\mathbb{H}}_\lambda^j)^{**}(\boldsymbol{\varepsilon}(\widetilde{\mathbf{u}})_{|\overline{\Omega}}, \widetilde{\boldsymbol{\mu}} \otimes_s \boldsymbol{\nu}) = \infty$. Then, by Step 4 of the proof of Theorem 18 from [2], for every net $\{(\mathbf{u}_p, \boldsymbol{\mu}_p)\}_{p \in P}$ such that $(\mathbf{u}_p, \boldsymbol{\mu}_p) \rightharpoonup (\widetilde{\mathbf{u}}, \widetilde{\boldsymbol{\mu}})$ in the weak* $BD(\Omega) \times \mathbb{M}_b(\Gamma_1, \mathbb{R}^n)$ topology, we have

$$(2.60) \quad \liminf_{p \in P} \lceil SQ_{\lambda,j} \rceil (\mathbf{u}_t, \boldsymbol{\mu}_t) = \infty = \lceil SP_{\lambda,j}^{**} \rceil (\widetilde{\mathbf{u}}, \widetilde{\boldsymbol{\mu}}).$$

Next, we proceed similarly to Step 5 of the proof of [2, Theorem 18]. ∎

COROLLARY 9. $\lceil SQ_{\lambda,j} \rceil$ *is coercive over* $BD(\Omega) \times \mathbb{M}_b(\Gamma_1, \mathbb{R}^n)$ *if and only if* $\lceil SP_{\lambda,j}^{**} \rceil$ *is coercive.*

Proof. We prove this similarly to [2, Corollary 19]. ∎

COROLLARY 10. *The functional* (3.8) *from* [2] *is coercive over* $BD(\Omega)$ *if and only if* $\lceil SP^{**}_{\lambda,j} \rceil$ *is coercive, where* $\mathbf{u}^0 = \mathbf{0}$ *on* Γ_0 *(in the functional* [2, (3.8)]*).*

Proof. Indeed, by [2, (3.11)], (2.51) and (2.54), the functional $\lceil SQ_{\lambda,j} \rceil$ is coercive over $BD(\Omega) \times \mathbb{M}_b(\Gamma_1, \mathbb{R}^n)$ if and only if (3.8) from [2] is coercive over $BD(\Omega)$. ∎

THEOREM 11. *Let* $ds(\Gamma_0) \neq 0$ *and* $\mathbf{f} \in L^{n+\delta}(\Omega, \mathbb{R}^n)$, *where* $\delta \geq 0$. *Moreover, let* $j^* : \Omega \times \mathbf{E}^n_s \to \mathbb{R} \cup \{\infty\}$ *be a nonnegative normal integrand. If the functional* (3.8) *from* [2] *is coercive over* $BD(\Omega)$ *and* $\mathbf{u}^0 = \mathbf{0}$ *on* Γ_0 *(in* [2, (3.8)]*), then the couple* $(\widetilde{\mathbf{u}}, \widetilde{\boldsymbol{\mu}}) \in BD(\Omega) \times \mathbb{M}_b(\Gamma_1, \mathbb{R}^n)$ *is a solution of the problem* $(SP^{**}_{\lambda,j})$ *if and only if* $\widetilde{\boldsymbol{\mu}} = \boldsymbol{\gamma}_B(\widetilde{\mathbf{u}})_{|\Gamma_1} \, ds \in \mathbb{M}_b(\Gamma_1, \mathbb{R}^n)$ *and* $\widetilde{\mathbf{u}}$ *is a solution of the problem* $(RP^{**}_{\lambda,j})$ *(cf.* [2, (5.2)–(5.4)], (2.52) *and* (2.53)*).*

Proof. Since the normal integrand j^* is nonnegative, $(RP^{**}_{\lambda,j})$ and $(SP^{**}_{\lambda,j})$ have finite infima, because $\lceil RP^{**}_{\lambda,j} \rceil(\mathbf{0}) < \infty$ and $\lceil SP^{**}_{\lambda,j} \rceil(\mathbf{0}, \mathbf{0}) < \infty$ (cf. [2, (5.6)]).

First, let $\widetilde{\mathbf{u}}$ be a solution of $(RP^{**}_{\lambda,j})$. Then the couple $(\widetilde{\mathbf{u}}, \boldsymbol{\gamma}_B(\widetilde{\mathbf{u}})_{|\Gamma_1} \, ds) \in BD(\Omega) \times \mathbb{M}_b(\Gamma_1, \mathbb{R}^n)$ is a solution of $(SP^{**}_{\lambda,j})$, since

$$(2.61) \qquad \inf(RP^{**}_{\lambda,j}) = \inf(P_{\lambda,j}) = \inf(SQ_{\lambda,j}) = \inf(SP^{**}_{\lambda,j}),$$

where $(P_{\lambda,j})$ is defined in [2, (5.1) and (3.8)] (cf. [2, Theorem 18], Theorem 8, [2, (3.8), (5.2), (5.1)] and (2.50), (2.52)).

