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PERTURBED NONLINEAR DEGENERATE PROBLEMS IN $\mathbb{R}^N$

Abstract. Via critical point theory we establish the existence and regularity of solutions for the quasilinear elliptic problem

$$\begin{cases} -\operatorname{div}\mathcal{A}(x,\nabla u) + a(x)|u|^{p-2}u = g(x)|u|^{p-2}u + h(x)|u|^{s-1}u & \text{in } \mathbb{R}^N,\\ u>0, \quad \lim\limits_{|x|\to\infty} u(x) = 0, \end{cases}$$

where $1<p<N$; $a(x)$ is assumed to satisfy a coercivity condition; $h(x)$ and $g(x)$ are not necessarily bounded but satisfy some integrability restrictions.

1. Introduction and notations. The purpose of this paper is to prove existence and regularity of solutions for the nonlinear degenerate problem

$$\text{(P)} \quad \begin{cases} -\operatorname{div}\mathcal{A}(x,\nabla u) + a(x)|u|^{p-2}u = g(x)|u|^{p-2}u + h(x)|u|^{s-1}u & \text{in } \mathbb{R}^N,\\ u>0, \quad \lim\limits_{|x|\to\infty} u(x) = 0, \end{cases}$$

where the mapping $\mathcal{A}:\mathbb{R}^N\times\mathbb{R}^N\to\mathbb{R}^N$, $(x,\xi)\mapsto\mathcal{A}(x,\xi)$, is measurable in x and continuous in ξ, and it satisfies the following assumptions for some constants $0<\alpha\le\beta<\infty$, for a.e. $x\in\mathbb{R}^N$ and all $\xi\in\mathbb{R}^N$:

$$\mathcal{A}(x,\xi).\xi \ge \alpha|\xi|^p, \tag{1.1}$$

$$|\mathcal{A}(x,\xi)| \le \beta|\xi|^{p-1}, \tag{1.2}$$

$$0 \le \mathcal{A}(x,\xi).\xi \le p\mathcal{B}(x,\xi), \tag{1.3}$$

where $\mathcal{B}:\mathbb{R}^N\times\mathbb{R}^N\to\mathbb{R}$ is a continuous and strictly convex function with respect to ξ such that $\mathcal{A}(x,\xi) = (d/d\xi)\mathcal{B}(x,\xi)$.

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Note that the divergence form operator $-\operatorname{div}\mathcal{A}(x,\nabla u)$ which can also be of degenerate type appears in many applications, e.g., non-Newtonian fluids, reaction-diffusion problems, porous media (a discussion of some physical background can be found in [5]). Several special cases of problem (P) have been considered in the literature. A prime example of the operators in question is the p-Laplacian $-\Delta_p = -\operatorname{div}(|\nabla u|^{p-2}\nabla u)$. Problems involving the p-Laplacian operator in unbounded domains have been intensively studied in the last decades. Let us mention the works [3] and [4] for $p=2$, [6], [7] and [8] for $p\neq 2$ and the references therein. Let us point out that the case of unbounded domains is more complicated; generally the main difficulty lies in the loss of Sobolev compact embedding.

In this note, to overcome the lack of compactness that arises from the critical exponent and the unboundedness of the domain, a perturbation a which is unbounded at infinity is introduced (see assumption (H_1)). By assuming some restrictions on the functions g and h, which are not necessarily bounded, we prove the existence result by using a variational approach based on the standard Mountain Pass Lemma [2]. To establish the regularity result, an effective iteration scheme is carefully constructed to bound the maximal norm of solutions. To prove Theorems 1.1 and 1.2 below, we impose the following assumptions:

(H_1) The function $a:\mathbb{R}^N\to\mathbb{R}$ is measurable with $a(x)\geq a_0>0$ for all $x\in\mathbb{R}^N$ and some a_0. Moreover, $a(x)$ satisfies the coercivity condition

$$\lim_{|x|\to\infty} a(x) = +\infty.$$

(H_2) $h(x), g(x)\geq 0$ for all $x\in\mathbb{R}^N$, $g\in L^{N/p}(\mathbb{R}^N)\cap L^{N/p(1-\varepsilon)}(\mathbb{R}^N)$ and $h\in L^{\omega}(\mathbb{R}^N)\cap L^{\omega/(1-\varepsilon)}(\mathbb{R}^N)$, where $p-1<s<p^*-1$ with $\omega = p^*/(p^*-(s+1))$ and $\varepsilon>0$ small enough.

