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ARBITRAGE IN MARKETS WITHOUT SHORTSELLING WITH PROPORTIONAL TRANSACTION COSTS

Abstract. We consider markets with proportional transaction costs and shortsale restrictions. We give necessary and sufficient conditions for the absence of arbitrage and also estimate the super-replication price.

1. Introduction. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space equipped with a finite discrete-time filtration $\mathbb{F} = (\mathcal{F}_t)_{t=0}^T$ such that $\mathcal{F}_T = \mathcal{F}$. Let $S = (S_t)_{t=0}^T = (S_t^1, \ldots, S_t^d)_{t=0}^T$ be an d-dimensional process adapted to $\mathbb{F}$, which has strictly positive components, i.e. $S_t^i > 0$, $\mathbb{P}$-a.s. We assume that there exists a bank account or a bond on the market, which is a process $B = (B_t)_{t=0}^T$ and all transactions are calculated in units of this process. For simplicity we assume that $B_t \equiv 1$ for all $t = 0, \ldots, T$. A trading strategy on the market is a d-dimensional process $H = (H_t)_{t=1}^T = (H_t^1, \ldots, H_t^d)_{t=1}^T$, which is predictable with respect to $\mathbb{F}$. We denote the set of such strategies by $\mathcal{P}$ and define the set of strategies without shortselling by $\mathcal{P}_+ = \{H \in \mathcal{P} \mid H \geq 0\}$.

Let $\lambda = (\lambda_1, \ldots, \lambda_d)$, $\mu = (\mu_1, \ldots, \mu_d)$ and

$$\varphi^i(x) = x + \lambda_i x^+ + \mu_i x^- \quad \text{for } i = 1, \ldots, d \text{ where } 0 < \lambda_i, \mu_i < 1.$$

The vectors λ, μ model proportional transaction costs for buying and selling respectively. We say that $\lambda < \mu$ if and only if $\lambda_i < \mu_i$ for all $i = 1, \ldots, d$. We use the notation

$$(H \cdot S)_t := \sum_{j=1}^{t} H_j \cdot \Delta S_j$$

where $\cdot$ is the inner product in $\mathbb{R}^d$.

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Let $x = (x_t^{\lambda,\mu})_{t=1}^T$ be the gain or loss process in the market with proportional transaction costs for the strategy H starting from 0 units in bank and stock accounts, i.e. $x_t^{\lambda,\mu}$ is defined as follows:

$$\begin{aligned} x_t^{\lambda,\mu} = x_t^{\lambda,\mu}(H) &= -\sum_{j=1}^{t} \varphi(\Delta H_j)\cdot S_{j-1} - \varphi(-H_t)\cdot S_t \\ &= -\sum_{j=1}^{t}\sum_{i=1}^{d} \varphi^i(\Delta H_j^i) S_{j-1}^i - \sum_{i=1}^{d} \varphi^i(-H_t^i) S_t^i \end{aligned}$$

where the function φ is of the form $\varphi = (\varphi_1, \ldots, \varphi_d)$ and $\Delta H_j^i = H_j^i - H_{j-1}^i$ for $i = 1, \ldots, d$ and $j = 1, \ldots, t$, with $H_0^i = 0$. We will usually omit the symbol of the inner product. We get

$$\begin{aligned} x_t^{\lambda,\mu} = &-\sum_{j=1}^{t} \Delta H_j S_{j-1} - \sum_{j=1}^{t}\sum_{i=1}^{d} \lambda_i (\Delta H_j^i)^+ S_{j-1}^i - \sum_{j=1}^{t}\sum_{i=1}^{d} \mu_i (\Delta H_j^i)^- S_{j-1}^i \\ &+ H_t S_t - \sum_{i=1}^{d} \lambda_i (-H_t^i)^+ S_t - \sum_{i=1}^{d} \mu_i (-H_t^i)^- S_t \\ = &\,(H\cdot S)_t - \sum_{j=1}^{t} \lambda (\Delta H_j)^+ S_{j-1} - \sum_{j=1}^{t} \mu (\Delta H_j)^- S_{j-1} \\ &- \lambda (H_t)^- S_t - \mu (H_t)^+ S_t. \end{aligned}$$

Notice that if $H \in \mathcal{P}_+$ then

$$x_t^{\lambda,\mu} = (H\cdot S)_t - \sum_{j=1}^{t} \lambda (\Delta H_j)^+ S_{j-1} - \sum_{j=1}^{t} \mu (\Delta H_j)^- S_{j-1} - \mu H_t S_t.$$

We use the notation $L_+^0(\mathcal{F}_t)$ for the set of non-negative, $\mathcal{F}_t$-measurable random variables and write L_+^0 for $t = T$. Let $\mathcal{R}_T^+(\lambda,\mu) := \{x_T^{\lambda,\mu}(H) \mid H \in \mathcal{P}_+\}$ and define the set of hedgeable claims to be

$$\mathcal{A}_T^+(\lambda,\mu) := \mathcal{R}_T^+(\lambda,\mu) - L_+^0.$$

Let $\overline{\mathcal{A}_T^+}(\lambda,\mu)$ be the closure of $\mathcal{A}_T^+(\lambda,\mu)$ in probability.

Lemma 1.1. *$\mathcal{A}_T^+(\lambda,\mu)$ is a convex cone.*

Proof. Notice that the function φ^i is convex for all $i = 1, \ldots, d$. ∎

Definition 1.2. We say that there is *no arbitrage* on the market if

$$(\mathrm{NA}_+^{\lambda,\mu}) \qquad \mathcal{R}_T^+(\lambda,\mu) \cap L_+^0 = \{0\}.$$

Notice that the condition $(\mathrm{NA}_+^{\lambda,\mu})$ is equivalent to $\mathcal{A}_T^+(\lambda,\mu) \cap L_+^0 = \{0\}$.

LEMMA 1.3. *Let* $0 < \lambda_1 < \lambda_2$ *and* $0 < \mu_1 < \mu_2$. *Then* $(\mathrm{NA}_+^{\lambda_1,\mu}) \Rightarrow (\mathrm{NA}_+^{\lambda_2,\mu})$ *and* $(\mathrm{NA}_+^{\lambda,\mu_1}) \Rightarrow (\mathrm{NA}_+^{\lambda,\mu_2})$.

Proof. Notice that $x_T^{\lambda_1,\mu} \geq x_T^{\lambda_2,\mu}$ and $x_T^{\lambda,\mu_1} \geq x_T^{\lambda,\mu_2}$. ∎

LEMMA 1.4. *Under* $(\mathrm{NA}_+^{\lambda,\mu})$, *i.e.* $\mathcal{A}_T^+(\lambda,\mu) \cap L_+^0 = \{0\}$, *there is no arbitrage on the market with any time horizon* $1 \leq t \leq T$, *i.e.* $\mathcal{A}_t^+(\lambda,\mu) \cap L_+^0(\mathcal{F}_t) = \{0\}$.

Proof. Notice that if H is an arbitrage strategy in the model with time horizon t (so at time t we liquidate all positions in stock) then there is also an arbitrage strategy in a model with larger time horizon, in particular with time horizon T. It is enough to take the same strategy H up to time t and later 0. ∎

Now similarly to [GRS] we introduce the definition of a consistent price system and some related notions.

DEFINITION 1.5 ((λ,μ)-consistent price system). We say that a pair $(\tilde{S},\tilde{P})$ is a (λ,μ)-*consistent price system* ((λ,μ)-CPS) if $\tilde{P}$ is a probability measure equivalent to $\mathbb{P}$ and $\tilde{S} = (\tilde{S}_t)_{t=0}^T$ is a d-dimensional process, adapted to the filtration $\mathbb{F}$, which is a $\tilde{P}$-martingale and satisfies

$$1-\mu_i \leq \frac{\tilde{S}_t^i}{S_t^i} \leq 1+\lambda_i, \quad \mathbb{P}\text{-a.s.},$$

for all $i = 1,\dots,d$ and $t = 0,\dots,T$. For $\lambda = \mu$ we write briefly λ-CPS.

