APPLICATIONES MATHEMATICAE
33,1 (2006), pp. 85–93

Rafał Kucharski (Warszawa)

CONVERGENCE OF OPTIMAL STRATEGIES IN A DISCRETE TIME MARKET WITH FINITE HORIZON

Abstract. A discrete-time financial market model with finite time horizon is considered, together with a sequence of investors whose preferences are described by a convergent sequence of strictly increasing and strictly concave utility functions. Existence of unique optimal consumption-investment strategies as well as their convergence to the limit strategy is shown.

Introduction. Recently, in a number of papers the following question was considered: does convergence of investors' preferences imply the convergence of their optimal strategies? In [2] a model with complete Brownian market model was described, while in [1] a discrete time model with finite horizon and utility functions defined on the whole real line was studied. Both papers gave a positive answer to the above problem under suitable assumptions.

In the present paper we prove a similar result for a discrete time market model with a finite horizon. We assume weaker regularity conditions on utility functions: strict concavity and strict monotonicity. The utility functions considered are defined on the positive axis.

In the first section we describe our model of financial market. Then we consider a one-step model and utilizing ideas from [4], we establish a few useful technical results. Finally, we prove the existence of optimal strategies for our model and their convergence together with the convergence of the investors' preferences.

1. Market model. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, \mathbb{P})$ be a discrete-time filtered probability space with finite time horizon $T \in \mathbb{N}$, with $\mathcal{F}_0 = \{\emptyset, \Omega\}$. Prices of

2000 *Mathematics Subject Classification*: 49L20, 91B16, 91B28, 93E20.

Key words and phrases: utility maximization, dynamic programming, optimal strategies.

d risky securities available on the market are represented by a d-dimensional, almost surely positive adapted process $S_t = (S_{t,1}, \dots, S_{t,d})'$, $0 \le t \le T$. For $t = 0, \dots, T-1$ we define

$$\zeta_{t,i} = \frac{S_{t+1,i}}{S_{t,i}}, \quad i = 1, \dots, d,$$

and $\zeta_t = (\zeta_{t,1}, \dots, \zeta_{t,d})'$. Let $D_t(\omega)$ be the smallest linear subspace containing the support of the regular conditional distribution of ζ_t with respect to $\mathcal{F}_t$ (it exists, cf. [6, Theorem 2.7.5]). Throughout the paper we assume that there are no redundant assets on the market, thus we have the following non-degeneracy assumption:

ASSUMPTION 1.1. *D_t is almost surely equal to $\mathbb{R}^d$ for $0 \le t \le T-1$.*

Let $\Delta_0 = \{\nu \in \mathbb{R}^d : \nu_i \ge 0, \sum_{i=1}^d \nu_i \le 1\}$, and $\Delta = \{\nu \in \Delta_0 : \sum_{i=1}^d \nu_i = 1\}$. We denote by $\langle \cdot, \cdot \rangle$ the usual scalar product in $\mathbb{R}^d$. Denote by X_t the wealth process at time t before consumption and possible transactions. Let $\pi_{t,i}$ and $\overline{\pi}_{t,i}$ be the portions of the wealth X_t invested in the ith asset at time t before and respectively after consumption and possible transactions. We do not allow short selling or short borrowing, so $\pi_t = (\pi_{t,1}, \dots, \pi_{t,d})' \in \Delta$ and $\overline{\pi}_t = (\overline{\pi}_{t,1}, \dots, \overline{\pi}_{t,d})' \in \Delta_0$.

At time $t = 0, \dots, T-1$, the investor who owns initial wealth X_t invested in portfolio π_t, consumes a part $\alpha_t \in [0,1]$ of his wealth and changes his portfolio composition to $\overline{\pi}_t$, according to the equation

$$X_t = X_t \alpha_t + X_t \sum_{i=1}^d \overline{\pi}_{t,i},$$

which implies that we are interested only in $\mathcal{F}_t$-measurable strategies such that $(\alpha_t, \overline{\pi}_t) \in [0,1] \times \Delta_0$ a.s. and

$$\alpha_t + \sum_{i=1}^d \overline{\pi}_{t,i} = 1 \quad \text{a.s.} \tag{1.1}$$

Denote the set of such strategies by $\mathcal{A}_t$.