(H_3) There exists a constant $p<\mu<s+1$ such that

$$\frac{1}{p}\,g(x)t^p \leq \left(\frac{1}{\mu}-\frac{1}{s+1}\right)h(x)t^{s+1}$$

for all $x\in\mathbb{R}^N$ and all $t\in\mathbb{R}$.

Here, as usual, $p^* = Np/(N-p)$ denotes the Sobolev conjugate of p.

Our main results are the following two theorems:

THEOREM 1.1. *Suppose* $1<p<N$ *and* (1.1)–(1.3) *and* (H_1)–(H_3) *are satisfied. Then problem* (P) *has at least one nontrivial weak solution.*

THEOREM 1.2. *Let* u *be a solution of* (P). *Then* $u\in L^{\sigma}(\mathbb{R}^N)$ *for all* $\sigma\in[p,\infty]$. *Moreover*, u *is positive and decays uniformly as* $|x|\to\infty$.

NOTATION. For simplicity we write L^p for $L^p(\mathbb{R}^N)$, $\|\cdot\|_p$ for the Lebesgue norm and $W^{1,p}$ for the space $W^{1,p}(\mathbb{R}^N)$; also we use the symbol $\int$ for integration over the whole $\mathbb{R}^N$. B_R denotes the ball in $\mathbb{R}^N$ centered at 0.

2. Preliminaries. Let us consider the function space $E \subset W^{1,p}$ defined by

$$E = \Big\{u \in W^{1,p} \Bigm| \int (|\nabla u|^p + a(x)|u|^p)\,dx < \infty\Big\}$$

equipped with the norm

$$\|u\| = \Big(\int (|\nabla u|^p + a(x)|u|^p)\,dx\Big)^{1/p}.$$

Since $a(x) \geq a_0 > 0$, E is continuously embedded in $W^{1,p}$. We also deduce from Sobolev's Theorem the continuous embeddings $E \hookrightarrow L^q$ for all $p \leq q \leq p^*$.

We say that $u \in E$ is a *weak solution* of problem (P) if

$$\int \mathcal{A}(x,\nabla u)\nabla\varphi\,dx + \int a(x)|u|^{p-2}u\varphi\,dx = \int g(x)|u|^{p-2}u\varphi\,dx + \int h(x)|u|^{s-1}u\varphi\,dx$$

for all $\varphi \in E$.

Now define the functional $I : E \to \mathbb{R}$ by

$$\begin{aligned} I(u) &= \int \mathcal{B}(x,\nabla u)\,dx + \frac{1}{p}\int a(x)|u|^p\,dx \\ &\quad - \frac{1}{p}\int g(x)|u|^p\,dx - \frac{1}{s+1}\int h(x)|u|^{s+1}\,dx \\ &= \int \mathcal{B}(x,\nabla u)\,dx + \frac{1}{p}\int a(x)|u|^p\,dx - K(u), \end{aligned}$$

where

$$K(u) = \frac{1}{p}\int g(x)|u|^p\,dx + \frac{1}{s+1}\int h(x)|u|^{s+1}\,dx.$$

By assumption (H_2) and Sobolev's inequality, we deduce easily that the functional K is well defined and of class $C^1(E,\mathbb{R})$ on the space E with

$$\langle \nabla K(u), v\rangle = \int g(x)|u|^{p-2}uv\,dx + \int h(x)|u|^{s-1}uv\,dx, \qquad \forall u,v \in E,$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing of E and E^*.

LEMMA 2.1. *Suppose $a(x)$ satisfies* (H_1). *Then E is compactly embedded in L^p.*

Proof. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence such that

$$u_n \rightharpoonup 0 \quad \text{weakly in } E.$$

Then $\|u\| \leq c$ for some constant $c > 0$.

From (H_1), given $\varepsilon > 0$ and $R > 0$ such that $a(x) \geq 2c^p/\varepsilon$ for all $|x| \geq R$, we have

$$u_n \rightharpoonup 0 \quad \text{weakly in } W^{1,p}(B_R).$$

The compact embedding $W^{1,p}(B_R) \hookrightarrow L^p(B_R)$ implies

$$\int_{B_R} |u_n|^p \, dx \leq \frac{\varepsilon}{2} \quad \forall n \geq n_0, \tag{2.1}$$

for some $n_0 \in \mathbb{N}$. On the other hand, we also have

$$\frac{2}{\varepsilon} \int_{\mathbb{R}^N/B_R} |u_n|^p \, dx \leq \frac{1}{c^p} \int_{\mathbb{R}^N/B_R} a(x)|u_n|^p \, dx \leq \frac{1}{c^p} \|u\|^p \leq 1. \tag{2.2}$$

Combining (2.1) and (2.2), we obtain

$$\int |u_n|_p^p \, dx \leq \varepsilon \quad \text{for all } n \geq n_0,$$

i.e. $u_n \to 0$ strongly in L^p.

