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PROBABILISTIC COMPARISON OF WEIGHTED MAJORITY RULES

Abstract. This paper studies a bi-parametric family of decision rules, so-called *restricted distinguished chairman rules*, which contains several one-parameter classes of rules considered previously in the literature. Roughly speaking, these rules apply to a variety of situations where the original committee appoints a subcommittee. Moreover, the chairman of the subcommittee, who is supposed to be the most competent committee member, may have more voting power than other jurors. Under the assumption of exponentially distributed decision skills, we obtain an analytic formula for the probability of any restricted distinguished chairman rule being optimal. We also study, for arbitrary fixed voting power of the chairman, the connection between the probability of the rule being optimal and the size of the subcommittee.

1. Introduction. There is a variety of situations where a group of n experts is required to select one of two alternatives, of which exactly one is correct. All decision makers share a common goal—identifying the correct alternative. This model is known as the *dichotomous choice model*, and goes back more than two centuries, as far as Condorcet. His famous work *Essai sur l'application de l'analyse à la probabilité des décisions rendues à la pluralité des voix* [14] plays a key role at the junction of decision making and philosophy. Applications of this model are relevant to a wide variety of areas, such as medicine, law, management and others (cf. McLean and Hewitt [24], Nurmi [28], Grofman and Owen [17], Miller [25], Urken [33], Berg [13], Ben-Yashar and Paroush [2], Berend, Sapir and Sapir [12]).

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A *decision rule* is a rule for translating the individual opinions into a group decision. The number of all possible group decision rules is 2^{2^n}. The most well-known group decision rule is the simple majority rule, strictly defined for odd-sized committees. However, for a committee of jurors with different decision skills, the simple majority rule is not necessarily the best procedure to follow (cf. Nitzan and Paroush [27], Karotkin and Schaps [23]). If the alternatives are symmetric, the experts are independent and the probability of each committee member to make the correct choice is known, then the optimal decision rule is a weighted majority rule (Nitzan and Paroush [26, 27], Shapley and Grofman [32]).

In the study of the dichotomous choice model, we focus on the direction concerned with the identification of the optimal decision rule under partial information on the decision skills. The probabilities of the expert and majority rules being optimal were calculated or estimated in a series of papers for a variety of distributions (cf. Nitzan and Paroush [26], Berend and Harmse [7], Berend and Sapir [8, 9, 11], Sapir [29, 30, 31]). Moreover, various families of weighted majority rules were considered by several authors (cf. Gradstein and Nitzan [16], Karotkin [18, 19], Karotkin and Nitzan [20], Berend and Sapir [10], Karotkin and Schaps [23], Berend, Chernyavsky and Sapir [6], Berend and Chernyavsky [5]). Gradstein and Nitzan [16] explored the families of balanced expert rules and restricted majority rules. For given correctness probabilities, they found a simple criterion for the optimality of such rules. Berend and Sapir [10] continued their study and calculated the probabilities of the restricted majority rule and of the balanced expert rule being optimal under the assumption of exponentially distributed decision skills. A bit later, for arbitrary correctness probabilities, Berend, Chernyavsky and Sapir [6] explored all possible rankings among the rules of each of the following families: balanced expert rules, restricted majority rules and distinguished chairman rules. For any group size, Karotkin [19] arranged all the weighted majority rules in a graph, whose nodes are the rules and whose edges correspond (not in a 1-1 manner) to voting profiles. This graph, together with the lengths of distinguishing profiles, for any given correctness probabilities, yields the ranking of the rules by their efficiency.

However, there are situations in which one cannot use every decision rule. For example, there may be certain institutional constraints dictating the usage of rules from a certain class. This paper studies a natural bi-parametric family of rules—restricted distinguished chairman rules—which contains as particular cases several more restricted families considered previously in the literature. Roughly speaking, these rules apply to the case where the original committee appoints a subcommittee. Moreover, while all members of this subcommittee are assumed to have the same decisional power, the chairman may have extra power. Under the assumption of exponential distribution, we generalize the results of Berend and Sapir [10], and provide an explicit formula for the probability of restricted distinguished chairman rule being optimal. In addition, we prove that, for an arbitrary fixed power of the most competent decision maker, the probability of the rule being optimal decreases as the size of the subcommittee increases.

The paper is organized as follows. In Section 2 we describe the setup more carefully and define all relevant families of rules. Section 3 contains the main results, and Section 4 their proofs. In Section 5 we summarize the paper and raise some questions for further research.

2. Model and main definitions. We study the dichotomous choice model. In this model, a committee consisting of n members is required to select one of two alternatives, denoted by 1 and -1, of which exactly one is correct. Suppose that the *a priori* probabilities of the two alternatives being correct are the same, and the losses associated with the two types of incorrect decisions are also the same. Thus, in the terminology of Nitzan and Paroush [27], the alternatives are assumed to be *symmetric*.

Suppose that each expert i, $1 \le i \le n$, selects independently of the others, with correctness probability p_i, where $1/2 \le p_i \le 1$, $1 \le i \le n$. Denote by $f(p_i) = \ln \frac{p_i}{1-p_i}$ the *log odds* of the ith member of the committee. (Strictly speaking, $f(p_i)$ is not well defined if $p_i = 1$. However, if any of the p_i's is 1, then we always know with certainty what the best decision is, so the situation becomes trivial.)

