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Dmitry Treschev and Oleg Zubelevich (Moscow)

ON WEAK SOLUTIONS TO THE LAGRANGE–D'ALEMBERT EQUATION

Abstract. We consider nonholonomic systems with collisions and propose a concept of weak solutions to Lagrange–d'Alembert equations. Using this concept we describe the dynamics of collisions. Collisions of a rotating ball and a rough floor are considered.

1. The description of the problem. Let us start from the following model example. Consider a solid ball B of radius r and mass m and let its centre of mass coincide with the geometric centre S. The moment of inertia relative to any axis passing through the point S is equal to J.

Here is an informal description of the problem. Being subjected to some potential forces the ball moves in three-dimensional space and sometimes collides with a floor. After the collision the ball bounces from the floor. The floor and the ball are rough: the ball cannot slide on the floor. That is, at the time of collision the ball obeys a nonholonomic constraint.

We wish to construct a theory of such a motion in the Lagrangian frame. In particular, we wish to give a meaning to the term "superelastic collision" in nonholonomic context.

In physical space we introduce a Cartesian coordinate system $Oxyz$. Let (x_S, y_S, z_S) be the coordinates of the point S.

Suppose that the plane Oxy is a solid rough floor. Then for all time $t \geq 0$ we have $z_S \geq r$.

Denote by $C \in B$ the contact point of the ball and the floor. The ball cannot slide on the floor, so

$$\overline{v}_C = \overline{v}_S + [\overline{\omega}, \overline{SC}] = 0, \tag{1.1}$$

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where $[\cdot,\cdot]$ is vector product, $\overline{v}_C$ is the velocity of the point C, and $\overline{\omega}$ is the angular velocity of the ball.

However the ball can rotate about the vertical axis at the contact point.

The configuration manifold of the system is $M = \mathbb{R}^3 \times \mathrm{SO}(3)$, where $(x_S, y_S, z_S) \in \mathbb{R}^3$ and an element of SO(3) determines the orientation of the ball.

In the general construction we suppose that the system is nonholonomic not only at the time of collision but also outside the unilateral constraint.

Thus the general construction is as follows. We have a smooth configuration manifold M with $\dim M = m$ and a smooth submanifold $N \subset M$ with $\dim N = m-1$ (the floor: $\{z_S = r\}$). We assume that both manifolds carry some distributions.

In the example under consideration the manifold M does not carry a nonholonomic constraint but the manifold N does: the constraint on N is given by (1.1).

Let

$$x = (x_1, \dots, x_m)^T \in M$$

be local coordinates on M.

Denote the distribution on M by $E(x) \subseteq T_xM$, $x \in M$, and let $F(x) \subseteq T_xN$, $x \in N$, be the distribution on $N \subset M$. Assume also that

$$F(x) \subseteq E(x), \qquad x \in N.$$

The dynamics of the system is described by a smooth Lagrangian $L(x, \dot{x})$.

From the viewpoint of the configuration manifold's geometry, collisions of rigid bodies are considered in [4].

The manifold M is endowed with the Riemannian metric generated by the kinetic energy of the system. The evolution of the system is expressed by a function $t \mapsto x(t) \in M$.

When the moving point $x(t)$ collides with the submanifold N, i.e. $x(\tau) \in N$ for some τ, it bounces, obeying the law of reflection: "the angle of incidence is equal to the angle of reflection". This law is obtained by means of a limit process where the constraint is replaced by a strong potential force field [4].

These results have been obtained in the absence of nonholonomic constraints. We generalize them to the nonholonomic case.

Another concept of generalized solutions to the Lagrange–d'Alembert equations in non-Lagrangian form has been investigated in [5].

In Section 5 we consider a rough ball colliding with a floor and obtain formulas which in particular describe the following effect [3]: "A perfectly rough ball which conserves kinetic energy behaves in such an unexpected way that it is difficult to pick up after it has bounced twice upon the floor, and, more bizarre, it returns to the hand on being thrown to the floor in such a way that it bounces from the underside of a table."