DEFINITION 1.6 (right-sided λ-consistent price system). We say that a pair $(\tilde{S},\tilde{P})$ is a *right-sided* λ-*consistent price system* (λ-CPS$^+$) if $\tilde{P}$ is a probability measure equivalent to $\mathbb{P}$ and $\tilde{S} = (\tilde{S}_t)_{t=0}^T$ is a d-dimensional, strictly positive process, adapted to $\mathbb{F}$, which is a $\tilde{P}$-martingale and

$$\frac{\tilde{S}_t^i}{S_t^i} \leq 1+\lambda_i, \quad \mathbb{P}\text{-a.s.},$$

for all $i = 1,\dots,d$ and $t = 0,\dots,T$.

When the process $\tilde{S}$ above is only a supermartingale or submartingale we can formulate similar definitions.

DEFINITION 1.7 ((λ,μ)-supCPS, (λ,μ)-subCPS). We say that a pair $(\tilde{S},\tilde{P})$ is a (λ,μ)-*supermartingale* (resp. *submartingale*) *consistent price system* if $\tilde{P}$ is a probability measure equivalent to $\mathbb{P}$ and $\tilde{S} = (\tilde{S}_t)_{t=0}^T$ is a d-dimensional process, adapted to $\mathbb{F}$, which is a $\tilde{P}$-supermartingale (resp. submartingale) and

$$1-\mu_i \leq \frac{\tilde{S}_t^i}{S_t^i} \leq 1+\lambda_i, \quad \mathbb{P}\text{-a.s.},$$

for all $i = 1, \dots, d$ and $t = 0, \dots, T$. When $\lambda = \mu$ we write briefly λ-supCPS (resp. λ-subCPS).

Definition 1.8 (λ-supCPS$^+$, λ-subCPS$^+$). We say that a pair $(\tilde{S}, \tilde{P})$ is a *right-sided λ-supermartingale* (resp. *submartingale*) *consistent price system* if $\tilde{P}$ is a probability measure equivalent to $\mathbb{P}$ and $\tilde{S} = (\tilde{S}_t)_{t=0}^T$ is a d-dimensional, strictly positive process, adapted to $\mathbb{F}$, which is a $\tilde{P}$-supermartingale (resp. submartingale) and

$$\frac{\tilde{S}_t^i}{S_t^i} \le 1 + \lambda_i, \quad \mathbb{P}\text{-a.s.},$$

for all $i = 1, \dots, d$ and $t = 0, \dots, T$.

Now we give the definition of robust no-arbitrage, similar to that introduced in [S].

Definition 1.9. We say that there is *robust no-arbitrage* on the market if

(rNA$_+$)
$$\exists\, \varepsilon > 0\colon (\varepsilon < \lambda,\ \mathcal{A}_T^+(\varepsilon, \mu) \cap L_+^0 = \{0\}) \text{ or } (\varepsilon < \mu,\ \mathcal{A}_T^+(\lambda, \varepsilon) \cap L_+^0 = \{0\}).$$

2. Main results

Theorem 2.1. *The implications* (a)$\Rightarrow$(b)$\Rightarrow$(c)$\Rightarrow$(d)$\Rightarrow$(e) *are true where:*

(a) $\mathcal{A}_T^+(\lambda, \mu) \cap L_+^0 = \{0\}$*;*
(b) $\mathcal{A}_T^+(\lambda, \mu) \cap L_+^0 = \{0\}$ *and* $\mathcal{A}_T^+(\varepsilon, \mu) = \overline{\mathcal{A}_T^+}(\varepsilon, \mu)$ *for any* $\varepsilon > \lambda$*;*
(c) $\overline{\mathcal{A}_T^+}(\varepsilon, \mu) \cap L_+^0 = \{0\}$ *for any* $\varepsilon > \lambda$*;*
(d) *for any* $\varepsilon > \lambda$ *there exists an* ε*-CPS*$^+$ $(\tilde{S}, \mathbb{Q})$ *with* $d\mathbb{Q}/d\mathbb{P} \in L^\infty$*;*
(e) *for any* $\varepsilon > \lambda$ *there exists,* ε*-supCPS*$^+$ $(\tilde{S}, \mathbb{Q})$ *with* $d\mathbb{Q}/d\mathbb{P} \in L^\infty$*.*

Remark 2.2. Notice that the conditions (d), (e) of Theorem 2.1 mean that there exists an ε-CPS$^+$ (resp. ε-supCPS$^+$) in the model with transaction cost vectors $\varepsilon > \lambda$ for buying and μ for selling.

In the proof of Theorem 2.1 will use the following lemmas whose proofs can be found e.g. in [KS].

Lemma 2.3. *Let X_n be a sequence of random vectors taking values in $\mathbb{R}^d$ such that for almost all $\omega \in \Omega$ we have* $\liminf \|X_n(\omega)\|_d < \infty$*. Then there is a sequence of random vectors Y_n taking values in $\mathbb{R}^d$ satisfying the following conditions:*

(1) *Y_n converges pointwise almost surely to a random vector Y taking values in $\mathbb{R}^d$,*
(2) *$Y_n(\omega)$ is a convergent subsequence of $X_n(\omega)$ for almost all $\omega \in \Omega$.*

Proof. See e.g. [KS, Lemma 2] or [KRS, Lemma 1]. ∎

REMARK 2.4. The above claim can be formulated as follows: there exists an increasing sequence of integer-valued random variables σ_k such that X_{σ_k} converges a.s.

LEMMA 2.5 (Kreps–Yan). *Let $K \supseteq -L^1_+$ be a closed convex cone in L^1 such that $K \cap L^1_+ = \{0\}$. Then there is a probability $\widetilde{P} \sim P$ with $d\widetilde{P}/dP \in L^\infty$ such that $E_{\tilde{P}}\xi \le 0$ for all $\xi \in K$.*

Proof. See e.g. [KS, Lemma 3] or [KSaf, Theorem 2.1.4]. ∎

Proof of Theorem 2.1. (a)⇒(b). Let

$$\begin{aligned} x^{\lambda,\mu}_{t,t+k}(H,\tilde{H}) = {} & \sum_{j=t}^{t+k} H_j \Delta S_j - \sum_{j=t}^{t+k} \lambda(\Delta H_j)^+ S_{j-1} - \sum_{j=t}^{t+k} \mu(\Delta H_j)^- S_{j-1} \\ & - \mu H_{t+k} S_{t+k} \end{aligned}$$

where $1 \le t \le t+k \le T$, H is predictable and $H \ge 0$, $\tilde{H} \in L^0(\mathbb{R}^d_+, \mathcal{F}_{t-1})$ and $\Delta H_t = H_t - \tilde{H}$. Define

$$\mathcal{R}^+_{t,t+k}(\tilde{H},\lambda) := \{x^{\lambda,\mu}_{t,t+k}(H,\tilde{H}) \mid H \text{ is predictable and } H \ge 0\}$$

and let $\mathcal{A}^+_{t,t+k}(\tilde{H},\lambda) := \mathcal{R}^+_{t,t+k}(\tilde{H},\lambda) - L^0_+(\mathcal{F}_{t+k})$. We will show that the set $\mathcal{A}^+_{t,t+k}(\tilde{H},\varepsilon)$ is closed for any $\varepsilon > \lambda$, any $\tilde{H} \in L^0(\mathbb{R}^d_+, \mathcal{F}_{t-1})$ and all t, k such that $1 \le t \le t+k \le T$. We prove this by induction on k.