LEMMA 2.2. *Suppose* (H_2) *is satisfied. Then* ∇K *is a compact map from* E *to* E^*.

Proof. We have

$$\begin{aligned}\langle \nabla K(u) - \nabla K(\overline{u}), v\rangle = {} & \int g(x)(|u|^{p-2}u - |\overline{u}|^{p-2}\overline{u})v \, dx \\ & + \int h(x)(|u|^{s-1}u - |\overline{u}|^{s-1}\overline{u})v \, dx.\end{aligned}$$

By (H_2) and Hölder's inequality, we obtain

$$\begin{aligned}\langle \nabla K(u) - \nabla K(\overline{u}), v\rangle \leq {} & \|g\|_{N/p(1-\varepsilon)} \|u^{p-1} - \overline{u}^{p-1}\|_{p_1} \|v\|_{p^*} \\ & + \|h\|_{\omega/(1-\varepsilon)} \|u^s - \overline{u}^s\|_{p_2} \|v\|_{p^*}\end{aligned}$$

where

$$p_1 = \frac{1}{1 - \frac{1}{p^*} - \frac{p(1-\varepsilon)}{N}} \quad \text{and} \quad p_2 = \frac{1}{1 - \frac{1}{p^*} - \frac{1-\varepsilon}{\omega}}.$$

By Lemma 2.1, the embedding $E \hookrightarrow L^p$ is compact, and it follows from the interpolation inequality

$$\|u\|_q \leq \|u\|_p^\eta \|u\|_{p^*}^{1-\eta},$$

where $1/q = \eta/p + (1-\eta)/p^*$, $0 < \eta < 1$, that the embedding $E \hookrightarrow L^q$ is compact for $p \leq q < p^*$.

On the other hand, calculations show that $p \leq p_1(p-1) < p^*$ and $p \leq p_2 s < p^*$ for ε small enough, namely, $\varepsilon \in]0, \min\{1/p', (s\omega)/p(1-p/p^*)\}[$, where $p' = p/(p-1)$ is the conjugate of p.

Therefore, for a sequence u_n such that $u_n \rightharpoonup \overline{u}$ weakly in E, we have

$$\nabla K(u_n) \to \nabla K(\overline{u}) \quad \text{strongly in } E^*.$$

This completes the proof.

Now consider the functional $\Phi : E \to \mathbb{R}$ defined by

$$\Phi(u) = \int \mathcal{B}(x, \nabla u)\, dx + \frac{1}{p}\int a(x)|u|^p\, dx$$

for all $u \in E$. Since $\mathcal{B}(x, \cdot)$ is a strictly convex function, it is easy to deduce that Φ is weakly lower semicontinuous, Fréchet differentiable and the derivative of Φ is continuous.

LEMMA 2.3. *Φ' belongs to the class (S_+), that is, for any sequence $(u_n) \subset E$ which is weakly convergent to $u \in E$ and*

$$\limsup_{n\to\infty} \langle \Phi'(u_n), u_n - u\rangle \le 0,$$

it follows that $u_n \to u$ strongly in E.

Proof. By (1.2) we have, for all $x, \xi \in \mathbb{R}^N$,

$$\mathcal{B}(x,\xi) = \int_0^1 \frac{d}{dt}\mathcal{B}(x, t\xi)\, dt = \int_0^1 \mathcal{A}(x, t\xi).\xi\, dt \le \beta \int_0^1 t^{p-1}|\xi|^{p-1}|\xi|\, dt \le \frac{\beta}{p}\, |\xi|^p.$$

This implies that

$$\int \mathcal{B}(x, \nabla u_n)\, dx \le \frac{\beta}{p}\int |\nabla u_n|^p\, dx \le \frac{\beta}{p}\, \|u_n\|^p.$$

