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ON JEFFREYS MODEL OF HEAT CONDUCTION

Abstract. The Jeffreys model of heat conduction is a system of two partial differential equations of mixed hyperbolic and parabolic character. The analysis of an initial-boundary value problem for this system is given. Existence and uniqueness of a weak solution of the problem under very weak regularity assumptions on the data is proved. A finite difference approximation of this problem is discussed as well. Stability and convergence of the discrete problem are proved.

0. Introduction. The Jeffreys model of heat conduction has recently been discussed by several authors [2], [5]–[7]. Its equations may be written in the following general form [5]:

$$T_t + \operatorname{div}(Q) = 0, \qquad Q_t + D\nabla T + Q - \kappa\Delta Q = 0. \tag{0.1}$$

Here D and κ are positive (in general constant) coefficients, the scalar function T is the temperature, and the vector-valued function Q represents the so-called *heat flux*. We are interested in application of the one-dimensional Jeffreys model to describe heat waves in a thin metallic layer under a very short laser impulse [5]. Generalizations to more space dimensions are certainly possible and do not seem to be very difficult.

Let us discuss the initial-boundary value problem for the one-dimensional Jeffreys model in the form appearing in [5]. We are looking for two scalar functions $T = T(t,x)$ and $Q = Q(t,x)$, subject to the following equations:

$$T_t + Q_x = 0, \qquad Q_t + DT_x + Q - \kappa Q_{xx} = 0 \tag{0.2}$$

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for $t \in (0, t_{\max})$, $x \in (L, P)$ (where $0 < t_{\max}$, $L < P$), and satisfying the following Dirichlet boundary conditions:

$$
\begin{aligned}
T(t, L) &= \phi(t), \quad & T(t, P) &= 0, \\
Q(t, L) &= \psi(t), \quad & Q(t, P) &= 0,
\end{aligned}
\tag{0.3}
$$

where the given functions ϕ and ψ describe the physical conditions defining the laser heat impulse. Moreover, the initial conditions are

$$
T(0, x) = T^0(x), \qquad Q(0, x) = Q^0(x),
\tag{0.4}
$$

with given functions T^0 and Q^0. More complex boundary conditions, for example in the form of linear combinations of the values of T and Q at each end of the interval $[L, P]$, can be considered without any essential change in the following text.

Our goal is to define and analyse a weak formulation of the problem (0.2), $(0,3)$, (0.4) which admits Dirichlet boundary conditions defined by functions ϕ and ψ of $L^2(0, t_{\max})$ regularity only.

The existence and uniqueness of the solution of the problem (0.2)–(0.4) in the weak formulation may be proved by one of the standard methods, however the weak regularity assumption on ϕ and ψ requires a special care. Moreover, a close inspection makes it possible to explain a certain phenomenon observed during the numerical treatment of the model [5]. Namely we can clearly see why the solution (T, Q) depends weakly on the boundary conditions imposed on the function T.

In Section 2, a finite difference approximation of the problem is discussed. We prove the stability and convergence of the finite difference scheme proposed.

In the report [3] under the same title that appeared in the proceedings of the FVCA Conference in Duisburg we give some information about the first version of this paper. This version was based on a different weak formulation, implying different conclusions on existence and uniquenes of the solution than in the present paper. The report contains only theorems without proofs.

1. Weak formulation. Let us first transform the differential problem (0.2)–(0.4) in order to obtain homogeneous Dirichlet boundary conditions. To do that we define two auxiliary functions

$$
\phi(t, x) = \phi(t)\frac{P - x}{P - L}, \qquad \psi(t, x) = \psi(t)\frac{P - x}{P - L}.
$$

Let us now introduce functions $\widetilde{T}$ and $\widetilde{Q}$ by

$$
T(t, x) = \widetilde{T}(t, x) + \phi(t, x), \qquad Q(t, x) = \widetilde{Q}(t, x) + \psi(t, x).
$$

The functions $\widetilde{T}$ and $\widetilde{Q}$ satisfy the following non-homogeneous equations:

$$
\widetilde{T}_t + \widetilde{Q}_x = f(t, x),
\tag{1.1}
$$

$$\widetilde{Q}_t + D\widetilde{T}_x + \widetilde{Q} - \kappa\widetilde{Q}_{xx} = g(t,x), \tag{1.2}$$

where

$$f(t,x) = \frac{1}{P-L}\Big[\psi(t) - (P-x)\frac{d}{dt}\phi(t)\Big],$$
$$g(t,x) = \frac{1}{P-L}\Big[D\phi(t) - (P-x)\Big(\frac{d}{dt}\psi(t) + \psi(t)\Big)\Big].$$

Note that the Dirichlet boundary condition is homogeneous:

$$\widetilde{T}(t,L) = \widetilde{Q}(t,L) = \widetilde{T}(t,P) = \widetilde{Q}(t,P) = 0,$$

and the initial conditions take the form

$$\begin{aligned}\widetilde{T}(0,x) &= \widetilde{T}^0(x) = T^0(x) - \phi(0)\frac{P-x}{P-L},\\ \widetilde{Q}(0,x) &= \widetilde{Q}^0(x) = Q^0(x) - \psi(0)\frac{P-x}{P-L}.\end{aligned} \tag{1.3}$$

The functions $\widetilde{T}$ and $\widetilde{Q}$ are not yet satisfactory: the initial conditions for them involve $\phi(0)$ and $\psi(0)$, which may not exist. Therefore we introduce new functions R and S:

$$R(t,x) = \int_0^t \widetilde{T}(s,x)\,ds, \qquad S(t,x) = \int_0^t \widetilde{Q}(s,x)\,ds.$$

They are more regular, and satisfy the *zero* initial and boundary conditions.

Let us derive equations for R and S. It is easy to see that

$$R_t + S_x = F, \tag{1.4}$$

$$S_t + DR_x + S - \kappa S_{xx} = G, \tag{1.5}$$

where

$$F(t,x) = \frac{1}{P-L}\Big[\int_0^t \psi(s)\,ds - (P-x)\phi(t)\Big] + T^0(x), \tag{1.6}$$

$$G(t,x) = \frac{1}{P-L}\Big[D\int_0^t \phi(s)\,ds - (P-x)\Big(\psi(t) + \int_0^t \psi(s)\,ds\Big)\Big] + Q^0(x). \tag{1.7}$$

To define the weak formulation of our problem we introduce a space $\mathbf{H}$:

Definition. Let $\mathbf{H}$ denote the Hilbert space of all pairs (V,W) such that

$$V, V_t, W_t \in L^2(0,t_{\max}; L^2(L,P)),$$
$$W \in L^2(0,t_{\max}; H_0^1(L,P)).$$

Its norm is defined by

$$\|(V,W)\|_{\mathbf{H}}^2 = \int_0^{t_{\max}} [\|V(s,\cdot)\|_0^2 + \|W(s,\cdot)\|_0^2 + \|W_x(s,\cdot)\|_0^2 + \|V_t(s,\cdot)\|_0^2 + \|W_t(s,\cdot)\|_0^2]\, ds,$$

where $\|\cdot\|_0 = \|\cdot\|_{L^2(L,P)}$.

Let R, S, V and W be sufficiently regular. Using the formulas

$$R_x W + R W_x = [RW]_x \quad \text{and} \quad S_{xx} W + S_x W_x = [S_x W]_x,$$

from (1.4) and (1.5) we obtain

$$DR_t V + DS_x V = DFV$$

and

$$S_t W - DRW_x + [RW]_x + SW + \kappa S_x W_x - \kappa [S_x W]_x = GW.$$

Note that W satisfies the homogeneous boundary condition. Adding the last two equations and integrating over $[L,P]$, we obtain

$$\int_L^P [DR_t V + S_t W]\, dx + \int_L^P [D(S_x V - RW_x) + SW + \kappa S_x W_x]\, dx = \int_L^P (DFV + GW)\, dx. \tag{1.8}$$

Define now bilinear forms

$$a, b : (L^2(L,P) \times H_0^1(L,P)) \times (L^2(L,P) \times H_0^1(L,P)) \to \mathbb{R}$$

by

$$a(U,Z;V,W) = \int_L^P [D(Z_x V - UW_x) + ZW + \kappa Z_x W_x]\, dx,$$

$$b(U,Z;V,W) = \int_L^P [DUV + ZW]\, dx.$$

Observe that

$$a(U,Z;U,Z) = \int_L^P [D(Z_x U - UZ_x) + Z^2 + \kappa Z_x^2]\, dx$$
$$= \int_L^P [Z^2 + \kappa Z_x^2]\, dx \geq \gamma \int_L^P [Z^2 + Z_x^2]\, dx = \gamma \|Z\|_1^2$$

where $\gamma = \min\{1, \kappa\}$. This means the $H_0^1(L,P)$-*ellipticity* of the form a with respect to the second argument of the pair (U,Z). Similarly

$$b(U,Z;U,Z) = D\|U\|_0^2 + \|Z\|_0^2.$$

Thus the form $a+b$ is $L^2(L,P)\times H_0^1(L,P)$-elliptic. Moreover, a is continuous:

$$\begin{aligned}|a(U,Z;V,W)|&\\ &\le C[\|Z\|_0\|W\|_0+\|Z_x\|_0\|W_x\|_0+\|U\|_0\|W_x\|_0+\|Z_x\|_0\|V\|_0]\\ &\le C\sqrt{\|U\|_0^2+\|Z\|_0^2+\|Z_x\|_0^2}\sqrt{\|V\|_0^2+\|W\|_0^2+\|W_x\|_0^2}\\ &\le C_1\sqrt{\|Z\|_1^2+\|U\|_0^2}\sqrt{\|W\|_1^2+\|V\|_0^2};\end{aligned}$$

here and below $(\cdot,\cdot)_0$, $(\cdot,\cdot)_1$, $\|\cdot\|_0$ and $\|\cdot\|_1$ are the $L^2(L,P)$ and $H_0^1(L,P)$ scalar products and norms, respectively, and C, C_1 are positive constants.