A decision rule translates all the individual opinions of the members into a group decision. More formally, a decision rule φ is a function from the set of all possible decision profiles $\{1, -1\}^n$ to the set $\{1, -1\}$. A decision rule is *optimal* if it maximizes the probability of the group to make a correct choice, for all possible combinations of opinions.

For known values of correctness probabilities p_i, $1 \le i \le n$, Nitzan and Paroush [27] and Shapley and Grofman [32] obtained the following criterion for identifying the optimal decision rule.

CRITERION 1. *The optimal decision rule is to choose the first alternative if and only if*

$$\sum_{i\in A} f(p_i) \ge \sum_{i\in B} f(p_i).$$

where $A \subseteq \{1, \dots, n\}$ is the set of group members recommending the first alternative, and $B = \{1, \dots, n\} \backslash A$ the set of those recommending the second.

In other words, the optimal decision rule is a *weighted majority rule*, defined by the system of weights $(f(p_1), \dots, f(p_n))$. We shall always assume that the group members are sorted, $p_1 \ge \dots \ge p_n$, so the optimal weights are non-increasing. Note that a weighted majority rule may be represented by numerous systems of weights. In general, two systems of weights define the same rule if they yield the same "winning coalitions". For instance, for $n = 3$ both systems of weights $(1, 1, 1)$ and $(3, 2, 2)$ define the simple majority rule, since in each of them the sum of any two of the weights exceeds the remaining weight. Similarly, both $(1, 0, 0)$ and $(5, 2, 2)$ define the expert rule, since the weight of the most qualified expert is larger than the sum of weights of the other two.

As explained earlier, we may be restricted to using rules of some type. In this paper we study the following bi-parametric family of rules.

Definition 1. Let k, a be of the same parity, where $1 \le a \le k \le n$. The *restricted distinguished chairman rule of order* (k, a) (henceforth $\mathrm{RDC}_{k,a}$) is given by the vector of weights

$$(a, 1, \overset{(k-1)}{\dots}, 1, 0, \overset{(n-k)}{\dots}, 0).$$

Each of the parameters k and a has a clear intuitive interpretation. Namely, the parameter k designates (unless $k = a$) the number of influential decision makers in the committee (or, alternatively, the size of the subcommittee selected to deal with the issue), while a provides the decisional power of the most competent member. Note that, for $k = a$, all systems of weights correspond to a single rule—the expert rule.

The particular case $a = 1$ gives the extensively studied class of restricted majority rules (cf. Gradstein and Nitzan [16], Karotkin [18], Karotkin and Nitzan [20], Berend and Sapir [10], Karotkin [19], Karotkin and Schaps [23], Karotkin and Paroush [22]), defined as follows.

Definition 2. Let k be odd, $1 \le k \le n$. The *restricted majority rule of order* k is denoted by $\mathrm{RMR}_{n,k}$ and is obtained by applying the simple majority rule to the subgroup of the k most competent experts. Namely, the rule is given by the system of weights $(1, \overset{(k)}{\dots}, 1, 0, \overset{(n-k)}{\dots}, 0)$.

Note that the family of restricted majority rules contains both polar decision rules (i.e., the simple majority rule for $k = n$, relevant for odd n, and the expert rule for $k = 1$) as particular instances.

Another special case of the restricted distinguished chairman rule, obtained for $a = k - 2$, is the family of balanced expert rules (cf. Gradstein and Nitzan [16], Berend and Sapir [10]), defined as follows:

Definition 3. Let $2 \le k \le n$. The *balanced expert rule of order* k is denoted by $\mathrm{BER}_{n,k}$ and is given by the system of weights

$$(k - 2, 1, \overset{(k-1)}{\dots}, 1, 0, \overset{(n-k)}{\dots}, 0).$$

The following 1-parameter family, studied by Berend, Chernyavsky and Sapir [6], is also a special case of the restricted distinguished chairman rule.

DEFINITION 4. Let a be of the same parity as n, where $1 \le a \le n$. The *distinguished chairman rule of order* a is denoted by $\mathrm{DCR}_{n,k}$ and is given by the system of weights $(a, 1, \overset{(n-1)}{\dots}, 1)$.

Note that this family also "connects" the simple majority and the expert rules.

Our class of rules contains another 1-parameter family of rules, the *restricted chairman rules.* This family is analogous to that of restricted majority rules for an even number of influential decision makers.

DEFINITION 5. Let k be even, $2 \le k \le n$. The *restricted chairman rule of order* k is denoted by $\mathrm{RCR}_{n,k}$ and is given by the system of weights $(2, 1, \overset{(k-1)}{\dots}, 1, 0, \overset{(n-k)}{\dots}, 0)$.

Similarly to Gradstein and Nitzan [16], we obtain the following simple criterion for optimality of the rules from the class of restricted distinguished chairman rules.

CRITERION 2. *The restricted distinguished chairman family rule is optimal if and only if*

$$w_1 + \sum_{i=(k+a)/2+1}^{k} w_i \ge \sum_{i=2}^{(k+a)/2} w_i + \sum_{i=k+1}^{n} w_i \tag{1}$$

and

$$\sum_{i=(k-a)/2+1}^{k} w_i \ge \sum_{i=1}^{(k-a)/2} w_i + \sum_{i=k+1}^{n} w_i, \tag{2}$$

where $w_i = f(p_i)$, $1 \le i \le n$.

