Agnieszka Rygiel (Tarnów)
Łukasz Stettner (Warszawa)

ARBITRAGE FOR SIMPLE STRATEGIES

Abstract. Various aspects of arbitrage on finite horizon continuous time markets using simple strategies consisting of a finite number of transactions are studied. Special attention is devoted to transactions without shortselling, in which we are not allowed to borrow assets. The markets without or with proportional transaction costs are considered. Necessary and sufficient conditions for absence of arbitrage are shown.

1. Introduction. Consider a market consisting of d risky assets with the prices given by an $\mathbb{R}^d$-valued adapted process $X = (X_t)_{t \in [0,T]}$ with strictly positive components, and of a bank account. We will assume for simplicity that the bank interest rate is equal to zero and we shall use the notation $\bar{X}_t = (1, X_t) \in \mathbb{R}^{d+1}$ for $t \in [0, T]$, where 1 stands for a unit amount in a bank account. Let $(\Omega, \mathcal{F}, P, \mathbb{F} = (\mathcal{F}_t)_{t \in [0,T]})$ be a filtered probability space satisfying the usual conditions, i.e. the filtration $(\mathcal{F}_t)_{t \in [0,T]}$ is right continuous, and $\mathcal{F}_0$ contains all the P-null sets of $\mathcal{F}$. Assume that the set of trading dates is a set of $\mathbb{F}$-stopping times $\{\tau_i : i = 1, \ldots, n\}$ such that $0 \le \tau_1 \le \cdots \le \tau_n \equiv T$ and $n \ge 2$. We admit only a finite number of transactions over a finite time horizon T, which is bounded by a deterministic number $n \ge 2$. In other words every admissible strategy should consist of a finite number of transactions, and for every strategy there is a deterministic integer $n \ge 2$ which bounds the number of transactions. Let the vector $\theta_i = (x_i, y_i) := (x_i, y_i^1, \ldots, y_i^d) \in \mathbb{R}^{d+1}$ be the position of the investor after transactions at time τ_i, where x_i is the position in a bank account and y_i^j is the number of jth risky assets held in the portfolio. We allow y_i^j to have an arbitrary sign (a negative quantity means shortselling), and assume it is $\mathcal{F}_{\tau_i}$-measurable, i.e. determined using information available at time τ_i. The

value of the portfolio at time t is modeled by the process

$$\sum_{i=1}^{n-1}\sum_{j=0}^{d}\theta_i^j \bar{X}_t^j \chi_{(\tau_i,\tau_{i+1}]}(t) = \sum_{i=1}^{n-1} x_i \chi_{(\tau_i,\tau_{i+1}]}(t) + \sum_{i=1}^{n-1}\sum_{j=1}^{d} y_i^j X_t^j \chi_{(\tau_i,\tau_{i+1}]}(t),$$

where for $\tau_i = \tau_{i+1}$ the interval $(\tau_i, \tau_{i+1}]$ is understood to consist of the point $\tau_i = \tau_{i+1}$ only. Then the random variable

$$\sum_{i=1}^{n-1}\sum_{j=0}^{d}\theta_i^j (\bar{X}_{\tau_{i+1}\wedge t}^j - \bar{X}_{\tau_i\wedge t}^j) = \sum_{i=1}^{n-1}\sum_{j=1}^{d} y_i^j (X_{\tau_{i+1}\wedge t}^j - X_{\tau_i\wedge t}^j)$$

models the gain or loss of our portfolio at time t. Consequently, the gain (or loss) $(Y \cdot X)_T$ over the whole time period $[0,T]$ is equal to

$$(1.1) \qquad (Y \cdot X)_T := \sum_{i=1}^{n-1}\sum_{j=1}^{d} y_i^j (X_{\tau_{i+1}}^j - X_{\tau_i}^j).$$

We start with the following definition:

DEFINITION 1.1. We call an $\mathbb{R}^d$-valued process $Y = (Y_t)_{t\in[0,T]}$ a *simple investment strategy* if there exists a positive integer $n \geq 2$ and a sequence of $\mathbb{F}$-stopping times $0 \leq \tau_1 \leq \cdots \leq \tau_n \equiv T$ such that

$$(1.2) \qquad Y_t = \sum_{i=1}^{n-1} y_i \chi_{(\tau_i,\tau_{i+1}]}(t),$$

where $y_i \in \mathbb{R}^d$ are $\mathcal{F}_{\tau_i}$-measurable random vectors.

The class of such strategies will be denoted by $\mathcal{S}_T$. Note that a simple strategy may consist of a random number of transactions over the time horizon T, but this number should be bounded by a deterministic constant. In this case we are allowed to have multiple transactions at time T and consequently we may restrict ourselves to a deterministic number of transactions.

We now restrict our investment strategies to the following class of simple strategies over time horizon T without shortselling:

DEFINITION 1.2. We call an $\mathbb{R}^d$-valued process $Y = (Y_t)_{t\in[0,T]}$ a *simple investment strategy without shortselling* if $Y \in \mathcal{S}_T$ and $y_i \in \mathbb{R}^d$ are nonnegative $\mathcal{F}_{\tau_i}$-measurable random vectors.

The set of simple investment strategies without shortselling is denoted by $\mathcal{S}_T^+$.

DEFINITION 1.3. We say that X admits a *simple arbitrage strategy* if there exists $Y \in \mathcal{S}_T$ with the properties

$$(1.3) \qquad (Y \cdot X)_T \geq 0 \ P\text{-a.e.} \quad \text{and} \quad P\{(Y \cdot X)_T > 0\} > 0.$$

Otherwise we say for brevity that X satisfies *condition* (AA) (absence of arbitrage over simple strategies).

DEFINITION 1.4. We say that X admits a *simple arbitrage strategy without shortselling* if there exists $Y \in \mathcal{S}_T^+$ such that (1.3) holds. Otherwise we say that X satisfies *condition* (AA$^+$) (absence of arbitrage over simple strategies without shortselling).

In what follows we shall consider conditions (AA) or (AA$^+$) for simple strategies only. Furthermore everywhere in the paper the terminal time T is assumed to be fixed.

Part of the paper is devoted to a finite horizon continuous time financial market with proportional transaction costs. Assume that investing in the jth assets an amount l^j we have to pay a portion of λ^j ($\lambda^j \in (0,1)$) as transaction costs so that in fact we have to spend $(1 + \lambda^j)l^j$. Similarly, selling the jth assets for an amount m^j we only obtain $(1 - \mu^j)m^j$ with $\mu^j \in (0,1)$. Assume that the set of trading dates is a set of stopping times $\{\tau_i : i = 1, \ldots, n\}$ such that $0 \le \tau_1 \le \cdots \le \tau_n \equiv T$ and $n \ge 2$. Consequently, each trading strategy consists of transactions whose number is bounded by a deterministic constant n (which may depend on the strategy). We assume furthermore that at terminal time T we may have multiple transactions so that practically we shall study investments with a deterministic number of transactions. In our approach the investor positions are measured in "physical" units. Let $(x_i, y_i^1, \ldots, y_i^d) \in \mathbb{R}^{d+1}$ be the position of the investor after transactions at time τ_i, which consists of x_i, the amount in the bank account, and "physical" quantities y_i^j in the jth risky assets for $j = 1, \ldots, d$ respectively. Denoting by $(\Delta y_i^j)^+$, $(\Delta y_i^j)^-$ the number of assets with which we increase or decrease the jth assets position at time τ_i (given by $(\Delta y_i^j)^+ = (y_i^j - y_{i-1}^j)^+$ and $(\Delta y_i^j)^- = (y_i^j - y_{i-1}^j)^-$) we obtain the following formula for the investment position $(x_i, y_i^1, \ldots, y_i^d)$ at time τ_i (after transactions):

$$(1.4) \quad \begin{cases} x_i = x_0 + \sum_{j=1}^{d}\left[(1 - \mu^j)\sum_{k=1}^{i}(\Delta y_k^j)^- X_{\tau_k}^j - (1 + \lambda^j)\sum_{k=1}^{i}(\Delta y_k^j)^+ X_{\tau_k}^j\right], \\ y_i^1 = y_0^1 + \sum_{k=1}^{i}\Delta y_k^1, \\ \vdots \\ y_i^d = y_0^d + \sum_{k=1}^{i}\Delta y_k^d, \end{cases}$$

with the initial position $(x_0, y_0^1, \ldots, y_0^d)$.

DEFINITION 1.5. We call an $\mathbb{R}^d$-valued process $Y = (Y_t)_{t \in [0,T]}$ a *simple investment strategy with transaction costs* if there exists a positive integer $n \geq 2$ and a sequence of $\mathbb{F}$-stopping times $0 \leq \tau_1 \leq \cdots \leq \tau_n \equiv T$ such that

$$Y_t = \sum_{i=1}^{n-2} y_i \chi_{(\tau_i, \tau_{i+1}]}(t) + y_{n-1} \chi_{(\tau_{n-1}, \tau_n)}(t) + y_n \chi_{\{T\}}(t),$$

where $y_i \in \mathbb{R}^d$ are $\mathcal{F}_{\tau_i}$-measurable random vectors.

The class of such simple strategies will be denoted by $\mathcal{B}_T$. As above, $(\tau_k, \tau_{k+1}] = \{\tau_{k+1}\}$ for $\tau_k = \tau_{k+1}$, and additionally $(\tau_k, \tau_{k+1}) = \{\tau_{k+1}\}$ whenever $\tau_k = \tau_{k+1} = T$.