The sequence (u_n) is bounded in E since $u_n \rightharpoonup u$ weakly, hence there exists a constant M such that $\Phi(u_n) \le M$ for all n. Then we may assume that $\Phi(u_n) \to \nu$. Using the fact that the function Φ is weakly lower semicontinuous, we deduce that

$$\Phi(u) \le \liminf_{n\to\infty} \Phi(u_n) = \nu.$$

Now, since Φ is strictly convex, we have

$$\Phi(u) > \Phi(u_n) + \langle \Phi'(u_n), u_n - u\rangle, \qquad \forall n.$$

By assumption the above inequality implies $\Phi(u) \ge \nu$ and thus $\Phi(u) = \nu$. On the other hand, also $\mu u_n + (1-\mu)u$ $(0 < \mu < 1)$ converges weakly to u in E. Using again the fact that the function Φ is weakly lower semicontinuous, we get

$$\nu = \Phi(u) \le \liminf_{n\to\infty} \Phi(\mu u_n + (1-\mu)u).$$

Suppose for contradiction that $\|u_n - u\|$ does not converge to 0. Then there exists $\varepsilon > 0$ and a subsequence still denoted by (u_n) such that $\|u_n - u\| > \varepsilon$. Let u_{n_0} be a fixed element of this sequence. The strict convexity of Φ gives

$$\mu\Phi(u_n) + (1-\mu)\Phi(u) - \Phi(\mu u_n + (1-\mu)u) \ge k\|u_{n_0} - u\|,$$

where k is a positive constant. Letting $n \to \infty$ we obtain

$$\limsup_{n\to\infty} \Phi(\mu u_n + (1-\mu)u) \le \nu - k\|u_{n_0} - u\|,$$

which gives a contradiction.

REMARK 2.1. Our result remains true if we suppose $\mathcal{B}(x,\xi)$ is p-uniformly convex instead of strictly convex, that is, there exists $k>0$ such that

$$\mathcal{B}(x,\mu\xi_1+(1-\mu)\xi_2)\le\mu\mathcal{B}(x,\xi_1)+(1-\mu)\mathcal{B}(x,\xi_2)-k|\xi_1-\xi_2|^p$$

for all $x,\xi_1,\xi_2\in\mathbb{R}^N$ and some $0<\mu<1$.

A weak solution of problem (P) is a critical point u of the functional I, i.e.

$$\begin{aligned}\langle I'(u),\varphi\rangle=\int\mathcal{A}(x,\nabla u)\nabla\varphi\,dx+\int a(x)|u|^{p-2}u\varphi\,dx\\ -\int g(x)|u|^{p-2}u\varphi\,dx-\int h(x)|u|^{s-1}u\varphi\,dx=0\end{aligned}$$

for all $\varphi\in E$. Let us remark that under the assumptions (1.1), (1.2), and (H_2) the integrals in I' are well defined. To find the critical points, we shall consider the Palais–Smale (PS) compactness condition. We recall that $(u_n)\subset E$ is a *Palais–Smale sequence* if there exists $M>0$ such that $I(u_n)\le M$ and $I'(u_n)\to0$ strongly in E^* as n goes to ∞.

LEMMA 2.4. *Let (u_n) be a Palais–Smale sequence. Then (u_n) converges strongly in E.*

Proof. We suppose $I(u_n)\to l>0$ and we prove that (u_n) is bounded in E. Suppose for contradiction that $\|u_n\|\to\infty$ as $n\to\infty$. Then, using (1.1)–(1.3) and (H_3), we deduce for n large enough the following estimates:

$$\begin{aligned}l+1+\|u_n\|&\ge I(u_n)-\frac{1}{\mu}\langle I'(u_n),u_n\rangle\\ &=\int\mathcal{B}(x,\nabla u_n)\,dx-\frac{1}{\mu}\int\mathcal{A}(x,\nabla u_n)\nabla u_n\,dx\\ &\quad+\left(\frac{1}{p}-\frac{1}{\mu}\right)\int a(x)|u_n|^p\,dx+\frac{1}{p}\int g(x)|u_n|^p\,dx\\ &\quad+\left(\frac{1}{\mu}-\frac{1}{s+1}\right)\int h(x)|u_n|^{s+1}\,dx-\frac{1}{p}\int g(x)|u_n|^p\,dx\\ &\ge\left(\frac{1}{p}-\frac{1}{\mu}\right)\alpha\int|\nabla u_n|^p\,dx+\left(\frac{1}{p}-\frac{1}{\mu}\right)\int a(x)|u_n|^p\,dx\\ &\ge C\|u_n\|^p,\end{aligned}$$