2. Weak solutions to the Lagrange–d'Alembert equation. In this section we assume that $L : M \times TM \to \mathbb{R}$ is a C^∞-smooth function.

In the absence of a unilateral constraint N, a smooth function

$$x(t) = (x_1, \dots, x_m)^T(t) \in M, \qquad x_i(\cdot) \in C^2[t_1, t_2],$$

is a motion of the system if and only if for any function

$$\begin{aligned} &\psi(t) = (\psi_1, \dots, \psi_m)^T(t), \qquad \psi_k \in \mathcal{D}(\mathbb{R}), \\ &\operatorname{supp} \psi_k \subset (t_1, t_2), \qquad \psi(t) \in E(x(t)), \end{aligned} \tag{2.1}$$

it satisfies the Lagrange–d'Alembert equation

$$\left(\frac{\partial L}{\partial x}(x(t), \dot{x}(t)) - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}(x(t), \dot{x}(t))\right)\psi(t) = 0, \qquad t \in [t_1, t_2], \tag{2.2}$$

and the equation of constraint

$$\dot{x}(t) \in E(x(t)). \tag{2.3}$$

Definition 1. We shall say that a function $x(\cdot) \in H^1[t_1, t_2]$ is a *weak solution* to the system of Lagrange–d'Alembert equations and the equations of constraint if the equation

$$\int_{t_1}^{t_2} \left(\frac{\partial L}{\partial x}(x(t), \dot{x}(t))\psi(t) + \frac{\partial L}{\partial \dot{x}}(x(t), \dot{x}(t))\dot{\psi}(t)\right) dt = 0 \tag{2.4}$$

holds for any ψ that satisfies (2.1), and condition (2.3) holds for almost all $t \in [t_1, t_2]$.

In formula (2.4) we suppose that the functions L and $x(t)$ are such that the integrand belongs to $L^1(t_1, t_2)$.

Note that by the Sobolev embedding theorem, the space $H^1[t_0, t_1]$ is contained in $C[t_0, t_1]$.

In the case of a smooth function $x(\cdot)$, equations (2.4), (2.3) are equivalent to (2.2), (2.3). This follows from integration by parts and the Lagrange–d'Alembert principle [1], [7].

If the motion $x(t)$ contains collisions it is piecewise differentiable: at the time of collision its first derivative is not continuous and the second derivative does not exist.

Equations (2.4) do not contain the second derivative of $x(t)$. Therefore the concept of weak solutions is a proper tool to describe the motion with collisions.

Let us turn to the details.

Consider a solution $x(\cdot)$ that collides with the wall at time $\tau \in (t_1, t_2)$, i.e. $x(\tau) \in N$. Correspondingly, one must put

$$\psi(\tau) \in F(x(\tau)). \tag{2.5}$$

We suppose that $x(\cdot) \in C[t_1, t_2]$ and

$$x(t) = \begin{cases} x^-(t), & t \in [t_1, \tau], \\ x^+(t), & t \in (\tau, t_2], \end{cases}$$

and $x^-(\cdot) \in C^2[t_1, \tau]$, $x^+(\cdot) \in C^2(\tau, t_2]$.

By definition put

$$x^+(\tau) = \lim_{t \to \tau+} x^+(t), \qquad \dot{x}^+(\tau) = \lim_{t \to \tau+} \dot{x}^+(t).$$

We assume that these limits exist.

The solution $x(\cdot)$ obeys a nonholonomic constraint, that is,

$$\dot{x}^\pm(t) \in E(x^\pm(t)). \tag{2.6}$$

If $\dot{x}^-(\tau) = \dot{x}^+(\tau)$ then the derivative $\dot{x}(\tau)$ is defined and

$$\dot{x}(\tau) \in F(x(\tau)). \tag{2.7}$$

2.1. Equations of collision. Introduce the notation

$$v^\pm = \dot{x}^\pm(\tau) \in E(x(\tau)).$$

Splitting the integral (2.4) into $\int_{t_1}^{\tau} + \int_{\tau}^{t_2}$ and integrating by parts, we obtain