DEFINITION 1.6. We call an $\mathbb{R}^d$-valued process $Y = (Y_t)_{t \in [0,T]}$ a *simple investment strategy without shortselling* if $Y \in \mathcal{B}_T$ and $y_i^j \geq 0$ for any $i = 1, \ldots, n$, $j = 1, \ldots, d$. The set of simple investment strategies without shortselling is denoted by $\mathcal{B}_T^+$.

Define the evaluation function

$$(1.5) \quad R_t(x, y^1, \ldots, y^d) := x + \sum_{j=1}^{d}[(1 - \mu^j)(y^j)^+ X_t^j - (1 + \lambda^j)(y^j)^- X_t^j]$$

and set

$$(1.6) \quad G_t := \{(x, y^1, \ldots, y^d) \in \mathbb{R}^{d+1} : R_t(x, y^1, \ldots, y^d) \geq 0\}.$$

Note that if $(x, y^1, \ldots, y^d) \in G_t$ we are able to repay possible debts in the bank account or risky asset accounts. In other words G_t is the set of all nonnegative positions. The set $-G_t$ is of the form

$$-G_t := \{(x, y^1, \ldots, y^d) \in \mathbb{R}^{d+1} : R_t(-x, -y^1, \ldots, -y^d) \geq 0\}$$
$$= \left\{x + \sum_{j=1}^{d}[(1 + \lambda^j)(y^j)^+ X_t^j - (1 - \mu^j)(y^j)^- X_t^j] \leq 0\right\}$$

and is the set of all positions which can be achieved starting from zero.

In the case of models with transaction costs the concept of arbitrage admits various natural generalizations. We say that we have a *weak arbitrage* at time T if starting from the position $(0, \ldots, 0)$ at time 0 we enter at time T the set of nonnegative positions a.s. and with positive probability the position $(x_n, y_n^1, \ldots, y_n^d) \neq (0, \ldots, 0)$. Denote by $A^T(\tau_1, \ldots, \tau_n)$ the set of all positions which can be achieved at time T starting at time 0 from the position $(0, \ldots, 0)$ with the use of simple strategies with transaction times $0 \leq \tau_1 \leq \cdots \leq \tau_n = T$. Clearly $A^T(\tau_1, \ldots, \tau_n) = \sum_{i=1}^{n} L^0(-G_{\tau_i}, \mathcal{F}_{\tau_i})$, where the sum stands for the algebraic sum and $L^0(-G_{\tau_i}, \mathcal{F}_{\tau_i})$ is the class of $\mathcal{F}_{\tau_i}$-measurable random variables taking values in $-G_{\tau_i}$ (in the case without shortselling an analog of this equality does not hold). Moreover, $A^T =$

$\bigcup_{n\in\mathbb{N}}\bigcup_{0\leq\tau_1\leq\cdots\leq\tau_n\equiv T} A^T(\tau_1,\ldots,\tau_n)$ is the class of all positions which can be achieved at time T starting from $(0,\ldots,0)$ and using simple strategies.

DEFINITION 1.7. We say that there is a *weak arbitrage opportunity* if

(WA) $$A^T \cap L^0(G_T, \mathcal{F}_T) \neq \{0\}.$$

If there is no weak arbitrage we have strict absence of arbitrage:

DEFINITION 1.8. We say that there is *strict absence of arbitrage* if

(AA$_\mathrm{s}$) $$A^T \cap L^0(G_T, \mathcal{F}_T) = \{0\}.$$

In the case of a model without transaction costs this definition coincides with the classical one. If starting from the position $(0,\ldots,0)$ we get at time T the position $(x_n, y_n^1, \ldots, y_n^d)$ such that

$$R_T(x_n, y_n^1, \ldots, y_n^d) \geq 0 \ P\text{-a.e.}, \quad P\{R_T(x_n, y_n^1, \ldots, y_n^d) > 0\} > 0,$$

we say that we have a *strict arbitrage opportunity*. Using the definition of the set G_T we can say that we enter the set G_T a.s. and int G_T with positive probability. If there is no strict arbitrage we have weak absence of arbitrage:

DEFINITION 1.9. We say that there is *weak absence of arbitrage* if

(AA$_\mathrm{w}$) $$A^T \cap L^0(G_T, \mathcal{F}_T) \subset L^0(\partial G_T, \mathcal{F}_T).$$

LEMMA 1.10. *The following conditions are equivalent:*

(i) $A^T \cap L^0(G_T, \mathcal{F}_T) \subset L^0(\partial G_T, \mathcal{F}_T)$, *i.e.* (AA$_\mathrm{w}$) *holds.*
(ii) $A^T \cap L^0(\mathbb{R}_+^{d+1}, \mathcal{F}_T) = \{0\}$.

Proof. Since $\mathbb{R}_+^{d+1} \cap \partial G_T = \{0\}$ the implication (i)$\Rightarrow$(ii) holds true. Suppose now that (i) does not hold. Then there exists $Y \in \mathcal{B}_T$, e.g. there exists $n \geq 2$, a sequence of $\mathbb{F}$-stopping times $0 \leq \tau_1 \leq \cdots \leq \tau_n \equiv T$ and sequences of $\mathbb{R}^d$-valued random vectors $y_1, \ldots, y_n$ such that

$$(x_n, y_n^1, \ldots, y_n^d) \in L^0(G_T, \mathcal{F}_T) \ P\text{-a.e.}, \quad P\{(x_n, y_n^1, \ldots, y_n^d) \in \text{int } G_T\} > 0.$$

Then we can construct the following simple strategy $\tilde{Y} \in \mathcal{B}_T$:

$(\Delta\tilde{y}_i^j)^- = (\Delta y_i^j)^-$ and $(\Delta\tilde{y}_i^j)^+ = (\Delta y_i^j)^+$ for $i = 1, \ldots, n-1; j = 1, \ldots, d$;
$(\Delta\tilde{y}_n^j)^- = (\Delta y_n^j)^- + (y_n^j)^+ \chi_{\{y_n^j > 0\}}$ for $j = 1, \ldots, d$;
$(\Delta\tilde{y}_n^j)^+ = (\Delta y_n^j)^+ + (y_n^j)^- \chi_{\{y_n^j < 0\}}$ for $j = 1, \ldots, d$,

which means that at time T we liquidate the asset account. Clearly

$$(\tilde{x}_n, \tilde{y}_n^1, \ldots, \tilde{y}_n^d) \in L^0(\mathbb{R}_+^{d+1}, \mathcal{F}_T)$$

and

$$\{(x_n, y_n^1, \ldots, y_n^d) \in \text{int } G_T\} \subset \{(\tilde{x}_n, \tilde{y}_n^1, \ldots, \tilde{y}_n^d) \neq 0\},$$

so (ii) does not hold. ∎

Consequently we have

COROLLARY 1.11. *There is a strict arbitrage opportunity if*

(SA) $$A^T \cap L^0(\mathbb{R}_+^{d+1}, \mathcal{F}_T) \neq \{0\}.$$

We can also characterize weak and strict absence of arbitrage using strategies without shortselling. Let A_+^T be the set of all positions which can be achieved at time T starting at time 0 from the position $(0, \ldots, 0)$ using simple strategies without shortselling. Denote

(1.7) $$\tilde{G}_t := \{(x, y^1, \ldots, y^d) \in \mathbb{R} \times \mathbb{R}_+^d : R_t(x, y^1, \ldots, y^d) \geq 0\}.$$

DEFINITION 1.12. We say that there is a *weak arbitrage opportunity without shortselling* if

(WA$^+$) $$A_+^T \cap L^0(\tilde{G}_T, \mathcal{F}_T) \neq \{0\}.$$

DEFINITION 1.13. We say that there is a *strict arbitrage opportunity without shortselling* if

(SA$^+$) $$A_+^T \cap L^0(\mathbb{R}_+^{d+1}, \mathcal{F}_T) \neq \{0\}.$$

DEFINITION 1.14. We say that there is *strict absence of arbitrage without shortselling* if

(AA$_s^+$) $$A_+^T \cap L^0(\tilde{G}_T, \mathcal{F}_T) = \{0\}.$$

DEFINITION 1.15. We say that there is *weak absence of arbitrage without shortselling* if

(AA$_w^+$) $$A_+^T \cap L^0(\mathbb{R}_+^{d+1}, \mathcal{F}_T) = \{0\}.$$

Clearly $\tilde{G}_T = \{(x, y^1, \ldots, y^d) \in \mathbb{R} \times \mathbb{R}_+^d : x + \sum_{j=1}^d (1 - \mu^j) y^j X_T^j \geq 0\}$. In other words, X admits a weak arbitrage opportunity with respect to the class of simple strategies without shortselling if there exists $Y \in \mathcal{B}_T^+$ such that $(x_n, y_n^1, \ldots, y_n^d)$ satisfies

$$x_n + \sum_{j=1}^d (1 - \mu^j) y_n^j X_T^j \geq 0 \ P\text{-a.e.}, \quad P\{(x_n, y_n^1, \ldots, y_n^d) \neq 0\} > 0.$$

Analogously, a simple investment strategy $Y \in \mathcal{B}_T^+$ with the final position $(x_n, y_n^1, \ldots, y_n^d)$ such that

$$x_n + \sum_{j=1}^d (1 - \mu^j) y_n^j X_T^j \geq 0 \ P\text{-a.e.}, \quad P\Big\{x_n + \sum_{j=1}^d (1 - \mu^j) y_n^j X_T^j > 0\Big\} > 0$$

is a strict arbitrage opportunity with respect to the class of simple strategies without shortselling.