At time $t+1$, due to price changes, the investor's wealth changes to

$$X_{t+1} = \sum_{i=1}^d X_t \overline{\pi}_{t,i} \zeta_{t,i} = X_t \langle \overline{\pi}_t, \zeta_t \rangle. \tag{1.2}$$

Equation (1.2) describes the dynamics of the control system we are dealing with: X_t is regarded as a state of the system, and $(\alpha_t, \overline{\pi}_t) \in [0,1] \times \Delta_0$ are its control parameters, constrained by (1.1) describing the admissible strategies. The initial condition is given by the endowment $x := X_0 > 0$.

We consider a sequence of investors with preferences described by utility functions $U_t^n\colon (0,\infty) \to \mathbb{R}$, $0 \le t \le T$, $n \in \overline{\mathbb{N}} := \mathbb{N} \cup \{\infty\}$.

ASSUMPTION 1.2. *The functions U_t^n are strictly increasing and strictly concave for $t \in \{0,\dots,T\}$ and $n \in \overline{\mathbb{N}}$. Moreover, for all $t \in \{0,\dots,T\}$ and $x \in (0,\infty)$,*

$$U_t^n(x) \to U_t^\infty(x) \quad \text{as } n \to \infty.$$

We are interested in maximization of the expected utility from consumption and terminal wealth, that is, we want to maximize the following reward functional:

$$J_T^n(x,(\alpha,\overline{\pi})) = \mathbb{E}\Big(\sum_{t=0}^{T-1} U_t^n(X_t\alpha_t) + U_T^n(X_T)\Big). \tag{1.3}$$

For our dynamic programming problem to be well posed and finite, we assume that the following conditions are satisfied:

ASSUMPTION 1.3. *For all $n \in \overline{\mathbb{N}}$, $k \in \{1,\dots,T\}$ and $x > 0$,*

$$\mathbb{E}(U_k^n)^+\Big(x\prod_{t=0}^{k-1}\max\{\zeta_{t,i} : i = 1,\dots,d\}\Big) < \infty,$$
$$\mathbb{E}(U_k^n)^-\Big(x\prod_{t=0}^{k-1}\min\{\zeta_{t,i} : i = 1,\dots,d\}\Big) < \infty.$$

REMARK 1.4. One can consume all or nothing of the wealth, so we need values of utility functions at 0. We deal with that problem by putting $U(0) := \lim_{x\to 0^+} U(x)$; if this limit is finite, the continuity and concavity properties are kept, and if not (e.g. for a logarithmic function), the agent will not choose such a strategy to maximize utility.

2. One-step case. We start with the case $T = 1$. Let $u, v\colon (0,\infty) \to \mathbb{R}$ be strictly increasing functions, u strictly concave and v concave. Let $\mathcal{H}$ be a sub-σ-field of $\mathcal{F}$, let $\zeta = (\zeta_1,\dots,\zeta_d)'$ be an $\mathbb{R}^d$-valued random variable with non-degenerate (in the sense of Assumption 1.1) conditional distribution with respect to $\mathcal{H}$, and let $\mathbb{E}(\cdot \mid \mathcal{H})$ denote conditional expectation with respect to $\mathcal{H}$. Denote by $\mathcal{A}$ the set of admissible strategies: $\mathcal{H}$-measurable random variables such that $(\alpha,\overline{\pi}) \in [0,1] \times \Delta_0$ a.s. and $\alpha + \sum_{j=1}^d \overline{\pi}_j = 1$. Define the value function by

$$w(x) := \operatorname*{ess\,sup}_{(\alpha,\overline{\pi})\in\mathcal{A}}\{u(x\alpha) + \mathbb{E}(v(x\langle\overline{\pi},\zeta\rangle) \mid \mathcal{H})\}, \qquad x > 0.$$

