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MAXIMUM LENGTH OF A SERIES IN A MARKOVIAN BINARY SEQUENCE WITH AN APPLICATION TO THE DESCRIPTION OF A BASKETBALL GAME

Abstract. The recurrence formulas for the probability distribution function of the maximum length of a series of 1's in a binary 0-1 Markovian sequence are analysed and the limiting distribution estimated. The result is used to test a semi-Markov model of basketball games.

1. Main recurrence formula. Consider a stationary Markovian binary 0-1 sequence $\{U_n, n \ge 0\}$ with transition matrix

$$(p_{ij}) = \begin{pmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{pmatrix}.$$

Define the bivariate sequence $\{(T_n, X_n),\, n \ge 1\}$, where T_n denotes the maximum length of a series of 1's in the sequence $\{U_k,\, 1 \le k \le n\}$ and X_n denotes the length of the series of 1's located at the end of this sequence. Note that $T_n = 0$ if $U_k = 0$ for $1 \le k \le n$; $X_n = 0$ if $U_n = 0$. It is easy to see that $\{T_n, X_n\}$ is Markovian, and the set $\mathcal{S}_n$ of states (i,j) changes in consecutive steps. Let $p_{ij;mx}(n) = P(T_{n+1} = m,\, X_{n+1} = x \,|\, T_n = i,\, X_n = j)$. These transition probabilities depend upon n. They are as follows:

$$\begin{aligned}
&p_{00;00}(n) = p_{00}, && p_{00;11}(n) = p_{01}, \\
&p_{ii;i0}(n) = p_{10}, && p_{ii;i+1,i+1}(n) = p_{11}, \quad 1 \le i \le n, \\
&p_{i0;i0}(n) = p_{00}, && p_{i0;i,1}(n) = p_{01}, \quad 1 \le i \le n-1, \\
&p_{ij;i0}(n) = p_{10}, && 1 \le j < i,\ i+j \le n-1, \\
&p_{ij;i,j+1}(n) = p_{11}, && 1 \le j < i-1,\ i+j \le n-1.
\end{aligned}$$

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Table 1. The distribution of T_n and its moments

n	0	1	2	3	4	5	6	7	$\mathrm{E}(T_n)$	$\mathrm{D}^2(T_n)$
1	0.50	0.50							0.500	0.250
2	0.25	0.50	0.25						1.000	0.707
3	0.13	0.50	0.25	0.13					1.375	0.857
4	0.06	0.44	0.31	0.13	0.06				1.688	0.982
5	0.03	0.38	0.34	0.16	0.06	0.03			1.938	1.088
6	0.02	0.31	0.36	0.19	0.08	0.03	0.02		2.156	1.176
7	0.01	0.26	0.37	0.21	0.09	0.04	0.02	0.01	2.344	1.246
8	0.00	0.21	0.37	0.23	0.11	0.05	0.02	0.01	2.512	1.305
9	0.00	0.17	0.36	0.25	0.12	0.05	0.02	0.01	2.662	1.352
10	0.00	0.14	0.35	0.26	0.14	0.06	0.03	0.01	2.799	1.392

Let $p_{ij}(n) = P(T_n = i,\ X_n = j)$, $(i,j) \in \mathcal{S}_n$. The initial state of the sequence $(T_n,\ X_n)$ for the stationary process $\{U_n\}$ has the following probability distribution function:

$$p_{00}(1) = P(U_1 = 0) = p_0 = \frac{1 - p_{11}}{2 - p_{00} - p_{11}},$$
$$p_{11}(1) = P(U_1 = 1) = p_1 = 1 - p_0.$$

The Markov dependence leads to the recursive formula

$$p_{n+1}(m, x) = \sum_{(i,j)\in\mathcal{S}_n} p_n(i,j)p_{ij;mx}, \qquad (m,x) \in \mathcal{S}_{n+1}.$$

Next, having these distributions, the boundary distribution of T_n and its moments may be calculated. The results for $1 \le n \le 10$ assuming $p_{00} = p_{11} = 1/2$ are shown in Table 1. This suggests that the expectation tends to infinity and the variances are limited.