Let $k = 0$. Fix t, $\tilde{H} \in L^0(\mathbb{R}^d_+, \mathcal{F}_{t-1})$ and a vector $\varepsilon > \lambda$, i.e. $\varepsilon_i > \lambda_i$ for all $i = 1, \dots, d$. By Lemmas 1.3 and 1.4 we have $\mathcal{A}^+_t(\varepsilon,\mu) \cap L^0_+(\mathcal{F}_t) = \{0\}$. Suppose that $v^n_{t,t} \to \zeta$ in probability where $v^n_{t,t} \in \mathcal{A}^+_{t,t}(\tilde{H},\varepsilon)$. The sequence $v^n_{t,t}$ contains a subsequence convergent to ζ a.s. (see e.g. [JP, Theorem 17.3]). Thus, possibly restricting to this subsequence we can assume that $v^n_{t,t} \to \zeta$, $\mathbb{P}$-a.s. Assume that

$$v^n_{t,t} = H^n_t \Delta S_t - \varepsilon(\Delta H^n_t)^+ S_{t-1} - \mu(\Delta H^n_t)^- S_{t-1} - \mu H^n_t S_t - r_n$$

where $\Delta H^n_t = H^n_t - \tilde{H}$ and $H^n_t \in L^0(\mathbb{R}^d_+, \mathcal{F}_{t-1})$, $r_n \in L^0_+(\mathcal{F}_t)$.

Consider first the situation on the set $\Omega_1 := \{\liminf \|H^n_t\| < \infty\} \in \mathcal{F}_{t-1}$. By Lemma 2.3 there exists an increasing sequence of integer-valued, $\mathcal{F}_{t-1}$-measurable stopping times τ_n such that $H^{\tau_n}_t$ is a.s. convergent on Ω_1, and for almost all $\omega \in \Omega_1$ the sequence $H^{\tau_n(\omega)}_t(\omega)$ is a convergent subsequence of $H^n_t(\omega)$. Notice that $H^{\tau_n}_t \in L^0(\mathbb{R}^d_+, \mathcal{F}_{t-1})$ and $r_{\tau_n} \in L^0_+(\mathcal{F}_t)$. Furthermore r_{τ_n} is convergent a.s. on Ω_1. Let $\tilde{H}_t := \lim_{n\to\infty} H^{\tau_n}_t$ and $\tilde{r} := \lim_{n\to\infty} r_{\tau_n}$. Then

$$\begin{aligned} \zeta &= \lim_{n\to\infty} \big(H^n_t \Delta S_t - \varepsilon(\Delta H^n_t)^+ S_{t-1} - \mu(\Delta H^n_t)^- S_{t-1} - \mu H^n_t S_t - r_n\big) \\ &= \lim_{n\to\infty} \big(H^{\tau_n}_t \Delta S_t - \varepsilon(\Delta H^{\tau_n}_t)^+ S_{t-1} - \mu(\Delta H^{\tau_n}_t)^- S_{t-1} - \mu H^{\tau_n}_t S_t - r_{\tau_n}\big) \end{aligned}$$

where the last limit is equal to

$$\tilde{H}_t\Delta S_t - \varepsilon(\tilde{H}_t - \tilde{H})^+ S_{t-1} - \mu(\tilde{H}_t - \tilde{H})^- S_{t-1} - \mu\tilde{H}_t S_t - \tilde{r} \in \mathcal{A}^+_{t,t}(\tilde{H},\varepsilon).$$

Consider now the set $\Omega_2 := \{\liminf \|H^n_t\| = \infty\} \in \mathcal{F}_{t-1}$. Suppose that $\mathbb{P}(\Omega_2) > 0$. Define $G^n_t := H^n_t/\|H^n_t\|$, $h_n := r_n/\|H^n_t\|$ and notice that $G^n_t \in L^0(\mathbb{R}^d_+, \mathcal{F}_{t-1})$. We have

$$G^n_t\Delta S_t - \varepsilon\bigg(G^n_t - \frac{\tilde{H}}{\|H^n_t\|}\bigg)^+ S_{t-1} - \mu\bigg(G^n_t - \frac{\tilde{H}}{\|H^n_t\|}\bigg)^- S_{t-1} - \mu G^n_t S_t - h_n \to 0.$$

Just as on Ω_1, by Lemma 2.3 there exists an increasing sequence of integer-valued, $\mathcal{F}_{t-1}$-measurable stopping times σ_n such that $G^{\sigma_n}_t$ is convergent a.s. on Ω_2 and for almost all $\omega \in \Omega_2$ the sequence $G^{\sigma_n(\omega)}_t(\omega)$ is a convergent subsequence of $G^n_t(\omega)$. Let $\tilde{G}_t := \lim_{n\to\infty} G^{\sigma_n}_t$ and $\tilde{h} := \lim_{n\to\infty} h_{\sigma_n}$. Taking into account the absence of shortselling we get

$$\tilde{G}_t\Delta S_t - \varepsilon(\tilde{G}_t)^+ S_{t-1} - \mu(\tilde{G}_t)^- S_{t-1} - \mu\tilde{G}_t S_t = \tilde{G}_t\Delta S_t - \varepsilon\tilde{G}_t S_{t-1} - \mu\tilde{G}_t S_t = \tilde{h}$$

where $\tilde{h} \in L^0_+(\mathcal{F}_t)$. From the absence of arbitrage, $\tilde{G}_t\Delta S_t - \varepsilon\tilde{G}_t S_{t-1} - \mu\tilde{G}_t S_t = 0$ on Ω_2. Notice that

$$\tilde{G}_t\Delta S_t - \lambda\tilde{G}_t S_{t-1} - \mu\tilde{G}_t S_t \geq \tilde{G}_t\Delta S_t - \varepsilon\tilde{G}_t S_{t-1} - \mu\tilde{G}_t S_t = 0.$$

Using once again the fact that $\mathcal{A}^+_t(\lambda,\mu) \cap L^0_+(\mathcal{F}_t) = \{0\}$ we can replace the inequality by an equality. Hence $\sum_{i=1}^d (\lambda_i - \varepsilon_i)\tilde{G}^i_t S^i_{t-1} = 0$. Because S_{t-1} is strictly positive we obtain $\tilde{G}_t = 0$, $\mathbb{P}$-a.s. on Ω_2, which contradicts the fact that $\|\tilde{G}_t\| = 1$. It follows that $\mathbb{P}(\Omega_2) = 0$.

Assume now that the claim is true for $k-1$ where $k \geq 1$. We show that it is true for k. Fix t such that $1 \leq t \leq t+k \leq T$, $\tilde{H} \in L^0(\mathbb{R}^d_+, \mathcal{F}_{t-1})$ and a vector $\varepsilon > \lambda$. By Lemmas 1.3 and 1.4 we have $\mathcal{A}^+_{t+k}(\varepsilon,\mu) \cap L^0_+(\mathcal{F}_{t+k}) = \{0\}$. Let $v^n_{t,t+k} \to \zeta$ in probability where $v^n_{t,t+k} \in \mathcal{A}^+_{t,t+k}(\tilde{H},\varepsilon)$. The sequence $v^n_{t,t+k}$ contains a subsequence convergent to ζ a.s. (see e.g. [JP, Theorem 17.3]). Thus, possibly restricting to this subsequence we can assume that $v^n_{t,t+k} \to \zeta$, $\mathbb{P}$-a.s. Assume that

$$v^n_{t,t+k} = \sum_{j=t}^{t+k} H^n_j\Delta S_j - \sum_{j=t}^{t+k} \varepsilon(\Delta H^n_j)^+ S_{j-1} - \sum_{j=t}^{t+k} \mu(\Delta H^n_j)^- S_{j-1} - \mu H^n_{t+k} S_{t+k} - r_n$$

where $\Delta H^n_t = H^n_t - \tilde{H}$, $H^n_j \in L^0(\mathbb{R}^d_+, \mathcal{F}_{j-1})$, $r_n \in L^0_+(\mathcal{F}_{t+k})$. The argument will be similar to the case $k = 0$.