Analogously to Assumption 1.3, we introduce

Assumption 2.1. *For all* $x > 0$,

$$\mathbb{E}v^+(x \max_{i\in\{1,\dots,d\}} \zeta_i) < \infty \quad \textit{and} \quad \mathbb{E}v^-(x \min_{i\in\{1,\dots,d\}} \zeta_i) < \infty.$$

The following technical lemmas are crucial:

Lemma 2.2. *There exists an almost surely continuous, strictly concave and strictly increasing (with respect to every coordinate) version of*

$$[0,\infty)^d \setminus \{0\} \ni \pi \mapsto \mathbb{E}(v(\langle \pi, \zeta\rangle) \,|\, \mathcal{H}).$$

Proof. Let κ denote the regular conditional distribution of ζ given $\mathcal{H}$. Then

$$\mathbb{E}(v(\langle \pi, \zeta\rangle) \,|\, \mathcal{H}) = \int_{\mathbb{R}^d} v(\langle \pi, x\rangle)\,\kappa(dx) \quad \text{a.s.},$$

and we take the right side as a definition of our version. By a routine calculation one checks it has the desired properties. We will show concavity. Fix $\pi^1, \pi^2 \in [0,\infty)^d \setminus \{0\}$, $\pi^1 \neq \pi^2$ and $t \in (0,1)$. Then

$$\begin{aligned} t\mathbb{E}(v(\langle \pi^1, \zeta\rangle) \,|\, \mathcal{H}) &+ (1-t)\mathbb{E}(v(\langle \pi^2, \zeta\rangle) \,|\, \mathcal{H}) \\ &= \int_{\mathbb{R}^d} [tv(\langle \pi^1, x\rangle) + (1-t)v(\langle \pi^2, x\rangle)]\,\kappa(dx) \\ &< \int_{\mathbb{R}^d} v(\langle t\pi^1 + (1-t)\pi^2, x\rangle)\,\kappa(dx) \\ &= \mathbb{E}(v(\langle t\pi^1 + (1-t)\pi^2, \zeta\rangle) \,|\, \mathcal{H}) \quad \text{a.s.} \end{aligned}$$

The strict inequality is justified by Assumption 1.1. ∎

Proposition 2.3. *For every* $x \in (0,\infty)$ *there exists a unique optimal pair* $(\widehat{\alpha}, \widehat{\pi}) \in \mathcal{A}$ *such that*

$$w(x) = u(x\widehat{\alpha}) + \mathbb{E}(v(x\langle \widehat{\pi}, \zeta\rangle) \,|\, \mathcal{H}) \quad \textit{a.s.} \tag{2.1}$$

Proof. We take the version of conditional expectation with the properties stated in Lemma 2.2, and consider the mapping

$$\Phi\colon [0,1] \times \Delta_0 \times \Omega \ni (\alpha, \overline{\pi}, \omega) \mapsto u(x\alpha) + \mathbb{E}(v(x\langle \overline{\pi}, \zeta\rangle) \,|\, \mathcal{H})(\omega) \in \mathbb{R}$$

which is continuous except on a $\mathbb{P}$-zero set N. Since the set

$$\Big\{(\alpha, \overline{\pi}) \in [0,1] \times \Delta_0 : \alpha + \sum_{j=1}^{d} \overline{\pi}_j = 1\Big\} \tag{2.2}$$

is compact, for any $\omega \in \Omega \setminus N$ there is a pair $(\widehat{\alpha}(\omega), \widehat{\pi}(\omega))$ attaining the supremum of Φ.