Figure 1 depicts the family of restricted distinguished chairman rules for $n = 8$. In this case, the family contains 13 rules. In particular, for $k = n$ we see the distinguished chairman rules; for $a = 1$ and odd k the restricted majority rules (for $k = 1$ the expert rule); for $a = 2$ and even k the restricted chairman rules; for $a = k - 2$ the balanced expert rules. Note that several rules belong in this case to two of the subfamilies.

In practice, the assumption of full information on the decision skills is usually far from being satisfied. Therefore, a model incorporating incomplete information seems more plausible. Suppose that the ranking of the members is known, but the exact values of correctness probabilities of the experts (or, equivalently, the log odds) are unknown. Namely, we assume the correctness probabilities of the group members to be independent random variables,

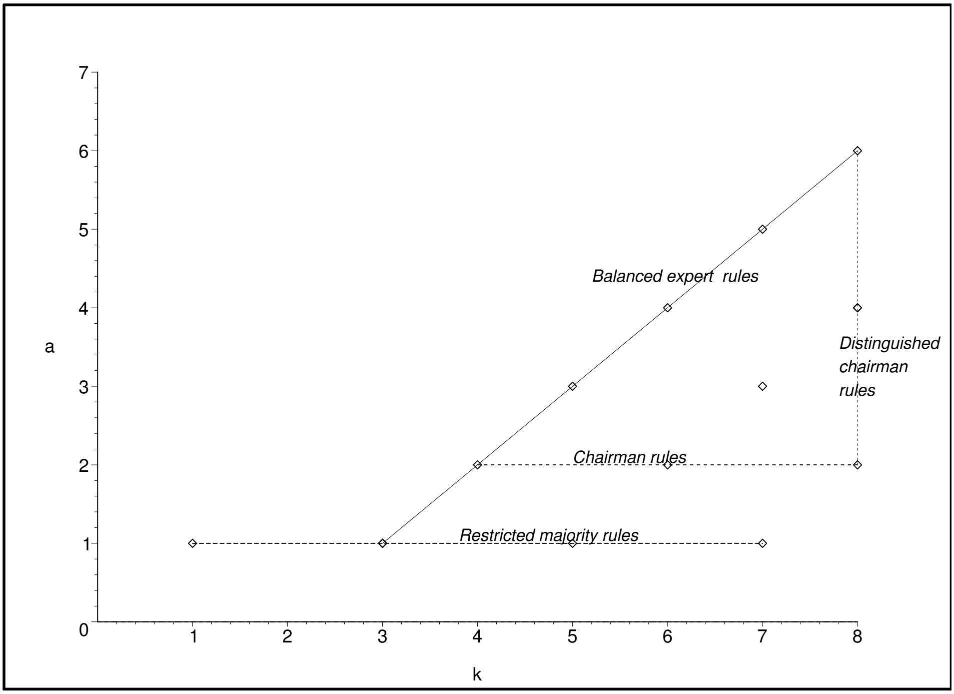

Fig. 1. All restricted distinguished chairman family rules for $n = 8$

distributed according to some distribution law. Thus, one can still follow rules based on the ranking of experts.

One of the commonly used measures of the efficiency of decision rules is their probability of being optimal. This may be roughly viewed as providing the performance of the rule in the worst case. For example, one verifies easily that the probability $P_{\mathrm{e}}(n)$ of the expert rule being optimal is the probability that, in case the top expert disagrees with all other experts, he is still more likely to be correct than all others. The probability $P_{\mathrm{m}}(n)$ of the majority rule being optimal is equal to the probability that the bottom $\lceil n/2 \rceil$ experts, when opposed by the top $\lfloor n/2 \rfloor$ experts, are more likely to be correct. The probabilities of the polar rules being optimal were compared in a series of papers for a variety of distributions (cf. Berend and Harmse [7], Berend and Sapir [8, 9], Sapir [29, 30, 31]). Under the assumption of exponentially distributed log odds, Berend and Sapir [10] calculated the probabilities of restricted majority rules and balanced expert rules being optimal. The present paper generalizes simultaneously all those results and provides an explicit formula for a wider class of decision rules being optimal. Namely, under the assumption of exponentially distributed log odds, we calculate the probability of the restricted distinguished chairman rule of order (k, a) being optimal.

3. Main result. The main results are formulated in terms of two natural parameters, $(k+a)/2$ and $(k-a)/2$, which have clear intuitive meaning. Namely, the first one is the minimal total weight required to accept one of the alternatives. The second parameter is the minimal number of influential experts the chairman needs for his opinion to be accepted.

In the following we use multinomial coefficients. These are denoted similarly to binomial coefficients, say $\binom{n}{n_1,\dots,n_r}$ denotes $\frac{n!}{n_1!\cdots n_r!}$ for $n_1+\dots+n_r = n$.