Consider first the situation on $\Omega_1 := \{\liminf \|H^n_t\| < \infty\} \in \mathcal{F}_{t-1}$. By Lemma 2.3 there exists an increasing sequence of integer-valued, $\mathcal{F}_{t-1}$-measurable stopping times τ_n such that $H^{\tau_n}_t$ is convergent a.s. on Ω_1 and for almost all $\omega \in \Omega_1$ the sequence $H^{\tau_n(\omega)}_t(\omega)$ is a convergent subsequence of $H^n_t(\omega)$. Notice that $H^{\tau_n}_t \in L^0(\mathbb{R}^d_+, \mathcal{F}_{t-1})$ and $r_{\tau_n} \in L^0_+(\mathcal{F}_{t+k})$. Define

$\tilde H_t := \lim_{n\to\infty} H_t^{\tau_n}$. Then ζ is equal to

$$\lim_{n\to\infty}\Big(\sum_{j=t}^{t+k} H_j^n\Delta S_j-\sum_{j=t}^{t+k}\varepsilon(\Delta H_j^n)^+S_{j-1}-\sum_{j=t}^{t+k}\mu(\Delta H_j^n)^-S_{j-1}-\mu H_{t+k}^n S_{t+k}-r_n\Big)$$

$$= \lim_{n\to\infty}\Big(H_t^{\tau_n}\Delta S_t-\varepsilon(H_t^{\tau_n}-\tilde H)^+S_{t-1}-\mu(H_t^{\tau_n}-\tilde H)^-S_{t-1}+\sum_{j=t+1}^{t+k} H_j^{\tau_n}\Delta S_j$$
$$-\sum_{j=t+1}^{t+k}\varepsilon(\Delta H_j^{\tau_n})^+S_{j-1}-\sum_{j=t+1}^{t+k}\mu(\Delta H_j^{\tau_n})^-S_{j-1}-\mu H_{t+k}^{\tau_n}S_{t+k}-r_{\tau_n}\Big)$$
$$= \tilde H_t\Delta S_t-\varepsilon(\tilde H_t-\tilde H)^+S_{t-1}-\mu(\tilde H_t-\tilde H)^-S_{t-1}+\lim_{n\to\infty}\Big(\sum_{j=t+1}^{t+k} H_j^{\tau_n}\Delta S_j$$
$$-\sum_{j=t+1}^{t+k}\varepsilon(\Delta H_j^{\tau_n})^+S_{j-1}-\sum_{j=t+1}^{t+k}\mu(\Delta H_j^{\tau_n})^-S_{j-1}-\mu H_{t+k}^{\tau_n}S_{t+k}-r_{\tau_n}\Big)$$

and by continuity this equals

$$\tilde H_t\Delta S_t-\varepsilon(\tilde H_t-\tilde H)^+S_{t-1}-\mu(\tilde H_t-\tilde H)^-S_{t-1}$$
$$+\lim_{n\to\infty}\Big(\sum_{j=t+1}^{t+k} H_j^{\tau_n}\Delta S_j-\varepsilon(H_{t+1}^{\tau_n}-\tilde H_t)^+S_t-\mu(H_{t+1}^{\tau_n}-\tilde H_t)^-S_t$$
$$-\sum_{j=t+2}^{t+k}\varepsilon(\Delta H_j^{\tau_n})^+S_{j-1}-\sum_{j=t+2}^{t+k}\mu(\Delta H_j^{\tau_n})^-S_{j-1}-\mu H_{t+k}^{\tau_n}S_{t+k}-r_{\tau_n}\Big).$$

Notice that $\tilde H_t \in L^0(\mathbb{R}_+^d,\mathcal{F}_{t-1})$ and by the induction hypothesis the above limit belongs to $\mathcal{A}_{t+1,t+k}^+(\tilde H_t,\varepsilon)$. Consequently, $\zeta\in\mathcal{A}_{t,t+k}^+(\tilde H,\varepsilon)$.

As previously, consider now the case $\Omega_2 := \{\liminf \|H_t^n\| = \infty\}\in\mathcal{F}_{t-1}$. Suppose that $\mathbb{P}(\Omega_2)>0$. For $j=t,\dots,t+k$ define $G_j^n := H_j^n/\|H_t^n\|$ and $h_n := r_n/\|H_t^n\|$. Notice that $G_j^n\in L^0(\mathbb{R}_+^d,\mathcal{F}_{j-1})$ and

$$\tilde v_{t,t+k}^n := \frac{v_{t,t+k}^n}{\|H_t^n\|} = \sum_{j=t}^{t+k} G_j^n\Delta S_j-\sum_{j=t+1}^{t+k}\varepsilon(\Delta G_j^n)^+S_{j-1}-\sum_{j=t+1}^{t+k}\mu(\Delta G_j^n)^-S_{j-1}$$
$$-\varepsilon\Big(G_t^n-\frac{\tilde H}{\|H_t^n\|}\Big)^+S_{t-1}-\mu\Big(G_t^n-\frac{\tilde H}{\|H_t^n\|}\Big)^-S_{t-1}-\mu G_{t+k}^nS_{t+k}-h_n\to 0.$$

Just as on Ω_1, by Lemma 2.3 there exists an increasing sequence of integer-valued, $\mathcal{F}_{t-1}$-measurable stopping times σ_n such that $G_t^{\sigma_n}$ is convergent a.s. on Ω_2, and for almost all $\omega\in\Omega_2$ the sequence $G_t^{\sigma_n(\omega)}(\omega)$ is a convergent subsequence of $G_t^n(\omega)$. Define $\tilde{G}_t := \lim_{n\to\infty} G_t^{\sigma_n}$ and notice that the sequence

$$\sum_{j=t+1}^{t+k} G_j^{\sigma_n}\Delta S_j - \sum_{j=t+1}^{t+k} \varepsilon(\Delta G_j^{\sigma_n})^+ S_{j-1} - \sum_{j=t+1}^{t+k} \mu(\Delta G_j^{\sigma_n})^- S_{j-1} - \mu G_{t+k}^{\sigma_n} S_{t+k} - h_{\sigma_n}$$

is convergent and its limit equals

$$\lim_{n\to\infty}\Big(\sum_{j=t+1}^{t+k} G_j^{\sigma_n}\Delta S_j - \varepsilon(G_{t+1}^{\sigma_n} - \tilde{G}_t)^+ S_t - \mu(G_{t+1}^{\sigma_n} - \tilde{G}_t)^- S_t$$
$$- \sum_{j=t+2}^{t+k} \varepsilon(\Delta G_j^{\sigma_n})^+ S_{j-1} - \sum_{j=t+2}^{t+k} \mu(\Delta G_j^{\sigma_n})^- S_{j-1} - \mu G_{t+k}^{\sigma_n} S_{t+k} - h_{\sigma_n}\Big).$$