Suppose that there are two such pairs, say $(\alpha^1, \overline{\pi}^1), (\alpha^2, \overline{\pi}^2) \in \mathcal{A}$. Take any $t \in (0,1)$. Putting $\alpha = t\alpha^1 + (1-t)\alpha^2$, $\overline{\pi} = t\overline{\pi}^1 + (1-t)\overline{\pi}^2$ we have

$\alpha \in (0,1)$, $\overline{\pi} \in \Delta_0$ a.s. Since $\sum_{i=1}^m \overline{\pi}_i = 1-\alpha$, it follows that $(\alpha, \overline{\pi}) \in \mathcal{A}$ and

$$\begin{aligned} w(x) &= tw(x) + (1-t)w(x) \\ &= t[u(x\alpha^1) + \mathbb{E}(v(x\langle \overline{\pi}^1, \zeta\rangle) \mid \mathcal{H})] \\ &\quad + (1-t)[u(x\alpha^2) + \mathbb{E}(v(x\langle \overline{\pi}^2, \zeta\rangle) \mid \mathcal{H})] \\ &\le u(x\alpha) + \mathbb{E}(v(tx\langle \overline{\pi}^1, \zeta\rangle + (1-t)x\langle \overline{\pi}^2, \zeta\rangle) \mid \mathcal{H}) \\ &= u(x\alpha) + \mathbb{E}(v(x\langle \overline{\pi}, \zeta\rangle) \mid \mathcal{H}) \le w(x) \quad \text{a.s.} \end{aligned}$$

Both u and v are strictly concave, thus the above inequality turns into an equality iff $\alpha^1 = \alpha^2$ and $\langle \overline{\pi}^1, \zeta\rangle = \langle \overline{\pi}^2, \zeta\rangle$ a.s. From the assumption we made on the support of the distribution of ζ, that implies $\overline{\pi}_i^1 = \overline{\pi}_i^2$ a.s., $i = 1, \ldots, d$, hence the proof of uniqueness is finished.

The optimal pair $(\widehat{\alpha}, \widehat{\pi})$ is an $\mathcal{H}$-measurable random variable, since for any open ball $B \subset \mathbb{R}^{d+1}$,

$$(\widehat{\alpha}, \widehat{\pi})(\omega) \in B \iff \bigvee_{(\alpha^*, \pi^*) \in C \cap B} \ \bigwedge_{(\alpha, \pi) \in C \setminus B} \Phi(\alpha^*, \pi^*)(\omega) > \Phi(\alpha, \pi)(\omega)$$

where C denotes a countable dense subset of (2.2), and therefore

$$\{(\widehat{\alpha}, \widehat{\pi}) \in B\} = \bigcup_{(\alpha^*, \pi^*) \in C \cap B} \ \bigcap_{(\alpha, \pi) \in C \setminus B} \{\Phi(\alpha^*, \pi^*) > \Phi(\alpha, \pi)\} \in \mathcal{H}. \ \blacksquare$$

LEMMA 2.4. *There is a version of the value function w which is almost surely strictly increasing and strictly concave.*

Proof. For every $q \in (0, \infty) \cap \mathbb{Q}$ fix a version of $w(q)$, which by Assumption 2.1 is almost surely finite. Fix $x, y \in (0, \infty) \cap \mathbb{Q}$. It is obvious that if $y < x$ then $w(y) < w(x)$ a.s. To show strict concavity, fix $t \in (0,1) \cap \mathbb{Q}$ and let $(\alpha^x, \overline{\pi}^x), (\alpha^y, \overline{\pi}^y) \in \mathcal{A}$ be optimal pairs for x and y respectively. Put $z = tx + (1-t)y$, $\beta = tx/z$, $\alpha = \beta\alpha^x + (1-\beta)\alpha^y$, $\overline{\pi} = \beta\overline{\pi}^x + (1-\beta)\overline{\pi}^y$. Obviously $\alpha \in [0,1]$, $\beta \in (0,1)$ a.s. Since $\sum_{i=1}^d \overline{\pi}_i = 1-\alpha$, we obtain

$$tx\overline{\pi}^x + (1-t)y\overline{\pi}^y = z(\beta\overline{\pi}^x + (1-\beta)\overline{\pi}^y) = z\overline{\pi},$$

