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THE SHORTEST CONFIDENCE INTERVAL FOR THE PROBABILITY OF SUCCESS IN A NEGATIVE BINOMIAL MODEL

Abstract. The existence of the shortest confidence interval for the probability of success in a negative binomial distribution is shown. The method of obtaining such an interval is presented as well. The interval obtained is compared with the Clopper–Pearson shortest confidence interval for the probability in the binomial model.

Consider the negative binomial (or Pascal) statistical model

$$(\{0,1,2,\ldots\},\{NB(r,\pi),0<\pi<1\}),$$

where $NB(r,\pi)$ denotes the negative binomial distribution with pdf

$$\binom{r+x-1}{x}\pi^r(1-\pi)^x,\quad x=0,1,2,\ldots.$$

It is known that

$$\sum_{x=0}^{t}\binom{r+x-1}{x}\pi^r(1-\pi)^x=F(r,t+1;\pi),$$

where $F(a,b;\cdot)$ denotes the cdf of the beta distribution with parameters (a,b).

Let X denote a negative binomial $NB(r,\pi)$ random variable. A confidence interval for the probability π at the confidence level γ is of the form (see Clopper and Pearson's (1934) construction of the confidence interval for π in a binomial statistical model)

$$(F^{-1}(r,X+1;\gamma_1);F^{-1}(r,X;\gamma_2)),$$

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where $\gamma_1, \gamma_2 \in (0,1)$ are such that $\gamma_2 - \gamma_1 = \gamma$ and $F^{-1}(a,b;\alpha)$ is the α quantile of the beta distribution with parameters (a,b), i.e.

$$P_\pi\{\pi \in (F^{-1}(r, X+1; \gamma_1); F^{-1}(r, X; \gamma_2))\} \geq \gamma, \qquad \forall \pi \in (0,1).$$

For $X = 0$ the right end is taken to be 1.

In the standard construction $\gamma_1 = (1-\gamma)/2$ is used, i.e. the rule of symmetric division of $1-\gamma$ to both sides of the interval is applied. It is of interest to find the shortest confidence interval. So we want to find γ_1 and γ_2 such that the confidence interval is the shortest possible.

Consider the length of the confidence interval when $X = x$ is observed:

$$d(\gamma_1, x) = F^{-1}(r, x; \gamma + \gamma_1) - F^{-1}(r, x+1; \gamma_1).$$

THEOREM 1. *For $x > 1$ there exists a two-sided shortest confidence interval.*

Proof. We have to show that for $x > 1$ there exists $0 < \gamma_1 < 1 - \gamma$ such that $d(\gamma_1, x)$ is minimal. The derivative of $d(\gamma_1, x)$ with respect to γ_1 equals (in what follows $B(\cdot,\cdot)$ denotes the beta function)

$$\begin{aligned}\frac{\partial d(\gamma_1, x)}{\partial \gamma_1} &= B(r,x)(1 - F^{-1}(r,x;\gamma+\gamma_1))^{1-x} F^{-1}(r,x;\gamma+\gamma_1)^{1-r} \\ &\quad - B(r,x+1)(1 - F^{-1}(r,x+1;\gamma_1))^{-x} F^{-1}(r,x+1;\gamma_1)^{1-r}.\end{aligned}$$

Let

$$\mathrm{LHS}(\gamma_1, x) = \frac{[1 - F^{-1}(r,x;\gamma+\gamma_1)]^{x-1} F^{-1}(r,x;\gamma+\gamma_1)^{r-1}}{B(r,x)},$$

$$\mathrm{RHS}(\gamma_1, x) = \frac{[1 - F^{-1}(r,x+1;\gamma_1)]^{x} F^{-1}(r,x+1;\gamma_1)^{r-1}}{B(r,x+1)}.$$

Then

$$\frac{\partial d(\gamma_1, x)}{\partial \gamma_1} = \frac{1}{\mathrm{LHS}(\gamma_1, x)} - \frac{1}{\mathrm{RHS}(\gamma_1, x)}.$$

For $x > 1$,

$$F^{-1}(r, x+1; 0) = 0 \quad \text{and} \quad F^{-1}(r, x; 1) = 1.$$

Hence

- if $\gamma_1 \to 0$, then $\mathrm{LHS}(\gamma_1, x) > 0$ and $\mathrm{RHS}(\gamma_1, x) \to 0$,
- if $\gamma_1 \to 1-\gamma$, then $\mathrm{LHS}(\gamma_1, x) \to 0$ and $\mathrm{RHS}(\gamma_1, x) > 0$.