In this paper we formulate a series of necessary and sufficient conditions for absence of arbitrage in markets without or with proportional transaction costs using simple strategies with or without shortselling. There are two

reasons for the interest in transactions using simple strategies. First of all, this is the only feasible class of investment strategies. Secondly, this class allows us to get rid of the restrictive assumption that the asset price process is a semimartingale (for more details see [7]). In this paper we are mainly interested in strategies without shortselling, since in many financial markets shortselling is restricted or even forbidden.

The paper consists of two sections devoted to markets without or with transaction costs respectively. In Section 2 we extend and modify results of [2], [7] and [1]. In particular, we introduce certain conditions (a) and (b) and their various equivalent forms, which are satisfied in the case of absence of arbitrage (AA) and absence of arbitrage without shortselling (AA$^+$) for one-dimensional markets (with only one risky asset). We also give two conditions (c) and (d) which characterize (AA) and (AA$^+$) in the multidimensional case (with $d > 1$ risky assets). We complete Section 2 with a modification of Kabanov and Stricker's result from [9], which allows a general supermartingale characterization of (AA$^+$).

Markets with proportional transaction costs (studied in Section 3) are more difficult. We cannot restrict ourselves to two transaction dates as in the case without transactions costs, which is shown by two illustrative examples. Following [13] we introduce condition (S) under which we show (AA$_\mathrm{w}$) for $d = 1$ and extend this to the multidimensional case. The main contributions in that section are theorems in which we introduce conditions (D) and (D^d) which guarantee (AA$_\mathrm{w}^+$) for one and multidimensional cases. We complete Section 3 by recalling some results of Kabanov, Rasonyi, Stricker and Grigoriev and formulating them in the embedded discrete time market approach. We also point out that similar characterizations for markets without shortselling are not available.

2. Markets without transaction costs

2.1. Preliminary results. The following lemma shows that there is no difference between an arbitrage opportunity with respect to the class of simple investment strategies and arbitrage opportunities for strategies with two transaction times only. Therefore absence of arbitrage can be reduced to the two transactions problem (see also Lemma 1 in [7]).

LEMMA 2.1. *The process X admits a simple arbitrage strategy if and only if there exists an investment strategy $G \in \mathcal{S}_T$ of the form $G = g\chi_{(\sigma_1, \sigma_2]}$ such that $g(X_{\sigma_2} - X_{\sigma_1}) \geq 0$ P-a.e. and $P\{g(X_{\sigma_2} - X_{\sigma_1}) > 0\} > 0$.*

Proof. Sufficiency: It is clear that an investment strategy $G \in \mathcal{S}_T$ of the form above satisfies $P\{(G \cdot X)_T \geq 0\} = 1$ and $P\{(G \cdot X)_T > 0\} > 0$, so X admits a simple arbitrage strategy.

Necessity: Assume that there exists a simple strategy $Y = \sum_{i=1}^{n-1} y_i \chi_{(\tau_i, \tau_{i+1}]}$ such that $(Y \cdot X)_T \geq 0$ P-a.e. and $P\{(Y \cdot X)_T > 0\} > 0$. Define

$$k = \min \left\{ l \in \{1, \ldots, n-1\} : P\left\{ \sum_{i=1}^{l} y_i (X_{\tau_{i+1}} - X_{\tau_i}) \geq 0 \right\} = 1 \right.$$

$$\left. \text{and } P\left\{ \sum_{i=1}^{l} y_i \cdot (X_{\tau_{i+1}} - X_{\tau_i}) > 0 \right\} > 0 \right\}.$$

Note that k is well-defined because we assume that the arbitrage strategy is given by Y. If $k = 1$ then $y_1(X_{\tau_2} - X_{\tau_1}) \geq 0$ P-a.e. and $y_1(X_{\tau_2} - X_{\tau_1}) > 0$ with positive probability. Take $\sigma_i = \tau_i$ for $i = 1, 2$ and $g = y_1$; then the condition holds true. So we assume $k > 1$. From the definition of k we have either

$$\sum_{i=1}^{k-1} y_i (X_{\tau_{i+1}} - X_{\tau_i}) \leq 0 \ P\text{-a.e.}, \quad \text{or} \quad P\left\{ \sum_{i=1}^{k-1} y_i (X_{\tau_{i+1}} - X_{\tau_i}) < 0 \right\} > 0.$$

We first consider the case $P\{ \sum_{i=1}^{k-1} y_i (X_{\tau_{i+1}} - X_{\tau_i}) < 0 \} > 0$. Let

$$A = \left\{ \sum_{i=1}^{k-1} y_i (X_{\tau_{i+1}} - X_{\tau_i}) < 0 \right\}.$$

Note that $A \in \mathcal{F}_{\tau_k}$ and on the set A we have

$$0 \leq \sum_{i=1}^{k} y_i (X_{\tau_{i+1}} - X_{\tau_i}) = \sum_{i=1}^{k-1} y_i (X_{\tau_{i+1}} - X_{\tau_i}) + y_k (X_{\tau_{k+1}} - X_{\tau_k})$$

$$< y_k (X_{\tau_{k+1}} - X_{\tau_k}).$$

This means that $P\{ y_k (X_{\tau_{k+1}} - X_{\tau_k}) > 0 \} \geq P(A) > 0$. Taking $g = y_k \chi_A$ and stopping times $\sigma_1 = \tau_k$, $\sigma_2 = \tau_{k+1}$ we get the given condition. Assume next that $\sum_{i=1}^{k-1} y_i (X_{\tau_{i+1}} - X_{\tau_i}) \leq 0$ P-a.e.; then $\sum_{i=1}^{k} y_i (X_{\tau_{i+1}} - X_{\tau_i}) \leq y_k (X_{\tau_{k+1}} - X_{\tau_k})$ P-a.e. By the definition of k,

$$P\{ y_k (X_{\tau_{k+1}} - X_{\tau_k}) \geq 0 \} = 1 \quad \text{and} \quad P\{ y_k (X_{\tau_{k+1}} - X_{\tau_k}) > 0 \} > 0.$$

We now take $g = y_k$, $\sigma_1 = \tau_k$, $\sigma_2 = \tau_{k+1}$. ∎

LEMMA 2.2. *The process X admits a simple arbitrage strategy without shortselling if and only if there exists $G \in \mathcal{S}_T^+$ of the form $G = g\chi_{(\sigma_1, \sigma_2]}$ such that $g(X_{\sigma_2} - X_{\sigma_1}) \geq 0$ P-a.e. and $P\{ g(X_{\sigma_2} - X_{\sigma_1}) > 0 \} > 0$.*

Proof. The proof is analogous to that of Lemma 2.1. ∎

We list without proofs two simple lemmas and a corollary which will be used later.

LEMMA 2.3. *If $X \geq 0$ almost surely and $P(C) > 0$ for $C = \{X > 0\}$, then $E(X \mid \mathcal{F}) > 0$ on $C \setminus N$, where N is some null set.*

LEMMA 2.4. *The following conditions are equivalent:*

(i) $P(B \mid \mathcal{F}) > 0$ *P-a.e.;*
(ii) $P(A \cap B) > 0$ *for any $A \in \mathcal{F}$ with $P(A) > 0$.*

COROLLARY 2.5. *Let B, C, D be any events. Suppose that either $P(B \mid \mathcal{F}) > 0$ P-a.e. and $P(C \mid \mathcal{F}) > 0$ P-a.e., or $P(D \mid \mathcal{F}) = 1$ P-a.e. Then for any $A \in \mathcal{F}$ with $P(A) > 0$, either $P(A \cap B) > 0$ and $P(A \cap C) > 0$, or $P(A \cap D) = P(A)$.*

2.2. One-dimensional characterizations. In this section we consider a market with one risky asset with price process $\{X_t\}_{t \in [0,T]}$ and a bank account with $X_t^0 = 1$ for $t \in [0, T]$. We provide conditions for absence of arbitrage over simple strategies with or without shortselling.

We begin with a lemma, which gives a necessary and sufficient condition for absence of arbitrage over the class $\mathcal{S}_T$. This is a slightly modified version of Lemma 1 from [7].

LEMMA 2.6. *The process X satisfies condition* (AA) (*over simple strategies*) *if and only if for any two stopping times $\tau_1 < \tau_2 \leq T$ and for any $A \in \mathcal{F}_{\tau_1}$ with $P(A) > 0$ we have either*

$$P(A \cap \{X_{\tau_2} - X_{\tau_1} > 0\}) > 0 \quad and \quad P(A \cap \{X_{\tau_2} - X_{\tau_1} < 0\}) > 0,$$

or $X_{\tau_2} = X_{\tau_1}$ a.s. on A.