and since u and v are strictly concave and ζ is almost surely positive, we have

$$\begin{aligned} tw(x) + (1-t)w(y) &= t[u(x\alpha^x) + \mathbb{E}(v(x\langle \overline{\pi}^x, \zeta\rangle) \mid \mathcal{H})] \\ &\quad + (1-t)[u(y\alpha^y) + \mathbb{E}(v(y\langle \overline{\pi}^y, \zeta\rangle) \mid \mathcal{H})] \\ &\le u(z\alpha) + \mathbb{E}(v(z\langle \overline{\pi}, \zeta\rangle \mid \mathcal{H})) \le w(z) \quad \text{a.s.} \end{aligned}$$

and moreover this inequality turns into an equality iff

$$x\alpha^x = y\alpha^y \quad \text{and} \quad x\langle \overline{\pi}^x, \zeta\rangle = y\langle \overline{\pi}^y, \zeta\rangle \quad \text{a.s.}$$

Once again using our assumption on the distribution of ζ, this implies

$$x\overline{\pi}_i^x = y\overline{\pi}_i^y, \quad i = 1, \ldots, d,$$

and summing those equalities up for $i = 1, \ldots, d$ we obtain

$$x[1-\alpha^x] = y[1-\alpha^y],$$

hence also $x = y$. This shows in particular that for all $x, y \in (0,\infty) \cap \mathbb{Q}$, $x \neq y$, we have

$$w\left(\frac{x+y}{2}\right) > \frac{w(x)+w(y)}{2} \quad \text{a.s.}$$

We can now extend this version of w to a function which is almost surely strictly increasing and strictly continuous for all $x \in (0,\infty)$. Finally, from monotone convergence, for fixed $x \in (0,\infty)$ and a sequence of rationals $q_n \uparrow x$ we have

$$w(x) = \lim_n w(q_n) = \lim_n \operatorname*{ess\,sup}_{(\pi,\alpha)\in\mathcal{A}} \{u(q_n\alpha) + \mathbb{E}(v(q_n\langle\overline{\pi},\zeta\rangle) \mid \mathcal{H})\}$$
$$= \operatorname*{ess\,sup}_{(\pi,\alpha)\in\mathcal{A}} \{u(x\alpha) + \mathbb{E}(v(x\langle\overline{\pi},\zeta\rangle) \mid \mathcal{H})\}. \ ∎$$

PROPOSITION 2.5. *There exists a selector of optimal strategies*

$$(0,\infty) \in x \mapsto (\widehat{\alpha},\widehat{\pi})(x) \in \mathcal{A}$$

which is continuous for almost all ω.

Proof. We fix a version of conditional expectation with the properties stated in Lemma 2.2. The random function

$$w(x,(\alpha,\overline{\pi})) := u(x\alpha) + \mathbb{E}(v(x\langle\overline{\pi},\zeta\rangle) \mid \mathcal{H})$$

is then almost surely continuous, jointly for all arguments. Suppose there exists $x \in (0,\infty)$ and a sequence $x_n \in (0,\infty)$, $n \in \mathbb{N}$, such that $x_n \to x$ and $(\widehat{\alpha},\widehat{\pi})(x_n) \not\to (\widehat{\alpha},\widehat{\pi})(x)$. Since all $(\widehat{\alpha},\widehat{\pi})(x_n)$ belong to the compact set (2.2), we may choose, using Lemma 2 from [3], a random subsequence $(\widehat{\alpha},\widehat{\pi})(x_{n_k})$ converging to some $(\widetilde{\alpha},\widetilde{\pi})$. Condition (1.1) holds for all $k \in \mathbb{N}$, so letting $k \to \infty$, we get $(\widetilde{\alpha},\widetilde{\pi}) \in \mathcal{A}$. By continuity,

$$\lim_{k\to\infty} w(x_{n_k},(\widehat{\alpha},\widehat{\pi})(x_{n_k})) = w(x,(\widetilde{\alpha},\widetilde{\pi})) =: \widetilde{w}, \tag{2.3}$$

$$\lim_{n\to\infty} w(x_n,(\widehat{\alpha},\widehat{\pi})(x)) = w(x,(\widehat{\alpha},\widehat{\pi})(x)) =: w, \tag{2.4}$$

and if $(\widetilde{\alpha},\widetilde{\pi}) \neq (\widehat{\alpha},\widehat{\pi})(x)$, then $\widetilde{w} < w$. If we fix $\varepsilon \in (0,(w-\widetilde{w})/2)$, then for k large enough

$$w(x_{n_k},(\widehat{\alpha},\widehat{\pi})(x)) > w - \varepsilon > \widetilde{w} + \varepsilon, \tag{2.5}$$

while from (2.3),

$$w(x_{n_k},(\widehat{\alpha},\widehat{\pi})(x_{n_k})) < \widetilde{w} + \varepsilon. \tag{2.6}$$