By the induction hypothesis the above limit belongs to $\mathcal{A}^+_{t+1,t+k}(\tilde{G}_t,\varepsilon)$ and finally $\lim_{n\to\infty}\tilde{v}^n_{t,t+k} \in \mathcal{A}^+_{t,t+k}(0,\varepsilon)$. Moreover $\lim_{n\to\infty}\tilde{v}^n_{t,t+k} = \lim_{n\to\infty}\tilde{v}^{\sigma_n}_{t,t+k} = 0$. We can assume that this limit is of the form

$$\sum_{j=t}^{t+k}\tilde{G}_j\Delta S_j - \sum_{j=t}^{t+k}\varepsilon(\Delta\tilde{G}_j)^+ S_{j-1} - \sum_{j=t}^{t+k}\mu(\Delta\tilde{G}_j)^- S_{j-1} - \mu\tilde{G}_{t+k}S_{t+k} - \tilde{h} = 0$$

where $\Delta\tilde{G}_t = \tilde{G}_t$. We get the equality

$$\sum_{j=t}^{t+k}\tilde{G}_j\Delta S_j - \sum_{j=t}^{t+k}\varepsilon(\Delta\tilde{G}_j)^+ S_{j-1} - \sum_{j=t}^{t+k}\mu(\Delta\tilde{G}_j)^- S_{j-1} - \mu\tilde{G}_{t+k}S_{t+k} = \tilde{h}$$

where $\tilde{h} \in L^0_+(\mathcal{F}_{t+k})$. From the absence of arbitrage we have

$$\sum_{j=t}^{t+k}\tilde{G}_j\Delta S_j - \sum_{j=t}^{t+k}\varepsilon(\Delta\tilde{G}_j)^+ S_{j-1} - \sum_{j=t}^{t+k}\mu(\Delta\tilde{G}_j)^- S_{j-1} - \mu\tilde{G}_{t+k}S_{t+k} = 0$$

on Ω_2. Notice that

$$\sum_{j=t}^{t+k}\tilde{G}_j\Delta S_j - \sum_{j=t}^{t+k}\lambda(\Delta\tilde{G}_j)^+ S_{j-1} - \sum_{j=t}^{t+k}\mu(\Delta\tilde{G}_j)^- S_{j-1} - \mu\tilde{G}_{t+k}S_{t+k}$$
$$\geq \sum_{j=t}^{t+k}\tilde{G}_j\Delta S_j - \sum_{j=t}^{t+k}\varepsilon(\Delta\tilde{G}_j)^+ S_{j-1} - \sum_{j=t}^{t+k}\mu(\Delta\tilde{G}_j)^- S_{j-1} - \mu\tilde{G}_{t+k}S_{t+k} = 0.$$

Using once again the fact that $\mathcal{A}^+_{t+k}(\lambda,\mu) \cap L^0_+(\mathcal{F}_{t+k}) = \{0\}$ we can replace the inequality by an equality. Hence $\sum_{j=t}^{t+k}\sum_{i=1}^{d}(\lambda_i - \varepsilon_i)(\Delta\tilde{G}^i_j)^+ S^i_{j-1} = 0$. Because S_{j-1} has strictly positive components we obtain $(\Delta\tilde{G}^i_j)^+ = 0$, $\mathbb{P}$-a.s., on Ω_2 for all $j = t,\ldots,t+k$ and $i = 1,\ldots,d$. Hence in particular $\tilde{G}_t = 0$, which contradicts the fact that $\|\tilde{G}_t\| = 1$. It follows that $\mathbb{P}(\Omega_2) = 0$.

As $\mathcal{A}^+_{1,t}(0,\varepsilon) = \mathcal{A}^+_t(\varepsilon,\mu)$, this ends the proof of closedness.

(b)⇒(c). Notice that $\mathcal{A}_T^+(\varepsilon,\mu)\cap L^0_+=\{0\}$ for any $\varepsilon>\lambda$ by Lemma 1.3.

(c)⇒(d). Fix any $\varepsilon>\lambda$. Notice that for any random variable η there exists a probability measure $P'\sim\mathbb{P}$ such that $dP'/d\mathbb{P}\in L^\infty$ and $\eta\in L^1(P')$. Property (c) is invariant under equivalent change of probability. This allows us to assume without loss of generality that all S_t are integrable. Define $\Lambda^\varepsilon:=\overline{\mathcal{A}_T^+}(\varepsilon,\mu)\cap L^1$, which is a closed, convex cone in L^1. Since $\Lambda^\varepsilon\cap L^1_+=\{0\}$, by Lemma 2.5 there exists a probability measure $\mathbb{Q}\sim\mathbb{P}$ such that $d\mathbb{Q}/d\mathbb{P}\in L^\infty$ and $E_{\mathbb{Q}}\xi\le 0$ for any $\xi\in\Lambda^\varepsilon$, in particular for

$$\xi_i=H^i_{t+1}(S^i_T-S^i_t)-\varepsilon_iH^i_{t+1}S^i_t-\mu_iH^i_{t+1}S^i_T,\qquad t=0,\dots,T-1,$$

where $H_{t+1}=(0,\dots,\mathbb{1}_A,\dots,0)$, $\mathbb{P}$-a.s., $A\in\mathcal{F}_t$ and the value $\mathbb{1}_A$ is in the ith position. This means that at time t if the event A holds we buy the ith share at the price S^i_t and liquidate the portfolio at time T. Hence

$$E_{\mathbb{Q}}[(S^i_T-S^i_t-\varepsilon_iS^i_t-\mu_iS^i_T)\mathbb{1}_A]\le 0.$$

Since $(1-\mu_i)E_{\mathbb{Q}}(S^i_T\mathbb{1}_A)\le(1+\varepsilon_i)E_{\mathbb{Q}}(S^i_t\mathbb{1}_A)$ for $i=1,\dots,d$ and any $A\in\mathcal{F}_t$, we have

$$(1-\mu_i)E_{\mathbb{Q}}(S^i_T\,|\,\mathcal{F}_t)\le(1+\varepsilon_i)E_{\mathbb{Q}}(S^i_t\,|\,\mathcal{F}_t)=(1+\varepsilon_i)S^i_t\quad\text{ for } t=0,\dots,T-1.$$

Define $\tilde S=(\tilde S_t)_{t=0}^T$ by $\tilde S_t:=(1-\mu)E_{\mathbb{Q}}(S_T\,|\,\mathcal{F}_t)$ and notice that $(\tilde S,\mathbb{Q})$ is a right-sided ε-consistent price system (ε-CPS$^+$).

(d)⇒(e). Trivial. ∎

3. Further theorems and examples

Corollary 3.1. *The implications* (a)⇒(b)⇒(c)⇒(d)⇒(e) *are true where:*

(a) $\mathcal{A}_T^+(\lambda,\mu)\cap L^0_+=\{0\}$;
(b) $\mathcal{A}_T^+(\lambda,\mu)\cap L^0_+=\{0\}$ *and* $\mathcal{A}_T^+(\lambda,\varepsilon)=\overline{\mathcal{A}_T^+}(\lambda,\varepsilon)$ *for any* $\varepsilon>\mu$;
(c) $\overline{\mathcal{A}_T^+}(\lambda,\varepsilon)\cap L^0_+=\{0\}$ *for any* $\varepsilon>\mu$;
(d) *for any* $\varepsilon>\mu$ *there exists a* λ-*CPS*$^+$ $(\tilde S,\mathbb{Q})$ *with* $d\mathbb{Q}/d\mathbb{P}\in L^\infty$;
(e) *for any* $\varepsilon>\mu$ *there exists a* λ-*supCPS*$^+$ $(\tilde S,\mathbb{Q})$ *with* $d\mathbb{Q}/d\mathbb{P}\in L^\infty$.