Next, we show the inverse implication. Let $(\widetilde{\mathbf{u}}, \widetilde{\boldsymbol{\mu}})$ be a solution of $(SP^{**}_{\lambda,j})$. Then there exists a sequence $\{(\mathbf{u}_m, \boldsymbol{\mu}_m)\}_{m \in \mathbb{N}}$ in $BD(\Omega) \times \mathbb{M}_b(\Gamma_1, \mathbb{R}^n)$ such that $\lim_{m \to \infty} \lceil SQ_{\lambda,j} \rceil(\mathbf{u}_m, \boldsymbol{\mu}_m) = \inf(SP^{**}_{\lambda,j})$ and $(\mathbf{u}_m, \boldsymbol{\mu}_m) \rightharpoonup (\widetilde{\mathbf{u}}, \widetilde{\boldsymbol{\mu}})$ in weak* $BD(\Omega) \times \mathbb{M}_b(\Gamma_1, \mathbb{R}^n)$ topology. By (2.50), there exists $m_0 \in \mathbb{N}$ such that for every $m > m_0$ we have $\boldsymbol{\mu}_m = \boldsymbol{\gamma}_B(\mathbf{u}_m) \, ds$ on Γ_1, because $\inf(SP^{**}_{\lambda,j}) < \infty$. Since $\inf(RP^{**}_{\lambda,j}) = \inf(SP^{**}_{\lambda,j})$, we obtain

$$(2.62) \quad \lim_{m \to \infty} \lceil SQ_{\lambda,j} \rceil(\mathbf{u}_m, \boldsymbol{\gamma}_B(\mathbf{u}_m)_{|\Gamma_1} \, ds) = \lim_{m \to \infty} P_{\lambda,j}(\mathbf{u}_m) = \inf \lceil RP^{**}_{\lambda,j} \rceil,$$

where $P_{\lambda,j}$ is defined in [2, (3.8)] (cf. (2.61)). Therefore, by [2, Theorem 18], we get

$$(2.63) \qquad\qquad \lceil RP^{**}_{\lambda,j} \rceil(\widetilde{\mathbf{u}}) = \inf(RP^{**}_{\lambda,j}) = \inf(SP^{**}_{\lambda,j}).$$

Reasoning by contradiction, we prove that $\widetilde{\boldsymbol{\mu}} = \boldsymbol{\gamma}_B(\widetilde{\mathbf{u}}) \, ds$ on Γ_1. Suppose $\widetilde{\boldsymbol{\mu}} \neq \boldsymbol{\gamma}_B(\widetilde{\mathbf{u}})_{|\Gamma_1} \, ds$. By [2, (5.2)], (2.52) and (2.63) we have $\lceil SP^{**}_{\lambda,j} \rceil(\widetilde{\mathbf{u}}, \boldsymbol{\gamma}_B(\widetilde{\mathbf{u}})_{|\Gamma_1} \, ds) = \lceil SP^{**}_{\lambda,j} \rceil(\widetilde{\mathbf{u}}, \widetilde{\boldsymbol{\mu}})$. The function

$$[0, \infty) \ni t \mapsto \int_{\Gamma_1} j_\infty \left(x, \frac{d(t(\widetilde{\boldsymbol{\mu}} - \boldsymbol{\gamma}_B(\widetilde{\mathbf{u}}) \, ds) \otimes_s \boldsymbol{\nu})}{d|(\widetilde{\boldsymbol{\mu}} - \boldsymbol{\gamma}_B(\widetilde{\mathbf{u}}) \, ds) \otimes_s \boldsymbol{\nu}|} \right) d|(\widetilde{\boldsymbol{\mu}} - \boldsymbol{\gamma}_B(\widetilde{\mathbf{u}}) \, ds) \otimes_s \boldsymbol{\nu}|$$

$$+ (1-t) \int_{\Gamma_1} \mathbf{g} \cdot \boldsymbol{\gamma}_B(\widetilde{\mathbf{u}}) \, ds + t \int_{\Gamma_1} \mathbf{g} \cdot \widetilde{\boldsymbol{\mu}}$$

is positively homogeneous. Then for every $t > 0$, $\lceil SP^{**}_{\lambda,j} \rceil(\widetilde{\mathbf{u}}, \widetilde{\boldsymbol{\mu}}) = \lceil SP^{**}_{\lambda,j} \rceil(\widetilde{\mathbf{u}}, t(\widetilde{\boldsymbol{\mu}} - \boldsymbol{\gamma}_B(\widetilde{\mathbf{u}})_{|\Gamma_1} ds) + \boldsymbol{\gamma}_B(\widetilde{\mathbf{u}})_{|\Gamma_1} ds)$. Therefore we obtain a contradiction with the coercivity of $\lceil SP^{**}_{\lambda,j} \rceil$ (cf. Corollary 10). ∎

Let $ds(\Gamma_0) = 0$ and $L(\widehat{\mathbf{u}}) = 0$ for every $\widehat{\mathbf{u}} \in \mathcal{R}_0 \equiv \{\mathbf{u} \in BD \mid \boldsymbol{\varepsilon}(\mathbf{u}) = \mathbf{0}\}$. Moreover, let the functionals (3.8) from [2] and $(RP^{**}_{\lambda,j})$ be defined over $BD(\Omega)/\mathcal{R}_0$, and let $(SQ_{\lambda,j})$ and $(SP^{**}_{\lambda,j})$ be defined over the space $(BD(\Omega) \times \mathbb{M}_b(\Gamma_1, \mathbb{R}^n))/\{(\widehat{\mathbf{u}}, \boldsymbol{\gamma}_B(\widehat{\mathbf{u}})_{|\Gamma_1} ds) \mid \widehat{\mathbf{u}} \in \mathcal{R}_0\}$. Then the assertions of Theorem 8, Corollary 9, Corollary 10 and Theorem 11 hold. Indeed, in their proofs, we can replace weak* $BD(\Omega)$ topology by $\sigma(\mathbf{Y}^1(\overline{\Omega}), C_{\text{div}}(\overline{\Omega}, \mathbf{E}^n_s))$, and $G_j(\boldsymbol{\varepsilon}(\cdot))$ does not depend on $\mathbf{u} \in \ker \boldsymbol{\varepsilon}$ (cf. [2, Lemma 15], (2.10) and (2.14)).