Similarly,

$$|b(U,Z;V,W)|\le C\sqrt{\|U\|_0^2+\|Z\|_0^2}\sqrt{\|V\|_0^2+\|W\|_0^2}.$$

Using the above notation we write (1.8) as

$$\begin{aligned}b(R_t(t,\cdot),S_t(t,\cdot);V,W)+a(R(t,\cdot),S(t,\cdot);V,W)&\\ =b(F(t,\cdot),G(t,\cdot);V,W),&\end{aligned}\tag{1.9}$$

with initial conditions

$$R(0,x)=0\quad\text{and}\quad S(0,x)=0.$$

We now define a weak formulation of the problem admitting the boundary conditions (0.3) with ϕ and ψ in $L^2(0,t_{\max})$. Note that F and G, as functions of the variable t, are also in $L^2(0,t_{\max})$.

Weak formulation. *Find a pair $(R,S)\in\mathbf{H}$ satisfying*

$$R(0,x)=0,\qquad S(0,x)=0$$

for a.e. x in (L,P) such that for a.e. t in $(0,t_{\max})$ and for all $(V,W)\in\mathbf{H}$,

$$\begin{aligned}b(R_t(t,\cdot),S_t(t,\cdot);V,W)+a(R(t,\cdot),S(t,\cdot);V,W)&\\ =b(F(t,\cdot),G(t,\cdot);V,W).&\end{aligned}\tag{1.10}$$

Our aim is to prove that the problem (1.10) is well posed. We first prove the existence of a solution by the Galerkin method [4].

Let $\xi_k\in L^2(L,P)$ and $\zeta_k\in H_0^1(L,P)$, $k=1,2,\ldots$, be such that $\{\xi_1,\ldots,\xi_n\}$ and $\{\zeta_1,\ldots,\zeta_n\}$ are linearly independent for each n, so that the subspaces $\operatorname{span}\{\xi_1,\ldots,\xi_n\}$ and $\operatorname{span}\{\zeta_1,\ldots,\zeta_n\}$ are of dimension n.

Definition. Let $r_j^n,s_j^n:[0,t_{\max}]\to\mathbb{R}$ for $j=1,\ldots,n$, $n=1,2,\ldots$ Set

$$R_n(t,x)=\sum_{j=1}^n r_j^n(t)\xi_j(x),\qquad S_n(t,x)=\sum_{j=1}^n s_j^n(t)\zeta_j(x).$$

The sequence $\{(R_n,S_n)\}$ is a *Galerkin solution* of (1.10) if:

- $r_j^n : [0, t_{\max}] \to \mathbb{R}$ and $s_j^n : (0, t_{\max}) \to \mathbb{R}$ are absolutely continuous,
- for a.e. t in $(0, t_{\max})$, $k = 1, \ldots, n$, and $n = 1, 2, \ldots$,

$$b(R_{nt}(t,\cdot), S_{nt}(t,\cdot); \xi_k, 0) + a(R_n(t,\cdot), S_n(t,\cdot); \xi_k, 0) = b(F, G; \xi_k, 0),$$

$$b(R_{nt}(t,\cdot), S_{nt}(t,\cdot); 0, \zeta_k) + a(R_n(t,\cdot), S_n(t,\cdot); 0, \zeta_k) = b(F, G, 0; \zeta_k).$$

Lemma 1.1. *Let us introduce the following notation*:

$$\begin{aligned}
\underline{r}^n &= [r_1^n, \ldots, r_n^n]^T, & \underline{s}^n &= [s_1^n, \ldots, s_n^n]^T,\\
G_1 &= ((\xi_j, \xi_k)_0)_{j,k=1,\ldots,n}, & G_2 &= ((\zeta_j, \zeta_k)_0)_{j,k=1,\ldots,n},\\
H_1 &= ((\zeta_j', \xi_k)_0)_{j,k=1,\ldots,n}, & H_2 &= ((\zeta_j', \zeta_k')_0)_{j,k=1,\ldots,n},\\
\underline{f}^n &= [f_1, \ldots, f_n]^T, & \underline{g}^n &= [g_1, \ldots, g_n]^T,\\
f_k &= \int_0^t (F, \xi_k)_0\, ds, & g_k &= \int_0^t (G, \zeta_k)_0\, ds.
\end{aligned}$$

If (R_n, S_n) *is a Galerkin solution of* (1.10), *then*

$$G_1 \underline{r}^n(t) + \int_0^t H_1 \underline{s}^n(t)\, ds = \underline{f}^n(t),$$

$$G_2 \underline{s}^n(t) + \int_0^t [(G_2 + \kappa H_2)\underline{s}^n(s) - D H_1 \underline{r}^n(s)]\, ds = \underline{g}^n(t).$$

Proof. It is enough to use the definitions of a and b and the equations

$$b(R_n(t,\cdot), S_n(t,\cdot); \xi_k, 0) + \int_0^t a(R_n(s,\cdot), S_n(s,\cdot); \xi_k, 0)\, ds = \int_0^t b(F, G; \xi_k, 0)\, ds,$$

$$b(R_n(t,\cdot), S_n(t,\cdot); 0, \zeta_k) + \int_0^t a(R_n(s,\cdot), S_n(s,\cdot); 0, \zeta_k)\, ds = \int_0^t b(F, G; 0, \zeta_k)\, ds. \blacksquare$$

Lemma 1.2. *If* $F \in L^2(0, t_{\max}; L^2(L, P))$ *and* $G \in L^2(0, t_{\max}; L^2(L, P))$, *then there exists a unique Galerkin solution of* (1.10).

Proof. By Lemma 1.1 we have

$$G_1 \underline{r}^n + \int_0^t H_1 \underline{s}^n\, ds = \underline{f}^n,$$

$$G_2 \underline{s}^n + \int_0^t [(G_2 + \kappa H_2)\underline{s}^n - D H_1 \underline{r}^n]\, ds = \underline{g}^n.$$

Note that $\underline{f}^n$ and $\underline{g}^n$ are absolutely continuous functions of $t \in [0, t_{\max}]$. Let $w = [\,\underline{r}^n, \underline{s}^n]^T$; then w satisfies the following integral equation:

$$w = A \int_0^t w\,ds + d,$$

where A is a constant $2n \times 2n$ matrix and $d : [0, t_{\max}] \to \mathbb{R}^{2n}$ is an absolutely continuous vector-valued function, defined by G_1, G_2, H_1, H_2, $\underline{f}^n$ and $\underline{g}^n$ respectively. To prove the existence of a solution, we proceed in a standard way. First, we construct a sequence of absolutely continuous functions:

$$\begin{aligned} w_0(t) &= 0, \\ w_1(t) &= A \int_0^t w_0(s)\,ds + d(t) = d(t), \\ w_2(t) &= A \int_0^t w_1(s)\,ds + d(t), \ \ldots \\ w_{k+1}(t) &= A \int_0^t w_k(s)\,ds + d(t), \ \ldots \end{aligned}$$

This sequence satisfies the estimate

$$|w_{k+1}(t) - w_k(t)| \le \frac{\|A\|^k t_{\max}^k}{k!} \sup_{t \in [0, t_{\max}]} |w_1(t)|$$

where $|\cdot|$ is a norm in $\mathbb{R}^{2n}$. From this inequality, using the standard procedure, we prove that the sequence $\{w_k(t)\}$ converges uniformly in $[0, t_{\max}]$ to a continuous function w, which is a solution of our integral equation. The equation implies that w is absolutely continuous since $\int_0^t w(s)\,ds$ and d are. Uniqueness is proved in the standard way. ■

LEMMA 1.3. *If* $F \in L^2(0, t_{\max}; L^2(L, P))$ *and* $G \in L^2(0, t_{\max}; L^2(L, P))$, *then the Galerkin solution* (R_n, S_n) *satisfies the following estimates*:

$$\|R_n(t, \cdot)\|_0^2 + \|S_n(t, \cdot)\|_0^2 \le K_1$$

for $t \in [0, t_{\max}]$, *and*

$$\int_0^{t_{\max}} (\|R_n(t, \cdot)\|_0^2 + \|S_n(t, \cdot)\|_0^2 + \|S_{nx}(t, \cdot)\|_0^2)\,dt \le K_2,$$

where K_1 *and* K_2 *are constants, determined by* F *and* G, *and independent of* n.