where $C=C(p,\mu,\alpha)>0$. But this contradicts our assumption, therefore (u_n) is bounded in E. It follows that there exists $u\in E$ such that a subsequence, still denoted by (u_n), converges to u weakly in E. On the other hand, we have

$$\begin{aligned}
&\int \mathcal{A}(x,\nabla u_n)(\nabla u_n-\nabla u)\,dx+\int a(x)|u_n|^{p-2}u_n(u_n-u)\,dx\\
&\quad=\langle I'(u_n),u_n-u\rangle+\int g(x)|u_n|^{p-2}u_n(u_n-u)\,dx+\int h(x)|u_n|^{s-1}u_n(u_n-u)\,dx\\
&\quad=\langle I'(u_n),u_n-u\rangle+\langle \nabla K(u_n),u_n-u\rangle.
\end{aligned}$$

By Lemma 2.2, we deduce that

$$\lim_{n\to\infty}\int \mathcal{A}(x,\nabla u_n)(\nabla u_n-\nabla u)\,dx+\int a(x)|u_n|^{p-2}u_n(u_n-u)\,dx=0,$$

that is,

$$\lim_{n\to\infty}\langle \Phi',u_n-u\rangle=0.$$

Then, in view to Lemma 2.3 we deduce that (u_n) converges strongly to u in E.

3. Existence and L^∞-estimate results

LEMMA 3.1. *There exist $\delta,\varrho>0$ such that $I(u)\ge\delta$ if $\|u\|=\varrho$. Moreover, $I(tu)\to-\infty$ as $t\to+\infty$, for all $u\in E\setminus\{0\}$.*

Proof. (1.1), (1.3), (H_2) and (H_3) imply

$$\begin{aligned}
I(u)&=\int \mathcal{B}(x,\nabla u)\,dx+\frac1p\int a(x)|u|^p\,dx\\
&\quad-\frac1p\int g(x)|u|^p\,dx-\frac1{s+1}\int h(x)|u|^{s+1}\,dx\\
&\ge\frac{\alpha}{p}\int|\nabla u|^p\,dx+\frac1p\int a(x)|u|^p\,dx\\
&\quad-\frac1p\int g(x)|u|^p\,dx-\frac1{s+1}\int h(x)|u|^{s+1}\,dx\\
&\ge\min\Big\{\frac{\alpha}{p},\frac1p\Big\}\|u\|^p-\frac1\mu\int h(x)|u|^{s+1}\,dx\\
&\ge\|u\|^p\Big[\min\Big\{\frac{\alpha}{p},\frac1p\Big\}-C'\|u\|^{s+1-p}\Big].
\end{aligned}$$

To prove the second part of the lemma, remark first that by studying the function $f(t)=\mathcal{B}(x,t\xi)$ and by using (1.2) and (1.3), we can easily see that

$$\mathcal{B}(x,t\xi)\le\mathcal{B}(x,\xi)t^p\qquad\forall t\ge1,\,x,\xi\in\mathbb{R}^N.$$

We have, for all $u \in E \setminus \{0\}$,

$$\begin{aligned} I(tu) &= \int \mathcal{B}(x, t\nabla u)\,dx + \frac{t^p}{p}\int a(x)|u|^p\,dx - \frac{t^p}{p}\int g(x)|u|^p\,dx \\ &\quad - t^{s+1}\frac{1}{s+1}\int h(x)|u|^{s+1}\,dx \\ &\le t^p\left[\int \mathcal{B}(x, \nabla u)\,dx + \frac{1}{p}\int a(x)|u|^p\,dx - \frac{1}{p}\int g(x)|u|^p\,dx\right] \\ &\quad - \frac{t^{s+1}}{s+1}\int h(x)|u|^{s+1}\,dx. \end{aligned}$$

Since $p < s+1$, we conclude that the second part of the lemma holds true.

In view of Lemmas 2.4 and 3.1, the Mountain Pass Lemma (cf. [2]) guarantees the existence of a weak solution of (P).

This concludes the proof of Theorem 1.1.

Using integrability conditions imposed on g and h, we prove the L^∞-estimate of u. We may choose $u \ge 0$ since we can show that the argument developed here is true for u^+ and u^- where $u^+ = \max(u, 0)$ and $u^- = \max(u, 0)$. For M a nonnegative real, we define $u_M(x) = \inf\{u(x), M\}$.