2. Rationalization of calculations. The transition matrix $(p_{ij;mx})$ has sparse positive elements, hence the complexity of the calculation of the distributions of the bivariate sequence can be reduced. Now we present the recurrence formulas in new notation. The initial distribution function is $p_{00}(1) = p_0$, $p_{01}(1) = p_{10}(1) = 0$, $p_{11}(1) = p_1$. From the boundary conditions

$$p_{n+1}(0,0) = p_n(0,0)p_{00},$$
$$p_{n+1}(1,0) = p_n(1,1)p_{10} + p_n(1,0)p_{00},$$
$$p_{n+1}(1,1) = p_n(1,0)p_{01} + p_n(0,0)p_{01},$$

for $i \ge 2$ we have:

$$p_{n+1}(i,0) = p_n(i,0)p_{00} + \sum_{j=1}^{m} p_n(i,j)p_{10} + p_n(i,i)p_{10},$$
$$i \le n-1,\ m = \min(i-1, n-i-1),$$

$$p_{n+1}(n,0) = p_n(n,n)p_{10},$$

$$p_{n+1}(i,1) = p_n(i,0)p_{01}, \quad i \le n-1,$$

$$p_{n+1}(i+1,i+1) = \begin{cases} p_n(i,i)p_{11} + p_n(i+1,i)p_{11}, & i+1 \le n/2, \\ p_n(i,i)p_{11}, & n/2 < i+1 \le n. \end{cases}$$

$$p_{n+1}(i,j+1) = p_n(i,j)p_{11}, \qquad i \le n-2,\ 1 \le j \le m = \min(i-2, n-i-1).$$

3. Distribution of T_n. The maximal length of a series of 1's in a Markovian binary sequence may also be analysed without explicitly applying the extended process to the bivariate process. We introduce the notation:

$$p_n(k) = P(T_n = k), \qquad P_n(k) = P(T_n \le k),$$

$$p_n^{(0)}(k) = P(T_n = k \,|\, U_0 = 0), \quad P_n^{(0)}(k) = P(T_n \le k \,|\, U_0 = 0), \quad 0 \le k \le n,\ n \ge 1.$$

Consider the series of the same symbols at the beginning of the sequence $\{U_n, n \ge 1\}$ and denote by Y the length of the initial series of 0's and by X the length of the initial series of 1's. Define

$$p^{(i|0)}(j) = P(Y = j \,|\, U_0 = i),$$

$$p^{(i|1)}(j) = P(X = j \,|\, U_0 = i), \quad i = 0,1,\ j \ge 0,$$

$$p^{(\cdot|0)}(j) = P(Y = j), \quad p^{(\cdot|1)}(j) = P(X = j).$$

We have

$$p^{(0|0)}(j) = p_{00}^{j} p_{01}, \quad p^{(1|1)}(j) = p_{11}^{j} p_{10}, \quad j \ge 0,$$

$$p^{(0|1)}(j) = p_{01} p_{11}^{j-1} p_{10}, \quad j \ge 1,$$

$$p^{(\cdot|0)}(0) = p_1, \quad p^{(\cdot|1)}(0) = p_0,$$

$$p^{(\cdot|0)}(j) = p_0 p_{00}^{j-1} p_{01}, \quad p^{(\cdot|1)}(j) = p_1 p_{11}^{j-1} p_{10}, \quad j \ge 1.$$

THEOREM 1. *Given the boundary values*

$$p_n(0) = p_0 p_{00}^{n-1}, \quad p_n(n) = p_1 p_{11}^{n-1},$$

and $1 \le k \le n-1$ *the following recurrence formulas involving* n *hold*:

$$p_n(k) = \sum_{j=0}^{k-1} p^{(\cdot|1)}(j) p_{n-j-1}^{(0)}(k) + p^{(\cdot|1)}(k) P_{n-k-1}^{(0)}(k),$$

$$p_n^{(0)}(k) = \sum_{j=0}^{k-1} p^{(0|1)}(j) p_{n-j-1}^{(0)}(k) + p^{(0|1)}(k) P_{n-k-1}^{(0)}(k).$$