Proof. Since R_n, S_n are absolutely continuous with respect to $t \in [0, t_{\max}]$, for fixed k have

$$
\begin{aligned}
&b(R_{nt}(t,\cdot),S_{nt}(t,\cdot);\xi_k,0)+a(R_n(t,\cdot),S_n(s,\cdot);\xi_k,0)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad=b(F(t,\cdot),G(t,\cdot);\xi_k,0),\\
&b(R_{nt}(t,\cdot),S_{nt}(t,\cdot);0,\zeta_k)+a(R_n(t,\cdot),S_n(s,\cdot);0,\zeta_k)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad=b(F(t,\cdot),G(t,\cdot);0,\zeta_k)
\end{aligned}
$$

a.e. in $(0,t_{\max})$. Take the linear combination of the first equations with the coefficients $r_k^n(t)$, $k=1,\ldots,n$, and the linear combination of the second equations with the coefficients $s_k^n(t)$, $k=1,\ldots,n$. Adding the resulting equations, we obtain

$$
\begin{aligned}
&b(R_{nt}(t,\cdot),S_{nt}(t,\cdot);R_n(t,\cdot),S_n(t,\cdot))+a(R_n(t,\cdot),S_n(\cdot);R_n(t,\cdot),S_n(t,\cdot))\\
&\qquad\qquad\qquad\qquad\qquad\qquad=b(F(t,\cdot),G(t,\cdot);R_n(t,\cdot),S_n(t,\cdot)).
\end{aligned}
$$

Using the definition of a and b, we have

$$
\begin{aligned}
&\frac{D}{2}\|R_n(t,\cdot)\|_{0t}^2+\frac{1}{2}\|S_n(t,\cdot)\|_{0t}^2+\|S_n(t,\cdot)\|_0^2+\kappa\|S_{nx}(t,\cdot)\|_0^2\\
&\qquad\qquad\qquad\qquad=D(F(t,\cdot),R_n(t,\cdot))_0+(G(t,\cdot),S_n(t,\cdot))_0
\end{aligned}
$$

a.e. in $(0,t_{\max})$. Integrating both sides over $(0,t)$ for $t\in[0,t_{\max}]$, we have

$$
\begin{aligned}
&\frac{D}{2}\|R_n(t,\cdot)\|_0^2+\frac{1}{2}\|S_n(t,\cdot)\|_0^2+\int_0^t[\|S_n(s,\cdot)\|_0^2+\kappa\|S_{nx}(s,\cdot)\|_0^2]\,ds\\
&\qquad\qquad=\int_0^t[D(F(s,\cdot),R_n(s,\cdot))_0+(G(s,\cdot),S_n(s,\cdot))_0]\,ds
\end{aligned}
$$

for $t\in[0,t_{\max}]$. Applying the ε-inequality to the right-hand side, we obtain

$$
\begin{aligned}
&\frac{D}{2}\|R_n(t,\cdot)\|_0^2+\frac{1}{2}\|S_n(t,\cdot)\|_0^2+\int_0^t[\|S_n(s,\cdot)\|_0^2+\kappa\|S_{nx}(s,\cdot)\|_0^2]\,ds\\
&\qquad\le\frac{D}{2}\int_0^t\frac{\|F(s,\cdot)\|_0^2}{\varepsilon^2}\,ds+\frac{1}{2}\int_0^t\frac{\|G(s,\cdot)\|_0^2}{\varepsilon^2}\,ds\\
&\qquad\quad+\frac{D}{2}\varepsilon^2\int_0^t\|R_n(s,\cdot)\|_0^2\,ds+\frac{\varepsilon^2}{2}\int_0^t\|S_n(s,\cdot)\|_0^2\,ds.
\end{aligned}
\tag{1.11}
$$

Using the notation

$$
\begin{aligned}
C&=\frac{D}{2}\int_0^{t_{\max}}\frac{\|F(s,\cdot)\|_0^2}{\varepsilon^2}\,ds+\frac{1}{2}\int_0^{t_{\max}}\frac{\|G(s,\cdot)\|_0^2}{\varepsilon^2}\,ds,\\
U(t)&=\frac{D}{2}\|R_n(t,\cdot)\|_0^2+\frac{1}{2}\|S_n(t,\cdot)\|_0^2,
\end{aligned}
$$

we get the following *Gronwall inequality* (see for example [1]):

$$0 \le U(t) \le C + \varepsilon^2 \int_0^t U(s)\, ds.$$

It yields, for example for $\varepsilon = 1$,

$$0 \le U(t) \le Ce^t \le Ce^{t_{\max}} = K_1.$$

This implies the first statement of the lemma. Moreover, for all n,

$$\int_0^{t_{\max}} \|R_n(s,\cdot)\|_0^2\, ds \le \int_0^{t_{\max}} [\|R_n(s,\cdot)\|_0^2 + \|S_n(s,\cdot)\|_0^2]\, ds \le K_3,$$

where K_3 is a constant which depends on F and G only. From (1.11) we have

$$\int_0^t [\|S_n(s,\cdot)\|_0^2 + \kappa\|S_{nx}(s,\cdot)\|_0^2]\, ds$$

$$\le \int_0^t \left[\frac{D}{2}\cdot\frac{\|F(s,\cdot)\|_0^2}{\varepsilon^2} + \frac{1}{2}\cdot\frac{\|G(s,\cdot)\|_0^2}{\varepsilon^2} + \frac{D}{2}\varepsilon^2\|R_n(s,\cdot)\|_0^2 + \frac{\varepsilon^2}{2}\|S_n(s,\cdot)\|_0^2\right] ds.$$

Using the estimate for $U(t)$, for $\varepsilon = 1$ we get

$$\int_0^{t_{\max}} [\|S_n(s,\cdot)\|_0^2 + \kappa\|S_{nx}(s,\cdot)\|_0^2]\, ds \le C + Ce^{t_{\max}} t_{\max} = K_4.$$

Hence,

$$\int_0^{t_{\max}} [\|R_n(s,\cdot)\|_0^2 + \|S_n(s,\cdot)\|_0^2 + \|S_{nx}(s,\cdot)\|_0^2]\, ds \le K_2,$$

where K_2 is a constant. ∎

LEMMA 1.4. *Let* $F \in L^2(0,t_{\max}; L^2(L,P))$ *and* $G \in L^2(0,t_{\max}; L^2(L,P))$. *Then there exists a constant* K, *determined by* F *and* G, *such that for the Galerkin solution* (R_n, S_n) *of* (1.10) *and for* $n = 1,2,\ldots$ *the following estimate holds*:

$$\int_0^{t_{\max}} [\|R_n(s,\cdot)\|_0^2 + \|S_n(s,\cdot)\|_0^2 + \|S_{nx}(s,\cdot)\|_0^2 + \|R_{nt}(s,\cdot)\|_0^2 + \|S_{nt}(s,\cdot)\|_0^2]\, ds \le K.$$

Proof. The proof reduces to showing that for some constant M,

$$\int_0^{t_{\max}} [\|R_{nt}(s,\cdot)\|_0^2 + \|S_{nt}(s,\cdot)\|_0^2]\, ds \le M. \tag{1.12}$$

Proceeding as in the proof of Lemma 1.3, we get

$$b(R_{nt}(t,\cdot), S_{nt}(t,\cdot); R_{nt}(t,\cdot), 0) + a(R_n(t,\cdot), S_n(t,\cdot,); R_{nt}(t,\cdot), 0) \\ = b(F(t,\cdot), G(t,\cdot); R_{nt}(t,\cdot), 0).$$

Hence,

$$\begin{aligned}|(R_{nt}(t,\cdot),R_{nt}(t,\cdot))_0| &\le |(S_{nx}(t,\cdot),R_{nt}(t,\cdot))_0| + |(F(t,\cdot),R_{nt}(t,\cdot))_0|\\ &\le \|S_{nx}(t,\cdot)\|_0\|R_{nt}(t,\cdot)\|_0 + \|F(t,\cdot)\|_0\|R_{nt}(t,\cdot)\|_0,\end{aligned}$$

and so

$$\|R_{nt}(t,\cdot)\|_0 \le \|S_{nx}(t,\cdot)\|_0 + \|F(t,\cdot)\|_0.$$

Integrating this inequality over $(0,t)$ for $t\in[0,t_{\max}]$, we get

$$\int_0^t \|R_{nt}(s,\cdot)\|_0\,ds \le \int_0^t \|S_{nx}(s,\cdot)\|_0\,ds + \int_0^t \|F(s,\cdot)\|_0\,ds, \tag{1.13}$$

which is bounded in view of Lemma 1.3. Similarly, by the definition of a Galerkin solution, we have

$$\begin{aligned}(S_{nt}(t,\cdot),S_{nt}(t,\cdot))_0 - D(R_n(t,\cdot),S_{nxt}(t,\cdot))_0 + (S_n(t,\cdot),S_{nt}(t,\cdot))_0&\\ + \kappa(S_{nx}(t,\cdot),S_{nxt}(s,\cdot))_0 = (G(t,\cdot),S_{nt}(t,\cdot))_0.&\end{aligned}$$