3. The limit analysis problem. Here we define the limit analysis problem $(SP^{**}_{0,j})_{AL}$ associated with $(SP^{**}_{\lambda,j})$. We prove that $(SP^{**}_{\lambda,j})$ is coercive if $\inf (SP^{**}_{0,j})_{AL} > \lambda$, under some assumptions given below. Moreover, we obtain an existence theorem for $(SP^{**}_{0,j})_{AL}$.

ASSUMPTION 8. j^* is a nonnegative function. ∎

The original problem $(P_{\lambda,j})$ is defined by (see [2, formula (3.8)–(3.11)])

$$(3.1) \qquad (P_{\lambda,j}) \quad \text{find } \inf\{\lambda F(\mathbf{u}) + G_j(\boldsymbol{\varepsilon}(\mathbf{u})) \mid \mathbf{u} \in BD(\Omega)\},$$

where

$$(3.2) \quad \lambda F(\mathbf{u}) \equiv -\lambda L(\mathbf{u}) + I_{C_a(\mathbf{0})}(\mathbf{u}), \qquad L(\mathbf{u}) \equiv \int_\Omega \mathbf{f} \cdot \mathbf{u} \, dx + \int_{\Gamma_1} \mathbf{g} \cdot \boldsymbol{\gamma}_B(\mathbf{u}) \, ds,$$

and

$$(3.3) \qquad C_a(\mathbf{0}) \equiv \{\mathbf{u} \in BD(\Omega) \mid \boldsymbol{\gamma}_B(\mathbf{u})_{|\Gamma_0} = \mathbf{0} \text{ on } \Gamma_0\}.$$

The elastic-plastic potential $G_j : \mathbb{M}_b(\Omega, \mathbf{E}^n_s) \to \mathbb{R} \cup \{\infty\}$ is defined by

$$(3.4) \qquad G_j(\boldsymbol{\mu}) \equiv \begin{cases} \int_\Omega j(x, \boldsymbol{\mu}) \, dx & \text{if } \boldsymbol{\mu} \in L^1(\Omega, \mathbf{E}^n_s), \\ \infty & \text{otherwise.} \end{cases}$$

With $(P_{\lambda,j})$ one associates the *limit analysis problem* $(P_{0,j})_{AL}$:

$$(3.5) \quad (P_{0,j})_{AL} \quad \text{find } \inf\left\{ \int_\Omega j_\infty(x, \boldsymbol{\varepsilon}(\mathbf{u})) \, dx \,\middle|\, \mathbf{u} \in LD(\Omega), \right.$$
$$\left. \boldsymbol{\gamma}_B(\mathbf{u}) = \mathbf{0} \text{ on } \Gamma_0, \, L(\mathbf{u}) = 1 \right\}.$$

Similarly, $(SP^{**}_{\lambda,j})$, defined in (2.52), is connected with the *relaxed limit analysis problem* $(SP^{**}_{0,j})_{AL}$:

$$(3.6) \quad (SP^{**}_{0,j})_{AL} \quad \text{find } \inf\left\{ \int_{\Gamma_0} j_\infty(x, -\boldsymbol{\gamma}_B(\mathbf{u}) \otimes_s \boldsymbol{\nu}) \, ds \right.$$
$$\left. + \int_{\Gamma_1} j_\infty\left(x, \frac{d((\boldsymbol{\mu} - \boldsymbol{\gamma}_B(\mathbf{u}) \, ds) \otimes_s \boldsymbol{\nu})}{d|(\boldsymbol{\mu} - \boldsymbol{\gamma}_B(\mathbf{u}) \, ds) \otimes_s \boldsymbol{\nu}|}\right) d|(\boldsymbol{\mu} - \boldsymbol{\gamma}_B(\mathbf{u}) \, ds) \otimes_s \boldsymbol{\nu}| \right.$$

$$+ \int_\Omega j_\infty(x, \boldsymbol{\varepsilon}(\mathbf{u})_a) \, dx + \int_\Omega j_\infty\left(x, \frac{d\boldsymbol{\varepsilon}(\mathbf{u})_s}{d|\boldsymbol{\varepsilon}(\mathbf{u})_s|}\right) d|\boldsymbol{\varepsilon}(\mathbf{u})_s| \, \Bigg|$$

$$\mathbf{u} \in BD(\Omega), \, \boldsymbol{\mu} \in \mathbb{M}_b(\Gamma_1, \mathbb{R}^n), \int_\Omega \mathbf{f} \cdot \mathbf{u} \, dx + \int_{\Gamma_1} \mathbf{g} \cdot \boldsymbol{\mu} = 1 \Bigg\}.$$

Directly from (3.5) and (3.6) we get

$$(3.7) \qquad \inf(P_{0,j})_{AL} \ge \inf(SP^{**}_{0,j})_{AL}.$$

If $ds(\Gamma_0) = 0$ and $\inf(P_{0,j})_{AL} > 0$ then for every $\mathbf{u} \in LD(\Omega)$ and $\overline{\mathbf{u}} \in LD(\Omega)$ such that $\boldsymbol{\varepsilon}(\overline{\mathbf{u}}) = \mathbf{0}$ in Ω, we have $L(\mathbf{u}) = L(\mathbf{u} + \overline{\mathbf{u}})$, since j^* is nonnegative (i.e. $j(x, \mathbf{0}) \le 0$ for dx-a.e. $x \in \Omega$); cf. Assumption 8 and (3.2)). We can prove the following result.

