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## ON THE NEWTON–KANTOROVICH THEOREM AND NONLINEAR FINITE ELEMENT METHODS

*Abstract.* Using a weaker version of the Newton–Kantorovich theorem, we provide a discretization result to find finite element solutions of elliptic boundary value problems. Our hypotheses are weaker and under the same computational cost lead to finer estimates on the distances involved and a more precise information on the location of the solution than before.

**1. Introduction.** In this study we are concerned with the problem of approximating a locally unique solution of the nonlinear equation

$$(1.1) \quad F(x) = 0,$$

where  $F$  is a Fréchet differentiable operator defined on an open convex subset  $D$  of a Banach space  $A$  with values in a Banach space  $B$ .

The famous Newton–Kantorovich theorem [4, Th. 6, (1,XVIII)] has been used to show existence and uniqueness of exact solutions of equation (1.1). Moreover, a priori and a posteriori estimates can be obtained as a direct consequence of the Newton–Kantorovich theorem. A survey of such recent results can be found in [2].

Tsuchiya [6] used that theorem to show existence of finite element solutions of strongly nonlinear elliptic boundary value problems. However, it can happen that the basic condition in that theorem, the so-called Newton–Kantorovich hypothesis given in [4], is violated and still Newton’s method converges [1], [2]. That is why we introduced a weaker hypothesis (see Theorem 2.1 that follows) originated in [1] which can always replace the Newton–Kantorovich hypothesis used in [4], [6] (see also (3.3)), and under the same computational cost. This way we can use Newton’s method to solve a wider

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range of problems than before. Moreover, finer estimates on the distances involved and a more precise information on the location of the solution are obtained in [1], [2].

Finally, we provide examples of elliptic boundary value problems where our results apply.

**2. Convergence analysis.** We state the version of our main result in [2] needed in this study (see also [1, p. 132, Case 3 for  $\delta = \delta_0$ ]).

**THEOREM 2.1.** *Let  $F : D \subseteq A \rightarrow B$  be a nonlinear Fréchet differentiable operator, where  $D$  is open, convex, and  $A, B$  are Banach spaces. Assume:*

- *there exists a point  $x_0 \in D$  such that the Fréchet derivative  $F'(x_0) \in L(A, B)$  is an isomorphism and  $F(x_0) \neq 0$ ;*
- *there exist positive constants  $\ell_0$  and  $\ell$  such that the following center Lipschitz and Lipschitz conditions are satisfied:*

$$(2.1) \quad \|F'(x_0)^{-1}[F'(x) - F'(x_0)]\| \leq \ell_0 \|x - x_0\|,$$

$$(2.2) \quad \|F'(x_0)^{-1}[F'(x) - F'(y)]\| \leq \ell \|x - y\|,$$

for all  $x, y \in D$ .

Setting

$$\eta = \|F'(x_0)^{-1}F(x_0)\|$$

and

$$h_1 = \frac{1}{4}(\ell + 4\ell_0 + \sqrt{\ell^2 + 8\ell_0\ell})\eta,$$

we further assume

$$(2.3) \quad h_1 \leq 1,$$

$$\bar{U}(x_1, t^* - \eta) = \{x \in D : \|x - x_1\| \leq t^* - \eta\} \subseteq D,$$

where  $x_1 = x_0 - F'(x_0)^{-1}F(x_0)$  and  $t^* \geq \eta$  is the unique least upper bound of the nondecreasing majorizing sequence  $\{t_n\}$  given by

$$(2.4) \quad t_0 = 0, \quad t_1 = \eta, \quad t_{n+2} = t_{n+1} + \frac{\ell_1(t_{n+1} - t_n)^2}{2(1 - \ell_0 t_{n+1})} \quad (n \geq 0),$$

where

$$\ell_1 = \begin{cases} \ell_0 & \text{if } n = 0, \\ \ell & \text{if } n > 0. \end{cases}$$

Then the equation  $F(x) = 0$  has a solution  $x^* \in \bar{U}(x_1, t^* - \eta)$  and this solution is unique in  $U(x_0, t^*) \cap D$  if  $\ell_0 = \ell$  and  $h_1 < 1$ , and in  $\bar{U}(x_0, t^*) \cap D$  if  $\ell_0 = \ell$  and  $h_1 = 1$ . If  $\ell_0 \neq \ell$  the solution  $x^*$  is unique in  $U(x_0, R)$  provided that  $\frac{1}{2}(t^* + R)\ell_0 \leq 1$  and  $U(x_0, R) \subseteq D$ .