After integration over $(0,t)$ for $t\in[0,t_{\max}]$, we obtain

$$\begin{aligned}&\int_0^t \|S_{nt}(s,\cdot)\|_0^2\,ds - D\int_0^t (R_n(s,\cdot),S_{nxt}(s,\cdot))_0\,ds\\ &\quad+\int_0^t (S_n(s,\cdot),S_{nt}(s,\cdot))_0\,ds + \kappa\int_0^t (S_{nx}(s,\cdot),S_{nxt}(s,\cdot))_0\,ds\\ &\qquad= \int_0^t (G(s,\cdot),S_{nt}(s,\cdot))_0\,ds.\end{aligned} \tag{1.14}$$

Observe that

$$\int_0^t (S_n(s,\cdot),S_{nt}(s,\cdot))_0\,ds = \frac{1}{2}\|S_n(t,\cdot)\|_0^2$$

and

$$\int_0^t (S_{nx}(s,\cdot),S_{nxt}(s,\cdot))_0\,ds = \frac{1}{2}\|S_{nx}(t,\cdot)\|_0^2,$$

since $S_n(0,\cdot)=0$ and $S_{nx}(0,\cdot)=0$. We also have

$$-\int_0^t (R_n(s,\cdot),S_{nxt}(s,\cdot))_0\,ds = \int_0^t (R_{nt}(s,\cdot),S_{nx}(s,\cdot))_0\,ds - (R_n(t,\cdot),S_{nx}(t,\cdot))_0.$$

Using the ε-inequality for the second term, we get

$$\begin{aligned}&\Big|\int_0^t (R_n(s,\cdot),S_{nxt}(s,\cdot))_0\,ds\Big|\\ &\quad\le 2\int_0^t \|S_{nx}(s,\cdot)\|_0^2\,ds + 2\int_0^t \|R_{nt}(s,\cdot)\|_0^2\,ds + C\|R_n(t,\cdot)\|_0^2 + \frac{\varepsilon^2}{2}\|S_{nx}(t,\cdot)\|_0^2.\end{aligned}$$

Using (1.13), we obtain

$$\Big|\int_0^t (R_n(s,\cdot), S_{nxt}(s,\cdot))_0\,ds\Big|$$
$$\leq 4\int_0^t \|S_{nx}(s,\cdot)\|_0^2\,ds + 2\int_0^t \|F(s,\cdot)\|_0^2\,ds + \frac{\varepsilon^2}{2}\|S_{nx}(t,\cdot)\|_0^2 + C\|R_n(t,\cdot)\|_0^2$$

where C is a constant introduced by the ε-inequality. To estimate the term $\int_0^t (G(s,\cdot), S_{nt}(s,\cdot))_0\,ds$, we again use the ε-inequality:

$$\Big|\int_0^t (G(s,\cdot), S_{nt}(s,\cdot))_0\,ds\Big| \leq C\int_0^t \|G(s,\cdot)\|_0^2\,ds + \varepsilon^2\int_0^t \|S_{nt}(s,\cdot)\|_0^2\,ds.$$

Using these estimates in (1.14), we get

$$(1-\varepsilon^2)\int_0^t \|S_{nt}(s,\cdot)\|_0^2\,ds + \frac{1}{2}\|S_n(t,\cdot)\|_0^2 + \frac{1}{2}(\kappa - D\varepsilon^2)\|S_x(t,\cdot)\|_0^2$$
$$\leq C\Big[\int_0^t \|F(s,\cdot)\|_0^2\,ds + \int_0^t \|G(s,\cdot)\|_0^2\,ds + \|R_n(t,\cdot)\|_0^2 + \int_0^t \|S_{nx}(s,\cdot)\|_0^2\,ds\Big],$$

where C is a positive constant. The terms of the right-hand side with R_n and S_{nx} are estimated by Lemma 1.3. From this, (1.13) and Lemma 1.3, the inequality (1.12) follows. ■

Let us comment on the results obtained. Lemmas 1.3 and 1.4 imply that if $F, G \in L^2(0, t_{\max}; L^2(L,P))$, then any Galerkin solution is in the space $\mathbf{H}$. This justifies our choice of the space $\mathbf{H}$ for the weak formulation of the problem.

THEOREM 1.1. *Assume that*

- ϕ, $\psi \in L^2(0, t_{\max})$,
- T^0, $Q^0 \in L^2(L,P)$.

Then in $\mathbf{H}$ *there exists a solution* (R,S) *of the problem* (1.10).

Proof. Let

$$X_n = \Big\{\sum_{j=1}^n \xi_j c_j \,\Big|\, c_j \in H^1(0,t_{\max})\Big\},$$
$$Y_n = \Big\{\sum_{j=1}^n \zeta_j d_j \,\Big|\, d_j \in H^1(0,t_{\max})\Big\},$$

where ξ_j and ζ_j are as in the Galerkin solution. Observe that $X_n \subset X_{n+1}$ and $Y_n \subset Y_{n+1}$ for all n, and that

$$X_n \times Y_n \subset \mathbf{H}.$$

We can assume that the subspaces X_n and Y_n satisfy the following approximation condition:

For all $(x,y) \in \mathbf{H}$ *and for all* n *there exist* $x_n(x) \in X_n$ *and* $y_n(y) \in Y_n$ *such that*

$$\|(x_n(x), y_n(y)) - (x,y)\|_{\mathbf{H}} \to 0 \quad \textit{as } n \to \infty.$$

Let (R_n, S_n) be the Galerkin solution of the problem (1.10). By Lemmas 1.3 and 1.4 it follows that the sequence $\{(R_n, S_n)\}$ is bounded in the Hilbert space $\mathbf{H}$, hence it has a subsequence, denoted also by $\{(R_n, S_n)\}$, weakly converging to some $(R,S) \in \mathbf{H}$. We prove that (R,S) is a solution of the problem (1.10). Fix m, and take $n \ge m$. For $k = 1, \ldots, m$ we have

$$\begin{aligned} b(R_{nt}(t,\cdot), S_{nt}(t,\cdot); \xi_k, 0) + a(R_n(t,\cdot), S_n(t,\cdot); \xi_k, 0) \\ = b(F(t,\cdot), G(t,\cdot); \xi_k, 0), \\ b(R_{nt}(t,\cdot), S_{nt}(t,\cdot); 0, \zeta_k) + a(R_n(t,\cdot), S_n(t,\cdot); 0, \zeta_k) \\ = b(F(t,\cdot), G(t,\cdot); 0, \zeta_k), \end{aligned}$$

a.e. in $(0, t_{\max})$. Taking linear combinations with coefficients $c_k, d_k \in H^1(0, t_{\max})$ for the first and second equations respectively and $k = 1, \ldots, m$, and adding the resulting equations, we get

$$\begin{aligned} b(R_{nt}(t,\cdot), S_{nt}(t,\cdot); V, W) + a(R_n(t,\cdot), S_n(t,\cdot); V, W) \\ = b(F(t,\cdot), G(t,\cdot); V, W), \end{aligned}$$

a.e. in $(0, t_{\max})$, for all $V \in X_m$ and $W \in Y_m$. Letting $n \to \infty$ and taking into account the continuity of the forms a and b in $\mathbf{H}$, we see that

$$b(R_t(t,\cdot), S_t(t,\cdot); V, W) + a(R(t,\cdot), S(t,\cdot); V, W) = b(F(t,\cdot), G(t,\cdot); V, W)$$

a.e. in $(0, t_{\max})$, for all $V \in X_m$ and $W \in Y_m$ and any m, and hence for all $(V,W) = \mathbf{H}$, by the approximation property of the subspaces X_n and Y_n. ■

THEOREM 1.2. *Under the assumptions of Theorem* 1.1, *the problem* (1.10) *has a unique solution in* $\mathbf{H}$.

Proof. Suppose that (R,S) and (R_1, S_1) are two solutions of (1.10) in $\mathbf{H}$. For $(U,Z) = (R - R_1, S - S_1)$ we have

$$b(U_t(t,\cdot), Z_t(t,\cdot); V, W) + a(U(s,\cdot), Z(s,\cdot); V, W)\, ds = 0,$$

where $U(0,\cdot) = Z(0,\cdot) = 0$ for all $(V,W) \in \mathbf{H}$, a.e. in $(0, t_{\max})$.