For $k > 0$, let us choose $\varphi = u_M^{kp+1}$ as a test function in the first equation of (P) (note that $\varphi \in E \cap L^\infty$). It follows that

$$\begin{aligned} &\int [\mathcal{A}(x, \nabla u)\nabla(u_M^{kp+1}) + a(x)|u|^{p-2}uu_M^{kp+1}]\,dx \\ &\qquad = \int g(x)|u|^{p-2}uu_M^{kp+1}\,dx + \int h(x)|u|^{s-1}uu_M^{kp+1}\,dx. \end{aligned}$$

On the one hand, we have

$$\begin{aligned} \int \mathcal{A}(x, \nabla u)\nabla(u_M^{kp+1})\,dx &= (kp+1)\int \mathcal{A}(x, \nabla u)\nabla u_M u_M^{kp}\,dx \\ &\ge \alpha(kp+1)\int |\nabla u_M|^p u_M^{kp}\,dx \\ &= \alpha\,\frac{kp+1}{(k+1)^p}\int |\nabla(u_M^{k+1})|^p\,dx \\ &\ge \alpha\,\frac{kp+1}{C_0^p(k+1)^p}\left(\int u_M^{(k+1)p^*}\,dx\right)^{p/p^*}. \end{aligned}$$

On the other hand, for $0 < \varepsilon < 1$ small enough, let

$$t = \frac{p^* p\omega}{p\omega + \varepsilon p^*}.$$

Observe that $t/p > 1$, $t < p^*$ and $(t/p)' \in [N/p, N/(p(1-\varepsilon))]$, where $(t/p)'$ is the exponent conjugate to t/p.

Using (H_1), (H_2) and Hölder's inequality we obtain

$$\begin{aligned}
&\int g(x)|u|^{p-2}uu_M^{kp+1}\,dx + \int h(x)|u|^{s-1}uu_M^{kp+1}\,dx \\
&\leq \int g(x)|u|^{(k+1)p}\,dx + \int h(x)u^{(k+1)p}u^{s+1-p}\,dx \\
&\leq \int g(x)|u|^{(k+1)p}\,dx + \Big(\int (h(x))^{\frac{\omega}{1-\varepsilon}}\,dx\Big)^{\frac{1-\varepsilon}{\omega}}\Big(\int u^{p^*}\,dx\Big)^{\frac{s+1-p}{p^*}}\Big(\int u^{(k+1)t}\,dx\Big)^{p/t} \\
&\leq \Big(\int g(x)^{(t/p)'}\,dx\Big)^{\frac{1}{(t/p)'}}\Big(\int u^{(k+1)t}\,dx\Big)^{p/t} \\
&\quad + \Big(\int (h(x))^{\frac{\omega}{1-\varepsilon}}\,dx\Big)^{\frac{1-\varepsilon}{\omega}}\Big(\int u^{(k+1)t}\,dx\Big)^{p/t}.
\end{aligned}$$

Hence there exists a constant $C_1 > 0$ independent of $M > 0$ and $k > 0$ such that

$$\Big(\int u_M^{(k+1)p^*}\,dx\Big)^{p/p^*} \leq C_1\,\frac{(k+1)^p}{(kp+1)}\Big(\int u^{(k+1)t}\,dx\Big)^{p/t},$$

i.e.

$$\|u_M\|_{(k+1)p^*} \leq C_2^{\frac{1}{k+1}}\left[\frac{k+1}{(kp+1)^{1/p}}\right]^{\frac{1}{k+1}}\|u\|_{(k+1)t} \tag{3.1}$$

with $C_2 = C_1^{1/p}$. Since $u \in E$ (and hence $u \in L^{p^*}$), we can choose $k = k_1$ in (3.1) such that $(k_1+1)t = p^*$, i.e. $k_1 = p^*/t - 1$, and we obtain

$$\|u_M\|_{(k_1+1)p^*} \leq C_2^{\frac{1}{k_1+1}}\left[\frac{k_1+1}{(k_1p+1)^{1/p}}\right]^{\frac{1}{k_1+1}}\|u\|_{p^*} \qquad \forall M > 0.$$

We have $\lim_{M\to\infty} u_M(x) = u(x)$, and Fatou's lemma implies

$$\|u\|_{(k_1+1)p^*} \leq C_2^{\frac{1}{k_1+1}}\left[\frac{k_1+1}{(k_1p+1)^{1/p}}\right]^{\frac{1}{k_1+1}}\|u\|_{p^*}.$$