Theorem 1. *Let a,k be of the same parity. Suppose $f(p_i)$, $i=1,\dots,n$, are independent exponentially distributed random variables. Denote $s = (k+a)/2$ and $d=(k-a)/2$. The probability of the restricted distinguished chairman rule of order (k,a) to be optimal is*

$$P_{\mathrm{rdc}}(n,k,a) = \begin{cases} \dfrac{n\binom{n-1}{d,d,n-k}}{2^{n-1}ks^{n-s}d^d}, & \begin{array}{l} a=1, \\ a\le k\le n, \end{array} \\ \dfrac{n\binom{n-1}{d,d-2,n-k,a,1}}{2^{n-1}s^{n-s-1}(s-1)^{d+1}\binom{d(s-1)-1}{a}}, & \begin{array}{l} 2\le a\le n-4, \\ a+4\le k\le n, \end{array} \\ \dfrac{n\binom{n-1}{k-1}}{2^{n-1}(s-1)s^{n-k}}, & \begin{array}{l} 2\le a\le n-2, \\ k=a+2. \end{array} \end{cases} \tag{3}$$

The first case of (3) provides the optimality probability of restricted majority rules, and the third case—of balanced expert rules. The results in these cases coincide with those of Berend and Sapir [10]. (Note that, for $k=a=1$, the factor d^d in the denominator becomes 0^0, which should be taken as 1.) The subcase $a=2$ of the second case corresponds to restricted chairman rules, and the case $k=n$ (which intersects all three cases in the theorem)—to distinguished chairman rules.

Figure 2 exemplifies the optimality probabilities of the restricted distinguished chairman rules for $n=8$. In this case, we have 2470 weighted majority rules (cf. Karotkin [19]), 13 of which belong to our family. It is interesting to note that thus, on the average, the optimality probabilities of the rules in our family are much higher than those of typical weighted majority rules.

Note that the results depicted in Figure 2 hint that, for an arbitrary fixed a, the optimality probability is monotonic as a function of k. The following proposition formalizes this observation.

Proposition 1. *Let a,k be of the same parity. For arbitrary fixed n and a, the optimality probability of the restricted distinguished chairman rule of order (k,a) decreases as the number k of influential decision makers in the committee increases:*

$$P_{\mathrm{rdc}}(n,k,a) > P_{\mathrm{rdc}}(n,k+2,a), \qquad a\le k\le n-2. \tag{4}$$

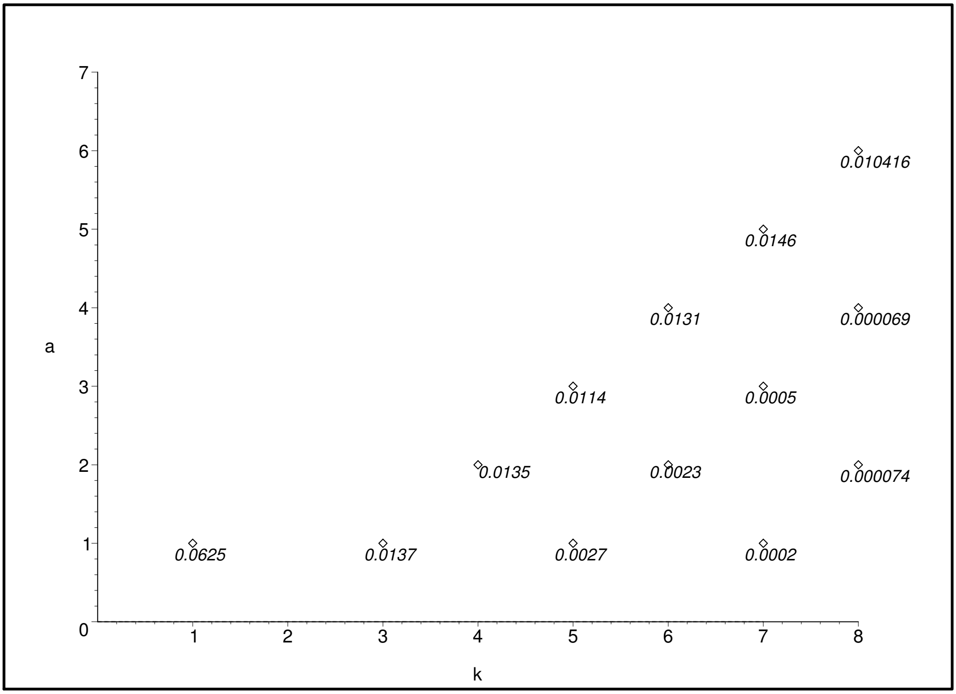

Fig. 2. Optimality probability of all restricted distinguished chairman rules for $n = 8$

Proposition 1 immediately implies

Corollary 1. *The expert rule has the maximal optimality probability among all restricted distinguished chairman rules. Namely,*

$$(5)\qquad P_{\mathrm{e}}(n) > P_{\mathrm{rdc}}(n,k,a), \quad 1 \le a \le n-2,\ a+2 \le k \le n.$$

Remark 1. The corollary does not necessarily hold if we replace the exponential distribution by some other distribution. In fact, it follows in particular from Theorems 1 and 3 of [11] that $P_{\mathrm{e}}(n)$ may be much smaller than the probability $P_{\mathrm{m}}(n)$ of the majority rule being optimal. Since the majority rule is just one special member of the family of restricted distinguished chairman rules, the corollary is not valid in general. For a specific example, we mention that, if the exponential distribution is replaced by the uniform distribution, then $P_{\mathrm{m}}(n)$ is much larger than $P_{\mathrm{e}}(n)$ (see [4, Example 3]).