4. Series of 0's and 1's. Let Z_n denote the maximum length of a series of 0's in the sequence $\{U_k, 1 \le k \le n\}$, and let T_n as before denote the maximum length of a series of 1's in this sequence. Let $(Z_n^{(0)}, T_n^{(0)}) = ((Z_n, T_n) \,|\, U_1 = 0)$. The mutual dependence of the random variables Z_n and T_n is clearly visible for small n. Therefore, we now extend the recurrence formulas given in Section 3 to the joint distribution of this pair of random variables. We introduce the notation for the fourth version of partially cumulated distributions:

$$\begin{aligned}
\mathrm{pp}_n(k,l) &= P(Z_n = k,\ T_n = l), && 0 \le k, l, k+l \le n,\\
\mathrm{Pp}_n(k,l) &= P(Z_n \le k,\ T_n = l), && k \ge 0,\ 0 \le l \le n,\\
\mathrm{pP}_n(k,l) &= P(Z_n = k,\ T_n \le l), && 0 \le k \le n,\ l \ge 0,\\
\mathrm{PP}_n(k,l) &= P(Z_n \le k,\ T_n \le l), && k, l \ge 0,\ n \ge 1.
\end{aligned}$$

Moreover, we also define the functions $\mathrm{pp}^{(0)}$, $\mathrm{Pp}^{(0)}$, $\mathrm{pP}^{(0)}$, $\mathrm{PP}^{(0)}$ indexed by $^{(0)}$ under the condition that $U_1 = 0$. The following theorem may be easily proved by considering the initial series of 0's and of 1's in $\{U_n\}$.

THEOREM 2. *Given the boundary values*

$$\begin{aligned}
&\mathrm{pp}_n(0,n) = g(0,n,n), \qquad \mathrm{pp}_n(n,0) = g(n,0,n),\\
&\mathrm{pp}_n^{(0)}(n,0) = g_0(n,0,n),\\
&\mathrm{pp}_n(k,n-k) = g(k,n-k,n) + g(0,n-k,n-k)p_{10}g_0(k,0,k),\\
&\mathrm{pp}_n^{(0)}(k,n-k) = g_0(k,n-k,n), \qquad 1 \le k \le n-1,
\end{aligned}$$

and $1 \le k, l, k+l \le n-1$ *the following recurrence formulas involving* n *hold*:

$$\begin{aligned}
\mathrm{pp}_n(k,l) = {} & \sum_{i=0}^{k-1}\sum_{j=1}^{l-1} g(i,j,n)\,\mathrm{pp}_{n-i-j}^{(0)}(k,l) + \sum_{i=0}^{k-1} g(i,l,n)\,\mathrm{pP}_{n-i-l}^{(0)}(k,l)\\
& + \sum_{j=1}^{l-1} g(k,j,n)\,\mathrm{Pp}_{n-k-j}^{(0)}(k,l) + g(k,l,n)\,\mathrm{PP}_{n-k-l}^{(0)}(k,l),\\
\mathrm{pp}_n^{(0)}(k,l) = {} & \sum_{i=1}^{k-1}\sum_{j=1}^{l-1} g_0(i,j,n)\,\mathrm{pp}_{n-i-j}^{(0)}(k,l) + \sum_{i=1}^{k-1} g_0(i,l,n)\,\mathrm{pP}_{n-i-l}^{(0)}(k,l)\\
& + \sum_{j=1}^{l-1} g_0(k,j,n)\,\mathrm{Pp}_{n-k-j}^{(0)}(k,l) + g_0(k,l,n)\,\mathrm{PP}_{n-k-l}^{(0)}(k,l),
\end{aligned}$$

where, *for* $0 \le i, j, i+j \le n$,

$$g(i,j,n)=\begin{cases} p_0p_{00}^{n-1}, & i=n,\ j=0,\\ p_0p_{00}^{i-1}p_{01}p_{11}^{n-i-1}, & i>0,\ j>0,\ i+j=n,\\ p_0p_{00}^{i-1}p_{01}p_{11}^{j-1}p_{10}, & i>0,\ j>0,\ i+j\le n-1,\\ p_1p_{11}^{n-1}, & i=0,\ j=n,\\ p_1p_{11}^{j-1}p_{10}, & i=0,\ j>0,\ j\le n-1,\\ 0, & i=j=0, \end{cases}$$

$$g_0(i,j,n)=\begin{cases} p_{00}^{n-1}, & i=n,\ j=0,\\ p_{00}^{i-1}p_{01}p_{11}^{n-i-1}, & i>0,\ j>0,\ i+j=n,\\ p_{00}^{i-1}p_{01}p_{11}^{j-1}p_{10}, & i>0,\ j>0,\ i+j\le n-1,\\ 0, & \textit{otherwise}, \end{cases}$$

and if the arguments are beyond the allowed domain the expressions are equal to zero.