Setting now $V = U$ and $W = Z$ and using the definition of a and b, we get

$$D(U_t, U)_0 + (Z_t, Z)_0 + \|Z\|_0^2 + \kappa \|Z_x\|_0^2 = 0.$$

Integrating this equation over $(0,t)$ for $t \in (0, t_{\max})$, we obtain

$$\frac{D}{2}\|U(t,\cdot)\|_0^2+\frac{1}{2}\|Z(t,\cdot)\|_0^2+\int_0^t\|Z(s,\cdot)\|_0^2\,ds+\kappa\int_0^t\|Z_x(s,\cdot)\|_0^2\,ds=0$$

for all $t\in[0,t_{\max}]$, which implies that $R=R_1$ and $S=S_1$. ■

REMARK 1. It follows from Lemmas 1.3 and 1.4 that the problem (1.10) is stable with respect to F and G, i.e.

$$\|(R,S)\|_{\mathbf{H}}\le C[\|F\|_{L^2(0,t_{\max};L^2(L,P))}+\|G\|_{L^2(0,t_{\max};L^2(L,P))}],$$

where C is independent of F and G.

REMARK 2. Theorems 1.1 and 1.2 imply that we may attribute to the original problem (0.2)–(0.4) a generalized solution (T,Q) satisfying

$$R(t,x)=\int_0^t[T(s,x)-\Phi(s,x)]\,ds,$$

$$S(t,x)=\int_0^t[Q(s,x)-\Psi(s,x)]\,ds,$$

where $(R,S)\in\mathbf{H}$ is the solution of (1.10). It follows that

$$T,\ Q,\ \int_0^t T(s,\cdot)\,ds\in L^2(0,t_{\max};L^2(L,P)),$$

however

$$\int_0^t Q(s,\cdot)\,ds\in L^2(0,t_{\max};H^1(L,P)).$$

Note that T and even $\int_0^t T(s,\cdot)\,ds$ are in $L^2(0,t_{\max};L^2(L,P))$. In fact the initial and boundary values for (T,Q), $T^0(x)$, $Q^0(x)$, $\Phi(t,0)$, and $\Psi(t,0)$ enter into the problem via the right-hand sides of the equations for (R,S), but $T(0,x)$, $Q(0,x)$, $T(t,0)$ are not well defined. This explains why the computed approximate solution (T,Q) of the original problem only *weakly* depends on the boundary condition on T. The function Q is more regular, hence its behaviour is slightly different.

2. Numerical model. On the rectangle $\Omega=[0,t_{\max}]\times[L,P]$, we define the grid

$$t_n=\tau n,\quad n=0,1,\ldots,N,\quad x_k=L+hk,\quad k=0,1,\ldots,M+1,$$

where $t_{\max}=\tau N$ and $h(M+1)=P-L$.

Let $f=\{f_k\}_{k=0,1,\ldots,M+1}$ be a real grid-function. The following notation is used for finite differences:

$$\Delta f_k=f_{k+1}-f_k\quad\text{(forward difference)},$$
$$\nabla f_k=f_k-f_{k-1}\quad\text{(backward difference)}.$$

We denote by $(\cdot,\cdot)_h$ and $\|\cdot\|_h$ the $L^2_h(L,P)$ inner product and norm, respectively:

$$(f,g)_h = h\sum_{k=1}^{M} f_k g_k, \qquad \|f\|_h^2 = h\sum_{k=1}^{M} f_k^2.$$

The values $T_k^n, Q_k^n, \ldots$ of grid-functions correspond to $T(t_n,x_k), Q(t_n,x_k), \ldots$ where $(T,\,Q)$ is the solution of (0.2)–(0.4) (see Section 1).

Let us now define the relevant discrete problem. The difference equations for the grid-functions $\{T_k^n\}$ and $\{Q_k^n\}$ are

$$T_k^{n+1} - T_k^n + \frac{\lambda}{4}(\Delta+\nabla)(Q_k^n + Q_k^{n+1}) = 0, \tag{2.1}$$

$$\begin{aligned} Q_k^{n+1} - Q_k^n + \frac{D\lambda}{4}(\Delta+\nabla)(T_k^n + T_k^{n+1}) + \frac{\tau}{2}(Q_k^n + Q_k^{n+1}) \\ - \frac{\kappa\mu}{2}(\nabla\Delta)(Q_k^n + Q_k^{n+1}) = 0, \end{aligned} \tag{2.2}$$

where $\lambda = \tau/h$ and $\mu = \tau/h^2$.

For these equations we impose initial conditions for the grid-functions

$$T_k^0,\ Q_k^0, \qquad k = 1,\ldots M, \tag{2.3}$$

and the following boundary conditions:

$$T_0^n = \phi^n, \quad T_{M+1}^n = 0, \quad Q_0^n = \psi^n, \quad Q_{M+1}^n = 0, \quad n = 0,1,\ldots N. \tag{2.4}$$

The right-hand sides of these conditions are given by certain averages of the functions T^0, Q^0 in $L^2(L,P)$ or ϕ and ψ in $L^2(0,t_{\max})$ respectively. For example

$$\phi^n = \frac{1}{\tau}\int_{t_n}^{t_{n+1}} \phi(s)\,ds, \qquad \psi^n = \frac{1}{\tau}\int_{t_n}^{t_{n+1}} \psi(s)\,ds,$$

and so on. Introducing, as in Section 1, the grid-functions

$$\phi_k^n = \phi^n\left(1 - \frac{kh}{P-L}\right) \quad \text{and} \quad \psi_k^n = \psi^n\left(1 - \frac{kh}{P-L}\right), \\ k = 0,1,\ldots,M+1,\ n = 0,1,\ldots,N,$$

and

$$\widetilde{T}_k^n = T_k^n - \phi_k^n, \qquad \widetilde{Q}_k^n = Q_k^n - \psi_k^n,$$

we write down equations for $\widetilde{T}$ and $\widetilde{Q}$:

$$\widetilde{T}_k^{n+1} - \widetilde{T}_k^n + \frac{\lambda}{4}(\Delta+\nabla)(\widetilde{Q}_k^n + \widetilde{Q}_k^{n+1}) = F_k^n,$$

$$\begin{aligned} \widetilde{Q}_k^{n+1} - \widetilde{Q}_k^n + \frac{D\lambda}{4}(\Delta+\nabla)(\widetilde{T}_k^n + \widetilde{T}_k^{n+1}) + \frac{\tau}{2}(\widetilde{Q}_k^n + \widetilde{Q}_k^{n+1}) \\ - \frac{\kappa\mu}{2}(\nabla\Delta)(\widetilde{Q}_k^n + \widetilde{Q}_k^{n+1}) = G_k^n. \end{aligned}$$

Here

$$F_k^n = \frac{\tau}{2}\cdot\frac{\psi^n+\psi^{n+1}}{(P-L)} - (\phi^{n+1}-\phi^n)\Big(1-\frac{kh}{P-L}\Big),$$

$$G_k^n = \frac{\tau}{2}\cdot\frac{D}{P-L}(\phi^n+\phi^{n+1}) = \Big[\psi^{n+1}-\psi^n+\frac{\tau}{2}(\psi^n+\psi^{n+1})\Big]\Big(1-\frac{kh}{P-L}\Big).$$

Let us finally introduce grid-functions R_k^n and S_k^n:

$$R_k^n = \tau\sum_{j=0}^{n}\widetilde{T}_k^j, \qquad R_k^0 = \tau\Big[T_k^0 - \phi(0)\Big(1-\frac{kh}{P-L}\Big)\Big],$$

$$S_k^n = \tau\sum_{j=0}^{n}\widetilde{Q}_k^j, \qquad S_k^0 = \tau\Big[Q_k^0 - \psi(0)\Big(1-\frac{kh}{P-L}\Big)\Big].$$

Multiplying both sides of the last two equations by τ and then adding them for $j=0,1,\ldots,n$, we get equations for $\{R_k^n\}$ and $\{S_k^n\}$:

$$R_k^{n+1} - R_k^n + \frac{\lambda}{4}(\Delta+\nabla)(S_k^n+S_k^{n+1}) = \tau\Phi_k^n, \tag{2.5}$$

$$S_k^{n+1} - S_k^n + \frac{D\lambda}{4}(\Delta+\nabla)(R_k^n+R_k^{n+1}) + \frac{\tau}{2}(S_k^n+S_k^{n+1}) - \frac{\kappa\mu}{2}(\nabla\Delta)(S_k^n+S_k^{n+1}) = \tau\Psi_k^n, \tag{2.6}$$

where

$$\begin{aligned}\Phi_k^n &= T_k^0 + \frac{\lambda}{4}(Q_{k+1}^0 - Q_{k-1}^0) + \frac{\lambda}{M+1}\sum_{j=0}^{n}\psi(t_j)\\ &\quad + \frac{\lambda}{2(M+1)}\psi(t_{n+1}) - \Big(1-\frac{k}{M+1}\Big)\phi(t_{n+1}),\\ \Psi_k^n &= \frac{\lambda D}{2(M+1)}\Big[\phi(0)+2\sum_{j=0}^{n}\phi(t_j)+\phi(t_{j+1})\Big]\\ &\quad + \Big(1-\frac{k}{M+1}\Big)\Big[\Big(1-\frac{\tau}{2}\Big)\phi(0) - \tau\sum_{j=1}^{n}\psi(t_j) - \Big(1+\frac{\tau}{2}\Big)\psi(t_{n+1})\Big]\\ &\quad + \Big(1+\frac{\tau}{2}\Big)\Big[Q_k^0 - \Big(1-\frac{k}{M+1}\Big)\psi(0)\Big]\\ &\quad + \frac{\lambda D}{4}\Big[T_{k+1}^0 - T_{k-1}^0 + \frac{2}{M+1}\phi(0)\Big] - \frac{\kappa\mu}{2}\nabla\Delta Q_k^0\end{aligned}$$

for $n=1,\ldots,N$.