Inequalities (2.5) and (2.6) lead to

$$w(x_{n_k},(\widehat{\alpha},\widehat{\pi})(x)) > w(x_{n_k},(\widehat{\alpha},\widehat{\pi})(x_{n_k}))$$

contradicting the optimality of $(\widehat{\alpha},\widehat{\pi})(x_{n_k})$. ∎

3. Convergence of optimal strategies. We are now going to use the results of the previous section in the general case. We define the Bellman functions:

$$\begin{aligned} &V_T^n(x) := U_T(x), \\ (3.1)\qquad &V_t^n(x) := \operatorname*{ess\,sup}_{(\alpha,\overline{\pi})\in\mathcal{A}} \{U_t^n(\alpha x) + \mathbb{E}(V_{t+1}^n(x\langle\overline{\pi},\zeta_t\rangle)\,|\,\mathcal{F}_t)\}, \end{aligned}$$

for $x \in (0,\infty)$ and $t = 0,\dots,T-1$.

THEOREM 3.1. *For all* $n \in \overline{\mathbb{N}}$ *and* $t = 0,\dots,T$:

(i) *the function* V_t^n *has a version which is strictly increasing and strictly concave almost surely*,
(ii) *there exists a unique* $\mathcal{B}(0,\infty)\otimes\mathcal{F}_t$*-measurable function* $(\widehat{\alpha}_t^n, \widehat{\pi}_t^n) \in \mathcal{A}_t$ *such that for all* $x \in (0,\infty)$,

$$V_t^n(x) = U_t^n(x\widehat{\alpha}_t^n(x)) + \mathbb{E}(V_{t+1}^n(x\langle\widehat{\pi}_t^n(x),\zeta_t\rangle)\,|\,\mathcal{F}_t).$$

Proof. Fix $n \in \overline{\mathbb{N}}$ and use backward induction. It is clear that V_T^n is strictly concave and strictly increasing since U_T^n is. Then decreasing t from $T-1$ to 0 and applying Lemma 2.4 and Proposition 2.3 with $w := V_t^n$, $u := U_t^n$, $v := V_{t+1}^n$, $\mathcal{A} := \mathcal{A}_t$, $\mathcal{H} := \mathcal{F}_t$ and $\zeta := \zeta_t$, we find that V_t^n has a strictly increasing and strictly concave version, and there is a unique optimal strategy $(\widehat{\alpha}_t^n, \widehat{\pi}_t^n) := (\widehat{\alpha}, \widehat{\pi})$ which is $\mathcal{F}_t$-measurable for all $x \in (0,\infty)$ and almost surely continuous, hence $\mathcal{B}(0,\infty)\otimes\mathcal{F}_t$-measurable. This proves the theorem. ∎

In this section we will make repeated use of the following elementary fact. It may be derived e.g. from pages 90 and 248 of [5], but we include an easy proof for completeness.

LEMMA 3.2. *Let* $U \subset \mathbb{R}$ *be an open set and* $f_n\colon U \to \mathbb{R}$ *be a sequence of increasing functions such that* f_n *converges pointwise on* U *to a continuous function* f. *Then* f_n *converges to* f *uniformly on each compact subset of* U.