Proof. Let X, Y denote the length of the series of 0's and the length of the series of 1's at the beginning of $\{U_n,\, n\ge 1\}$, $(X_0,Y_0)=((X,Y)\,|\,U_1=0)$. Let us introduce the notations for the probability distribution functions:

$$g(i,j,n)=P(X=i,\,Y=j),$$
$$g_0(i,j,n)=P(X_0=i,\,Y_0=j),\quad 0\le i,j,i+j\le n,\ n\ge 1.$$

We have the recurrence formulas:

$$Z_0^{(0)}=0,\quad T_0^{(0)}=0,\quad Z_1^{(0)}=1,\quad T_1^{(0)}=0,$$
$$(Z_n,T_n)\stackrel{d}{=}(\max(X,Z^{(0)}_{n-X-Y}),\max(Y,T^{(0)}_{n-X-Y})),$$
$$(Z_n^{(0)},T_n^{(0)})\stackrel{d}{=}(\max(X_0,Z^{(0)}_{n-X_0-Y_0}),\max(Y_0,T^{(0)}_{n-X_0-Y_0})).$$

Therefore

$$P(Z_n=0,\,T_n=n)=p_1p_{11}^{n-1},\quad P(Z_n=n,\,T_n=0)=p_0p_{00}^{n-1},$$
$$P(Z_n^{(0)}=0,\,T_n^{(0)}=n)=0,\quad P(Z_n^{(0)}=n,\,T_n^{(0)}=0)=p_{00}^{n-1},$$
$$P(Z_n=k,\,T_n=n-k)=p_0p_{00}^{k-1}p_{01}p_{11}^{n-k-1}+p_1p_{11}^{n-k-1}p_{10}p_{00}^{k-1},$$
$$P(Z_n^{(0)}=k,\,T_n^{(0)}=n-k)=p_{00}^{k-1}p_{01}p_{11}^{n-k-1},\quad 1\le k\le n-1,$$

and for $1\le k,l,k+l\le n-1$ we have:

$$P(Z_n=k,\,T_n=l)=\sum_{i=0}^{k-1}\sum_{j=1}^{l-1}P(X=i,\,Y=j)P(Z^{(0)}_{n-i-j}=k,\,T^{(0)}_{n-i-j}=l)$$
$$+\sum_{i=0}^{k-1}P(X=i,\,Y=l)P(Z^{(0)}_{n-i-l}=k,\,T^{(0)}_{n-i-l}\le l)$$

$$+\sum_{j=1}^{l-1} P(X=k,\, Y=j)P(Z^{(0)}_{n-k-j}\le k,\, T^{(0)}_{n-k-j}=l)$$
$$+P(X=k,\, Y=l)P(Z^{(0)}_{n-k-l}\le k,\, T^{(0)}_{n-k-l}\le l),$$
$$P(Z_n^{(0)}=k, T_n^{(0)}=l)=\sum_{i=1}^{k-1}\sum_{j=1}^{l-1} P(X_0=i,\, Y_0=j)P(Z^{(0)}_{n-i-j}=k,\, T^{(0)}_{n-i-j}=l)$$
$$+\sum_{i=1}^{k-1} P(X_0=i,\, Y_0=l)P(Z^{(0)}_{n-i-l}=k,\, T^{(0)}_{n-i-l}\le l)$$
$$+\sum_{j=1}^{l-1} P(X_0=k,\, Y_0=j)P(Z^{(0)}_{n-k-j}\le k,\, T^{(0)}_{n-k-j}=l)$$
$$+P(X_0=k,\, Y_0=l)P(Z^{(0)}_{n-k-l}\le k,\, T^{(0)}_{n-k-l}\le l).$$

Substituting notations we obtain Theorem 1. ∎

Next, having the distributions, we can calculate the correlations $c(T_n, Z_n)$. The results for $3\le n\le 10$ assuming $p_{00}=p_{11}=1/2$ are shown in Table 2.