$$\int_{t_1}^{\tau} \left(\frac{\partial L}{\partial x}(x(t), \dot{x}(t)) - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}(x(t), \dot{x}(t)) \right) \psi(t)\, dt = 0, \tag{2.8}$$

$$\int_{\tau}^{t_2} \left(\frac{\partial L}{\partial x}(x(t), \dot{x}(t)) - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}(x(t), \dot{x}(t)) \right) \psi(t)\, dt = 0, \tag{2.9}$$

$$\left(\frac{\partial L}{\partial \dot{x}}(x(\tau), v^+) - \frac{\partial L}{\partial \dot{x}}(x(\tau), v^-) \right) \psi(\tau) = 0. \tag{2.10}$$

Indeed, to obtain (2.10) one must take a sequence of functions $\psi(\cdot)$ with support shrinking to the point τ. Then to obtain (2.8) one must employ functions ψ with support lying in (t_1, τ).

Equations (2.8), (2.9) express the fact that the functions $x^\pm(\cdot)$ satisfy the Lagrange–d'Alembert equations. By assumption they also satisfy the equations of constraint (2.6). That is, before and after collisions the system behaves in the standard way.

In particular, if the system is holonomic outside N (i.e. $E(x) = T_x M$) then in their domains the functions $x^\pm(\cdot)$ satisfy the Lagrange equations

$$\frac{\partial L}{\partial x}(x^\pm(t), \dot{x}^\pm(t)) - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}}(x^\pm(t), \dot{x}^\pm(t)) = 0.$$

Equation (2.10) describes the behaviour of the system at the time of collision (see also [6]). This equation is of main importance to us.

3. A lemma from vector algebra. The following lemma is mainly used in Section 5. But we place it here because it provides an introduction to the geometry of the next section.

LEMMA 1. *Let $X = \mathbb{R}^m$ be a Euclidean vector space with scalar product given by its Gram matrix G. And let B be the matrix of a linear operator* (*we denote operators and their matrices by the same letter*)

$$B : X \to \mathbb{R}^{m-s}, \qquad \operatorname{rank} B = m - s.$$

Let

$$X = \ker B \oplus W, \qquad W \perp \ker B,$$

be the orthogonal decomposition.

Then the square matrix of the orthogonal projector $P : X \to X$ onto W is

$$P = G^{-1}B^T(BG^{-1}B^T)^{-1}B. \tag{3.1}$$

If an operator $A : X \to \mathbb{R}^k$ is such that $\ker B \subseteq \ker A$ *then*

$$AP = A. \tag{3.2}$$

In particular, this implies that $P(\ker A) \subseteq \ker A$.

Proof. To obtain formula (3.1) fix an arbitrary vector $x \in X$ and introduce the linear function $f(\xi) = (Px)^T G\xi$. It is clear

$$\ker B \subseteq \ker f.$$

This implies that there is an operator $\lambda : \mathbb{R}^{m-s} \to \mathbb{R}$ such that $(Px)^T G = \lambda B$ and $Px = G^{-1}B^T\lambda^T$. It remains to find λ^T from the equation $B(x-Px) = 0$.

To obtain formula (3.2) note that there exists an operator

$$\gamma : \mathbb{R}^{m-s} \to \mathbb{R}^k$$

such that $A = \gamma B$. Consequently, formula (3.2) follows from (3.1).

The lemma is proved.

4. The natural Lagrangian system. In applications, distributions are determined by means of linear operators in the following way. Introduce a linear operator

$$A(x) : T_xM \to \mathbb{R}^{m-l}, \qquad \dim \operatorname{im} A(x) = m - l, \qquad x \in M,$$

with a smooth mapping $x \mapsto A(x)$ such that $E(x) = \ker A(x)$ is an l-dimensional distribution on M. To determine an s-dimensional distribution

$F(x)$ on N introduce a linear operator

$$B(x) : T_xM \to \mathbb{R}^{m-s}, \quad \dim \operatorname{im} B(x) = m - s, \quad x \in N, \quad F(x) = \ker B(x).$$

The operator $B(x)$ is also assumed to be a smooth function of x.