PROPOSITION 12. *Assume that $\overline{\lambda}_r$ satisfies Assumption 5, where λ is replaced by $\overline{\lambda}_r$. Moreover, let $\mathbf{u}^0 = \mathbf{0}$ in $(P_{\lambda_r,j})$. If $\inf(P_{0,j})_{AL} > \overline{\lambda}_r > \lambda_r \ge 0$ then $\inf(P_{\lambda_r,j}) > -\infty$. The inverse implication holds in the following form: if $\inf(P_{\lambda_r,j}) > -\infty$ then $\inf(P_{0,j})_{AL} \ge \lambda_r$. Moreover, if $\inf(P_{0,j})_{AL} > \overline{\lambda}_r > \lambda_r \ge 0$ then any sequence $\{\mathbf{u}_m\}_{m\in\mathbb{N}} \subset BD(\Omega)$ such that*

$$(3.8) \qquad \inf_{\mathbf{z}}\{\|\mathbf{u}_m + \mathbf{z}\|_{BD} \mid \mathbf{z} \in BD(\Omega) \text{ and } \boldsymbol{\varepsilon}(\mathbf{z}) = \mathbf{0}\} \to \infty$$

satisfies $\lim_{m\to\infty}(\lambda_r F(\mathbf{u}_m) + G_j(\boldsymbol{\varepsilon}(\mathbf{u}_m))) = \infty$, i.e. the functional $BD(\Omega) \ni \mathbf{u} \mapsto (\lambda_r F(\mathbf{u}) + G_j(\boldsymbol{\varepsilon}(\mathbf{u})))$ is coercive. ∎

Below we study the relation between $(SP^{**}_{\lambda,j})$ and the relaxed limit analysis problem $(SP^{**}_{0,j})_{AL}$.

REMARK 1. If $\inf(P_{0,j})_{AL} > 0$, then the assumptions of Corollaries 9, 10 and Theorem 8 hold (in the sense of (3.8)) if $ds(\Gamma_0) = 0$. Indeed, $L(\widehat{\mathbf{u}}) = 0$ for $\boldsymbol{\varepsilon}(\widehat{\mathbf{u}}) = \mathbf{0}$ (cf. Assumption 8 and (3.2)).

PROPOSITION 13. *Assume that $\overline{\lambda}_r$ satisfies Assumption 5, where λ is replaced by $\overline{\lambda}_r$, and suppose $\inf(P_{0,j})_{AL} > \overline{\lambda}_r > \lambda_r \ge 0$. Problem $(SP^{**}_{\lambda_r,j})$ is coercive if and only if $\inf(SP^{**}_{0,j})_{AL} > \lambda_r$.*

Proof. By Corollary 10, problem $(SP^{**}_{\lambda,j})$ is coercive if and only if $(P_{\lambda,j})$ is coercive. By Proposition 12, if $\overline{\lambda}_r$ satisfies Assumption 5 and $\inf(P_{0,j})_{AL} > \overline{\lambda}_r > \lambda_r \ge 0$, then $(P_{\lambda_r,j})$ is coercive if and only if (P_{λ_r,j_∞}) is coercive. From Corollary 10, (P_{λ,j_∞}) is coercive if and only if $(SP^{**}_{\lambda,j_\infty})$ is coercive. Therefore, if $\overline{\lambda}_r$ satisfies Assumption 5 and $\inf(P_{0,j})_{AL} > \overline{\lambda}_r > \lambda_r \ge 0$, then $(SP^{**}_{\lambda_r,j})$ is coercive if and only if $(SP^{**}_{\lambda_r,j_\infty})$ is coercive. Moreover, $(SP^{**}_{\lambda,j_\infty})$ is coercive if and only if $(SP^{**}_{0,j})_{AL} > \lambda$, where $\lambda \ge 0$. ∎

THEOREM 14. *There exists a solution $(\widehat{\mathbf{u}}, \widehat{\boldsymbol{\mu}}) \in BD(\Omega) \times \mathbb{M}_b(\Gamma_1, \mathbb{R}^n)$ of the problem $(SP^{**}_{0,j})_{AL}$, i.e. $(SP^{**}_{0,j})_{AL}$ achieves its infimum at the point $(\widehat{\mathbf{u}}, \widehat{\boldsymbol{\mu}})$. If $ds(\Gamma_0) = 0$ then we assume that $L(\widehat{\mathbf{u}}) = 0$ for every $\widehat{\mathbf{u}} \in \mathcal{R}_0 \equiv \{\mathbf{u} \in BD \mid \boldsymbol{\varepsilon}(\mathbf{u}) = \mathbf{0}\}$.*

Proof. The functional $(SP^{**}_{\lambda,j_\infty})$ is l.s.c. in the weak* $BD(\Omega)\times\mathbb{M}_b(\Gamma_1,\mathbb{R}^n)$ topology over bounded subsets of $BD(\Omega)\times\mathbb{M}_b(\Gamma_1,\mathbb{R}^n)$, for $\mathbf{f}=\mathbf{0}$ and $\mathbf{g}=\mathbf{0}$. The functional

$$(3.9)\qquad \mathbb{M}_b(\Gamma_1,\mathbb{R}^n)\ni\boldsymbol{\mu}\mapsto\int_{\Gamma_1}\mathbf{g}\cdot\boldsymbol{\mu}\in\mathbb{R}$$

is continuous in the weak* $\mathbb{M}_b(\Gamma_1,\mathbb{R}^n)$ topology, because $\mathbf{g}\in C(\overline{\Gamma}_1,\mathbb{R}^n)$ (cf. Assumption 5). By [2, Lemma 15] the functional

$$(3.10)\qquad BD(\Omega)\ni\mathbf{u}\mapsto\int_\Omega\mathbf{f}\cdot\mathbf{u}\,dx\in\mathbb{R}$$

is continuous in the weak* $BD(\Omega)$ topology on bounded subsets of $BD(\Omega)$. If $ds(\Gamma_0)\neq 0$, then by Assumption 7, every minimizing sequence for $(SP^{**}_{0,j})_{AL}$ is bounded.