Moreover, we have the estimate

$$(2.5) \quad \|x^* - x_0\| \leq t^*.$$

We will simply use  $\|\cdot\|$  if the norm involved is understood. Otherwise we will use  $\|\cdot\|_X$  for the norm on a particular set  $X$ .

We assume the following:

(A<sub>1</sub>) There exist Banach spaces  $Z \subseteq X$  and  $U \subseteq Y$  such that the inclusions are continuous, and the restriction of  $F$  to  $Z$ , denoted again by  $F$ , is a Fréchet differentiable operator from  $Z$  to  $U$ .

(A<sub>2</sub>) For any  $v \in Z$  the derivative  $F'(v) \in L(Z, U)$  can be extended to  $F'(v) \in L(X, Y)$  and it is:

- *locally Lipschitz continuous* on  $Z$ , i.e., for any bounded convex set  $T \subseteq Z$  there exists a positive constant  $c_1$  depending on  $T$  such that

$$(2.6) \quad \|F'(v) - F'(w)\| \leq c_1 \|v - w\| \quad \text{for all } v, w \in T;$$

- *center locally Lipschitz continuous* at a fixed  $u_0 \in Z$ , i.e., for any bounded convex set  $T \subseteq Z$  with  $u_0 \in T$  there exists a positive constant  $c_0$  depending on  $u_0$  and  $T$  such that

$$(2.7) \quad \|F'(v) - F'(u_0)\| \leq c_0 \|v - u_0\| \quad \text{for all } v \in T.$$

(A<sub>3</sub>) There are Banach spaces  $V \subseteq Z$  and  $W \subseteq U$  such that the inclusions are continuous. We suppose that there exists a subset  $S \subseteq V$  for which the following holds: “if  $F'(u) \in L(V, W)$  is an isomorphism between  $V$  and  $W$  at  $u \in S$ , then there exists an extension  $F'(u) \in L(X, Y)$  which is an isomorphism between  $X$  and  $Y$  as well”.

To define discretized solutions of  $F(u) = 0$ , we introduce the finite-dimensional subspaces  $S_d \subseteq Z$  and  $S_d \subseteq U$  parametrized by  $d$ ,  $0 < d < 1$ , with the following properties:

(A<sub>4</sub>) There exists  $r \geq 0$  and a positive constant  $c_2$  independent of  $d$  such that

$$(2.8) \quad \|v_d\|_Z \leq \frac{c_2}{d^r} \|v_d\|_X \quad \text{for all } v_d \in S_d.$$

(A<sub>5</sub>) There exists a projection  $\Pi_d : X \rightarrow S_d$  for each  $S_d$  such that, if  $u_0 \in S$  is a solution of  $F(u) = 0$ , then

$$(2.9) \quad \lim_{d \rightarrow 0} d^{-r} \|u_0 - \Pi_d u_0\|_X = 0,$$

$$(2.10) \quad \lim_{d \rightarrow 0} d^{-r} \|u_0 - \Pi_d u_0\|_Z = 0.$$

We show the following result concerning the existence of locally unique solutions of discretized equations.

**THEOREM 2.2.** *Assume that conditions (A<sub>1</sub>)–(A<sub>5</sub>) hold. Suppose  $F'(u_0) \in L(V, W)$  is an isomorphism, and  $u_0 \in S$ . Moreover, assume  $F'(u_0)$  can be decomposed into  $F'(u_0) = Q + R$ , where  $Q \in L(X, Y)$  and  $R \in L(X, Y)$*

is compact. The discretized nonlinear operator  $F_d : Z \rightarrow U$  is defined by

$$(2.11) \quad F_d(u) = (I - P_d)Q(u) + P_dF(u)$$

where  $I$  is the identity of  $Y$ , and  $P_d : Y \rightarrow S_d$  is a projection such that

$$(2.12) \quad \lim_{d \rightarrow 0} \|v - P_d v\|_Y = 0 \quad \text{for all } v \in Y$$

and

$$(2.13) \quad (I - P_d)Q(v_d) = 0 \quad \text{for all } v_d \in S_d.$$

Then, for sufficiently small  $d > 0$ , there exists  $u_d \in S_d$  such that  $F_d(u_d) = 0$ , and  $u_d$  is locally unique.