Proof. Necessity: Assume X does not admit an arbitrage opportunity with respect to the class $\mathcal{S}_T$. Let $\tau_1 < \tau_2 \leq T$ be two stopping times and $A \in \mathcal{F}_{\tau_1}$ with $P(A) > 0$. Suppose to the contrary that either

$$P(A \cap \{X_{\tau_2} - X_{\tau_1} > 0\}) = 0 \quad and \quad P(A \cap \{X_{\tau_2} \neq X_{\tau_1}\}) > 0,$$

or

$$P(A \cap \{X_{\tau_2} - X_{\tau_1} < 0\}) = 0 \quad and \quad P(A \cap \{X_{\tau_2} \neq X_{\tau_1}\}) > 0.$$

Without loss of generality we can assume that the first case holds. It follows that $X_{\tau_2} \leq X_{\tau_1}$ on A and $P(A \cap \{X_{\tau_2} \neq X_{\tau_1}\}) > 0$. Then $-\chi_A \chi_{(\tau_1, \tau_2]} \in \mathcal{S}_T$ is an arbitrage strategy for X, a contradiction to absence of arbitrage.

Sufficiency: Assume the second equivalent condition in Lemma 2.6 is satisfied, but X does admit a simple arbitrage strategy. By Lemma 2.1 there is $G \in \mathcal{S}_T$ of the form $G = g\chi_{(\sigma_1, \sigma_2]}$ such that $g(X_{\sigma_2} - X_{\sigma_1}) \geq 0$ P-a.e. and $P(g(X_{\sigma_2} - X_{\sigma_1}) > 0) > 0$. Let $B = \{g(X_{\sigma_2} - X_{\sigma_1}) > 0\}$ and $A_1 = \{g > 0\} \in \mathcal{F}_{\tau_1}$, $A_2 = \{g < 0\} \in \mathcal{F}_{\tau_1}$. Note that $P(g \neq 0) > 0$, since otherwise $g(X_{\sigma_2} - X_{\sigma_1}) = 0$ P-a.e., contrary to $P(g(X_{\sigma_2} - X_{\sigma_1}) > 0) > 0$. Since $P(B) > 0$ we have either $P(B \cap A_1) > 0$ or $P(B \cap A_2) > 0$. We need to reach a contradiction in both cases.

If $P(B \cap A_1) > 0$ then clearly $P(A_1) > 0$ and by our assumption either $P(A_1 \cap \{X_{\sigma_2} - X_{\sigma_1} > 0\}) > 0$ and $P(A_1 \cap \{X_{\sigma_2} - X_{\sigma_1} < 0\}) > 0$, or $X_{\sigma_2} = X_{\sigma_1}$ on A_1. On the other hand, since $P(B \cap A_1) > 0$ then $P(A_1 \cap \{X_{\sigma_2} \neq X_{\sigma_1}\}) > 0$ and therefore we have $P(A_1 \cap \{X_{\sigma_2} - X_{\sigma_1} > 0\}) > 0$ and $P(A_1 \cap \{X_{\sigma_2} - X_{\sigma_1} < 0\}) > 0$. But then $g(X_{\sigma_2} - X_{\sigma_1}) < 0$ on A_1 with positive probability, which contradicts $P(g(X_{\sigma_2} - X_{\sigma_1}) \geq 0) = 1$.

If $P(B \cap A_2) > 0$ we come to a contradiction analogously. ∎

Using Corollary 2.5 we show the following implication:

COROLLARY 2.7. *Suppose that for any two stopping times $\tau_1 < \tau_2 \leq T$ either*

$$P(X_{\tau_2} - X_{\tau_1} > 0 \mid \mathcal{F}_{\tau_1}) > 0 \ \textit{P-a.e.} \quad \text{and} \quad P(X_{\tau_2} - X_{\tau_1} < 0 \mid \mathcal{F}_{\tau_1}) > 0 \ \textit{P-a.e.},$$

or

$$P(X_{\tau_2} = X_{\tau_1} \mid \mathcal{F}_{\tau_1}) = 1 \quad \textit{P-a.e.}$$

Then X satisfies condition (AA).

Proof. From Corollary 2.5 we know that for any $A \in \mathcal{F}$ with $P(A) > 0$ either $P(A \cap \{X_{\tau_2} - X_{\tau_1} > 0\}) > 0$ and $P(A \cap \{X_{\tau_2} - X_{\tau_1} < 0\}) > 0$, or $P(A \cap \{X_{\tau_2} = X_{\tau_1}\}) = P(A)$, which in view of Lemma 2.6 implies (AA). ∎

The next lemma provides an absence of arbitrage characterization in the case of shortsale restrictions (compare with Proposition 3 in [2]).

LEMMA 2.8. *The process X satisfies condition* (AA^+) (*over simple strategies without shortselling*) *if and only if for any two stopping times $\tau_1 < \tau_2 \leq T$ and for any $A \in \mathcal{F}_{\tau_1}$ with $P(A) > 0$ we have either $P(A \cap \{X_{\tau_2} - X_{\tau_1} < 0\}) > 0$, or $X_{\tau_2} = X_{\tau_1}$ a.s. on A.*

Proof. Necessity: Assume X satisfies (AA^+). Let $\tau_1 < \tau_2 \leq T$ be stopping times and $A \in \mathcal{F}_{\tau_1}$ with $P(A) > 0$ such that $P(A \cap \{X_{\tau_2} - X_{\tau_1} < 0\}) = 0$ and $P(A \cap \{X_{\tau_2} \neq X_{\tau_1}\}) > 0$. Then $X_{\tau_2} \geq X_{\tau_1}$ on A and $P(A \cap \{X_{\tau_2} > X_{\tau_1}\}) > 0$. This means that $\chi_A \chi_{(\tau_1, \tau_2]} \in \mathcal{S}_T^+$ is an arbitrage strategy for X, a contradiction.

Sufficiency: Assume that for any two stopping times $\tau_1 < \tau_2 \leq T$ and for any $A \in \mathcal{F}_{\tau_1}$ with $P(A) > 0$ we have either $P(A \cap \{X_{\tau_2} - X_{\tau_1} < 0\}) > 0$ or $X_{\tau_2} = X_{\tau_1}$ a.s. on A, but X admits an arbitrage opportunity with respect to the class $\mathcal{S}_T^+$. By Lemma 2.2, there exists $G \in \mathcal{S}_T^+$ of the form $G = g \chi_{(\sigma_1, \sigma_2]}$ such that $g(X_{\sigma_2} - X_{\sigma_1}) \geq 0$ P-a.e. and $P(g(X_{\sigma_2} - X_{\sigma_1}) > 0) > 0$. Clearly the set $B = \{g > 0\}$ has positive probability and $B \in \mathcal{F}_{\sigma_1}$. Since $g(X_{\sigma_2} - X_{\sigma_1}) \geq 0$ P-a.e. we have $X_{\sigma_2} - X_{\sigma_1} \geq 0$ a.s. on B. Observe that $X_{\sigma_2} - X_{\sigma_1} > 0$ with positive probability on B, since otherwise $g(X_{\sigma_2} - X_{\sigma_1}) \leq 0$ a.s. on B. Since $g(X_{\sigma_2} - X_{\sigma_1}) \geq 0$ P-a.e., we see that $g(X_{\sigma_2} - X_{\sigma_1}) = 0$ a.s. on B and by the definition of B, $g(X_{\sigma_2} - X_{\sigma_1}) = 0$ a.s. on B^c. This contradicts $P(g(X_{\sigma_2} - X_{\sigma_1}) > 0) > 0$. Hence, we have shown that $X_{\sigma_2} - X_{\sigma_1} \geq 0$ a.s.

on B and $X_{\sigma_2} - X_{\sigma_1} > 0$ with positive probability on B. In other words, $P(B \cap \{X_{\sigma_2} - X_{\sigma_1} < 0\}) = 0$ and $P(B \cap \{X_{\sigma_2} \neq X_{\sigma_1}\}) > 0$, which contradicts our assumption. ∎

COROLLARY 2.9. *If for any two stopping times* $\tau_1 < \tau_2 \leq T$ *either* $P(X_{\tau_2} - X_{\tau_1} < 0 \mid \mathcal{F}_{\tau_1}) > 0$ *or* $P(X_{\tau_2} = X_{\tau_1} \mid \mathcal{F}_{\tau_1}) = 1$ *P-a.e., then* X *satisfies condition* (AA$^+$).

Proof. We use the same arguments as in the proof of Corollary 2.7. ∎

LEMMA 2.10. *If for any stopping time* $\tau \leq T$ *either* $P(\forall_{t \in (\tau,T]} : X_\tau > X_t \mid \mathcal{F}_\tau) > 0$ *or* $P(\forall_{t \in [\tau,T]} : X_\tau = X_t \mid \mathcal{F}_\tau) = 1$ *P-a.e., then* X *satisfies condition* (AA$^+$).

Proof. Let σ, τ be stopping times such that $\sigma < \tau \leq T$. By the assumption we have either $P(\forall_{t \in (\sigma,T]} : X_\sigma > X_t \mid \mathcal{F}_\sigma) > 0$ or $P(\forall_{t \in [\sigma,T]} : X_\sigma = X_t \mid \mathcal{F}_\sigma) = 1$ P-a.e. Observe that $\{\forall_{t \in (\sigma,T]} : X_\sigma > X_t\} \subset \{X_\sigma > X_\tau\}$ and hence $P(X_\sigma - X_\tau > 0 \mid \mathcal{F}_\sigma) > 0$ or $P(X_\sigma = X_\tau \mid \mathcal{F}_\sigma) = 1$ P-a.e. Consequently, Corollary 2.9 gives absence of arbitrage over simple strategies without short-selling. ∎

2.3. Sticky type conditions. Let us recall the definition of stickiness from [6] and the definition of condition ($\star$) from [2] for a given finite horizon $T > 0$.