Table 2. The correlations of T_n, Z_n

n	3	4	5	6	7	8	9	10
$c(T_n, Z_n)$	−0.87	−0.75	0.64	−0.56	0.50	−0.45	−0.41	−0.38

5. Limiting distributions. Let $X(p)$ denote a geometrically distributed random variable with parameter p: $P(X(p)=k)=p^{k-1}q$, $0<p<1$, $q=1-p$, $k\ge 1$. Note that $\mathrm{E}(X(p))=1/p$. The binary Markovian sequence $\{U_n\}$ generates alternating series of 1's of length $X_j(p_{11})$ and of 0's of length $Y_j(p_{00})$, $j\ge 1$, which are independent geometrically distributed random variables with the given parameters. Consider the maximum

$$T_n=\max(X_1(p),\dots,X_n(p)),\qquad n\ge 1.$$

Obviously, $P(T_n\le k)=(1-p^k)^n$, $k\ge 1$. We prove that the random variable T_n does not have a limiting distribution. Let $[x]$ denote the integer part of x, and $\{x\}=x-[x]$ denote the fractional part of x.

LEMMA 3. *For every $k\ge 0$, as $n\to\infty$ the following asymptotical formula holds*:

$$P(T_n+[\log_p n]\le k)-\exp(-\exp(k_n\log p))\to 0,$$

where $k_n=k+\{\log_p n\}$.

Proof. For every $k \ge 0$ we have

$$P(T_n + [\log_p n] \le k) = P(T_n + \log_p n - \{\log_p n\} \le k)$$
$$= \left(1 - \frac{1}{n} p^{k_n}\right)^n \sim \exp(-p^{k_n}) = \exp(-\exp(k_n \log p)). \ \blacksquare$$

Note that in the case $p = 1/2$, $n = 2^m$, $m \ge 0$, as $m \to \infty$ it follows that

$$P(T_n - m \le k) \to \Phi(k) = \exp(-\exp(-k \log 2)), \qquad -\infty < k < \infty,$$

where the distribution Φ is the discrete version of the Gompertz distribution. Numerical calculations show that the expected value is 1.332 and the variance is 3.500. For $n = 32$ the exact moments are $\mathrm{E}(T_n) = 6.355$ and $\mathrm{Var}(T_n) = 3.443$.

Lemma 3 shows that $T_n + [\log_p n]$ is a nondegenerate integer-valued random variable for which, as $n \to \infty$, the probabilities $P(T_n + [\log_p n] = k)$ oscillate for every k. If we consider $T_n + \log_p n$, then the following formula for the distribution function holds:

$$P(T_n + \log_p n \le x) = P(T_n \le [x - \log_p n]) = P(T_n \le x - \log_p n - \Delta_n(x))$$
$$\sim \exp(-p^{x - \Delta_n(x)}), \qquad -\infty < x < \infty,$$

where $\Delta_n(x) = \{x - \log_p n\}$.

6. Alternating process. Consider the delayed alternating renewal process defined by consecutive series of 1's of length X_k and of 0's of length Y_k, $k \ge 1$, which, together, form the sequence $\{U_n, n \ge 1\}$. Here, the parameter of the geometrically distributed random variable is omitted. Let $Z_k = X_k + Y_k$ denote the length of cycles in this alternating process and define the renewal process $n(t)$ by the inequality $S_{n(t)} \le t < S_{n(t)+1}$, where $S_n = Z_1 + \cdots + Z_n$. We have $\mathrm{E}(Z_n) = 1/p_{11} + 1/p_{00} = \mu_1$. From renewal theory it follows that $n(t)/t \overset{p.1}{\to} 1/\mu_1 = p_{00}p_{11}/(p_{00} + p_{11})$.

THEOREM 4. *Let T_t^* denote the maximum length of a series of* 1*'s in the sequence $\{U_n, 1 \le n \le t\}$. As $t \to \infty$ the following approximation holds*:

$$P(T_t^* - \log_{p_{11}} \mathrm{E}(n(t)) \le k) - \exp(-p^{x(t)}) \to 0, \qquad -\infty < x < \infty,$$

where $x(t) = k + \{\log_{p_{11}} \mathrm{E}(n(t))\}$.