Proof. Notice that in our model when we buy some shares we must sell them up to time T, so using analogous arguments to the proof of Theorem 2.1 we get the theorem for transaction costs $\varepsilon>\mu$. ∎

Remark 3.2. Notice that the conditions (d), (e) of Corollary 3.1 mean that there exists a λ-CPS$^+$ (λ-supCPS$^+$) in the model with transaction cost vectors λ for buying and $\varepsilon\in(\mu,1)$ for selling.

By Theorem 2.1 and Corollary 3.1 we obtain a straightforward corollary.

Corollary 3.3. (rNA$_+$) ⇒ $\exists\lambda$-*CPS*$^+$.

The theorem below gives a sufficient condition for the absence of arbitrage.

THEOREM 3.4. *Let* $(\tilde{S}, \mathbb{Q})$ *be a* (λ,μ)*-supCPS and define* $\tilde{R}_T^+ := \{(H\cdot\tilde{S})_T \mid H \in \mathcal{P}_+\}$. *Then* $\tilde{R}_T^+ \cap L_+^0 = \{0\}$ *and we have the absence of arbitrage in our model, i.e.* $\mathcal{A}_T^+(\lambda,\mu)\cap L_+^0 = \{0\}$.

LEMMA 3.5. *Let* $\tilde{R}_T^{+,b} := \{(H\cdot\tilde{S})_T \mid H \in \mathcal{P}_+$ *and* H *is bounded*$\}$. *The condition* $\tilde{R}_T^+ \cap L_+^0 = \{0\}$ *is equivalent to* $\tilde{R}_T^{+,b}\cap L_+^0 = \{0\}$.

Proof. Notice that the condition $\tilde{R}_T^+ \cap L_+^0 = \{0\}$ is equivalent to the absence of arbitrage for any one-step model, i.e. $\{\eta\Delta\tilde{S}_t \mid \eta \in L_+^0(\mathcal{F}_{t-1})\}\cap L_0^+ = \{0\}$ for any $t = 1,\dots,T$ (see e.g. [KSaf, Chapter 2.1.1]). Hence assume that there exists $H_t \in L_+^0(\mathcal{F}_{t-1})$ satisfying

$$(\mathrm{A}_+) \qquad H_t\Delta\tilde{S}_t \ge 0,\ \mathbb{P}\text{-a.s.}, \quad \text{and} \quad \mathbb{P}(H_t\Delta\tilde{S}_t > 0) > 0.$$

It is enough to show that there exists $\tilde{H}_t \in L_+^0(\mathcal{F}_{t-1})$ which is bounded and satisfies (A_+). One can take

$$\tilde{H}_t := \begin{cases} H_t/\|H_t\|, & H_t \neq 0,\\ 0, & H_t = 0.\end{cases}$$

It is also possible to use the arguments from [KSaf, Chapter 2.1.1]. Define $H_t^n := H_t\mathbb{1}_{\{\|H_t\|\le n\}}$. Then there exists a sufficiently large $n\in\mathbb{N}$ such that H_t^n satisfies (A_+). ∎

Proof of Theorem 3.4. By Lemma 3.5 it is enough to prove that $\tilde{R}_T^{+,b}\cap L_+^0 = \{0\}$. Let $x = (H\cdot\tilde{S})_T \in \tilde{R}_T^{+,b}\cap L_+^0$. Then $(H\cdot\tilde{S})_T \ge 0$ and in particular H is a bounded strategy. We show that $E_{\mathbb{Q}}(H\cdot\tilde{S})_T \le 0$. Using the assumption that $\tilde{S}$ is a $\mathbb{Q}$-supermartingale and taking into account shortsale restrictions we get $E_{\mathbb{Q}}(H_t\Delta\tilde{S}_t \,|\, \mathcal{F}_{t-1}) = H_tE_{\mathbb{Q}}(\Delta\tilde{S}_t \,|\, \mathcal{F}_{t-1}) \le 0$. Consequently, $E_{\mathbb{Q}}(H\cdot\tilde{S})_T \le 0$. Hence $x = (H\cdot\tilde{S})_T = 0$, $\mathbb{Q}$-a.s., and from the equivalence of measures $x = 0$, $\mathbb{P}$-a.s.

We now show that $\mathcal{A}_T^+(\lambda,\mu)\cap L_+^0 = \{0\}$. Take any $\xi \in \mathcal{A}_T^+(\lambda,\mu)\cap L_+^0$. Then

$$0 \le \xi \le -\sum_{t=1}^T \Delta H_tS_{t-1} + (1-\mu)H_TS_T - \sum_{t=1}^T \lambda(\Delta H_t)^+S_{t-1} - \sum_{t=1}^T \mu(\Delta H_t)^-S_{t-1}.$$

Notice that $-\mu_iS_t^i \le \tilde{S}_t^i - S_t^i \le \lambda_iS_t^i$, $\mathbb{P}$-a.s., for any $t = 0,\dots,T$ and

$i = 1, \dots, d$. Hence

$$\begin{aligned}\xi \le & -\sum_{t=1}^{T} \Delta H_t S_{t-1} + (1-\mu) H_T S_T - \sum_{t=1}^{T} \lambda (\Delta H_t)^+ S_{t-1} - \sum_{t=1}^{T} \mu (\Delta H_t)^- S_{t-1} \\ \le & -\sum_{t=1}^{T} \Delta H_t \tilde{S}_{t-1} + H_T \tilde{S}_T + \sum_{t=1}^{T} \lambda (\Delta H_t)^+ S_{t-1} + \sum_{t=1}^{T} \mu (\Delta H_t)^- S_{t-1} \\ & -\sum_{t=1}^{T} \lambda (\Delta H_t)^+ S_{t-1} - \sum_{t=1}^{T} \mu (\Delta H_t)^- S_{t-1} = (H \cdot \tilde{S})_T.\end{aligned}$$

Due to the condition $\tilde{R}_T^+ \cap L_+^0 = \{0\}$ we get $(H \cdot \tilde{S})_T = 0$, $\mathbb{P}$-a.s., and hence $\xi = 0$, $\mathbb{P}$-a.s. ■

Lemma 3.6. *Assume that the process $(x_t^{\lambda,\mu})_{t=1}^T$ is a $\mathbb{Q}$-supermartingale with respect to a measure $\mathbb{Q} \sim \mathbb{P}$. Then there exists a stochastic process $\tilde{S} = (\tilde{S}_t)_{t=0}^T$ such that $(\tilde{S}, \mathbb{Q})$ is a λ-CPS$^+$. Moreover, there is no arbitrage in the model, i.e. $\mathcal{A}_T^+(\lambda, \mu) \cap L_+^0 = \{0\}$.*

Proof. Let $\tilde{S}_t := (1-\mu) E_{\mathbb{Q}}(S_T \mid \mathcal{F}_t)$ for $t = 0, \dots, T$. We show first that the process $\tilde{S} = (\tilde{S}_t)_{t=0}^T$ is a λ-CPS$^+$. It is enough to take a strategy where at time $t < T$ we buy one share S_t^i and sell it at time T. Then for any $i = 1, \dots, d$ taking into account that $(x_t)_{t=1}^T$ is a $\mathbb{Q}$-supermartingale we have

$$E_{\mathbb{Q}}(x_T \mid \mathcal{F}_t) = E_{\mathbb{Q}}(S_T^i - S_t^i - \lambda_i S_t^i - \mu_i S_T^i \mid \mathcal{F}_t) = (\tilde{S}_t^i - S_t^i) - \lambda_i S_t^i \le 0.$$

Clearly $\tilde{S}$ is a $\mathbb{Q}$-martingale.