Moreover, the following estimate holds:

$$(2.14) \quad \|u_d - \Pi_d(u_0)\| \leq \ell_1 \|u_0 - \Pi_d(u_0)\|$$

where  $\ell_1$  is a positive constant independent of  $d$ .

*Proof.* The proof is similar to the corresponding one in [6, Th. 2.1, p. 126]. However, there are some crucial differences where the weaker (2.7) is used (needed) instead of the stronger condition (2.6).

STEP 1. We claim that there exists a positive constant  $c_3$ , independent of  $d$ , such that, for sufficiently small  $h > 0$ ,

$$(2.15) \quad \|F'_d(\Pi_d(u_0))v_d\|_Y \geq c_3 \|v_d\|_X \quad \text{for all } v_d \in S_d.$$

From (A<sub>3</sub>) and  $u_0 \in S$ ,  $F'(u_0) \in L(X, Y)$  is an isomorphism. Set  $B_0 = \|F'(u_0)^{-1}\|$ .

We have

$$(2.16) \quad \begin{aligned} F'_d(\Pi_d(u_0))v_d &= F'(u_0)v_d + P_d(F'(\Pi_d(u_0)) - F'(u_0))v_d \\ &\quad - (I - P_d)(-Q + F'(u_0))v_d. \end{aligned}$$

Since  $-Q + F'(u_0) \in L(X, Y)$  is compact, by (2.12) we get

$$(2.17) \quad \lim_{d \rightarrow 0} \|(I - P_d)(-Q + F'(u_0))\| = 0.$$

By (2.12) there exists a positive constant  $c_4$  such that

$$(2.18) \quad \sup_{d > 0} \|P_d\| \leq c_4.$$

That is, using (2.7) we get

$$(2.19) \quad \|P_d(F'(\Pi_d(u_0)) - F'(u_0))\| \leq c_0 c_4 \|\Pi_d(u_0) - u_0\|.$$

Hence, by (2.10) we have

$$(2.20) \quad \|F'_d(\Pi_d(u_0))v_d\| \geq (1/B_0 - \delta(d))\|v_d\|,$$

where  $\lim_{d \rightarrow 0} \delta(d) = 0$ , and so (2.15) holds with  $c_3 = B_0^{-1}/2$ .

STEP 2. We shall show

$$(2.21) \quad \lim_{d \rightarrow 0} d^{-r} \|F'_d(\Pi_d(u_0))^{-1} F_d(\Pi_d(u_0))\| = 0.$$

Note that

$$(2.22) \quad \|F_d(\Pi_d(u_0))\| \leq c_4 \|F_d(\Pi_d(u_0)) - F_d(u_0)\| \\ \leq c_4 \int_0^1 \|G_t\| dt \|\Pi_d(u_0) - u_0\| \leq c_4 c_5 \|\Pi_d(u_0) - u_0\|,$$

where

$$(2.23) \quad G_t = F'((1-t)u_0 + t\Pi_d(u_0))$$

and we used

$$(2.24) \quad \|G_t\| \leq \|G_t - F'(u_0)\| + \|F'(u_0)\| \\ \leq c_0 t \|\Pi_d(u_0) - u_0\| + \|F'(u_0)\| \leq c_5,$$

where  $c_5$  is independent of  $d$ .

The claim is proved.

STEP 3. We use our modification of the Newton–Kantorovich theorem with the following choices:

$$A = S_d \subseteq Z \quad \text{with norm } d^{-r} \|w_d\|_X, \\ B = S_d \subseteq U \quad \text{with norm } d^{-r} \|w_d\|_Y, \\ x_0 = \Pi_d(u_0), \\ F = F_d.$$

Notice that  $\|S\|_{L(A,B)} = \|S\|_{L(X,Y)}$  for any linear operator  $S \in L(S_d, S_d)$ .