DEFINITION 2.11. We say that a progressively measurable process X is *sticky* with respect to the filtration $\mathbb{F}$ and the probability measure P if for all $\epsilon > 0$ and all stopping times τ such that $P(\tau < T) > 0$ we have

$$P\Big(\sup_{t \in [\tau,T]} |X_\tau - X_t| < \epsilon, \, \tau < T \Big) > 0.$$

DEFINITION 2.12. We say that an adapted càdlàg process X satisfies *condition* ($\star$) with respect to the filtration $\mathbb{F}$ if for any stopping time τ such that $\tau \leq T$ a.s. and any $A \in \mathcal{F}_\tau$ with $P(A) > 0$ we have

$$P\Big(A \cap \Big\{ \inf_{t \in [\tau,T]} (X_t - X_\tau) > -\epsilon \Big\} \Big) > 0$$

for any $\epsilon > 0$.

REMARK 2.13. It is worth pointing out that by Lemma 2.4 condition ($\star$) is equivalent to a condition (a) defined below. We say that an adapted càdlàg process X satisfies *condition* (a) with respect to the filtration $\mathbb{F}$ if for any stopping time τ with $\tau \leq T$ a.s. and any $\epsilon > 0$ we have

$$P\Big(\inf_{t \in [\tau,T]} (X_t - X_\tau) > -\epsilon \,\Big|\, \mathcal{F}_\tau \Big) > 0 \quad \text{P-a.e.}$$

Lemma 2.14. *Assume X is an adapted càdlàg process. If X is sticky with respect to the filtration $\mathbb{F}$ and the probability measure P, then X satisfies condition $(\star)$.*

Proof. Let τ be any stopping time with $\tau \leq T$ a.s. and A be any set from $\mathcal{F}_\tau$ with $P(A) > 0$. Define

$$\tau^A(\omega) = \begin{cases} \tau(\omega) & \text{for } \omega \in A, \\ T & \text{for } \omega \notin A. \end{cases}$$

Note that τ^A is a stopping time with respect to the filtration $\mathbb{F}$ and if $\tau^A(\omega) < T$, then $\omega \in A$, so we have

$$\left\{ \sup_{t \in [\tau, T]} |X_\tau - X_t| < \epsilon, \, \tau^A < T \right\} \subset A \cap \left\{ \sup_{t \in [\tau, T]} |X_\tau - X_t| < \epsilon \right\}$$

for any $\epsilon > 0$. We have to show that $P(A \cap \{\sup_{t \in [\tau,T]} |X_\tau - X_t| < \epsilon\}) > 0$. Consider two cases: $P(\tau^A < T) > 0$ and $P(\tau^A < T) = 0$. If $P(\tau^A < T) > 0$, then since X is sticky we have $P(\{\sup_{t \in [\tau,T]} |X_\tau - X_t| < \epsilon, \tau^A < T\}) > 0$, which implies that $P(A \cap \{\sup_{t \in [\tau,T]} |X_\tau - X_t| < \epsilon\}) > 0$. Now if $P(\tau^A < T) = 0$, then $\tau^A = T$ a.s. and $\tau = T$ a.s., so $P(A \cap \{\sup_{t \in [\tau,T]} |X_\tau - X_t| < \epsilon\}) = P(A) > 0$. Note that

$$A \cap \left\{ \sup_{t \in [\tau, T]} |X_\tau - X_t| < \epsilon \right\} \subset A \cap \left\{ \inf_{t \in [\tau, T]} (X_t - X_\tau) > -\epsilon \right\},$$

so $P(A \cap \{\inf_{t \in [\tau,T]} (X_t - X_\tau) > -\epsilon\}) > 0$. ∎

Lemma 2.15. *All martingales satisfy condition $(\star)$.*

Proof. Assume X is a martingale and X does not satisfy condition $(\star)$. Then there is a stopping time $\tau \leq T$ a.s. and $A \in \mathcal{F}_\tau$ with $P(A) > 0$ such that $P(A \cap \{\inf_{t \in [\tau,T]} (X_t - X_\tau) > -\epsilon\}) = 0$ for some $\epsilon > 0$. Therefore $\inf_{t \in [\tau,T]} (X_t - X_\tau) \leq -\epsilon$ a.s. on A. Let $\sigma = \inf\{t > \tau : X_t \leq X_\tau - \epsilon\}$ on A and $\sigma = \tau$ on the complement of A. Observe that σ is a stopping time and $\sigma \leq T$ a.s. Since X is a martingale we have $E(X_\sigma \mid \mathcal{F}_\tau) = X_\tau$ and

$$E(X_\sigma \mid \mathcal{F}_\tau) \leq \chi_A E(X_\tau - \epsilon \mid \mathcal{F}_\tau) + \chi_{A^c} E(X_\tau \mid \mathcal{F}_\tau) = \chi_A(X_\tau - \epsilon) + \chi_{A^c} X_\tau.$$

This implies that $X_\tau \leq X_\tau - \epsilon$ on A, the required contradiction. ∎

2.4. Absence of arbitrage over simple strategies. The following lemma shows that property (a) (defined in Remark 2.13) is a necessary condition for absence of arbitrage over simple strategies.

Lemma 2.16. *Let X be an adapted càdlàg process. If X satisfies condition* (AA), *then it satisfies condition* (a).

Proof (see proof in [2]). Assume X does not admit a simple arbitrage strategy and X does not satisfy (a). Then there is a stopping time τ with

$\tau \leq T$ a.s. and $\epsilon > 0$ such that $P(\inf_{t \in [\tau, T]}(X_t - X_\tau) > -\epsilon \mid \mathcal{F}_\tau) = 0$ with positive probability. Let

$$A = \Big\{ P\Big(\inf_{t \in [\tau, T]}(X_t - X_\tau) > -\epsilon \mid \mathcal{F}_\tau \Big) = 0 \Big\}.$$

Then clearly $A \in \mathcal{F}_\tau$ and $P(A) > 0$. So $P(A \cap \{\inf_{t \in [\tau, T]}(X_t - X_\tau) > -\epsilon\}) = 0$ and

(2.1) $$\inf_{t \in [\tau, T]}(X_t - X_\tau) \leq -\epsilon \quad \text{on } A \text{ with probability one.}$$

Define

$$\tilde{\tau} = \begin{cases} \tau & \text{on } A, \\ T & \text{on } A^c. \end{cases}$$

Since $A \in \mathcal{F}_\tau$, $\tilde{\tau}$ is a stopping time. If we define $\sigma = \inf\{t \geq \tilde{\tau} : X_t - X_{\tilde{\tau}} < -\epsilon/2\}$ then since X has right continuous paths, $X_\sigma - X_{\tilde{\tau}} \leq -\epsilon/2$ on A with probability one. Moreover $\sigma \leq T$ almost surely on A: otherwise $X_t - X_{\tilde{\tau}} \geq -\epsilon/2$ on A with positive probability for any $t \in [\tilde{\tau}, T]$, contrary to (2.1). This means that $X_{\tilde{\sigma}} - X_{\tilde{\tau}} < 0$ on A, so $\chi_A(X_{\tilde{\sigma}} - X_{\tilde{\tau}}) \leq 0$ almost surely and $P(\chi_A(X_{\tilde{\sigma}} - X_{\tilde{\tau}}) < 0) > 0$ for

$$\tilde{\sigma} = \begin{cases} \sigma & \text{on } A, \\ T & \text{on } A^c. \end{cases}$$

In other words, the investment strategy $-\chi_A \chi_{(\tilde{\tau}, \tilde{\sigma}]}$ is an arbitrage strategy in $\mathcal{S}_T$, which contradicts our assumption (note that $\tilde{\tau} = \tau$ on A). ∎

LEMMA 2.17. *Let X be an adapted càdlàg positive process. Then the following statements are equivalent:*

(i) *For any stopping time τ with $\tau \leq T$ a.s. and for any $\epsilon > 0$ we have*

$$P\Big(\inf_{t \in [\tau, T]}(X_t - X_\tau) > -\epsilon \mid \mathcal{F}_\tau \Big) > 0 \quad P\text{-a.e.}$$

(ii) *For any $\epsilon > 0$ and for any two stopping times $\tau_1 \leq \tau_2 \leq T$ we have*

$$P\big(X_{\tau_2} - X_{\tau_1} > -\epsilon \mid \mathcal{F}_{\tau_1}\big) > 0 \quad P\text{-a.e.}$$

(iii) *For any $\delta > 0$ and for any two stopping times $\tau_1 \leq \tau_2 \leq T$ we have*

$$P\big(X_{\tau_2}/X_{\tau_1} > 1 - \delta \mid \mathcal{F}_{\tau_1}\big) > 0 \quad P\text{-a.e.}$$

Proof. (i)⇒(ii). Assume X satisfies (i). Let τ_1, τ_2 be two stopping times with $\tau_1 \leq \tau_2 \leq T$ a.s. Note that

$$\Big\{ \inf_{t \in [\tau_1, T]}(X_t - X_{\tau_1}) > -\epsilon \Big\} \subset \{X_{\tau_2} - X_{\tau_1} > -\epsilon\}$$

for any given $\epsilon > 0$. Hence $0 < P(\inf_{t \in [\tau_1, T]}(X_t - X_{\tau_1}) > -\epsilon \mid \mathcal{F}_{\tau_1}) \leq P(X_{\tau_2} - X_{\tau_1} > -\epsilon \mid \mathcal{F}_{\tau_1})$ and X satisfies (ii).