For *the original* functions T_k^n and Q_k^n we have

$$T_k^{n+1} = \frac{R_k^{n+1} - R_k^n}{\tau} + \phi_k^n, \qquad Q_k^{n+1} = \frac{S_k^{n+1} - S_k^n}{\tau} + \psi_k^n.$$

Let now $\{f_k\}_{k=0,1,\dots,M+1}$ and $\{g_k\}_{k=0,1,\dots,M+1}$ be two grid-functions. We need the following facts:

LEMMA 2.1. *If* $f_{M+1} = g_0 = 0$, *then*

$$\sum_{k=1}^{M} g_k \Delta f_k = -\sum_{k=1}^{M} f_k \nabla g_k.$$

Proof. We have

$$\begin{aligned}\sum_{k=1}^{M} g_k \Delta f_k &= \sum_{k=1}^{M} f_{k+1} g_k - \sum_{k=1}^{M} f_k g_k = \sum_{k=2}^{M+1} f_k g_{k-1} - \sum_{k=1}^{M} f_k g_k \\ &= \sum_{k=2}^{M} f_k (g_{k-1} - g_k) + f_{M+1} g_M - f_1 g_1 \\ &= \sum_{k=2}^{M} f_k (g_{k-1} - g_k) + f_1 (g_0 - g_1) = -\sum_{k=1}^{M} f_k \nabla g_k. \ \blacksquare\end{aligned}$$

LEMMA 2.2. *If* $f_0 = g_{M+1} = 0$, *then*

$$\sum_{k=1}^{M} g_k \nabla f_k = -\sum_{k=1}^{M} f_k \Delta g_k.$$

Proof. Exchange f_k and g_k. ■

LEMMA 2.3. *If* $g_0 = g_{M+1}$, *then*

$$\sum_{k=1}^{M} g_k \nabla \Delta f_k = -\sum_{k=0}^{M} \Delta f_k \Delta g_k.$$

Proof. Let $F_k = \Delta f_k$. We have

$$\begin{aligned}\sum_{k=1}^{M} g_k \nabla \Delta f_k &= \sum_{k=1}^{M} g_k F_k - \sum_{k=0}^{M-1} F_k g_{k+1} \\ &= \sum_{k=1}^{M-1} F_k (g_k - g_{k+1}) + g_m F_M - g_1 F_0 \\ &= -\sum_{k=1}^{M-1} F_k (g_{k+1} - g_k) + g_M F_M - g_{M+1} F_M \\ &= -F_0 g_1 + F_0 g_0 = -\sum_{k=0}^{M} F_k (g_{k+1} - g_k) = -\sum_{k=0}^{M} \Delta f_k \Delta g_k. \ \blacksquare\end{aligned}$$

LEMMA 2.4 (Gronwall inequality). *Let* $u_n \geq 0$, $\gamma_n \geq 0$, $\gamma_n \leq \gamma_{n+1}$, $c \geq 0$ *for all* $n = 0, 1, 2, \ldots$ *If*

$$0 \leq u_n \leq \gamma_n + c \sum_{j=0}^{n-1} u_j \quad \text{for all } n = 0, 1, \ldots,$$

then

$$0 \leq u_n \leq e^{nc}(\gamma_n + u_0) \quad \text{for all } n = 0, 1, \ldots$$

Proof. Let us proceed by induction. We have $0 \leq u_0 \leq (\gamma_0 + u_0) = e^{0c}(\gamma_0 + u_0)$. Assume the statement to be true for u_j, $0 \leq j \leq n$. Then

$$\begin{aligned} 0 \leq u_{n+1} &\leq \gamma_{n+1} + c \sum_{j=0}^{n} u_j \leq \gamma_{n+1} + c \sum_{j=0}^{n} e^{jc}(\gamma_j + u_0) \\ &\leq \gamma_{n+1} + c \sum_{j=0}^{n} e^{jc}(\gamma_{n+1} + u_0) \\ &\leq \gamma_{n+1}\Big[1 + c \sum_{j=0}^{n} (e^c)^j\Big] + u_0 c \sum_{j=0}^{n} (e^c)^j. \end{aligned}$$

Observe that

$$c \sum_{j=0}^{n} (e^c)^j = c\frac{e^{c(n+1)} - 1}{e^c - 1} \quad \text{and} \quad 1 + c\frac{e^{c(n+1)} - 1}{c} = e^{c(n+1)}.$$

But $c \leq e^c - 1$, hence

$$c \sum_{j=0}^{n} (e^c)^j \leq c\frac{e^{c(n+1)} - 1}{c} \leq e^{c(n+1)}.$$

Hence

$$u_{n+1} \leq \gamma_{n+1}\Big[1 + c \sum_{j=0}^{n} (e^c)^j\Big] + u_0 c \sum_{j=0}^{n} (e^c)^j \leq e^{c(n+1)}(\gamma_{n+1} + u_0). \ \blacksquare$$

We are now in a position to state the first stability result. For an arbitrary grid-function $\{U_k^n\}_{k=1,\ldots,M}$, let

$$\underline{U}^n = [U_1^n, \ldots, U_M^n]^T.$$

THEOREM 2.1. *There exists a positive constant* K, *independent of* τ *and* h, *such that for* $m = 0, 1, \ldots, N-1$ ($N \leq t_{\max}/\tau$),

$$D\|\underline{R}^m\|_h^2 + \|\underline{S}^m\|_h^2 \leq K\Big[\|\underline{R}^0\|_h^2 + \|\underline{S}^0\|_h^2 + \tau \sum_{k=0}^{m} (\|\underline{\Phi}^k\|_h^2 + \|\underline{\Psi}^k\|_h^2)\Big].$$

Proof. Multiply (2.5) and (2.6) by $Dh(R_k^{n+1} + R_k^n)$ and $h(S_k^{n+1} + S_k^n)$ respectively and sum each of them over $k = 1, \ldots, M$. Summing the resulting two equations for $n = 0, 1, \ldots, m$, we get

$$D[\|\underline{R}^{m+1}\|_h^2 - \|\underline{R}^0\|_h^2] + \frac{\lambda D}{4}\sum_{n=0}^{m}((\Delta+\nabla)(\underline{S}^n + \underline{S}^{n+1}), \underline{R}^n + \underline{R}^{n+1})_h = \tau D \sum_{n=0}^{m}(\underline{\Phi}^n, \underline{R}^n + \underline{R}^{n+1})_h,$$

and

$$\|\underline{S}^{m+1}\|_h^2 - \|\underline{S}^0\|_h^2 + \frac{\lambda D}{4}\sum_{n=0}^{m}((\Delta+\nabla)(\underline{R}^n + \underline{R}^{n+1}), \underline{S}^n + \underline{S}^{n+1})_h + \frac{\tau}{2}\sum_{n=0}^{m}\|\underline{S}^n + \underline{S}^{n+1}\|_h^2 - \frac{\kappa\mu}{2}\sum_{n=0}^{m}((\nabla\Delta)(\underline{S}^n + \underline{S}^{n+1}), \underline{S}^n + \underline{S}^{n+1})_h = \tau\sum_{n=0}^{m}(\underline{\Psi}^n, \underline{S}^n + \underline{S}^{n+1})_h.$$

Adding the resulting equations, by Lemmas 2.1–2.3 we have

$$D\|\underline{R}^{m+1}\|_h^2 + \|\underline{S}^{n+1}\|_h^2 + \frac{\tau}{2}\sum_{n=0}^{m}\|\underline{S}^n + \underline{S}^{n+1}\|_h^2 + \frac{\kappa\mu}{2}\sum_{n=0}^{m}\sum_{k=0}^{M} h[\Delta(S_k^n + S_k^{n+1})]^2 = \tau D\sum_{n=0}^{m}(\underline{\Phi}^n, \underline{R}^n + \underline{R}^{n+1})_h + \tau\sum_{n=0}^{m}(\underline{\Psi}^n, \underline{S}^n + \underline{S}^{n+1})_h + D\|\underline{R}^0\|_h^2 + \|\underline{S}^0\|_h^2.$$

Applying now the ε-inequality to the terms $(\underline{\Phi}^n, \underline{R}^n + \underline{R}^{n+1})_h$ and $(\underline{\Psi}^n, \underline{S}^n + \underline{S}^{n+1})_h$, we obtain

$$\begin{aligned}
(2.7)\quad & D(1-2\varepsilon\tau)\|\underline{R}^{m+1}\|_h^2 + (1-2\varepsilon\tau)\|\underline{S}^{m+1}\|_h^2 \\
&\quad + \frac{\tau}{2}\sum_{n=0}^{m}\|\underline{S}^n + \underline{S}^{n+1}\|_h^2 + \frac{\kappa\tau}{2}\sum_{n=0}^{m}\sum_{k=0}^{M} h\left[\frac{\Delta(S_k^n + S_k^{n+1})}{h}\right]^2 \\
&\le 2\varepsilon\tau\sum_{n=0}^{m}[D\|\underline{R}^n\|_h^2 + \|\underline{S}^n\|_h^2] \\
&\quad + D\|\underline{R}^0\|_h^2 + \|\underline{S}^n\|_h^2 + K_1\tau\sum_{n=0}^{m}[D\|\underline{\Phi}^n\|_h^2 + \|\underline{\Psi}^n\|_h^2].
\end{aligned}$$