Therefore, the equation

$$\frac{\partial d(\gamma_1, x)}{\partial \gamma_1} = 0 \qquad (*)$$

has a solution.

It is easy to see that $\mathrm{LHS}(\cdot, x)$ and $\mathrm{RHS}(\cdot, x)$ are unimodal and concave functions on the interval $(0, 1-\gamma)$. Hence, the solution of $(*)$ is unique. Let γ_1^* denote the solution. Because $\partial d(\gamma_1, x)/\partial \gamma_1 < 0$ for $\gamma_1 < \gamma_1^*$ and

$\partial d(\gamma_1, x)/\partial\gamma_1 > 0$ for $\gamma_1 > \gamma_1^*$, we have $d(\gamma_1^*, x) = \inf\{d(\gamma_1, x) : 0 < \gamma_1 < 1-\gamma\}$.

THEOREM 2. *For $x = 1$ the shortest confidence interval is one-sided.*

Proof. For $x = 1$ we have

$$\mathrm{LHS}(\gamma_1, 1) = \frac{F^{-1}(r, 1; \gamma+\gamma_1)^{r-1}}{B(r, 1)},$$
$$\mathrm{RHS}(\gamma_1, 1) = \frac{[1 - F^{-1}(r, 2; \gamma_1)]F^{-1}(r, 2; \gamma_1)^{r-1}}{B(r, 2)}.$$

It can be seen that

$$\mathrm{RHS}(\gamma_1, 1) < \mathrm{LHS}(\gamma_1, 1) \quad \text{for } 0 < \gamma_1 < 1-\gamma.$$

So $\partial d(\gamma_1, 1)/\partial\gamma_1 < 0$ and hence $d(\gamma_1, 1)$ achieves its minimal value for $\gamma_1 = 1-\gamma$.

The value of γ_1^* for given γ, r and x may be found numerically. Tables 1–3 give those values for $\gamma = 0.95$. The classical symmetric confidence intervals are also shown.

Table 1. $r = 5$

	Shortest c.i.				Classical c.i.		
x	γ_1^*	$\mathrm{left}_{\mathrm{short}}$	$\mathrm{right}_{\mathrm{short}}$	$\mathrm{length}_{\mathrm{short}}$	$\mathrm{left}_{\mathrm{sym}}$	$\mathrm{right}_{\mathrm{sym}}$	$\mathrm{length}_{\mathrm{sym}}$
1	0.05000	0.41820	1.00000	0.58180	0.35877	0.99495	0.63618
5	0.02303	0.18339	0.78408	0.60070	0.18709	0.78799	0.60091
10	0.01515	0.10436	0.56211	0.45775	0.11824	0.58104	0.46279
15	0.01263	0.07289	0.43500	0.36210	0.08657	0.45565	0.36908
20	0.01141	0.05600	0.35417	0.29817	0.06831	0.37384	0.30553
25	0.01069	0.04546	0.29849	0.25302	0.05642	0.31664	0.26022
30	0.01021	0.03826	0.25786	0.21960	0.04806	0.27450	0.22644
35	0.00988	0.03303	0.22694	0.19391	0.04186	0.24221	0.20035
40	0.00963	0.02905	0.20262	0.17356	0.03708	0.21669	0.17961
45	0.00944	0.02593	0.18300	0.15706	0.03328	0.19601	0.16274
50	0.00928	0.02342	0.16684	0.14342	0.03018	0.17893	0.14875

Below we give a short program in the `R` language for calculating γ_1^* and the ends of the shortest confidence interval. Of course, one can also use other mathematical or statistical packages (in a similar way) to find the values of γ_1^* (cf. Zieliński 2010).

```
Qbet = function(a,b,q){qbeta(q,a,b)} #Beta quantile
Lower = function(r,x,q){Qbet(r,x+1,q)}
Upper = function(r,x,q){Qbet(r,x,q)}
Leng = function(r,x,q,s){Upper(r,x,q+s)-Lower(r,x,s)}
```

```
FindMinimumLeng = function(r,x,q)
  {
    optimize(Leng,interval=c(0,1-q),r=r,x=x,q=q,tol=1e-09)$minimum
  }
r = 10; #input r
x = 20; #input x greater then 1
q = 0.95; #input confidence level
ss = FindMinimumLeng(r,x,q)
ss #output γ1*
Lower(r,x,ss) #output left end
Upper(r,x,q+ss) #output right end
```