Now we show the absence of arbitrage by induction on T. Notice that there exists a λ-CPS$^+$ of the form constructed above. Let $T = 1$ and $\xi \in \mathcal{A}_1^+(\lambda, \mu) \cap L_+^0(\mathcal{F}_1)$. Then

$$0 \le \xi \le x_1 = H_1(S_1 - S_0) - \lambda H_1 S_0 - \mu H_1 S_1 = H_1(1-\mu)S_1 - H_1(1+\lambda)S_0.$$

From the form of λ-CPS$^+$ we have $\tilde{S}_0^i \le (1+\lambda) S_0^i$ and $\tilde{S}_1^i = (1-\mu_i) S_1^i$ for $i = 1, \dots, d$. Hence

$$E_{\mathbb{Q}}(x_1) \le E_{\mathbb{Q}}(H_1 \Delta \tilde{S}_1) \le 0.$$

Finally $\xi = 0$, $\mathbb{Q}$-a.s., and from the equivalence of measures $\xi = 0$, $\mathbb{P}$-a.s.

Now let $T > 1$ and $\mathcal{A}_{T-1}^+(\lambda, \mu) \cap L_+^0(\mathcal{F}_{T-1}) = \{0\}$. We show that $\mathcal{A}_T^+(\lambda, \mu) \cap L_+^0(\mathcal{F}_T) = \{0\}$. Take any $\xi \in \mathcal{A}_T^+(\lambda, \mu) \cap L_+^0(\mathcal{F}_T)$. We have $0 \le \xi \le x_T$ and hence

$$0 \le E_{\mathbb{Q}}(\xi \mid \mathcal{F}_{T-1}) \le E_{\mathbb{Q}}(x_T \mid \mathcal{F}_{T-1}) \le x_{T-1}.$$

Notice that $x_{T-1} \ge 0$, $\mathbb{P}$-a.s. From the absence of arbitrage in the model with time horizon $T-1$ we get $x_{T-1} = 0$, $\mathbb{P}$-a.s., and $E_{\mathbb{Q}}(\xi \mid \mathcal{F}_{T-1}) = 0$, $\mathbb{P}$-a.s. Hence from the equivalence of measures $E_{\mathbb{Q}}(\xi \mid \mathcal{F}_{T-1}) = 0$, $\mathbb{Q}$-a.s., and consequently $E_{\mathbb{Q}} \xi = 0$. Finally $\xi = 0$, $\mathbb{P}$-a.s. ■

Example 3.7. Notice that the existence of a λ-CPS$^+$ is not a sufficient condition for the absence of arbitrage. Consider the following market. Let $T=2$, $d=1$, $\lambda=\mu<1/3$ and $S_0=1$, $S_1=1+\mathbb{1}_A$, $S_2=(1+\lambda)/(1-\lambda)$ where $A\in\mathcal{F}_1$ and $0<\mathbb{P}(A)<1$. Furthermore, assume that $\mathcal{F}_0=\{\emptyset,\Omega\}$, $\mathcal{F}_1=\{\emptyset,A,\Omega\setminus A,\Omega\}$. Notice that there exists a λ-CPS$^+$ in this model. We construct it as in the proof of Theorem 2.1, i.e. define $\tilde{S}_t:=(1-\mu)E_{\mathbb{Q}}(S_2\,|\,\mathcal{F}_t)$ where $\mathbb{Q}$ is a probability measure equivalent to $\mathbb{P}$ and $t\in\{0,1,2\}$. Here $\mathbb{Q}$ can be any probability measure equivalent to $\mathbb{P}$ due to the fact that $(1-\lambda)E_{\mathbb{Q}}(S_2\,|\,\mathcal{F}_1)=(1-\lambda)E_{\mathbb{Q}}(S_2\,|\,\mathcal{F}_0)=1+\lambda$ and the inequalities for λ-CPS$^+$ are satisfied. On the other hand notice that there exists arbitrage in the model. Define a strategy as follows: $\Delta H_1=H_1=1$ and $\Delta H_2=-\mathbb{1}_A$. Then

$$x_2^{\lambda,\mu}=-1-\lambda+(2-2\lambda)\mathbb{1}_A+\left(\frac{1+\lambda}{1-\lambda}-\lambda\frac{1+\lambda}{1-\lambda}\right)\mathbb{1}_{\Omega\setminus A}=(1-3\lambda)\mathbb{1}_A.$$

Finally $\mathcal{A}_2^+(\lambda,\mu)\cap L_+^0(\mathcal{F}_2)\neq\{0\}$ despite the existence of a λ-CPS$^+$.

Remark 3.8. Actually due to Theorem 3.4 and the above example the existence of a λ-CPS$^+$ does not imply the existence of a (λ,μ)-supCPS.

Example 3.9. Let $d=1, T=1, \lambda=\mu$ and $S_0=1$, $S_1=(1+\lambda)/(1-\lambda)+\mathbb{1}_A$ where $A\in\mathcal{F}_1$ and $0<\mathbb{P}(A)<1$. Let $\mathcal{F}_0=\{\emptyset,\Omega\}$. Then a strategy in the model is of the form $H_1=a$, $\mathbb{P}$-a.s., where $a\in\mathbb{R}_+$ by shortsale restrictions. Notice that for any transaction costs for buying shares $\varepsilon>\lambda$ we have the absence of arbitrage. Indeed,

$$\begin{aligned}x_1^{\varepsilon,\lambda}&=H_1\Delta S_1-\varepsilon H_1S_0-\lambda H_1S_1=a\left(\frac{1+\lambda}{1-\lambda}-1+\mathbb{1}_A\right)-\varepsilon a\\&\quad-\lambda\left(\frac{1+\lambda}{1-\lambda}+\mathbb{1}_A\right)\\&=a((1-\lambda)\mathbb{1}_A+\lambda-\varepsilon).\end{aligned}$$

On the other hand $\mathcal{A}_1^+(\lambda,\mu)\cap L_+^0\neq\{0\}$. It is enough to take $a=1$. Then $x_1^{\lambda,\lambda}=(1-\lambda)\mathbb{1}_A$.

4. Super-replication problem. Since we do not have equivalent conditions for the absence of arbitrage in our model, we only have some necessary and sufficient conditions, and we cannot give an exact formula for the super-replication price. In particular under the assumption of robust no-arbitrage we have only an upper bound for this price.

Definition 4.1. We say that C is a *contingent claim* when C is a random variable, i.e. $C\in L^0(\mathcal{F}_T)$.

Let us define the set of initial endowments which hedge the payoff of the contingent claim C:

$$\Gamma^+ = \Gamma^+(C) := \{x \in \mathbb{R} \mid \exists H \in \mathcal{P}_+ \colon x + x_T^{\lambda,\mu}(H) \geq C,\ \mathbb{P}\text{-a.s.}\}$$

and the sets

$$\mathcal{Q}^+ := \{\mathbb{Q} \sim \mathbb{P} \mid \exists \tilde{S} \colon (\tilde{S}, \mathbb{Q}) \text{ is a } \lambda\text{-CPS}^+\},$$

$$D^+ = D^+(C) := \{x \in \mathbb{R} \mid \forall \mathbb{Q} \in \mathcal{Q}^+ \colon E_{\mathbb{Q}} C \leq x\}.$$

Using similar arguments to those [KRS] we will prove the theorem below.

THEOREM 4.2. *Assume that the model satisfies* (rNA$_+$). *Then* $D^+ \subseteq \Gamma^+$.