Then $u \in L^{(k_1+1)p^*}$. By the same argument we can choose $k = k_2$ in (3.1) such that $(k_2+1)t = (k_1+1)p^*$, i.e. $k_2 = (p^*/t)^2 - 1$. Then we have

$$\|u\|_{(k_2+1)p^*} \leq C_2^{\frac{1}{k_2+1}}\left[\frac{k_2+1}{(k_2p+1)^{1/p}}\right]^{\frac{1}{k_2+1}}\|u\|_{(k_1+1)p^*}.$$

By iteration, we obtain $k_n = (p^*/t)^n - 1$ such that

$$\|u\|_{(k_n+1)p^*} \leq C_2^{\frac{1}{k_n+1}}\left[\frac{k_n+1}{(k_np+1)^{1/p}}\right]^{\frac{1}{k_n+1}}\|u\|_{(k_{n-1}+1)p^*} \qquad \text{for all } n \in \mathbb{N}.$$

It follows that

$$\|u\|_{(k_n+1)p^*} \leq C_2^{\sum_{i=1}^n \frac{1}{k_i+1}}\prod_{i=1}^n\left[\frac{k_i+1}{(k_ip+1)^{1/p}}\right]^{\frac{1}{k_i+1}}\|u\|_{p^*},$$

or equivalently

$$\|u\|_{(k_n+1)p^*} \le C_2^{\sum_{i=1}^{n}\frac{1}{k_i+1}} \prod_{i=1}^{n} \left[\left[\frac{k_i+1}{(k_i p+1)^{1/p}} \right]^{\frac{1}{\sqrt{k_i+1}}} \right]^{\frac{1}{\sqrt{k_i+1}}} \|u\|_{p^*}.$$

Since

$$\left[\frac{a+1}{(ap+1)^{1/p}} \right]^{\frac{1}{\sqrt{a+1}}} > 1 \quad \forall a>0 \quad \text{and} \quad \lim_{a\to\infty} \left[\frac{a+1}{(ap+1)^{1/p}} \right]^{\frac{1}{\sqrt{a+1}}} = 1,$$

there exists a constant $C_3 > 0$ independent of $n \in \mathbb{N}$ such that

$$\|u\|_{(k_n+1)p^*} \le C_2^{\sum_{i=1}^{n}\frac{1}{k_i+1}} C_3^{\sum_{i=1}^{n}\frac{1}{\sqrt{k_i+1}}} \|u\|_{p^*},$$

where

$$\begin{cases} \dfrac{1}{k_i+1} = \left(\dfrac{t}{p^*}\right)^i, \\ \dfrac{1}{\sqrt{k_i+1}} = \left(\sqrt{\dfrac{t}{p^*}}\right)^i \quad \text{and} \quad \dfrac{t}{p^*} < \sqrt{\dfrac{t}{p^*}} < 1. \end{cases}$$

Hence, there exists a constant $C_4 > 0$ independent of $n \in \mathbb{N}$ such that

$$\|u\|_{(k_n+1)p^*} \le C_4 \|u\|_{p^*} \quad \text{for all } n \in \mathbb{N}. \tag{3.2}$$

Letting n tend to infinity we get

$$\|u\|_\infty \le C_4 \|u\|_{p^*}. \tag{3.3}$$

Due to (3.2) and (3.3), we deduce

$$u \in L^\sigma \quad \text{for all } p^* \le \sigma \le \infty.$$

Thus $u \in L^\sigma$, $p \le \sigma \le \infty$. The positivity of u follows immediately from the weak Harnack type inequality of Trudinger [15]. Finally, the decay of u follows directly from Theorem 1 of Serrin [13].

Remark 3.1. 1. The functions g and h are not assumed to be bounded. Here, we impose more restrictions on the integrability of g and h.

2. Note that when $g \in L^{N/p(1-\varepsilon)}$ and $h \in L^{\omega/(1-\varepsilon)}$, we have the L^∞-estimate of solution. Moreover, if we suppose that $g, h \in L^\infty_{\mathrm{loc}}$, the regularity result of Tolksdorf [14] implies that $u \in \mathcal{C}^{1,\delta}(B_R(0))$ for any $R > 0$ with some $\delta(R) \in]0,1[$.

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