(ii)⇒(i). Assume X satisfies (ii) but not (i). Then by Lemma 2.4 there is a stopping time τ_1 with $\tau_1 \leq T$ a.s. and $A \in \mathcal{F}_{\tau_1}$ with $P(A) > 0$ such

that $P(A \cap \{\inf_{t\in[\tau_1,T]}(X_t - X_{\tau_1}) > -\epsilon\}) = 0$ for some $\epsilon > 0$. Consequently, $\inf_{t\in[\tau_1,T]}(X_t - X_{\tau_1}) \leq -\epsilon$ on A with probability one. If we define $\tau_2 = \inf\{s > \tau_1 : X_s \leq X_{\tau_1} - \epsilon\} \wedge T$ then since X has right continuous paths, $X_{\tau_2} - X_{\tau_1} \leq -\epsilon$ on A. Thus $P(A \cap \{X_{\tau_2} - X_{\tau_1} > -\epsilon\}) = 0$, contradicting (ii).

(ii)$\Rightarrow$(iii). Let $\tau_1 \leq \tau_2$ a.s. be two stopping times with $\tau_2 \leq T$, $A \in \mathcal{F}_{\tau_1}$ with $P(A) > 0$, and $\delta > 0$. Since X_{τ_1} takes values in $(0,\infty)$ and $P(A) > 0$, for sufficiently large $M \in \mathbb{N}$ the event $B = A \cap \{1/M \leq X_{\tau_1} \leq M\} \in \mathcal{F}_{\tau_1}$ has positive probability. As the function $f(x) = \ln x$ is nondecreasing and uniformly continuous on $[1/M, M]$, for any given $\gamma > 0$ there exists $\epsilon > 0$ such that whenever $x - y > -\epsilon$ and $x, y \in [1/M, M]$ then $\ln x - \ln y > -\gamma$. Hence for any given $\gamma > 0$ there exists $\epsilon > 0$ such that

$$B \cap \{X_{\tau_2} - X_{\tau_1} > -\epsilon\} \subset B \cap \{\ln X_{\tau_2} - \ln X_{\tau_1} > -\gamma\}.$$

Since the function $g(x) = \exp(x) - 1$ is continuous and nondecreasing, for any given $\delta > 0$ there exists $\gamma > 0$ such that for $x > -\gamma$ we have $\exp(x) - 1 > -\delta$. Note that $g(\ln X_{\tau_2} - \ln X_{\tau_1}) = X_{\tau_2}/X_{\tau_1} - 1$, so for a given $\delta > 0$ there exists $\gamma > 0$ such that

$$B \cap \{\ln X_{\tau_2} - \ln X_{\tau_1} > -\gamma\} \subset B \cap \{X_{\tau_2}/X_{\tau_1} > 1 - \delta\}.$$

Consequently, for any given $\delta > 0$ there exists $\epsilon > 0$ such that

(2.2) $$B \cap \{X_{\tau_2} - X_{\tau_1} > -\epsilon\} \subset B \cap \{X_{\tau_2}/X_{\tau_1} > 1 - \delta\}.$$

Since X satisfies (ii) and $B \in \mathcal{F}_{\tau_1}$ has positive probability, we have

$$P(B \cap \{X_{\tau_2} - X_{\tau_1} > -\epsilon\}) > 0.$$

The inclusion (2.2) implies that also $P(B \cap \{X_{\tau_2}/X_{\tau_1} > 1 - \delta\}) > 0$ and since $B \subset A$ we get $P(A \cap \{X_{\tau_2}/X_{\tau_1} > 1 - \delta\}) > 0$. Thus we have shown that for any two stopping times $\tau_1 \leq \tau_2 \leq T$ and for any $A \in \mathcal{F}_{\tau_1}$ such that $P(A) > 0$ we have

$$P(A \cap \{X_{\tau_2}/X_{\tau_1} > 1 - \delta\}) > 0$$

for any $\delta > 0$. We can now conclude from Lemma 2.4 that

$$P(X_{\tau_2}/X_{\tau_1} > 1 - \delta \mid \mathcal{F}_{\tau_1}) > 0 \quad P\text{-a.e.}$$

(iii)$\Rightarrow$(ii). In view of Lemma 2.4 we need to show that for any stopping times $\tau_1 \leq \tau_2 \leq T$ and for any $A \in \mathcal{F}_{\tau_1}$ with $P(A) > 0$, we have $P(A \cap \{X_{\tau_2} - X_{\tau_1} > -\epsilon\}) > 0$ for any $\epsilon > 0$. Let $B_a = \{X_{\tau_1} < a\}$. Then clearly $B_a \in \mathcal{F}_{\tau_1}$ and $P(B_a) \nearrow 1$ as $a \to \infty$. This implies that, for sufficiently large $a \in \mathbb{R}^+$, the event $A \cap B_a \in \mathcal{F}_{\tau_1}$ has positive probability. Since X satisfies (iii), Lemma 2.4 again yields

$$P(A \cap B_a \cap \{X_{\tau_2}/X_{\tau_1} > 1 - \delta\}) > 0$$

for any $\delta > 0$. Since $A \cap B_a \cap \{X_{\tau_2}/X_{\tau_1} > 1 - \delta\} \subset A \cap \{X_{\tau_2} - X_{\tau_1} > -\delta a\}$,

we have

$$P(A \cap \{X_{\tau_2} - X_{\tau_1} > -\delta a\}) > 0,$$

and it is sufficient to take $\delta = \epsilon/a$. ∎

Summing up, we have three conditions which are necessary for the absence of arbitrage over simple strategies. The following lemma shows that condition (a) (and any of the previous conditions) is necessary and sufficient for the absence of arbitrage over simple strategies for nonnegative local martingales.

THEOREM 2.18. *Assume X is a nonnegative càdlàg process that admits an equivalent local martingale measure Q. Then X satisfies condition* (AA) *if and only if X satisfies condition* (a).

This result is borrowed from Proposition 1 in [2]. We use the same technique with some simplifications.

Proof. Sufficiency: Assume X satisfies (a) and admits a simple arbitrage strategy. We may apply Corollary 2.7 to find two stopping times $\tau_1 < \tau_2 \leq T$ such that either

$$P(X_{\tau_2} - X_{\tau_1} \geq 0 \mid \mathcal{F}_{\tau_1}) = 1 \quad \text{and} \quad P(X_{\tau_2} - X_{\tau_1} > 0 \mid \mathcal{F}_{\tau_1}) > 0, \qquad P\text{-a.e.,}$$

or

$$P(X_{\tau_2} - X_{\tau_1} \leq 0 \mid \mathcal{F}_{\tau_1}) = 1 \quad \text{and} \quad P(X_{\tau_2} - X_{\tau_1} < 0 \mid \mathcal{F}_{\tau_1}) > 0, \qquad P\text{-a.e.}$$

Without loss of generality we can assume that the second case holds. It follows that $X_{\tau_2} \leq X_{\tau_1}$ almost surely and $P(X_{\tau_2} - X_{\tau_1} < 0) > 0$. Since X is a càdlàg process, it follows that $\inf_{t \in [0,T]} X_t$ is almost surely finite and $P(\inf_{t \in [\tau_1,T]} (X_t - X_{\tau_1}) > -\eta) \nearrow 1$ as $\eta \to \infty$. Then

$$\lim_{\eta \to \infty} \chi_{\{\inf_{t \in [\tau_1,T]} (X_t - X_{\tau_1}) > -\eta\}} = 1$$

and by the dominated convergence theorem for conditional expectation we obtain

$$\lim_{\eta \to \infty} E(\chi_{\{\inf_{t \in [\tau_1,T]} (X_t - X_{\tau_1}) > -\eta\}} \mid \mathcal{F}_{\tau_1}) = 1.$$

Clearly $P(X_{\tau_1} < M) \nearrow 1$ as $M \to \infty$. So letting

$$A_{M,\eta} = \Big\{ P\Big(\inf_{t \in [\tau_1,T]} (X_t - X_{\tau_1}) > -\eta \;\Big|\; \mathcal{F}_{\tau_1} \Big) > 0 \Big\} \cap \{X_{\tau_1} < M\}$$

we have

(2.3) $$P(A_{M,\eta} \cap \{X_{\tau_2} - X_{\tau_1} < 0\}) > 0$$

for M, η sufficiently large. Define the following two stopping times:

$$\tilde{\tau}_1 = \begin{cases} \tau_1 & \text{if } \omega \in A_{M,\eta}, \\ T & \text{if } \omega \notin A_{M,\eta}, \end{cases} \qquad \tilde{\tau}_2 = \begin{cases} \tau_2 & \text{if } \omega \in A_{M,\eta}, \\ T & \text{if } \omega \notin A_{M,\eta}. \end{cases}$$