If $ds(\Gamma_0)=0$, then the assertion of Corollary 10 holds for (3.8) from [2] and $(SP^{**}_{\lambda,j})$, where j is replaced by j_∞. Then $\inf(SP^{**}_{0,j})_{AL}$ is equal to the infimum in the limit analysis problem associated to (P_{λ,j_∞}), defined by

$$(3.11)\quad (P_{0,j})_{AL}\quad \text{Find}\inf\Big\{\int_\Omega j_\infty(x,\boldsymbol{\varepsilon}(\mathbf{u}))\,dx\,\Big|\,\mathbf{u}\in LD(\Omega),$$
$$\boldsymbol{\gamma}_B(\mathbf{u})=\mathbf{0}\text{ on }\Gamma_0\text{ and }L(\mathbf{u})=1\Big\}$$

(cf. (3.2)). There exists a bounded minimizing net $\{\widetilde{\mathbf{u}}_k\}_{k\in\mathbf{K}}$ for $(P_{0,j})_{AL}$, since $G_{j_\infty}(\boldsymbol{\varepsilon}(\cdot))$ does not depend on $u_s\in\ker\boldsymbol{\varepsilon}=\mathcal{R}_0$ and $L(\widehat{\mathbf{u}})=0$ for every $\widehat{\mathbf{u}}\in\mathcal{R}_0$. Therefore $\{(\widetilde{\mathbf{u}}_k,\boldsymbol{\gamma}_B(\widetilde{\mathbf{u}}_k)\}_{k\in\mathbf{K}}$ is a bounded minimizing net for $(SP^{**}_{0,j})_{AL}$.

Then by continuity (in the weak* $BD(\Omega)\times\mathbb{M}_b(\Gamma_1,\mathbb{R}^n)$ topology) of the functionals (3.9) and (3.10) over bounded sets in $BD(\Omega)\times\mathbb{M}_b(\Gamma_1,\mathbb{R}^n)$, we obtain the existence of the minimum for $(SP^{**}_{0,j})_{AL}$. ∎

4. Appendix. Below we introduce a family of perturbations studied in convex optimization (see [9, Chapter 3]). Next, we apply these results to Hencky plasticity.

Let V and V^* (resp. Y and Y^*) be two topological vector spaces in duality via the bilinear pairing $\langle\cdot,\cdot\rangle_V$ ($\langle\cdot,\cdot\rangle_Y$, respectively). We shall assume the existence of a continuous linear operator Λ from V into Y, with transpose Λ^*. Taking a function $V\ni\mathbf{u}\mapsto F(\mathbf{u})+G(\Lambda\mathbf{u})\in\mathbb{R}\cup\{\infty\}$, we are concerned with the minimization problem

$$(4.1)\qquad (P)\quad \inf\{F(\mathbf{u})+G(\Lambda\mathbf{u})\mid\mathbf{u}\in V\}.$$

We shall also consider a function $\Phi:V\times Y\to\mathbb{R}\cup\{\infty\}$ such that $\Phi(\mathbf{u},\mathbf{0})=F(\mathbf{u})+G(\Lambda\mathbf{u})$, and for every $\mathbf{z}\in Y$ we shall consider the mini-

mization problem

(4.2) $(P_{\mathbf{z}})$ $\inf\{\Phi(\mathbf{u}, \mathbf{z}) \mid \mathbf{u} \in V\}$,

where $\Phi(\mathbf{u}, \mathbf{z}) \equiv F(\mathbf{u}) + G(\Lambda(\mathbf{u}) + \mathbf{z})$ for every $\mathbf{u} \in V$ and $\mathbf{z} \in Y$. The problems $(P_{\mathbf{z}})$ will be said to be *perturbed problems* of (P). Let Φ^* be the conjugate function of Φ in the duality between $V \times Y$ and $V^* \times Y^*$, given by

(4.3) $\Phi^*(\mathbf{u}^*, \mathbf{z}^*) = \sup\{\langle \mathbf{u}^*, \mathbf{u}\rangle_V - F(\mathbf{u}) - \langle \mathbf{z}^*, \Lambda \mathbf{u}\rangle_Y + \langle \mathbf{z}^*, \Lambda(\mathbf{u}) + \mathbf{z}\rangle_Y$
$- G(\Lambda(\mathbf{u}) + \mathbf{z}) \mid \mathbf{u} \in V, \mathbf{z} \in Y\} = F^*(\mathbf{u}^* - \Lambda^*\mathbf{z}^*) + G^*(\mathbf{z}^*).$

The problem

(4.4) (P^*) $\sup\{-\Phi^*(\mathbf{0}, \mathbf{z}^*) \mid \mathbf{z}^* \in Y^*\}$

is termed the *dual problem* of (P) with respect to Φ.