Proof. First notice that f is increasing, being the limit of a sequence of increasing functions. Fix a compact set $C \subset U$ and an arbitrary $\varepsilon > 0$. Without loss of generality, we may assume that $C = [a,b]$ is an interval. On C, the function f is uniformly continuous, hence we can find $x_0,\dots,x_k \in C$ with $a := x_0 < x_1 < \dots < x_{k-1} < x_k =: b$ such that $|f(x_i) - f(x_{i-1})| < \varepsilon/2$ for $i \in \{1,\dots,k\}$. Let $N_i \in \mathbb{N}$ be such that $|f_n(x_i) - f(x_i)| < \varepsilon/2$ for $n \ge N_i$, and define $N := \max\{N_i : i \in \{0,\dots,k\}\}$. Then for any $x \in A$ there is $i \in \{0,\dots,k-1\}$ such that $x \in [x_i, x_{i+1}]$, and for $n \ge N$ we have

$$\begin{aligned} f(x) - \varepsilon \le f(x_{i+1}) - \varepsilon \le f_n(x_{i+1}) - \varepsilon/2 \le f_n(x) \le f_n(x_i) + \varepsilon/2 \\ \le f(x_i) + \varepsilon \le f(x) + \varepsilon. \end{aligned}$$

Since $x \in C$ was arbitrary, the assertion follows. ∎

Now we are ready to prove the convergence of optimal strategies. Again we will start with the one-step case.

PROPOSITION 3.3. *Assume that for every $n \in \overline{\mathbb{N}}$ functions u^n, v^n are strictly increasing and strictly concave, and moreover $\lim_{n\to\infty} u^n(x) = u^\infty(x)$ and $\lim_{n\to\infty} v^n(x) = v^\infty(x)$ for all $x \in (0,\infty)$. Let $(\widehat{\alpha}^n, \widehat{\pi}^n)$ denote the optimal strategy fulfilling (2.1) with u and v replaced by u^n and v^n. Then, for every $x \in (0,\infty)$,*

$$\lim_{n\to\infty} (\widehat{\alpha}^n, \widehat{\pi}^n)(x) = (\widehat{\alpha}^\infty, \widehat{\pi}^\infty)(x) \qquad a.s.$$

Proof. Suppose that, on the contrary, the convergence fails for some $x \in (0,\infty)$. Since $[0,1] \times \Delta_0$ is compact, by the use of Lemma 2 from [3] we choose a random subsequence $(n_k \in \mathbb{N} : k \in \mathbb{N})$ such that $\lim_{k\to\infty} (\widehat{\alpha}^{n_k}, \widehat{\pi}^{n_k})(x) = (\widetilde{\alpha}, \widetilde{\pi}) \in \mathcal{A}$, $(\widetilde{\alpha}, \widetilde{\pi}) \neq (\widehat{\alpha}^\infty, \widehat{\pi}^\infty)$. Define

$$w^n(\alpha, \overline{\pi}) := u^n(x\alpha) + \mathbb{E} v^n(x\langle \overline{\pi}, \zeta\rangle \mid \mathcal{H}), \qquad (\alpha, \overline{\pi}) \in \mathcal{A},\ n \in \overline{\mathbb{N}},$$

with a continuous version of the conditional expectation. Then the functions w^n depend continuously on $\overline{\pi}$ and α, the uniform convergence of u^n and v^n on compact sets gives

$$\lim_{k\to\infty} w^{n_k}(\widehat{\alpha}^{n_k}, \widehat{\pi}^{n_k}) = w^\infty(\widetilde{\alpha}, \widetilde{\pi}) \quad \text{a.s.,} \tag{3.2}$$

and by our hypothesis

$$\widetilde{w} := w^\infty(\widetilde{\alpha}, \widetilde{\pi}) < w^\infty(\widehat{\alpha}^\infty, \widehat{\pi}^\infty) =: w.$$

Fix $\varepsilon \in (0, (w - \widetilde{w})/2)$. Since pointwise convergence ensures

$$\lim_{n\to\infty} w^n(\widehat{\alpha}^\infty, \widehat{\pi}^\infty) = w,$$

for k large enough we have

$$w^k(\widehat{\alpha}^\infty, \widehat{\pi}^\infty) > w - \varepsilon > \widetilde{w} + \varepsilon, \tag{3.3}$$

while from (3.2) we get

$$w^{n_k}(\widehat{\alpha}^{n_k}, \widehat{\pi}^{n_k}) < \widetilde{w} + \varepsilon. \tag{3.4}$$