Set

$$u_{m+1} = D\|\underline{R}^{m+1}\|_h^2 + \|\underline{S}^{m+1}\|_h^2,$$

$$\gamma_{m+1} = \frac{K_1}{1-2\varepsilon\tau}\tau\sum_{n=0}^{m}[D\|\underline{\Phi}^n\|_h^2 + \|\underline{\Psi}^n\|_h^2] + \frac{D}{1-2\varepsilon\tau}\|\underline{R}^0\|_h^2 + \frac{1}{1-2\varepsilon\tau}\|\underline{S}^0\|_h^2,$$

$$c = \tau\frac{2\varepsilon}{1-2\varepsilon\tau},$$

and apply the Gronwall inequality (Lemma 2.4). This implies, for $0 \le m \le N \le t_{\max}/\tau$,

$$D\|\underline{R}^m\|_h^2 + \|\underline{S}^m\|_h^2 \le e^{m\tau 2\varepsilon/(1-2\varepsilon\tau)} \times \left[\frac{K_1}{1-2\varepsilon\tau}\tau\sum_{n=0}^{m}(D\|\underline{\Phi}^n\|_h^2 + \|\underline{\Psi}^n\|_h^2) + \frac{D}{1-2\varepsilon\tau}\|\underline{R}^0\|_h^2 + \frac{1}{1-2\varepsilon\tau}\|\underline{S}^0\|_h^2\right].$$

We can choose a constant K such that for all n with $0 \le n \le N$,

$$D\|\underline{R}^m\|_h^2 + \|\underline{S}^m\|_h^2 \le K\left[\tau\sum_{n=0}^{m}(\|\underline{\Phi}^n\|_h^2 + \|\underline{\Psi}^n\|_h^2) + \|\underline{R}^0\|_h^2 + \|\underline{S}^0\|_h^2\right],$$

since $m\tau \le N\tau \le t_{\max}$. ■

We now state the second stability result.

THEOREM 2.2. *There is a positive constant K independent of τ and h such that for $m = 1, \dots, N-1$ ($N \le t_{\max}/\tau$),*

$$D\|\underline{R}^{m+1}\|_h^2 + \|\underline{S}^{m+1}\|_h^2 + \frac{\tau}{2}\sum_{n=0}^{m}\|\underline{S}^n + \underline{S}^{n+1}\|_h^2 + \frac{\kappa\tau}{2}\sum_{n=0}^{m}\sum_{k=0}^{M} h\left[\frac{\Delta(S_k^n + S_k^{n+1})}{h}\right]^2 \le K\left[\tau\sum_{n=0}^{N}(\|\underline{\Phi}^n\|_h^2 + \|\underline{\Psi}^n\|_h^2) + \|\underline{R}^0\|_h^2 + \|\underline{S}^0\|_h^2\right].$$

Proof. For sufficiently small ε the inequality (2.7) implies

$$\begin{aligned}
D\|\underline{R}^{m+1}\|_h^2 + \|\underline{S}^{m+1}\|_h^2 + \frac{\tau}{2}\sum_{n=0}^{m}\|\underline{S}^n + \underline{S}^{n+1}\|_h^2 + \frac{\kappa}{2}\sum_{n=0}^{m}\sum_{k=0}^{M} h\tau\left[\frac{\Delta(S_k^n + S_k^{n+1})}{h}\right]^2 \\
\le D\|\underline{R}^{m+1}\|_h^2 + \|\underline{S}^{m+1}\|_h^2 + \frac{\tau}{2(1-2\varepsilon\tau)}\sum_{n=0}^{m}\|\underline{S}^n + \underline{S}^{n+1}\|_h^2 \\
+ \frac{\kappa}{2(1-2\varepsilon\tau)}\sum_{n=0}^{m}\sum_{k=0}^{M} h\tau\left[\frac{\Delta(S_k^n + S_k^{n+1})}{h}\right]^2 \\
\le \frac{K_1}{1-2\varepsilon\tau}\tau\sum_{n=0}^{m}[D\|\underline{\Phi}^n\|_h^2 + \|\underline{\Psi}^n\|_h^2] \\
+ \frac{D}{1-2\varepsilon\tau}\|\underline{R}^0\|_h^2 + \frac{1}{1-2\varepsilon\tau}\|\underline{S}^0\|_h^2 \\
+ \frac{2\varepsilon\tau}{1-2\varepsilon\tau}\sum_{n=0}^{m}[D\|\underline{R}^n\|_h^2 + \|\underline{S}^n\|_h^2].
\end{aligned}$$

Applying now Theorem 2.1 to the last sum and taking into account that $\tau m \le \tau N = t_{\max}$, we obtain the assertion with some constant K. ■

REMARK 3. Theorems 2.1 and 2.2 give the stability of the finite difference scheme (2.5), (2.6). Theorem 2.1 gives the stability in the discrete norm, the max norm with respect to t and the discrete L^2 norm with respect to x. Theorem 2.2 gives the stability in some stronger norm. By well-known theory, the convergence in the corresponding norms follows from the stability [8].

We now derive a stability result for the scheme with the grid-functions $\{T_k^n\}$ and $\{Q_k^n\}$, corresponding directly to the original problem (0.2)–(0,4). For that we introduce divided differences $(\underline{R}^{n+1} - \underline{R}^n)/\tau$ and $(\underline{S}^{n+1} - \underline{S}^n)/\tau$.

It is convenient to use the following simple estimate:

Let $f = [f_1, \ldots, f_M]^T$ *and* $f_0 = f_{M+1} = 0$. Then

$$\|(\Delta + \nabla)f\|_h^2 \le 4\sum_{k=0}^{M} h(\Delta f_k)^2. \tag{2.8}$$

THEOREM 2.3. *There exists a constant K independent of τ and h such that for all $m < N = t_{\max}/\tau$,*

$$\sum_{n=0}^{m-1}\tau\left\|\frac{\underline{R}^{n+1} - \underline{R}^n}{\tau}\right\|_h^2 + \sum_{n=0}^{m-1}\tau\left\|\frac{\underline{S}^{n+1} - \underline{S}^n}{\tau}\right\|_h^2 + \|S^m\|_h^2 + \sum_{k=0}^{M} h\left(\frac{\Delta S_k^m}{h}\right)^2$$
$$\le K\left[\|\underline{R}^0\|_h^2 + \|\underline{S}^0\|_h^2 + \sum_{k=0}^{M} h\left(\frac{\Delta S_k^0}{h}\right)^2 + \sum_{n=0}^{N}\tau\|\underline{\varPhi}^n\|_h^2 + \sum_{n=0}^{N}\tau\|\underline{\varPsi}^n\|_h^2\right].$$

Proof. From (2.5) we deduce

$$\left\|\frac{\underline{R}^{n+1} - \underline{R}^n}{\tau}\right\|_h^2 \le \frac{1}{8}\left\|\frac{\Delta + \nabla}{h}\right\|_h^2 + 2\|\underline{\varPhi}^n\|h^2,$$

and by (2.8), summing with respect to $m < N = t_{\max}/\tau$, we get

$$\tau\sum_{n=0}^{m}\left\|\frac{\underline{R}^{n+1} - \underline{R}^n}{\tau}\right\|_h^2 \le \frac{\tau}{2}\sum_{n=0}^{m} h\sum_{k=0}^{M}\left[\frac{\Delta(S_k^n + S_k^{n+1})}{h}\right]^2 + 2\tau\sum_{n=0}^{m}\|\underline{\varPhi}^n\|_h^2.$$

This gives the estimate

$$\tau\sum_{n=0}^{m}\left\|\frac{\underline{R}^{n+1} - \underline{R}^n}{\tau}\right\|_h^2 \le K\Big\{\tau\sum_{n=0}^{N}[\|\underline{\varPhi}^n\|_h^2 + \|\underline{\varPsi}^n\|_h^2] + \|\underline{R}^0\|_h^2 + \|\underline{S}^0\|_h^2\Big\} \tag{2.9}$$

for a constant K independent of τ and h. Using Lemmas 2.1 and 2.2, we deduce from (2.6) that

$$\sum_{n=0}^{m}\tau\left\|\frac{\underline{S}^{n+1}-\underline{S}^{n}}{\tau}\right\|_h^2+\frac{1}{2}\|\underline{S}^{m+1}\|_h^2+\frac{\kappa}{2}h\sum_{k=0}^{M}\left(\frac{\Delta S_k^{m+1}}{h}\right)^2$$
$$=\frac{1}{2}\|\underline{S}^{0}\|_h^2+\frac{\kappa}{2}h\sum_{k=0}^{M}\left(\frac{\Delta S_k^{0}}{h}\right)^2$$
$$+\frac{D}{4}\sum_{n=0}^{m}h\sum_{k=1}^{M}\left[\frac{\Delta+\nabla}{h}(S_k^{n+1}-S_k^{n})\right](R_k^{n+1}+R_k^{n})$$
$$+\tau\sum_{n=0}^{m}\left(\underline{\Phi}^{n},\frac{\underline{S}^{n+1}-\underline{S}^{n}}{\tau}\right)_h.$$