Let $\tau = \inf\{t \geq \tilde{\tau}_1 : X_t > M + \eta + 1\} \wedge \tilde{\tau}_2$. We claim that if $\tau = \tilde{\tau}_2$ almost surely on $A_{M,\eta}$ then $\chi_{A_{M,\eta}}(X_t - X_{\tau_1})$ is bounded in $[\tau_1, \tau_2]$. Indeed, if $\tau = \tilde{\tau}_2$, then $X_t \leq M + \eta + 1$ for $t \in [\tau_1, \tau_2)$. Moreover, $X_{\tau_1} < M$ and also $X_{\tau_2} < M$ a.s. on $A_{M,\eta}$ and X is a nonnegative process, hence $-M \leq \chi_{A_{M,\eta}}(X_t - X_{\tau_1}) \leq M + \eta + 1$ for $t \in [\tau_1, \tau_2]$. The boundedness implies that the local martingale $\chi_{A_{M,\eta}}(X_t - X_{\tau_1})$ is a martingale and $E_Q(\chi_{A_{M,\eta}}(X_t - X_{\tau_1}) \mid \mathcal{F}_{\tau_1}) = 0$. This means that $Q(A_{M,\eta} \cap \{X_{\tau_2} - X_{\tau_1} < 0\}) = 0$ and clearly $P(A_{M,\eta} \cap \{X_{\tau_2} - X_{\tau_1} < 0\}) = 0$, which contradicts (2.3). So we may assume that the event $A_{M,\eta} \cap \{\tau < \tilde{\tau}_2\}$ has positive probability. Note that $A_{M,\eta} \cap \{\tau < \tilde{\tau}_2\} \in \mathcal{F}_\tau$, because $A_{M,\eta} \in \mathcal{F}_{\tau_1} \subset \mathcal{F}_\tau$ and clearly $\{\tau < \tilde{\tau}_2\} \in \mathcal{F}_\tau$. Since X is càdlàg, $X_\tau \geq M + \eta + 1$ on $A_{M,\eta} \cap \{\tau < \tilde{\tau}_2\}$ and since $X_{\tau_2} \leq M$ on $A_{M,\eta}$ we have

$$A_{M,\eta} \cap \{\tau < \tilde{\tau}_2\} \cap \{X_{\tau_2} - X_\tau > -\eta\}$$
$$\subset A_{M,\eta} \cap \{\tau < \tilde{\tau}_2\} \cap \{M - (M + \eta + 1) > -\eta\}.$$

Hence $P(A_{M,\eta} \cap \{\tau < \tilde{\tau}_2\} \cap \{X_{\tau_2} - X_\tau > -\eta\}) = 0$, from which in view of the inclusion

$$\left\{ \inf_{t \in [\tau,T]} X_t - X_\tau > -\eta \right\} \subset \{X_{\tau_2} - X_\tau > -\eta\},$$

it follows that

$$P\left(A_{M,\eta} \cap \{\tau < \tilde{\tau}_2\} \cap \left\{ \inf_{t \in [\tau,T]} X_t - X_\tau > -\eta \right\} \right) = 0.$$

We now show that

$$P\left(\inf_{t \in [\tau,T]} X_t - X_\tau > -\eta \;\Big|\; \mathcal{F}_\tau \right) = 0 \quad \text{on } A_{M,\eta} \cap \{\tau < \tilde{\tau}_2\},$$

which contradicts condition (a). Indeed, otherwise $E(\chi_{\{\inf_{t \in [\tau,T]} X_t - X_\tau > -\eta\}} \mid \mathcal{F}_\tau) > 0$ on $A_{M,\eta} \cap \{\tau < \tilde{\tau}_2\}$ with positive probability and clearly

$$\chi_{A_{M,\eta} \cap \{\tau < \tilde{\tau}_2\}} E(\chi_{\{\inf_{t \in [\tau,T]} X_t - X_\tau > -\eta\}} \mid \mathcal{F}_\tau) > 0$$

with positive probability. Then, since the event $A_{M,\eta} \cap \{\tau < \tilde{\tau}_2\}$ belongs to $\mathcal{F}_\tau$, we have

$$P\left(A_{M,\eta} \cap \{\tau < \tilde{\tau}_2\} \cap \left\{ \inf_{t \in [\tau,T]} X_t - X_\tau > -\eta \right\} \right) > 0,$$

a contradiction.

Necessity: Apply Lemma 2.16. ∎

Note that the existence of an equivalent local martingale measure is not equivalent to absence of arbitrage over simple strategies. The following example, given in [3], exhibits a nonnegative local martingale that admits an arbitrage opportunity in $\mathcal{S}_T$.

EXAMPLE 2.19. Let $(B_t)_{t \geq 0}$ be a Brownian motion with $B_0 = 1$ and let $\tau = \inf\{t > 0 : B_t = 0\}$. Define $X_t = B_{\tan(t\pi/2) \wedge \tau}$ for $t < 1$ and

$X_1 = B_\tau = 0$. Then X is a nonnegative local martingale. The strategy H of the form $H = -1\chi_{(0,1]}$ yields an arbitrage opportunity: $(H \cdot X)_1 = 1$ a.s. and $(H \cdot X)_0 = 0$ a.s. Clearly X_t does not satisfy (a) since for $\tau = 0$ and $T = 1$, $\inf_{t \in [0,1]}(X_t - X_0) = -1$. Notice that the simple strategy is not a tame strategy (since X_t can take arbitrarily large values before reaching zero), so that we are not able to use the general arbitrage theory developed by F. Delbaen and W. Schachermayer (see [4]).

Below we give another example of a nonnegative process that admits an equivalent local martingale measure, but has a simple arbitrage strategy.

EXAMPLE 2.20. Let $y(0) = 1$ and $P\{y(n) = 2^{n-1}\} = P\{y(n) = -2^{n-1}\} = 1/2$ for $n = 1, 2, \ldots$. Let $z(t) = \sum_{i=0}^{k} y(i)$ for $k \leq t < k + 1$. Define $X_t = z(\tan(t\pi/2) \wedge \tau)$ for $t < 1$ and $X_1 = z(\tau)$ with $\tau = \inf\{t > 0 : z(t) = 0\}$. The arbitrage strategy is given by $H = -1\chi_{(0,1]}$. Clearly $(H \cdot X)_1 = 1$ a.s. and $(H \cdot X)_0 = 0$ a.s.

REMARK 2.21. The examples above show that without (a), absence of arbitrage may not hold. It would be interesting to find a condition weaker than (a) which together with the existence of an equivalent local martingale measure implies (AA).

2.5. Absence of arbitrage over simple strategies without short-selling. We now give necessary conditions for the absence of arbitrage over simple strategies with shortsale restrictions.

DEFINITION 2.22. We say that an adapted càdlàg process X satisfies *condition* (b) with respect to the filtration $\mathbb{F}$ if for any $\epsilon > 0$ and any stopping time τ such that $\tau \leq T$ a.s. we have

$$P\Big(\sup_{t \in [\tau,T]} (X_t - X_\tau) < \epsilon \,\Big|\, \mathcal{F}_\tau \Big) > 0 \quad P\text{-a.e.}$$

REMARK 2.23. Condition (b) is weaker than stickiness. The proof is the same as that of Lemma 2.14 with an obvious modification: it suffices to change the last display of the proof to

$$A \cap \Big\{ \sup_{t \in [\tau,T]} |X_\tau - X_t| < \epsilon \Big\} \subset A \cap \Big\{ \sup_{t \in [\tau,T]} (X_t - X_\tau) < \epsilon \Big\}$$

and apply Lemma 2.4.

LEMMA 2.24. *Let X be an adapted càdlàg positive process. Then the following statements are equivalent:*

(i) *For any stopping time τ with $\tau \leq T$ a.s. and for any $\epsilon > 0$ we have*

$$P\Big(\sup_{t \in [\tau,T]} (X_t - X_\tau) < \epsilon \,\Big|\, \mathcal{F}_\tau \Big) > 0 \quad P\text{-a.e.}$$

(ii) *For any $\epsilon > 0$ and for any two stopping times $\tau_1 \leq \tau_2 \leq T$ we have*

$$P\big(X_{\tau_2} - X_{\tau_1} < \epsilon \mid \mathcal{F}_{\tau_1}\big) > 0 \qquad P\text{-}a.e.$$

(iii) *For any $\delta > 0$ and for any two stopping times $\tau_1 \leq \tau_2 \leq T$ we have*

$$P(X_{\tau_2}/X_{\tau_1} < 1 + \delta \mid \mathcal{F}_{\tau_1}) > 0 \qquad P\text{-}a.e.$$

Proof. One can use similar arguments to those in the proof of Lemma 2.17. ∎

LEMMA 2.25. *Let X be an adapted càdlàg process. If X satisfies condition* (AA$^+$), *then it satisfies condition* (b).