Recall also that $\ker B(x) \subseteq \ker A(x)$ for $x \in N$.

The operators A, B are not uniquely defined: the same distributions can be generated by other operators A, B.

To proceed with our analysis put

$$L = T(x, \dot{x}, \dot{x}) - V(x).$$

The form

$$T(x, \xi, \eta) = \tfrac{1}{2}\xi^T G(x)\eta, \quad \xi = (\xi_1, \dots, \xi_m)^T, \quad \eta = (\eta_1, \dots, \eta_m)^T,$$

is the kinetic energy of the system, the matrix $G(x) \equiv G^T(x)$ is positive definite. It defines a Riemannian metric on M. The potential energy V is a smooth function on M.

By (2.5) equation (2.10) reduces to

$$T(x(\tau), v^+ - v^-, u) = 0 \quad \text{for any } u \in \ker B(x(\tau)). \tag{4.1}$$

From formula (4.1) it follows that the difference $v^+ - v^-$ is perpendicular to $\ker B(x(\tau))$.

Hypothesis 1. *The vector v^+ depends on v^- by means of a linear operator*

$$v^+ = R(x(\tau))v^-, \quad R(x(\tau)) : \ker A(x(\tau)) \to \ker A(x(\tau)).$$

This hypothesis is not the unique possible: see for example [8, 2] for nonlinear models of collision.

Hypothesis 2. *The energy is conserved during collisions:*

$$T(x(\tau), v^+, v^+) = T(x(\tau), v^-, v^-).$$

This is the simplest relation between energies before and after collision. There are various possibilities to relax this assumption, for example, introducing the restitution coefficient (see Section 6).

Hypothesis 3. *The system is reversible: if $x(t)$ is a motion of the system then $x(-t)$ is also a motion. For a collision this implies that*

$$v^- = R(x(\tau))v^+.$$

The third hypothesis implies $(R(x(\tau)))^2 = I$. All these hypotheses are consistent with Lagrangian theory of impact in holonomic systems [4].

Note that if $\dim \ker B(x(\tau)) = \dim \ker A(x(\tau)) - 1$ then the last hypothesis is automatically satisfied.

It is reasonable to consider the decomposition

$$T_{x(\tau)}M = \ker B(x(\tau)) \oplus W(x(\tau)),$$

where $W(x(\tau))$ is the orthogonal complement of $\ker B(x(\tau))$, and let

$$P : T_{x(\tau)}M \to W(x(\tau))$$

be the orthogonal projection.

Introduce the notation $Pv = v_\perp$, $(I-P)v = v_\| \in \ker B(x(\tau))$ and the norm $|\xi|^2 = T(x(\tau),\xi,\xi)$. Then write

$$v^\pm = v_\perp^\pm + v_\|^\pm.$$

THEOREM 4.1. *Under Hypotheses* 1–3 *the following formula holds:*

$$v^+ = (I-2P)v^-. \tag{4.2}$$

Formula (4.2) gives a physically correct model of collision.

Namely, denote by $x(t,\hat{x},\hat{v})$, $t \in [t_1,t_2]$, a solution such that

$$x(t_1,\hat{x},\hat{v}) = \hat{x}, \qquad \dot{x}(t_1,\hat{x},\hat{v}) = \hat{v}$$

and for some $\hat{x}',\hat{v}'$ and $\tau' \in (t_1,t_2)$ we have

$$\lim_{t\to\tau'-} \dot{x}(t,\hat{x}',\hat{v}') \notin T_{x(\tau',\hat{x}',\hat{v}')}N, \qquad x(\tau',\hat{x}',\hat{v}') \in N.$$

THEOREM 4.2. *The solution $x(t,\hat{x},\hat{v})$ is a continuous function of $t \in [t_1,t_2]$ for $(\hat{x},\hat{v})$ close to $(\hat{x}',\hat{v}')$, and there exists a continuous function*

$$\tau = \tau(\hat{x},\hat{v}), \qquad \tau(\hat{x}',\hat{v}') = \tau'$$

such that the collision occurs at time τ, i.e. $x(\tau(\hat{x},\hat{v}),\hat{x},\hat{v}) \in N$.