The monotonicity property of Proposition 1 does not hold for the parameter a. As exemplified by Figure 2 for $k = 8$, the probability in question does not vary monotonically as a function of a in general. For example, for $n = 8$ the rule with the minimal optimality probability is the distinguished chairman rule of order 4.

4. Proofs

Proof of Theorem 1. The first and third cases in (3) were proved by Berend and Sapir [10]. Thus we deal in the proof only with the second case, namely, $2 \le a \le n-4$, $a+4 \le k \le n$. Let Y_i, $i = 1, \dots, n$, be the order statistics of $f(p_i)$, which are Exp(1) distributed:

$$Y_1 \ge \cdots \ge Y_n.$$

According to Criterion 2,

$$(6)\qquad P_{\mathrm{rdc}}(n,k,a) = P\Big(Y_1 + \sum_{i=(k+a)/2+1}^{k} Y_i \ge \sum_{i=2}^{(k+a)/2} Y_i + \sum_{i=k+1}^{n} Y_i, \quad \sum_{i=(k-a)/2+1}^{k} Y_i \ge \sum_{i=1}^{(k-a)/2} Y_i + \sum_{i=k+1}^{n} Y_i, \Big).$$

Denote

$$Z_i = Y_i - Y_{i+1}, \quad i = 1, \dots, n-1, \qquad Z_n = Y_n.$$

Since the variables X_i, $1 \le i \le n$, are independent exponentially distributed, the differences Z_i, $1 \le i \le n$, are also independent exponentially distributed, with parameter i [15, Sec. 1.6].

Representing the order statistics in terms of the Z_i's,

$$Y_i = \sum_{j=i}^{n} Z_j, \qquad i = 1, \dots, n,$$

we may rewrite the right-hand side of (6) in the form

$$(7)\qquad P\Big(Z_1 \ge \sum_{i=3}^{s-1}(i-2)Z_i + \sum_{i=s}^{k-1}(k+a-2-i)Z_i - \sum_{i=k}^{n}(k-a+2-i)Z_i \quad Z_1 \le -\sum_{i=2}^{d-1} iZ_i + \sum_{i=d}^{k-2}(k-a-i)Z_i + \sum_{i=k}^{n}(k+a-i)Z_i\Big).$$

Introducing the events

$$E_1 = \Big\{Z_1 \ge \sum_{i=3}^{s-1}(i-2)Z_i + \sum_{i=s}^{k-1}(k+a-2-i)Z_i - \sum_{i=k}^{n}(k-a+2-i)Z_i\Big\},$$

$$E_2 = \Big\{Z_1 \le -\sum_{i=2}^{d-1} iZ_i + \sum_{i=d}^{k-2}(k-a-i)Z_i + \sum_{i=k}^{n}(k+a-i)Z_i\Big\},$$

we may rewrite (7) as $P(E_1 \cap E_2)$. Hence

$$(8)\qquad P_{\mathrm{rdc}}(n,k,a) = n! \int_{\mathcal{D}} e^{-z_1} \cdot e^{-2z_2} \cdot \ldots \cdot e^{-nz_n}\, dz_1\, dz_2 \cdots dz_n,$$

where $\mathcal{D} \subseteq \mathbb{R}^n$ is the polyhedron determined by the inequalities

$$(9) \qquad \begin{cases} z_1, \dots, z_n \geq 0, \\ z_1 \geq \sum_{i=2}^{s-1}(i-2)z_i + \sum_{i=s}^{k-1}(k+a-2-i)z_i - \sum_{i=k}^{n}(k-a+2-i)z_i, \\ z_1 \leq -\sum_{i=2}^{d-1} iz_i + \sum_{i=d}^{k-2}(k-a-i)z_i + \sum_{i=k}^{n}(k+a-i)z_i. \end{cases}$$

We use the right-hand sides of the last two inequalities in (9) to define functions of $z_2, \dots, z_n$ as follows:

$$g^-(z_2, \dots, z_n) = \sum_{i=2}^{s-1}(i-2)z_i + \sum_{i=s}^{k-1}(k+a-2-i)z_i - \sum_{i=k}^{n}(k-a+2-i)z_i,$$

$$g^+(z_2, \dots, z_n) = -\sum_{i=2}^{d-1} iz_i + \sum_{i=d}^{k-2}(k-a-i)z_i + \sum_{i=k}^{n}(k+a-i)z_i.$$

For all points of $\mathcal{D}$, we must have $g^-(z_2, \dots, z_n) \leq g^+(z_2, \dots, z_n)$. Hence the projection of $\mathcal{D}$ on the subspace $\{z_1 = 0\}$ of $\mathbb{R}^n$ is given by the inequalities

$$(10) \qquad \begin{cases} z_2, z_3, \dots, z_{k-1}, z_{k+1}, \dots, z_n \geq 0, \\ z_k \geq \sum_{i=2}^{d-1}(i-1)z_i + \sum_{i=d}^{s}(d-1)z_i \\ \qquad + \sum_{i=s+1}^{k-1}(k-i-1)z_i + \sum_{i=k+1}^{n}(i-k-1)z_i. \end{cases}$$