By Step 1,  $F'_d(\Pi_d(u_0)) \in L(S_d, S_d)$  is an isomorphism. It follows from (2.6) and (A<sub>4</sub>) that for any  $w_d, v_d \in S_d$ ,

$$(2.25) \quad \|F'_d(w_d) - F'_d(v_d)\| \leq c_1 c_4 \|w_d - v_d\|_Z \leq c_1 c_2 c_4 d^{-r} \|w_d - v_d\|_X$$

Similarly, using (2.7) and (A<sub>4</sub>) we get

$$\|F'_d(w_d) - F'_d(\Pi_d(u_0))\| \leq c_1 c_2 c_4 d^{-r} \|w_d - x_0\|_X.$$

Hence the assumptions are satisfied with

$$(2.26) \quad \ell = c_1 c_2 c_3^{-1} c_4 \quad \text{and} \quad \ell_0 = c_0 c_2 c_3^{-1} c_4.$$

From Step 2, we may take sufficiently small  $d > 0$  such that  $(\ell_0 + \ell)\eta \leq 1$ , where

$$\eta = d^{-r} \|F'_d(\Pi_d(u_0))^{-1} F_d(\Pi_d(u_0))\|_X.$$

That is, the assumption  $h_1 \leq 1$  is satisfied.

Hence for sufficiently small  $d > 0$  there exists a locally unique  $u_d \in S_d$  such that  $F_d(u_d) = 0$  and

$$\|u_d - \Pi_d(u_0)\|_X \leq 2d^r \eta \leq 2c_3^{-1} \|F_d(\Pi_d(u_0))\|_Y \leq 2c_3^{-1} c_4 c_5 \|u_0 - \Pi_d(u_0)\|_X.$$

It follows that (2.14) holds with  $\ell_1 = 2c_3^{-1} c_4 c_5$ .

That completes the proof of the Theorem.

### 3. Concluding remarks and applications

REMARK 3.1. In general

$$(3.1) \quad c_0 \leq c_1 \quad (\text{i.e., } \ell_0 \leq \ell)$$

and  $\ell/\ell_0$  can be arbitrarily large, where  $\ell$  and  $\ell_0$  are given by (2.26) [1], [2].

If  $\ell = \ell_0$  our Theorem 2.2 reduces to Theorem 2.1 in [6, p. 126].

Otherwise our condition  $h_1 \leq 1$  is weaker than the corresponding one in [6] using the Newton–Kantorovich hypothesis, famous for its simplicity and clarity:

$$(3.2) \quad h = 2\ell\eta \leq 1$$

(see [4], [1]). That is,

$$(3.3) \quad h \leq 1 \Rightarrow h_1 \leq 1$$

but not necessarily vice versa, unless  $\ell_0 = \ell$ .

As already shown in [2], finer error estimates on the distances  $\|u_d - \Pi_d(u_0)\|$  and a more precise information on the location of the solution are provided here and under the same computational cost since in practice the evaluation of  $c_1$  requires that of  $c_0$ .

Note also that our parameter  $d$  will be smaller than the corresponding one in [6], which in turn implies that fewer computations and smaller dimension subspaces  $S_d$  are used to approximate  $u_d$ . This observation is important in computational mathematics [1].

The above observations suggest that all results obtained in [6] can be improved if rewritten with weaker  $h_1 \leq 1$  instead of stronger  $h \leq 1$ .

However, we do not attempt this here (leaving this task to the motivated reader). Instead we provide examples of nonlinear problems already reported in [6] where finite element methods apply along the lines of our theorem above.

EXAMPLE 3.2 ([6]). Find  $u \in H_0^1(J)$  and  $J = (b, c) \subseteq \mathbb{R}$  such that

$$(3.4) \quad \langle F(u), v \rangle = \int_J [g_0(x, u, u')v' + g(x, u, u')v] dx = 0$$

for all  $v \in H_0^1(J)$ , where  $g_0$  and  $g_1$  are sufficiently smooth functions from  $J \times \mathbb{R} \times \mathbb{R}$  to  $\mathbb{R}$ .

EXAMPLE 3.3 ([6]). For the  $N$ -dimensional case ( $N = 2, 3$ ) let  $D \subseteq \mathbb{R}^N$  be a bounded domain with a Lipschitz boundary. Then consider the problem: find  $u \in H_0^1(D)$  such that

$$(3.5) \quad \langle F(u), v \rangle = \int_D [q_0(x, u, \nabla u) \cdot \nabla v + q(x, u, \nabla u) \cdot v] dx = 0$$

for all  $v \in H_0^1(D)$ , where  $q_0, q : D \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}$  are sufficiently smooth functions.

REMARK 3.4. Since equations (3.4) and (3.5) are in divergence form, their finite element solutions are defined in a natural way. Finite element methods applied to nonlinear elliptic boundary value problems have also been considered by other authors [3], [5]. Finally, more details on Examples 3.2 and 3.3 can be found in [6].

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