Proof. We have

$$P(T_{n(t)} + [\log_{p_{11}} \mathrm{E}(n(t))] \le k)$$
$$= P\left(T_{n(t)} + \log_{p_{11}} n(t) - \{\log_{p_{11}} \mathrm{E}(n(t))\} - \log_{p_{11}} \frac{n(t)}{\mathrm{E}(n(t))} \le k\right)$$
$$= \left(1 - \frac{1}{n(t)} \exp\left(\left(k + \{\log_{p_{11}} \mathrm{E}(n(t))\} + \log_{p_{11}} \frac{n(t)}{\mathrm{E}(n(t))}\right) \log p\right)\right)^{n(t)}.$$

Note that $T_t^* - T_{n(t)} \xrightarrow{p.1} 0$. Now the fact that $n(t) \xrightarrow{p.1} \infty$ may be used, together with Lemma 3 and the fact that $n(t)\mu_1/t \xrightarrow{p.1} 1$. ∎

7. Applications. In [2] a semi-Markovian model of the course of a basketball game is presented in which the points scored in a game form cycles governed by a binary Markov sequence. It is assumed that the control sequence is stationary, but the alternative case that teams have "hot and cold" periods may also be considered. Using the series of successes we prove that the hypothesis that the scoring sequence is Markovian is not rejected for each of the several games monitored.

We consider basketball games played by the following teams (in brackets we give the abbreviation of the name of the team): Nobiles Włocławek (*No*), Lineker Nicosia [Linel TEX Basket Imola] (*Li*), Split Zagreb [Croatia Osiguranje Split] (*Sp*), Śląsk Wrocław (*Śl*), Trefl Sopot (*Tr*), Ulker Istanbul (*Ul*), and five national teams: Greece (*Gr*), Yugoslavia (*Yu*), Lithuania (*Li*), Latvia (*La*) and Germany (*Ge*). Some games were played in the Suproliga and Saporta Cup in the 2000/01 season and one in the Polish basketball league in 2002. They are of different importance, since the teams present various standards. The games took place in different circumstances. In particular the Yugoslavia–Latvia game was played in Turkey, so there was no home team. All games, excluding one, were monitored by M. Truś and used in the diploma paper [3], the fourth game (see below) was monitored by J. Dembiński (see [1]). The same data have been used to test a semi-Markovian model in [2]. The results of the games are as follows:

Nobiles–Lineker 83:51,
Śląsk–Ulker 80:69,
Split–Śląsk 75:76,
Śląsk–Trefl 84:71,
Yugoslavia–Latvia 114:77,
Lithuania–Latvia 77:94,
Greece–Germany 80:75.

Table 3 shows data concerning the number of cycles in a game, the transition probabilities p_{ii} and the maximum length n_i of a scoring series for the home team ($i = 0$) and for the away team ($i = 1$). We also present the number of points $W^{(i)}$ scored in the series of maximum length.

We compare the realizations of the random variables Z and T with their distribution functions under the appropriate transition probabilities and the number of cycles. We note that the observed maximum lengths of series are not placed at the extremities of the domain of the distribution. A significant result seems to be the last case: we observe a maximum length of six 1's when $\mathrm{E}(T) = 3.38$ and $\mathrm{Var}(T) = 0.956$. However, $P(T \le 6) = 0.978$. In the case

Table 3. Maximum length of series in selected basketball games

Match	p_{00}	p_{11}	n_0	n_1	Z_n	$W^{(0)}$	T_n	$W^{(1)}$
No-Li	0.444	0.308	37	29	5	12	3	7
Śl-Ul	0.324	0.344	36	34	4	10	5	8
Sp-Śl	0.500	0.500	35	37	6	15	4	8
Śl-Tr	0.395	0.281	41	32	4	10	3	6
Yu-La	0.511	0.324	49	36	5	11	4	10
Li-La	0.324	0.368	37	39	4	6	4	10
Gr-Ge	0.500	0.379	37	32	4	9	6	15

where a test is repeated many times, such a probability should not be used as an argument for the rejection of the semi-Markovian model. It should be noted that this series of maximum length was realised by the defeated team.

References

[1] J. Dembiński and B. Kopociński, *Modelling of the course of a basketball match*, Human Movement 2 (2003), no. 8, 11–15 (in Polish, English summary).

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