Theorem 4.2 follows directly from the implicit function theorem and from the fact that the function $x \mapsto B(x)$ is smooth.

Proof of Theorem 4.1. Actually we deal with vectors $v \in \ker A(x(\tau))$ only. By Lemma 1 one has $P(\ker A(x(\tau))) \subseteq \ker A(x(\tau))$.

By Hypothesis 2 it follows that $|v^+| = |v^-|$ and $R(x(\tau))$ is an isometric operator.

Since the difference $v^+ - v^- = (v_\|^+ - v_\|^-) + (v_\perp^+ - v_\perp^-)$ is perpendicular to $\ker B(x(\tau))$, we have

$$v_\|^+ = v_\|^-$$

so that

$$R(x(\tau))|_{\ker B(x(\tau))} = I.$$

Introduce the space

$$H(x(\tau)) = W(x(\tau)) \cap \ker A(x(\tau)).$$

Then $\ker A(x(\tau)) = H(x(\tau)) \oplus \ker B(x(\tau))$ and $R(x(\tau))H(x(\tau)) = H(x(\tau))$.

We finally have

$$v^+ = Qv_\perp^- + v_\|^-, \qquad Q = R(x(\tau))|_{H(x(\tau))} : H(x(\tau)) \to H(x(\tau)). \tag{4.3}$$

Hypothesis 3 implies that $Q^2 = I$ and consequently each eigenvalue of Q is either equal to 1 or to -1.

If $Qv_\perp^- = v_\perp^-$ then $v_\perp^- = 0$. Indeed, the assumption implies $v^+ = Qv_\perp^- + v_\parallel^- = v^-$. According to (2.7) one has $v^- \in \ker B(x(\tau))$ so that $v_\perp^- = 0$. Consequently, $Q = -I$ and $v^+ = -v_\perp^- + v_\parallel^-$. In terms of the matrix P the same is written in (4.2).

The theorem is proved.

5. Superelastic ball. Introduce the notation $J' = J + r^2 m$.

We use the Euler angles local coordinates in SO(3).

Consequently, the position of the ball is determined by the vector

$$x = (x_S, y_S, z_S, -\varphi, \theta, \psi)^T.$$

Why we are writing φ with a negative sign will be clear below.

Being subjected to some potential forces the ball can move in the half-space $\{z_S > r\}$ and sometimes it can collide with the floor.

After the ball meets the floor ($z_S = r$) it bounces. The point of contact $C \in B$ has zero velocity (1.1).

In [3] the following hypotheses for collision of a rough ball with surfaces are used. The energy and the angular momentum are conserved during collisions and the modulus of the vertical component of the ball center's velocity is also conserved. Those hypotheses are enough for a two-dimensional model. Our case is substantially three-dimensional and this setting does not work. But formulas of [3] turn out to be a special case of Theorem 5.1. For a similar reason the result presented below does not follow from [6].

Let $\overline{v}_S^\pm, \overline{\omega}^\pm$ stand for the velocity of the point S and for the ball's angular velocity after $(+)$ and before $(-)$ collision respectively.

In the coordinates $Oxyz$,

$$\overline{v}_S^\pm = (v_1^\pm, v_2^\pm, v_3^\pm), \qquad \overline{\omega}^\pm = (\omega_1^\pm, \omega_2^\pm, \omega_3^\pm).$$

THEOREM 5.1. *At the time of collision,*

$$\begin{aligned}
v_1^+ &= \frac{mr^2 - J}{J'} v_1^- + \frac{2Jr}{J'} \omega_2^-, &\qquad \omega_1^+ &= -\frac{2rm}{J'} v_2^- + \frac{J - mr^2}{J'} \omega_1^-,\\
v_2^+ &= \frac{mr^2 - J}{J'} v_2^- - \frac{2Jr}{J'} \omega_1^-, &\qquad \omega_2^+ &= \frac{2rm}{J'} v_1^- + \frac{J - mr^2}{J'} \omega_2^-,\\
v_3^+ &= -v_3^-, &\qquad \omega_3^+ &= \omega_3^-.
\end{aligned}$$

Note that by these formulas the angular momentum about the point of contact C is conserved during collision:

$$m[\overline{CS}, \overline{v}_S^+] + J\overline{\omega}^+ = m[\overline{CS}, \overline{v}_S^-] + J\overline{\omega}^-.$$

Due to (2.10) this is not a surprise.