Table 2. $r = 10$

	Shortest c.i.				Classical c.i.		
x	γ_1^*	$\mathrm{left}_{\mathrm{short}}$	$\mathrm{right}_{\mathrm{short}}$	$\mathrm{length}_{\mathrm{short}}$	$\mathrm{left}_{\mathrm{sym}}$	right_{sym}	$\mathrm{length}_{\mathrm{sym}}$
1	0.05000	0.63564	1.00000	0.36436	0.58722	0.99747	0.41025
10	0.02421	0.27069	0.71006	0.43937	0.27196	0.71136	0.43940
20	0.01891	0.16563	0.49950	0.33387	0.17287	0.50832	0.33545
30	0.01696	0.11944	0.38350	0.26406	0.12692	0.39326	0.26635
40	0.01595	0.09341	0.31087	0.21746	0.10030	0.32022	0.21992
50	0.01533	0.07671	0.26127	0.18456	0.08293	0.26992	0.18699
60	0.01492	0.06507	0.22527	0.16020	0.07069	0.23322	0.16253
70	0.01461	0.05650	0.19798	0.14147	0.06160	0.20528	0.14368
80	0.01439	0.04993	0.17657	0.12664	0.05459	0.18330	0.12872
90	0.01421	0.04472	0.15934	0.11461	0.04900	0.16557	0.11656
100	0.01407	0.04050	0.14517	0.10466	0.04446	0.15096	0.10650

Table 3. $r = 50$

	Shortest c.i.				Classical c.i.		
x	γ_1^*	$\mathrm{left}_{\mathrm{short}}$	$\mathrm{right}_{\mathrm{short}}$	length_{short}	$\mathrm{left}_{\mathrm{sym}}$	$\mathrm{right}_{\mathrm{sym}}$	$\mathrm{length}_{\mathrm{sym}}$
1	0.05000	0.91033	1.00000	0.08967	0.89553	0.99949	0.10397
50	0.02489	0.39823	0.59721	0.19898	0.39832	0.59730	0.19898
100	0.02262	0.25708	0.40890	0.15182	0.25856	0.41050	0.15194
150	0.02170	0.19000	0.31034	0.12034	0.19161	0.31213	0.12052
200	0.02120	0.15072	0.24995	0.09923	0.15224	0.25168	0.09944
250	0.02089	0.12492	0.20921	0.08429	0.12631	0.21080	0.08450
300	0.02067	0.10666	0.17987	0.07321	0.10793	0.18134	0.07341
350	0.02051	0.09306	0.15774	0.06468	0.09422	0.15909	0.06487
400	0.02039	0.08254	0.14046	0.05792	0.08360	0.14170	0.05810
450	0.02030	0.07415	0.12659	0.05243	0.07514	0.12774	0.05260
500	0.02022	0.06732	0.11521	0.04789	0.06823	0.11628	0.04805

In a practical problem of estimation of π one of two models may be applied: the binomial model or the negative binomial model. In what follows those two approaches to the problem will be compared.

In the binomial model the confidence interval has the form

$$(F^{-1}(Y, n-Y+1; \gamma_B), F^{-1}(Y+1, n-Y; \gamma+\gamma_B)),$$

where γ_B is chosen in such a way that the confidence interval is the shortest one (Zieliński 2010). Here Y is a r.v. with binomial $\mathrm{Bin}(n, \pi)$ distribution.

In the negative binomial model we have the interval

$$(F^{-1}(r, X+1; \gamma_N), F^{-1}(r, X; \gamma+\gamma_N)),$$

where γ_N is chosen in such a way that the interval is the shortest possible.

The expected lengths of those intervals are, in the binomial model,

$$\begin{aligned}\Delta_B(\pi; n) &= E_\pi(F^{-1}(Y+1, n-Y; \gamma_B) - F^{-1}(Y, n-Y+1; \gamma+\gamma_B)) \\ &= \sum_{x=0}^{n}(F^{-1}(x+1, n-x; \gamma_B) - F^{-1}(x, n-x+1; \gamma+\gamma_B))\binom{n}{x}\pi^x(1-\pi)^{n-x},\end{aligned}$$

and in the negative binomial model,

$$\begin{aligned}\Delta_N(\pi; r) &= E_\pi(F^{-1}(r, X; \gamma_N) - F^{-1}(r, X+1; \gamma+\gamma_N)) \\ &= \sum_{x=0}^{\infty}(F^{-1}(r, x+1; \gamma_N) - F^{-1}(r, x; \gamma+\gamma_N))\binom{r+x-1}{x}\pi^r(1-\pi)^x.\end{aligned}$$