We have

$$\sum_{n=0}^{m}h\sum_{k=1}^{M}\left[\frac{\Delta+\nabla}{h}(S_k^{n+1}-S_k^{n})\right](R_k^{n+1}+R_k^{n})$$
$$=-\tau\sum_{n=1}^{m}h\sum_{k=1}^{M}\left[\frac{\Delta+\nabla}{h}S_k^{n}\right]\left(\frac{R_k^{n+1}-R_k^{n}}{\tau}-\frac{R_k^{n}-R_k^{n-1}}{\tau}\right)$$
$$-h\sum_{k=1}^{M}\left[\frac{\Delta+\nabla}{h}S_k^{0}\right](R_k^{0}+R_k^{1})+h\sum_{k=1}^{M}\left[\frac{\Delta+\nabla}{h}S_k^{m+1}\right](R_k^{m}+R_k^{m+1})$$

and hence

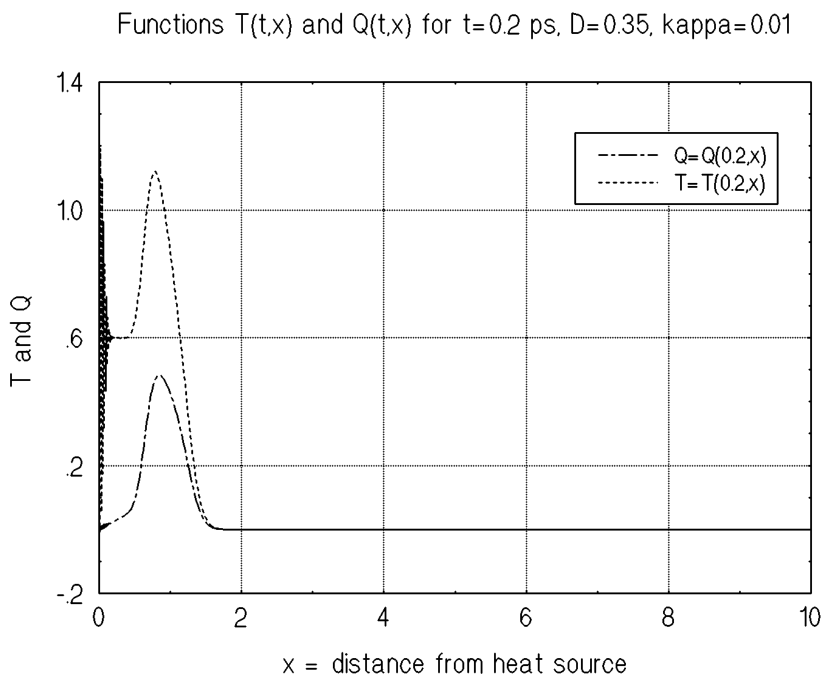
$$\sum_{n=0}^{m}\tau\left\|\frac{\underline{S}^{n+1}-\underline{S}^{n}}{\tau}\right\|_h^2+\frac{1}{2}\|\underline{S}^{m+1}\|_h^2+\frac{\kappa}{2}h\sum_{k=0}^{M}\left(\frac{\Delta S_k^{m+1}}{h}\right)^2$$
$$=\frac{1}{2}\|\underline{S}^{0}\|_h^2+\frac{\kappa}{2}h\sum_{k=0}^{M}\left(\frac{\Delta S_k^{0}}{h}\right)^2$$
$$-\frac{D}{4}\tau\sum_{n=1}^{m}\left(\frac{\Delta+\nabla}{h}\underline{S}^{n},\frac{\underline{R}^{n+1}-\underline{R}^{n}}{\tau}-\frac{\underline{R}^{n}-\underline{R}^{n-1}}{\tau}\right)_h$$
$$+\frac{D}{4}\left(\frac{\Delta+\nabla}{h}\underline{S}^{0},\underline{R}^{0}+\underline{R}^{1}\right)_h-\frac{D}{4}\left(\frac{\Delta+\nabla}{h}\underline{S}^{m+1},\underline{R}^{m+1}+\underline{R}^{m}\right)_h$$
$$+\tau\sum_{n=0}^{m}\left(\underline{\Phi}^{n},\frac{\underline{S}^{n+1}-\underline{S}^{n}}{\tau}\right)_h.$$

Using the ε-inequality twice, we obtain

$$(1-\varepsilon)\sum_{n=0}^{m}\tau\left\|\frac{\underline{S}^{n+1}-\underline{S}^{n}}{\tau}\right\|_h^2+\frac{1}{2}\|\underline{S}^{m+1}\|_h^2+\left(\frac{\kappa}{2}-\varepsilon\right)h\sum_{k=0}^{M}\left(\frac{\Delta S_k^{m+1}}{h}\right)^2$$
$$\le\frac{1}{2}\|\underline{S}^0\|_h^2+Kh\sum_{k=0}^{M}\left(\frac{\Delta S_k^0}{h}\right)^2$$
$$+\tau K\sum_{n=1}^{m}\sum_{k=0}^{M}h\left(\frac{\Delta}{h}S_k^n\right)^2$$
$$+K\tau\sum_{n=1}^{m}\left\|\frac{\underline{R}^{n+1}-\underline{R}^{n}}{\tau}\right\|_h^2+K\tau\sum_{n=1}^{m}\left\|\frac{\underline{R}^{n}-\underline{R}^{n-1}}{\tau}\right\|_h^2$$
$$+K\|\underline{R}^0\|_h^2+K\|\underline{R}^1\|_h^2+K\|\underline{R}^m\|_h^2+K\|\underline{R}^{m+1}\|_h^2+\tau K\sum_{n=0}^{m}\|\underline{\Psi}^n\|_h^2.$$

Using now the Gronwall inequality (Lemma 2.4) to the terms $\sum_{k=0}^{M}(\frac{\Delta}{h}S_k^n)^2$, inequality (2.9), Theorems 2.1 and 2.2, we get the final result. ■

Figure 1



REMARK 4. The last theorem gives the stability and convergence for the finite difference scheme, directly for the original grid-functions $\{T_k^n\}$ and $\{Q_k^n\}$.

Remark 5. We now briefly discuss certain numerical experiments which confirm our results. The numerical solution of the problem (0.2)–(0.4) with $D = 0.35$ and $\kappa = 0.01$ is obtained by the scheme (2.1)–(2.4) for the original problem. The graph shown in Fig. 1 corresponds to experiments described in [1] and [4]. Two curves $T(t, x)$ and $Q(t, x)$ are given, both for $t = 0.2$ picoseconds (time from the beginning of the experiment). For both functions the zero initial conditions and zero Dirichlet boundary conditions at $x = P$ are imposed. The heat impulse is modelled by the boundary conditions of the Dirichlet type at $x = L$, which depend on t. For T this impulse is constantly equal to zero, while $Q(t, L)$ is a discontinuous rectangular impulse lasting 0.096 picoseconds. A weaker regularity of the function T (temperature) compared to Q (heat flux) is visible: see oscillations near the left boundary, and then a sudden jump and stabilization at a positive level, before the wave peak. This means that the zero boundary condition imposed at $x = L$ for T does not play any role even very close to the start point of the heat impulse. The heat flux Q behaves in a more regular way. It is a travelling wave of the form of smoothed initial impulse.

References

[1] R. Bellman and E. Beckenbach, *Inequalities*, Ergeb. Math. Grenzgeb. 30, Springer, 1965.

[2] S. D. Brorson, J. G. Fujimoto and E. P. Ippen, *Femtosecond electronic heat-transport dynamics in thin gold films*, Phys. Rev. Lett. 59 (1987), 1962–1965.

[3] M. Dryja and K. Moszyński, *On Jeffreys model of heat conduction*, in: Proceedings of the Conference "Finite Volumes for Complex Applications" (Duisburg, 1999).

[4] J. L. Lions, *Problèmes aux limites dans les équations aux dérivées partielles*, Univ. de Montréal, 1967.

[5] K. Moszyński and A. Palczewski, *Asymptotic analysis of heat propagation models*, Arch. Mech. 52 (2000), 225–246.

[6] D. D. Joseph and L. Preziosi, *Heat waves*, Rev. Modern Phys. 61 (1989), 41–73.

[7] —, —, *Addendum to the paper "Heat waves"*, ibid. 62 (1990), 375–391.

[8] R. Richtmyer and K. W. Morton, *Difference Methods for Initial Value Problems*, 2nd ed., Interscience, 1967.

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