Proof of Theorem 5.1. Introduce the Euler angles so that at the time of collision one has $\varphi = \psi = 0$, $\theta = \pi/2$. Then it follows that $\overline{\omega} = \dot{\theta}\overline{e}_x + \dot{\psi}\overline{e}_z - \dot{\varphi}\overline{e}_y$. Thus at the time of collision we have

$$v^{\pm} = (v_1^{\pm}, v_2^{\pm}, v_3^{\pm}, \omega_1^{\pm}, \omega_2^{\pm}, \omega_3^{\pm})^T.$$

The formula $T = \frac{1}{2}m\overline{v}_S^2 + \frac{1}{2}J\overline{\omega}^2$ implies $G = \operatorname{diag}(m, m, m, J, J, J)$. From (1.1) one obtains

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & -r & 0 \\ 0 & 1 & 0 & r & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}, \qquad A = 0.$$

The matrix of the operator P is computed with the help of Lemma 1:

$$P = \frac{1}{J'} \begin{pmatrix} J & 0 & 0 & 0 & -Jr & 0 \\ 0 & J & 0 & Jr & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & rm & 0 & r^2 m & 0 & 0 \\ -rm & 0 & 0 & 0 & r^2 m & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Now Theorem 5.1 follows from (4.2).

Nonholonomic pendulum. Suppose that the ball moves in the standard gravity field $\overline{g} = -g\overline{e}_z$.

Throw the ball to the floor so that

$$\overline{v}_S^- = -v\overline{e}_x - u\overline{e}_z, \qquad \overline{\omega}^- = \frac{rmv}{J}\overline{e}_y, \qquad u, v > 0.$$

From Theorem 5.1 it follows that

$$\overline{v}_S^+ = -\overline{v}_S^-, \qquad \overline{\omega}^+ = -\overline{\omega}^-.$$

Thus after the ball bounces from the floor, its center S moves along the same parabola just in the opposite direction. Since the angular velocity also changes its direction, the motion is periodic.

6. A remark on inelastic collision. Analogously it is possible to construct models of inelastic collision.

For example, following Newton's law of restitution, we propose the following hypothesis:

$$v^+ = -\mu v_\perp^- + v_\parallel^-, \tag{6.1}$$

where $\mu \in [0, 1]$ is the restitution coefficient. This hypothesis is consistent with (4.1).

For a plastic collision we have $\mu = 0$, and for a superelastic one we have $\mu = 1$. In terms of the operator P formula (6.1) has the form

$$v^+ = (I - (1+\mu)P)v^-.$$

Under this hypothesis the corresponding formulas for the ball colliding with the floor take the form

$$\begin{aligned}
v_1^+ &= \frac{mr^2 - \mu J}{J'} v_1^- + \frac{Jr(1+\mu)}{J'}\omega_2^-, & \omega_1^+ &= -\frac{rm(1+\mu)}{J'}v_2^- + \frac{J - \mu mr^2}{J'}\omega_1^-,\\
v_2^+ &= \frac{mr^2 - \mu J}{J'} v_2^- - \frac{Jr(1+\mu)}{J'}\omega_1^-, & \omega_2^+ &= \frac{rm(1+\mu)}{J'}v_1^- + \frac{J - \mu mr^2}{J'}\omega_2^-,\\
v_3^+ &= -\mu v_3^-, & \omega_3^+ &= \omega_3^-.
\end{aligned}$$

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Dmitry Treschev, Oleg Zubelevich
Department of Theoretical Mechanics
Mechanics and Mathematics Faculty
M. V. Lomonosov Moscow State University
Vorob'evy gory
119899, Moscow, Russia
E-mail: ozubel@yandex.ru

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