Therefore, the polyhedron $\mathcal{D}$ is defined by the system of inequalities

$$\begin{cases} z_2, \dots, z_{k-1}, z_{k+1}, \dots, z_n \geq 0, \\ z_k \geq \sum_{i=2}^{d-1}(i-1)z_i + \sum_{i=d}^{s}(d-1)z_i \\ \qquad + \sum_{i=s+1}^{k-1}(k-i-1)z_i + \sum_{i=k+1}^{n}(i-k-1)z_i, \\ z_1 \geq \sum_{i=2}^{s-1}(i-2)z_i + \sum_{i=s}^{k-1}(k+a-2-i)z_i - \sum_{i=k}^{n}(k-a+2-i)z_i, \\ z_1 \leq -\sum_{i=2}^{d-1} iz_i + \sum_{i=d}^{k-2}(k-a-i)z_i + \sum_{i=k}^{n}(k+a-i)z_i. \end{cases}$$

With this ordering of the variables, the polyhedron $\mathcal{D}$ is repetitive according to the terminology of [3, Def. 1]. In other words, the probability we want to calculate can be expressed as a single repeated integral:

$$(11)\qquad P_{\mathrm{rdc}}(n,k,a) = n!\int_0^\infty \cdots \int_0^\infty \int_{g(z_2,\dots,z_{k-1},z_{k+1},\dots,z_n)}^\infty \int_{g^-(z_2,\dots,z_n)}^{g^+(z_2,\dots,z_n)} e^{-\sum_{i=1}^n iz_i}\, dz_1\, dz_k\, dz_2 \cdots dz_{k-1}\, dz_{k+1} \cdots dz_n.$$

For fixed non-negative $z_2,\dots,z_n$, denote

$$\widetilde{P}_1 = P(E_1 \cap E_2 \mid Z_i = z_i,\ 2 \le i \le n).$$

It is readily verified that $g^-(z_2,\dots,z_n) \ge 0$ for any non-negative z_i's. Hence

$$\begin{aligned}\widetilde{P}_1 &= \int_{g^-(z_2,\dots,z_n)}^{g^+(z_2,\dots,z_n)} e^{-z_1}\, dz_1 \\ &= \exp\Big(-\sum_{i=3}^{s-1}(i-2)z_i - \sum_{i=s}^{k-1}(k+a-2-i)z_i + \sum_{i=k}^{n}(k-a+2-i)z_i\Big) \\ &\quad - \exp\Big(\sum_{i=2}^{d-1} iz_i + \sum_{i=d}^{k-1}(k-a-i)z_i - \sum_{i=k}^{n}(k+a-i)z_i\Big).\end{aligned}$$

Thus

(12)

$$\begin{aligned}P(E_1 \cap E_2) &= \int_0^\infty \cdots \int_0^\infty \int_{g(z_2,\dots,z_{k-1},z_{k+1},\dots,z_n)}^\infty \widetilde{P}_1 n! \exp\Big(-\sum_{i=2}^n iz_i\Big)\, dz_2 \cdots dz_n \\ &= \int_0^\infty \cdots \int_0^\infty \int_{g(z_2,\dots,z_{k-1},z_{k+1},\dots,z_n)}^\infty H_1(z_2,\dots,z_n)\, dz_2\, dz_3 \cdots dz_n,\end{aligned}$$

where

$$\begin{aligned}H_1(z_2,\dots,z_n) &= n!\exp\Big(-2\sum_{i=d}^{k-1}(i-d)z_i - 2\sum_{i=k}^{n} sz_i\Big) \\ &\quad - n!\exp\Big(-2\sum_{3}^{s-1}(i-1)z_i - 2(s-1)\sum_{i=s}^{k-1} z_i + 2\sum_{i=k}^{n}(d+1-i)z_i\Big)\end{aligned}$$

and

$$\begin{aligned}g(z_2,\dots,z_{k-1},z_{k+1},\dots,z_n) &= \sum_{i=2}^{d-1}(i-1)z_i + \sum_{i=d}^{s}(d-1)z_i \\ &\quad + \sum_{i=s+1}^{k-1}(k-i-1)z_i + \sum_{i=k+1}^{n}(i-k-1)z_i.\end{aligned}$$

Clearly, $g(z_2,\dots,z_{k-1},z_{k+1},\dots,z_n)\ge 0$ for any non-negative $z_2,\dots,z_{k-1}$, $z_{k+1},\dots,z_n$. Denote

$$H_2(z_2,\dots,z_{k-1},z_{k+1},\dots,z_n) = \int\limits_{g(z_2,\dots,z_{k-1},z_{k+1},\dots,z_n)}^{\infty} H_1(z_2,\dots,z_n)\,dz_k = \frac{n!}{2(s-1)s}e^{Q},$$

where

$$Q = 2\Big(-s\sum_{i=2}^{d-1}(i-1)z_i - \sum_{i=d}^{s-1}((d-1)(s-1)+i-1)z_i - (s-1)\sum_{i=s}^{k-1}(k-i)z_i - s\sum_{i=k+1}^{n}(i-k)z_i\Big).$$