Proof. Notice that $\mathcal{Q}^+ \neq \emptyset$ by Corollary 3.3. Suppose that the inclusion $D^+ \subseteq \Gamma^+$ fails, so there exists $x \in D^+$ such that $x \notin \Gamma^+$. Then $C - x \notin \mathcal{A}_T^+(\lambda,\mu)$. The set $\mathcal{A}_T^+(\lambda,\mu)$ is a convex cone closed in probability (by Theorem 2.1 and Corollary 3.1). Notice that for any random variable η there exists a probability measure $\tilde{\mathbb{P}} \sim \mathbb{P}$ such that $d\tilde{\mathbb{P}}/d\mathbb{P} \in L^\infty$ and $\eta \in L^1(\tilde{\mathbb{P}})$. Hence we can assume that C is integrable with respect to $\tilde{\mathbb{P}}$. The set $\mathcal{A}_T^+(\lambda,\mu)$ is also closed in probability $\tilde{\mathbb{P}}$. Set $\Lambda_T^+ := \mathcal{A}_T^+(\lambda,\mu) \cap L^1(\tilde{\mathbb{P}})$, which is a closed convex cone in $L^1(\tilde{\mathbb{P}})$. Notice that $\Lambda_T^+ \cap L_+^1(\tilde{\mathbb{P}}) = \{0\}$ and $C - x \notin \Lambda_T^+$ since by the Hahn–Banach separation theorem (see [Ru] for more details) there exists $z_x \in L^\infty(\tilde{\mathbb{P}})$ such that

$$\forall \xi \in \Lambda_T^+ \colon \quad E_{\tilde{\mathbb{P}}} z_x \xi < E_{\tilde{\mathbb{P}}} z_x (C - x).$$

As Λ_T^+ is a cone we have $E_{\tilde{\mathbb{P}}} z_x \xi \leq 0$ for any $\xi \in \Lambda_T^+$. Furthermore, for $\xi = 0$ we get $E_{\tilde{\mathbb{P}}} z_x (C - x) > 0$.

Now we show that $z_x \geq 0$, $\tilde{\mathbb{P}}$-a.s. Define $A := \{z_x < 0\}$ and suppose that $\tilde{\mathbb{P}}(A) > 0$. Taking the sequence $\xi_n := -\lambda_n \mathbb{1}_A \in \Lambda_T^+$ where $\lambda_n \to \infty$ we obtain $E_{\tilde{\mathbb{P}}} z_x \xi_n \to \infty$, which contradicts the inequality $E_{\tilde{\mathbb{P}}} z_x \xi_n \leq 0$.

Normalizing we can assume that $z_x \leq 1$ and $\|z_x\| = 1$. Notice that $\mathbb{Q} := z_x \tilde{\mathbb{P}}$ is a probability measure equivalent to $\mathbb{P}$ such that $d\mathbb{Q}/d\mathbb{P} \in L^\infty$ and $E_{\mathbb{Q}} \xi \leq 0$ for any $\xi \in \Lambda_T^+$, in particular for

$$\xi_i = H_{t+1}^i (S_T^i - S_t^i) - \lambda_i H_{t+1}^i S_t^i - \mu_i H_{t+1}^i S_T^i, \qquad t = 0, \ldots, T-1,$$

where $H_{t+1} = (0, \ldots, \mathbb{1}_A, \ldots, 0)$, $\mathbb{P}$-a.s., $A \in \mathcal{F}_t$ and $\mathbb{1}_A$ is in the ith position. This means that at time t, if the event A holds we buy the ith share at the price S_t^i and liquidate the portfolio at time T. Hence

$$E_{\mathbb{Q}}[(S_T^i - S_t^i - \lambda_i S_t^i - \mu_i S_T^i)\mathbb{1}_A] \leq 0.$$

Since $(1 - \mu_i) E_{\mathbb{Q}}(S_T^i \mathbb{1}_A) \leq (1 + \lambda_i) E_{\mathbb{Q}}(S_t^i \mathbb{1}_A)$ for $i = 1, \ldots, d$ and for any $A \in \mathcal{F}_t$, all in all we have

$$(1 - \mu_i) E_{\mathbb{Q}}(S_T^i \mid \mathcal{F}_t) \leq (1 + \lambda_i) E_{\mathbb{Q}}(S_t^i \mid \mathcal{F}_t) = (1 + \lambda_i) S_t^i \quad \text{for } t = 0, \ldots, T-1.$$

Define $\tilde{S} = (\tilde{S}_t)_{t=0}^T$ by $\tilde{S}_t := (1-\mu)E_{\mathbb{Q}}(S_T \mid \mathcal{F}_t)$ and notice that $(\tilde{S}, \mathbb{Q})$ is a λ-CPS$^+$. Moreover $E_{\mathbb{Q}}(C-x) = E_{\tilde{P}}(z_x(C-x)) > 0$, which contradicts the fact that $x \in D^+$. ∎

Let us now define the *super-replication price*

$$p_s := \inf \Gamma^+ = \inf\{x \in \mathbb{R} \mid \exists H \in \mathcal{P}_+ \colon x + x_T^{\lambda,\mu}(H) \geq C,\ \mathbb{P}\text{-a.s.}\}.$$

By Theorem 4.2 we immediately get the following corollary.

Corollary 4.3. *Assume that the model satisfies* (rNA$_+$). *Then*

$$p_s \leq \sup_{\mathbb{Q} \in \mathcal{Q}^+} E_{\mathbb{Q}} C.$$

Proof. Notice that by Corollary 3.3 we have $\mathcal{Q}^+ \neq \emptyset$ and $D^+ \subseteq \Gamma^+$. ∎

As previously we can also define

$$\hat{\mathcal{Q}}^+ := \{\mathbb{Q} \sim \mathbb{P} \mid \exists \tilde{S} \colon (\tilde{S}, \mathbb{Q}) \text{ is a } (\lambda,\mu)\text{-supCPS}\},$$
$$\hat{D}^+ := \{x \in \mathbb{R} \mid \forall \mathbb{Q} \in \hat{\mathcal{Q}}^+ \colon E_{\mathbb{Q}} C \leq x\}.$$

Lemma 4.4. *Assume that there exists a* (λ,μ)*-supCPS. Then* $\Gamma^+ \subseteq \hat{D}^+$.

Proof. Take any $x \in \Gamma^+$. Let $(\tilde{S}, \mathbb{Q})$ be a (λ,μ)-supCPS. Then by the definition of Γ^+ and using the same arguments as in the proof of Theorem 3.4 there exists a strategy $H \in \mathcal{P}_+$ such that

$$C \leq x + x_T^{\lambda,\mu}(H) \leq x + (H \cdot \tilde{S})_T.$$

Notice that $E_{\mathbb{Q}}(H \cdot \tilde{S})_T \leq 0$. Hence for any $\mathbb{Q} \in \hat{\mathcal{Q}}^+$ we have $E_{\mathbb{Q}} C \leq x$. ∎

Let us define $\mathcal{Q} := \{\mathbb{Q} \sim \mathbb{P} \mid \exists \tilde{S} : (\tilde{S}, \mathbb{Q}) \text{ is } (\lambda,\mu)\text{-CPS}\}$. The following corollary is straightforward.

Corollary 4.5. *Assume that there exists a* (λ,μ)*-supCPS in the model. Then* $\sup_{\mathbb{Q} \in \hat{\mathcal{Q}}^+} E_{\mathbb{Q}} C \leq p_s$. *Moreover, if we assume that there exists a* (λ,μ)*-CPS then*

$$\sup_{\mathbb{Q} \in \mathcal{Q}} E_{\mathbb{Q}} C \leq \sup_{\mathbb{Q} \in \hat{\mathcal{Q}}^+} E_{\mathbb{Q}} C \leq p_s \leq \sup_{\mathbb{Q} \in \mathcal{Q}^+} E_{\mathbb{Q}} C.$$

Proof. Notice that $\Gamma^+ \subseteq \hat{D}^+$. In addition any (λ,μ)-CPS is in particular a (λ,μ)-supCPS and also a λ-CPS$^+$. ∎

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