Combining (3.3) and (3.4) we obtain $w^{n_k}(\widehat{\alpha}^\infty, \widehat{\pi}^\infty) > w^{n_k}(\widehat{\alpha}^{n_k}, \widehat{\pi}^{n_k})$, contradicting the optimality of $(\widehat{\alpha}^{n_k}, \widehat{\pi}^{n_k})$. ∎

Now we can prove the main theorem.

THEOREM 3.4. *Let $((\widehat{\alpha}_t^n, \widehat{\pi}_t^n) : t = 0, \ldots, T-1)$ be optimal strategies maximizing (1.3) with the corresponding functions $(U_0^n, \ldots, U_T^n)$, $n \in \overline{\mathbb{N}}$. Then for every $x \in (0,\infty)$,*

$$\lim_{n\to\infty} (\widehat{\alpha}_t^n, \widehat{\pi}_t^n)(x) = (\widehat{\alpha}_t^\infty, \widehat{\pi}_t^\infty)(x) \qquad a.s.,\ t = 0, \ldots, T-1.$$

Proof. The assertion follows from the foregoing proposition applied consecutively to the Bellman functions (3.1) with $u^n := U_t^n$ and $v^n := V_{t+1}^n$ for $t = T-1, \ldots, 0$. We only need to check that $\lim_{n\to\infty} V_t^n(x) = V_t^\infty(x)$

for $x \in (0,\infty)$ and $t = T,\dots,1$. For $t = T$ this is obvious since $V_T^n = U_T^n$, $n \in \overline{\mathbb{N}}$. If we have proved that $(\widehat{\alpha}_t^n, \widehat{\pi}_t^n) \to (\widehat{\alpha}_t^\infty, \widehat{\pi}_t^\infty)$ for some $t \le T$, then from uniform convergence on compact sets and the Lebesgue Theorem, for all $x \in (0,\infty)$,

$$\begin{aligned}\lim_{n\to\infty} V_t^n(x) &= \lim_{n\to\infty} (U_t^n(x\widehat{\alpha}_t^n(x)) + \mathbb{E}(V_{t+1}^n(x\langle \widehat{\pi}_t^n(x), \zeta_t\rangle) \mid \mathcal{F}_t)) \\ &= U_t^\infty(x\widehat{\alpha}_t^\infty(x)) + \mathbb{E}(V_{t+1}^\infty(x\langle \widehat{\pi}_t^\infty(x), \zeta_t\rangle) \mid \mathcal{F}_t) \\ &= V_t^\infty(x). \ \blacksquare\end{aligned}$$

References

[1] L. Carassus and M. Ràsonyi, *Optimal strategies and utility-based prices converge when agents' preferences do*, preprint.

[2] E. Jouini and C. Napp, *Convergence of utility functions and convergence of optimal strategies*, Finance Stoch. 8 (2004), 133–144.

[3] Yu. M. Kabanov and Ch. Stricker, *A teachers' note on no-arbitrage criteria*, in: Séminaire de Probabilités, XXXV, Lecture Notes in Math. 1755, Springer, Berlin, 2001, 149–152.

[4] M. Ràsonyi and Ł. Stettner, *On utility maximization in discrete-time financial market models*, Ann. Appl. Probab. 15 (2005), 1367–1395.

[5] R. T. Rockafellar, *Convex Analysis*, Princeton Math. Ser. 28, Princeton Univ. Press, Princeton, NJ, 1970.

[6] A. N. Shiryaev, *Probability*, Grad. Texts in Math. 95, Springer, New York, 1996.

Institute of Mathematics
Polish Academy of Sciences
Śniadeckich 8
00-956 Warszawa, Poland
E-mail: R.Kucharski@impan.gov.pl

Received on 24.1.2006;
revised version on 10.3.2006 (1804)