Let the number n of trials in the binomial model be fixed. Let the number $r(n)$ of successes in the negative binomial model be such that the mean lengths of the shortest confidence intervals are equal:

$$\int_0^1 \Delta_B(\pi; n)\, d\pi = \int_0^1 \Delta_N(\pi; r)\, d\pi.$$

Analytical solution of the above equation with respect to $r(n)$ is impossible. Below we give exemplary numerical solutions (the confidence level is 0.95):

n	10	50	100	500
$r(n)$	4	23	45	227

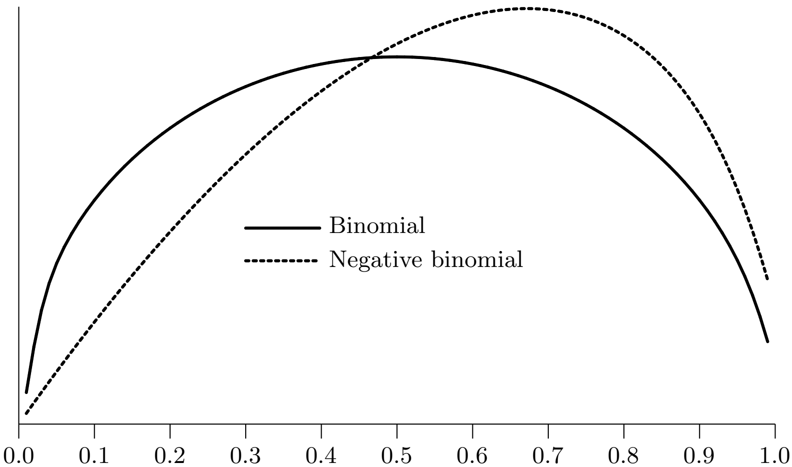
In Table 4 the lengths of the shortest confidence intervals in the binomial model ($n = 100$) and in the negative binomial model ($r(100) = 45$) are compared. Those results are shown in the figure. It is seen that there exists a probability π_0 such that for $\pi \le \pi_0$ the confidence interval in the negative binomial model is shorter than in the binomial model. At π_0 the expected lengths in both models are equal, i.e. π_0 is the solution of the equation

$$\Delta_B(\pi_0; n) = \Delta_N(\pi_0; r).$$

In the case of $n = 100$ and $r(100) = 45$ we obtain $\pi_0 = 0.45643$.

Table 4. $n = 100$ and $r(n) = 45$

π	$\Delta_B(\pi)$	$\Delta_N(\pi)$	$\frac{r(n)}{\pi}$	π	$\Delta_B(\pi)$	$\Delta_N(\pi)$	$\frac{r(n)}{\pi}$
0.05	0.08857	0.02891	900	0.55	0.20139	0.21885	82
0.10	0.12340	0.05624	450	0.60	0.19842	0.22533	75
0.15	0.14613	0.08195	300	0.65	0.19335	0.22864	70
0.20	0.16309	0.10597	225	0.70	0.18600	0.22836	65
0.25	0.17609	0.12822	180	0.75	0.17609	0.22383	60
0.30	0.18600	0.14863	150	0.80	0.16309	0.21407	57
0.35	0.19335	0.16709	129	0.85	0.14613	0.19747	53
0.40	0.19842	0.18350	113	0.90	0.12341	0.17106	50
0.45	0.20139	0.19771	100	0.95	0.09020	0.12930	48
0.50	0.20238	0.20956	90				



Lengths of confidence intervals

The question arises which model should be chosen to conduct an experiment. The comparison of the number of observations in the whole experiment may help in making an appropriate decision. Assume that in the binomial model there are n experiments. Then in the negative binomial model with $r(n)$ the expected number of observations equals

$$\frac{r(n)}{\pi}.$$

The expected number of observations in the negative binomial model is a decreasing function of π such that

$$\lim_{\pi\to 0} \frac{r(n)}{\pi} = \infty \quad \text{and} \quad \lim_{\pi\to 1} \frac{r(n)}{\pi} = r(n).$$

Because $r(n) < n$ there exists π_* such that

$$\frac{r(n)}{\pi} \begin{cases} > n, & \text{for } \pi < \pi_*, \\ = n, & \text{for } \pi = \pi_*, \\ < n, & \text{for } \pi > \pi_*. \end{cases}$$

In the last column of Table 4 we give the expected lengths of the experiment in the negative binomial model for $r(100) = 45$ (here $\pi_* = 0.45$).

Because calculating confidence intervals is very easy with the aid of computer software, using the shortest confidence interval is recommended, especially for small values of r. Comparison of the two models suggests that the negative binomial model is better for small values of π, while for large values the binomial model is the better one.

References

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