Thus

$$P_{\mathrm{rdc}}(n,k,a) = \int\limits_0^{\infty}\cdots\int\limits_0^{\infty} H_2(z_2,\dots,z_{k-1},z_{k+1},\dots,z_n)\,dz_2\cdots dz_{k-1}\,dz_{k+1}\cdots dz_n.$$

Since the integrand is a product of $n-2$ functions of a single variable each, we immediately obtain

$$\begin{aligned}P_{\mathrm{rdc}}(n,k,a) &= \frac{n!}{2(s-1)s}\prod_{i=2}^{d-1}\int\limits_0^{\infty} e^{-2s\sum_{i=2}^{d-1}(i-1)z_i}\,dz_i \\ &\quad\times\prod_{i=d}^{s-1}\int\limits_0^{\infty} e^{-2\sum_{i=d}^{s-1}((d-1)(s-1)+i-1)z_i}\,dz_i \\ &\quad\times\prod_{i=s}^{k-1}\int\limits_0^{\infty} e^{-2(s-1)\sum_{i=s}^{k-1}(k-i)z_i}\,dz_i\cdot\prod_{i=k+1}^{n}\int\limits_0^{\infty} e^{-2s\sum_{i=k+1}^{n}(i-k)z_i}\,dz_i \\ &= \frac{n!2^{-n+1}}{s(s-1)}\prod_{i=2}^{d-1}\frac{1}{s(i-1)}\prod_{i=d}^{s-1}\frac{1}{(d-1)(s-1)+i-1} \\ &\quad\times\prod_{i=s}^{k-1}\frac{1}{(s-1)(k-i)}\prod_{i=k+1}^{n}\frac{1}{s(i-k)}.\end{aligned}$$

Routine calculations give

$$\begin{aligned}P_{\mathrm{rdc}}(n,k,a) &= \frac{n!(d(s-1)-a-1)!}{2^{n-1}s^{n-s-1}(s-1)^{d+1}(n-k)!d!(d-2)!(d(s-1)-1)!} \\ &= \frac{n\binom{n-1}{d,d-2,n-k,a,1}}{2^{n-1}s^{n-s-1}(s-1)^{d+1}\binom{d(s-1)-1}{a}}.\ \blacksquare\end{aligned}$$

Before proving Proposition 1, we need the following lemma.

LEMMA 1. *For any integer $m \geq 2$ and real $1/2 \leq p \leq 1$ we have*

$$\binom{m}{2} p^{m-2}(1-p)^2 \leq \frac{3}{8}. \tag{13}$$

Proof. The inequality is readily verified for $2 \leq m \leq 9$.

For fixed $m \geq 10$, put

$$g(p) = p^{m-2}(1-p)^2, \qquad 1/2 \leq p \leq 1. \tag{14}$$

A routine calculation shows that $p_0 = (m-2)/m$ is the maximum point of g. Hence

$$g(p) \leq g\left(\frac{m-2}{m}\right) = \left(\frac{m-2}{m}\right)^{m-2}\left(\frac{2}{m}\right)^2, \qquad 1/2 \leq p \leq 1.$$

Denote

$$h(m) = \binom{m}{2} \cdot \left(\frac{m-2}{m}\right)^{m-2}\left(\frac{2}{m}\right)^2, \qquad m \geq 10.$$

Then

$$\begin{aligned} h(m) &= 2\left(1-\frac{1}{m}\right)\left(1-\frac{2}{m}\right)^{m-2} \leq 2\left(1-\frac{1}{m}\right)\left(1-\frac{1}{m}\right)^{2m-4} \\ &= 2\left(1-\frac{1}{m}\right)^{2m-3} \leq 2e^{3/m-2} \leq \frac{3}{8}, \qquad m \geq 10. \end{aligned} \tag{15}$$

The lemma now follows from (14) and (15). ■

Proof of Proposition 1. The case of $a = 1$ and general k and the case of general a and $k = a$ are covered by Propositions 1 and 7, respectively, of Berend and Sapir [10]. In the other cases, denote

$$R(n,k,a) = \frac{P_{\mathrm{rdc}}(n,k+2,a)}{P_{\mathrm{rdc}}(n,k,a)}, \qquad a \leq k \leq n-2. \tag{16}$$

Starting with $k = a+2$ we have

$$\begin{aligned} R(n,a+2,a) &= \frac{P_{\mathrm{rdc}}(n,a+4,a)}{P_{\mathrm{rdc}}(n,a+2,a)} \\ &= 2\binom{n-a-2}{2}\left(\frac{a+1}{a+2}\right)^{n-a-4}\left(\frac{1}{a+2}\right)^2 \frac{a(a+2)}{\binom{2a+2}{a+1}}, \qquad a \geq 2. \end{aligned} \tag{17}$$

By Lemma 1 with $m = n-a-2$ and $p = \frac{a+1}{a+2}$ we have

$$R(n,a+2,a) \leq \frac{3}{4} \cdot \frac{(a+2)!(a+1)!a}{(2a+2)!} = \frac{3}{4}\left(\prod_{j=3}^{a+1} \frac{j}{a+j}\right)\frac{2a}{2a+2} < 1, \tag{18}$$

which implies the result for $k = a+2$.