It is natural to associate the perturbed problems $(\mathbf{u}^* \in V^*)$

(4.5) $(P^*_{\mathbf{u}^*})$ $\sup\{-\Phi^*(\mathbf{u}^*, \mathbf{z}^*) \mid \mathbf{z}^* \in Y^*\}$

with the dual problem (P^*), and to determine the dual problem of (P^*) with respect to these perturbations; we easily arrive at the following problem which will be termed the *bidual problem* of (P):

(4.6) (P^{**}) $\inf\{\Phi^{**}(\mathbf{u}, \mathbf{0}) \mid \mathbf{u} \in V\}$,

where $\Phi^{**}(\mathbf{u}, \mathbf{z}) = F^{**}(\mathbf{u}) + G^{**}(\Lambda(\mathbf{u}) + \mathbf{z})$ for every $\mathbf{u} \in V, \mathbf{z} \in Y$ and

(4.7) $F^{**}(\mathbf{u}) = \sup\{\langle \mathbf{u}^*, \mathbf{u}\rangle_V - F^*(\mathbf{u}^*) \mid \mathbf{u}^* \in V^*\}$,

(4.8) $G^{**}(\Lambda(\mathbf{u}) + \mathbf{z}) = \sup\{\langle \mathbf{z}^*, \Lambda(\mathbf{u}) + \mathbf{z}\rangle_Y - G^*(\mathbf{z}^*) \mid \mathbf{z}^* \in Y^*\}$.

Below we study Hencky plasticity with the Mises (or Tresca) yield condition (see [10]). We assume that $\Omega \subset \mathbb{R}^n$ and $n \geq 2$.

ASSUMPTION 9. Let $\mathcal{K}^D : \overline{\Omega} \to 2^{(\mathbf{E}_s^n)^D}$ be a multifunction such that $\mathcal{K}^D(x) \subset (\mathbf{E}_s^n)^D$ and

(4.9) $\mathcal{K}(x) = \mathcal{K}^D(x) \oplus \{\mathbf{w}^* \in \mathbf{E}_s^n \mid \exists t \in \mathbb{R}, \ w_{ij}^* = t\delta_{ij}\}$

for every $x \in \overline{\Omega}$, where $(\mathbf{E}_s^n)^D$ denotes the set of symmetric real $n \times n$ matrices (whose trace is 0) and w_{ij}^* are the components of $\mathbf{w}^*$. Here $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise. There exist $r_1, r_2 > 0$ such that

(4.10) $B_{(\mathbf{E}_s^n)^D}(0, r_1) \subset \mathcal{K}^D(x) \subset B_{(\mathbf{E}_s^n)^D}(0, r_2).$ ■

ASSUMPTION 10. The normal integrand j^* satisfies

(4.11) $j^*(x, \mathbf{w}^*) = j_D^*(x, (\mathbf{w}^*)^D) + j_{\mathrm{tr}}^*\left(x, \frac{1}{n} \operatorname{tr} \mathbf{w}^*\right)$

for every $\mathbf{w}^* \in \mathbf{E}_s^n$ and dx-a.e. $x \in \Omega$, where j_D^* and j_{tr}^* are nonnegative functions, and the deviator $(\mathbf{w}^*)^D$ of $\mathbf{w}^*$ is given by

$$(4.12) \qquad\qquad w_{ij}^{*D} = w_{ij}^* - \frac{1}{n}\delta_{ij}(\operatorname{tr} \mathbf{w}^*).$$

There exist $\widehat{k} > 0$, $a \geq a_1 > 0$ and $b \geq 0$ such that

$$(4.13) \qquad at^2 + b \geq j_{tr}^*(x,t) \geq a_1 t^2 \quad \forall t \in \mathbb{R}, \text{ for } dx\text{-a.e. } x \in \Omega,$$

$$(4.14) \quad j_D^*(x,(\mathbf{w}^*)^D) \leq \widehat{k} \quad \forall (\mathbf{w}^*)^D \in \mathcal{K}^D(x), \text{ for } dx\text{-a.e. } x \in \Omega. \ \blacksquare$$

The vector spaces

$$(4.15) \qquad\qquad U(\Omega) \equiv \{\mathbf{u} \in BD(\Omega) \mid \operatorname{div}\mathbf{u} \in L^2(\Omega)\},$$

$$(4.16) \qquad\qquad U_p(\Omega) \equiv \{\mathbf{u} \in BD(\Omega) \mid \operatorname{div}\mathbf{u} \in L^{n/(n-1)}(\Omega)\}$$

are Banach spaces with the natural norms

$$(4.17) \qquad\qquad \|\mathbf{u}\|_{U(\Omega)} = \|\mathbf{u}\|_{BD(\Omega)} + \|\operatorname{div}\mathbf{u}\|_{L^2(\Omega)},$$

$$(4.18) \qquad\qquad \|\mathbf{u}\|_{U_p(\Omega)} = \|\mathbf{u}\|_{BD(\Omega)} + \|\operatorname{div}\mathbf{u}\|_{L^{n/(n-1)}(\Omega)}$$

(see [13]). We obtain

$$(4.19) \qquad\qquad U(\Omega) \subset U_p(\Omega) \subset BD(\Omega),$$

since $\Omega \subset \mathbb{R}^n$ is a bounded set and $n \geq 2$. Moreover, we consider the space

$$(4.20) \quad \Sigma(\Omega) \equiv \{\boldsymbol{\sigma} \in L^2(\Omega, \mathbf{E}_s^n) \mid \boldsymbol{\sigma}^D \in L^\infty(\Omega, \mathbf{E}_s^n), \operatorname{div}\boldsymbol{\sigma} \in L^{n+1}(\Omega, \mathbb{R}^n)\}$$

endowed with the natural norm

$$(4.21) \quad \|\boldsymbol{\sigma}\|_{\Sigma(\Omega)} = \|\boldsymbol{\sigma}^D\|_{L^\infty(\Omega, \mathbf{E}_s^n)} + \|\boldsymbol{\sigma}\|_{L^2(\Omega, \mathbf{E}_s^n)} + \|\operatorname{div}\boldsymbol{\sigma}\|_{L^{n+1}(\Omega, \mathbb{R}^n)}$$

(cf. [13, Chapter 2, Section 7]). Define

$$(4.22) \qquad \Sigma_s(\Omega) \equiv \{\boldsymbol{\sigma} \in \Sigma(\Omega) \mid \boldsymbol{\gamma}_\Sigma(\boldsymbol{\sigma}) \in L^\infty(\operatorname{Fr}\Omega, \mathbb{R}^n)\},$$

where the trace $\boldsymbol{\gamma}_\Sigma : \Sigma(\Omega) \to C^1(\operatorname{Fr}\Omega, \mathbb{R}^n)'$ is given by

$$(4.23) \qquad \boldsymbol{\gamma}_\Sigma(\boldsymbol{\sigma}) = \boldsymbol{\sigma} \cdot \boldsymbol{\nu} \quad \text{on } \operatorname{Fr}\Omega, \text{ for all } \boldsymbol{\sigma} \in C^1(\overline{\Omega}, \mathbf{E}_s^n).$$

Let

$$(4.24) \qquad\qquad V \equiv U_p(\Omega), \quad Y^* \equiv \Sigma_s(\Omega).$$

Let $\mathbf{I}$ denote the $n \times n$ unit matrix and set

$$(4.25) \qquad Y(\Omega) \equiv \operatorname{span}(\boldsymbol{\varepsilon}(U_p(\Omega)), L^1(\Omega, (\mathbf{E}_s^n)^D) \oplus L^{n/(n-1)}(\Omega, \mathbb{R})\mathbf{I})$$
$$= \{\boldsymbol{\mu} \in \mathbb{M}_b(\Omega, \mathbf{E}_s^n) \mid \exists\, \mathbf{u} \in U_p(\Omega) \text{ and}$$
$$\exists \mathbf{w} \in L^1(\Omega, (\mathbf{E}_s^n)^D) \oplus L^{n/(n-1)}(\Omega, \mathbb{R})\mathbf{I}$$
$$\text{such that } \boldsymbol{\mu} = \boldsymbol{\varepsilon}(\mathbf{u}) + \mathbf{w}dx\}.$$

The dual space V^* (to $U_p(\Omega)$) is

$$(4.26) \qquad V^* \equiv \{(\mathbf{u}^*, \boldsymbol{\psi}^*) \in L^n(\Omega, \mathbb{R}^n) \times L^\infty(\operatorname{Fr}\Omega, \mathbb{R}^n)\}.$$

The bilinear form between $U_p(\Omega)$ and V^* is

$$(4.27) \quad U_p(\Omega) \times V^* \ni (\mathbf{u}, (\mathbf{u}^*, \boldsymbol{\psi}^*)) \mapsto - \int_\Omega \mathbf{u}^* \cdot \mathbf{u}\, dx$$

$$+ \int_{\mathrm{Fr}\,\Omega} \boldsymbol{\psi}^* \cdot \boldsymbol{\gamma}_B(\mathbf{u})\, ds \equiv \langle \mathbf{u}; (\mathbf{u}^*, \boldsymbol{\psi}^*) \rangle_{U_p \times V^*}.$$

The space $U_p(\Omega)$ is endowed with the topology $\sigma(U_p(\Omega), V^*)$. Similarly V^* is endowed with the topology $\sigma(V^*, U_p(\Omega))$. Therefore $[U_p(\Omega), \sigma(U_p(\Omega), V^*)]^* = V^*$ and $[V^*, \sigma(V^*, U_p(\Omega))]^* = U_p(\Omega)$ (see [8, Theorem V.3.9]).

The bilinear form between Y and Y^* is given by

$$(4.28) \quad \langle \boldsymbol{\mu}; \boldsymbol{\sigma} \rangle_{Y \times Y^*} = \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\mu} = \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\mathbf{u}) + \int_\Omega \boldsymbol{\sigma} : \mathbf{w}\, dx$$

for $\boldsymbol{\sigma} \in \Sigma_s(\Omega)$ and $\boldsymbol{\mu} \in Y$, where $\boldsymbol{\mu} = \boldsymbol{\varepsilon}(\mathbf{u}) + \mathbf{w}$, $\mathbf{u} \in U_p(\Omega)$ and $\mathbf{w} \in L^1(\Omega, (\mathbf{E}_s^n)^D) \oplus L^{n/(n-1)}(\Omega, \mathbb{R})\mathbf{I}$. Then we have $[Y, \sigma(Y, Y^*)]^* = Y^*$ and $[Y^*, \sigma(Y^*, Y)]^* = Y$.