For $a+4 \le k \le n-2$ a routine calculation yields

$$R(n,k,a) = AG \prod_{j=0}^{a-1} \frac{\beta+j}{\beta+k+j}, \tag{19}$$

where

$$\beta = \frac{(k+a)(k-a-2)}{4},$$

$$G = \frac{(k+a+2)^2(k+a-2)}{(k-a+2)(k-a-2)(k+a)} \left(\frac{k+a-2}{k+a+2}\right)^{(k-a)/2}, \tag{20}$$

$$A = 2\binom{n-k}{2} \left(\frac{k+a}{k+a+2}\right)^{n-k-2} \left(1 - \frac{k+a}{k+a+2}\right)^2. \tag{21}$$

Now

$$\begin{aligned} \prod_{j=0}^{a-1} \frac{\beta+j}{\beta+k+j} &\le \frac{\beta}{\beta+k} \left(\frac{\beta+a}{\beta+k+a}\right)^{a-1} \\ &= \frac{(k+a)(k-a-2)}{(k+a+2)(k-a)} \left(\frac{(k+a-2)(k-a)}{(k+a)(k-a+2)}\right)^{a-1}. \end{aligned} \tag{22}$$

By Lemma 1 with $m = n-k$ and $p = (k+a)/(k+a+2)$ we have

$$A \le \frac{3}{4}. \tag{23}$$

Using (23) and (22) to estimate the right-hand side of (19), we obtain

$$R(n,k,a) \le \frac{3}{4} G \frac{(k+a)(k-a-2)}{(k+a+2)(k-a)} \left(\frac{(k+a-2)(k-a)}{(k+a)(k-a+2)}\right)^{a-1}. \tag{24}$$

By (20) we have therefore

$$\begin{aligned} &R(n,k,a) \\ &\le B \left(\frac{k-a+2}{k-a}\right) \left(\frac{k+a-2}{k+a+2}\right)^{\frac{k-a}{2}} \left(\frac{k-a+2}{k-a}\right)^{\frac{k-a}{2}} \left(\frac{k+a-2}{k+a}\right)^{a}, \end{aligned} \tag{25}$$

where

$$B = \frac{3}{2} \binom{\frac{k+a}{2}+1}{2} \left(\frac{k-a}{k-a+2}\right)^{\frac{k+a}{2}-1} \left(1 - \frac{k-a}{k-a+2}\right)^2.$$

Now clearly

$$\frac{k-a+2}{k-a} = 1 + \frac{2}{k-a} \le \frac{3}{2}, \qquad a+4 \le k \le n-2, \tag{26}$$

and by Lemma 1 with $m = (k+a)/2+1$ and $p = (k-a)/k-a+2$ we have

$$B \le \frac{9}{16}. \tag{27}$$

Employing (27) and (26) to bound the right-hand side of (25), we get

$$\begin{aligned} R(n,k,a) &\le \frac{9}{16}\cdot\frac{3}{2}\left(\frac{k+a-2}{k+a+2}\right)^{\frac{k-a}{2}}\left(\frac{k-a+2}{k-a}\right)^{\frac{k-a}{2}}\left(\frac{k+a-2}{k+a}\right)^{a} \\ &\le \frac{27}{32}e^{-\frac{2(k-a)}{(k+a+2)}}\cdot e\cdot e^{-\frac{2a}{k+a}} \\ &\le \frac{27}{32}e^{-\frac{2(k-a)}{(k+a+2)}+1-\frac{2a}{k+a+2}} \\ &\le \frac{27}{32}e^{-\frac{2}{k+a+2}} < 1, \end{aligned}$$

for $a+4 \le k \le n-2$, which completes the proof of the proposition. ■

5. Concluding remarks. Situations where a committee appoints a certain subcommittee to handle a particular issue occur frequently in a variety of institutions and organizations. What is the optimal size of the subcommittee? Should all the members of the subcommittee have the same voting power? This paper investigates a bi-parametric family of rules, with parameters corresponding to the subcommittee size and the chairman power. The paper provides a probabilistic comparison of the rules under the assumption of exponentially distributed log odds. Namely, Theorem 1 presents an analytical formula for the probability of any rule from this family being optimal. Proposition 1 claims that, increasing the number of subcommittee members, we decrease the optimality probability of the rule. It would be interesting to know whether the conclusion is robust, namely, whether this pattern is valid for a large variety of distributions.

Note that the optimality probability of a certain rule provides a view of the performance of the rule in its worst case, composed of its *borderline* cases. While we may usually expect a "reasonable" decision rule to lead to the correct decision, one should hesitate using the simple majority rule if, say, in a committee comprising 11 members, the 6 members known to be least qualified happen to favor one view while all the 5 more qualified members hold the opposite view. Similarly, employing the expert rule would seem strange if we happen to be in its borderline case—the top expert is opposed by all the others. To claim that, in a specific case, the majority rule, or the expert rule, is optimal, is tantamount to asserting that we should indeed favor the opinion of the 6 against the 5 in the first example, or of the top expert in the second example. Consequently, by comparing the probabilities of the rules being optimal we are provided with a view of the performance of the rules in question in some extreme cases, and hints to what extent we should rather modify them in those cases.

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