The linear operator $\boldsymbol{\varepsilon} : [U_p, \sigma(U_p, V^*)] \to [Y, \sigma(Y, \Sigma_s(\Omega))]$ is continuous. The explicit form of $\boldsymbol{\varepsilon}^* : \Sigma_s(\Omega) \to V^*$ reads

$$(4.29) \quad \langle \boldsymbol{\varepsilon}(\mathbf{u}), \boldsymbol{\sigma} \rangle_{Y \times Y^*} = \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\mathbf{u}) = \int_\Omega \boldsymbol{\sigma}^D : \boldsymbol{\varepsilon}^D(\mathbf{u}) + \frac{1}{n} \int_\Omega \mathrm{tr}\,\boldsymbol{\sigma}\,\mathrm{div}\,\mathbf{u}\, dx$$

$$= - \int_\Omega (\mathrm{div}\,\boldsymbol{\sigma}) \cdot \mathbf{u}\, dx + \int_{\mathrm{Fr}\,\Omega} \boldsymbol{\gamma}_\Sigma(\boldsymbol{\sigma}) \cdot \boldsymbol{\gamma}_B(\mathbf{u})\, ds$$

$$= \langle \mathbf{u}; \boldsymbol{\varepsilon}^*(\boldsymbol{\sigma}) \rangle_{U_p \times V^*}.$$

Hence the function $\boldsymbol{\varepsilon}^* : \Sigma_s(\Omega) \to V^*$ is given by $\boldsymbol{\varepsilon}^*(\boldsymbol{\sigma}) = (\mathrm{div}\,\boldsymbol{\sigma}, \boldsymbol{\gamma}_\Sigma(\boldsymbol{\sigma}))$.

The original problem $(P_{\lambda,j})$ is defined by (3.1)–(3.4).

Let the functional of relaxed elastic-plastic energy $\lceil RP_{\lambda,j} \rceil : BD(\Omega) \to \mathbb{R} \cup \{\infty\}$ be defined by

$$(4.30) \quad \lceil RP_{\lambda,j} \rceil(\mathbf{u}) = \lambda F_R(\mathbf{u}) + G_j(\boldsymbol{\varepsilon}(\mathbf{u})),$$

where $\lambda F_R : BD(\Omega) \to \mathbb{R} \cup \{\infty\}$ is defined by

$$(4.31) \quad \lambda F_R(\mathbf{u}) \equiv -\lambda L(\mathbf{u}) + \int_{\Gamma_0} j_\infty(x, -\boldsymbol{\gamma}_B(\mathbf{u}) \otimes_s \boldsymbol{\nu})\, ds$$

if $\mathbf{u} \in LD(\Omega)$, and $\lambda F_R(\mathbf{u}) \equiv \infty$ otherwise (cf. (3.2)). The elastic-plastic potential G_j is given by (3.4). The relaxed problem $(RP_{\lambda,j})$ is

$$(4.32) \quad (RP_{\lambda,j}) \quad \text{find } \inf\{\lceil RP_{\lambda,j} \rceil(\mathbf{u}) \mid \mathbf{u} \in U_p(\Omega)\}.$$

We can show that the dual relaxed problem $(RP^*_{\lambda,j})$ is

$$(4.33) \quad (RP^*_{\lambda,j}) \quad \sup\Big\{ -\int_\Omega j^*(x, \boldsymbol{\sigma})\, dx \,\Big|\, \boldsymbol{\sigma} \in \Sigma_s(\Omega),$$

$$\mathrm{div}\,\boldsymbol{\sigma} = -\lambda\mathbf{f} \text{ in } \Omega \text{ and } \boldsymbol{\gamma}_\Sigma(\boldsymbol{\sigma}) = \lambda\mathbf{g} \text{ on } \Gamma_1 \Big\}.$$

The bidual relaxed problem $(RP_{\lambda,j}^{**})$ is

(4.34) $(RP_{\lambda,j}^{**})$ find $\inf\{(\lambda F_R)^{**}(\mathbf{u}) + G_j^{**}(\varepsilon(\mathbf{u})) \mid \mathbf{u} \in U_p(\Omega)\},$

where $(\lambda F_R)^{**}(\mathbf{u}) = \lambda F_R(\mathbf{u})$ for every $\mathbf{u} \in U_p(\Omega)$ and

(4.35) $G_j^{**}(\varepsilon(\mathbf{u})) = \int\limits_{\Omega} j(x, \varepsilon(\mathbf{u})_a)\, dx + \int\limits_{\Omega} j_\infty\left(x, \dfrac{d\varepsilon(\mathbf{u})_s}{d|\varepsilon(\mathbf{u})_s|}\right) d|\varepsilon(\mathbf{u})_s|$

for every $\mathbf{u} \in U_p(\Omega)$.

Similarly to [14], we prove that the supremum in the problem $(RP_{\lambda,j}^*)$ exists in the space $\Sigma(\Omega)$. Moreover, if $ds(\Gamma_0) = 0$, the solution of $(RP_{\lambda,j}^*)$ belongs to the space $\Sigma_s(\Omega)$, since $\mathbf{g} \in C^1(\Gamma_1, \mathbb{R}^n)$.

In [13] and [1] an existence theorem for $(RP_{\lambda,j}^{**})$ is proved in the space $U(\Omega)$ (cf. (4.11), (4.13), (4.14) and (4.19)).

By [9, Chapter 3], the dual problem to $(RP_{\lambda,j}^{**})$ is equal to $(RP_{\lambda,j}^*)$.

If $\widehat{\sigma}$ is a solution of $(RP_{\lambda,j}^*)$, $\widehat{\mathbf{u}}$ is a solution of $(RP_{\lambda,j}^{**})$, $\inf(RP_{\lambda,j}^{**}) = \sup(RP_{\lambda,j}^*)$ and this value is finite, then the couple $(\widehat{\sigma}, \widehat{\mathbf{u}})$ satisfies the extremality relation (see [9, Chapter 3, Proposition 2.4]).

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