Proof. Assume X satisfies (AA$^+$) but not (b). Then there is a stopping time τ with $\tau \leq T$ a.s. and $\epsilon > 0$ such that $P(\sup_{t \in [\tau, T]}(X_t - X_\tau) < \epsilon \mid \mathcal{F}_\tau) = 0$ with positive probability. Define

$$A = \Big\{ P\Big(\sup_{t \in [\tau, T]} (X_t - X_\tau) < \epsilon \ \Big| \ \mathcal{F}_\tau \Big) = 0 \Big\} \in \mathcal{F}_\tau.$$

Then $E(\chi_A \chi_{\{\sup_{t \in [\tau,T]}(X_t - X_\tau) < \epsilon\}} \mid \mathcal{F}_\tau) = 0$ almost surely. Hence $P(A \cap \{\sup_{t \in [\tau,T]}(X_t - X_\tau) < \epsilon\}) = 0$ and since

$$\Big\{ \sup_{t \in [\tau, T]} (X_t - X_\tau) < \epsilon \Big\} = \Big\{ \inf_{t \in [\tau, T]} (X_\tau - X_t) > -\epsilon \Big\}$$

we obtain

(2.4) $$\inf_{t \in [\tau, T]} (X_\tau - X_t) \leq -\epsilon \quad \text{on } A \text{ with probability one.}$$

Let $\tilde{\tau} = \tau$ on A and $\tilde{\tau} = T$ on A^c. Then $\tilde{\tau}$ is a stopping time since $A \in \mathcal{F}_\tau$. Define $\sigma = \inf\{t \geq \tilde{\tau} : X_{\tilde{\tau}} - X_t < -\epsilon/2\}$. Clearly σ is a stopping time, and since X is càdlàg, $X_{\tilde{\tau}} - X_\sigma \leq -\epsilon/2$ on A with probability one. We claim that $\sigma \leq T$ almost surely on A. Indeed, otherwise $P(\{\sigma > T\} \cap A) > 0$, so $P(\{X_{\tilde{\tau}} - X_t \geq -\epsilon/2\} \cap A) > 0$ for any $t \in [\tilde{\tau}, T]$, which contradicts (2.4). Finally, we define the stopping time $\tilde{\sigma} = \sigma$ on A and $\tilde{\sigma} = T$ on A^c. Then we have $\chi_A(X_{\tilde{\sigma}} - X_{\tilde{\tau}}) \geq 0$ almost surely and $P(\chi_A(X_{\tilde{\sigma}} - X_{\tilde{\tau}}) > 0) > 0$. Hence $\chi_A \chi_{(\tilde{\tau}, \tilde{\sigma}]}$ is an arbitrage strategy. Note that $\chi_A \chi_{(\tilde{\tau}, \tilde{\sigma}]} \in \mathcal{S}_T^+$, which contradicts (AA$^+$). ∎

LEMMA 2.26. *Let X be a nonnegative càdlàg process. If there exists an equivalent probability measure Q such that X is a Q-supermartingale, then X satisfies* (AA$^+$).

Proof. Assume that X admits a simple arbitrage strategy without short-selling. Then by Lemma 2.2 there exists $G \in \mathcal{S}_T^+$ of the form $G = g\chi_{(\sigma_1, \sigma_2]}$ such that $g(X_{\sigma_2} - X_{\sigma_1}) \geq 0$ P-a.e. and $P\{g(X_{\sigma_2} - X_{\sigma_1}) > 0\} > 0$. Since $G \in \mathcal{S}_T^+$, g is a nonnegative random variable, hence $X_{\sigma_2} - X_{\sigma_1} \geq 0$ P almost everywhere and clearly $E_Q(X_{\sigma_2} - X_{\sigma_1} \mid \mathcal{F}_{\sigma_1}) \geq 0$. Moreover $\sigma_1 < \sigma_2$

are bounded stopping times and X is a Q-supermartingale, so by the optional sampling theorem we have $E_Q(X_{\sigma_2} - X_{\sigma_1} \mid \mathcal{F}_{\sigma_1}) \leq 0$. This means that $E_Q(X_{\sigma_2} - X_{\sigma_1} \mid \mathcal{F}_{\sigma_1}) = 0$ and $X_{\sigma_2} - X_{\sigma_1} \geq 0$ Q-a.e., which contradicts $P\{g(X_{\sigma_2} - X_{\sigma_1}) > 0\} > 0$. ∎

REMARK 2.27. We remark that existence of a local martingale measure implies (AA$^+$) but not (AA) (it is well known, see e.g. [11, 1.5.19(ii)], that a nonnegative local martingale is a supermartingale, which means that if there exists an equivalent probability measure Q such that a nonnegative process X is a Q-local martingale, then X is a Q-supermartingale; as we showed in Examples 2.19, 2.20, the existence of an equivalent local martingale measure does not imply (AA)).

2.6. Multidimensional characterization. In this section we extend the absence of arbitrage characterization to the case of a market that consists of a bank account and several risky assets ($d > 1$). First, we introduce a sufficient condition for absence of arbitrage over simple strategies with shortsale restrictions.

DEFINITION 2.28. We say that an adapted càdlàg process X satisfies *condition* (c) with respect to the filtration $\mathbb{F}$ if for any two stopping times $\tau_1 < \tau_2 \leq T$ we have

$$P\Big(\bigcap_{j \in J} \{X^j_{\tau_2} < X^j_{\tau_1}\} \,\Big|\, \mathcal{F}_{\tau_1}\Big) > 0 \quad P\text{-a.e.},$$

where $J = \{j \in \{1, \ldots, d\} : P(X^j_{\tau_2} \neq X^j_{\tau_1} \mid \mathcal{F}_{\tau_1}) > 0$ with positive probability\}.

LEMMA 2.29. *Let X be an adapted càdlàg process. If X satisfies condition* (c), *then it satisfies condition* (AA$^+$).

Proof. Assume that X admits a simple arbitrage strategy without short-selling. We may apply Lemma 2.2 to find $G = (G^1, \ldots, G^d) \in \mathcal{S}_T^+$ of the form $G^j = g^j \chi_{(\tau_1, \tau_2]}$ such that $\sum_{j=1}^d g^j (X^j_{\tau_2} - X^j_{\tau_1}) \geq 0$ P-a.e. and $P(\sum_{j=1}^d g^j (X^j_{\tau_2} - X^j_{\tau_1}) > 0) > 0$. Then there exists $k \in \{1, \ldots, d\}$ such that $P(\{g^k > 0\} \cap \{X^k_{\tau_2} \neq X^k_{\tau_1}\}) > 0$. Indeed, otherwise $P(\{g^k = 0\} \cup \{X^k_{\tau_2} = X^k_{\tau_1}\}) = 1$ for all $k \in \{1, \ldots, d\}$ and $\sum_{j=1}^d g^j (X^j_{\tau_2} - X^j_{\tau_1}) = 0$ almost surely. Note that if $P(\{g^k > 0\} \cap \{X^k_{\tau_2} \neq X^k_{\tau_1}\}) > 0$ then $P(\{g^k > 0\} \cap \{X^k_{\tau_2} \neq X^k_{\tau_1}\} \mid \mathcal{F}_{\tau_1}) > 0$ with positive probability and because $\{g^k > 0\} \in \mathcal{F}_{\tau_1}$ it follows that $P(X^k_{\tau_2} \neq X^k_{\tau_1} \mid \mathcal{F}_{\tau_1}) > 0$ with positive probability, which means that $k \in J$. We now show that

$$P\Big(\{g^k > 0\} \cap \bigcap_{j \in J} \{X^j_{\tau_2} < X^j_{\tau_1}\}\Big) =: P(B) > 0.$$

Since X satisfies condition (c) and $\{g^k > 0\} \in \mathcal{F}_{\tau_1}$, we have $P(\{g^k > 0\} \cap \bigcap_{j \in J}\{X_{\tau_2}^j < X_{\tau_1}^j\} \mid \mathcal{F}_{\tau_1}) > 0$ on $\{g^k > 0\}$, which implies that $P(B) > 0$. If $j \notin J$, then $P(X_{\tau_2}^j \neq X_{\tau_1}^j \mid \mathcal{F}_{\tau_1}) = 0$ almost surely, so $X_{\tau_2}^j = X_{\tau_1}^j$ almost surely and

$$\sum_{i=1}^{d} g^i (X_{\tau_2}^i - X_{\tau_1}^i) = \sum_{j \in J} g^j (X_{\tau_2}^j - X_{\tau_1}^j).$$

Moreover, we have $g^k(X_{\tau_2}^k - X_{\tau_1}^k) < 0$ a.s. on B and $g^j(X_{\tau_2}^j - X_{\tau_1}^j) \leq 0$ a.s. on B for $j \in J$. It follows that

$$\sum_{j \in J} g^j (X_{\tau_2}^j - X_{\tau_1}^j) = g^k(X_{\tau_2}^k - X_{\tau_1}^k) + \sum_{\substack{j \in J \\ j \neq k}} g^j(X_{\tau_2}^j - X_{\tau_1}^j) < 0.$$

So $P(\sum_{j=1}^{d} g^j(X_{\tau_2}^j - X_{\tau_1}^j) < 0) \geq P(B) > 0$ and we get a contradiction to $\sum_{j=1}^{d} g^j(X_{\tau_2}^j - X_{\tau_1}^j) \geq 0$ almost surely. ∎

We now give a sufficient condition for absence of arbitrage over $\mathcal{S}_T$.

DEFINITION 2.30. We say that an adapted càdlàg process X satisfies *condition* (d) with respect to the filtration $\mathbb{F}$ if for any two stopping times $\tau_1 < \tau_2 \leq T$ we have

$$\prod_{(k_1,\ldots,k_d) \in \{0,1\}^d} P\Big(\bigcap_{j \in J} \{(-1)^{k_j}(X_{\tau_2}^j - X_{\tau_1}^j) > 0\} \,\Big|\, \mathcal{F}_{\tau_1} \Big) > 0 \quad P\text{-a.e.},$$

where $J = \{j \in \{1,\ldots,d\} : P(X_{\tau_2}^j \neq X_{\tau_1}^j \mid \mathcal{F}_{\tau_1}) > 0$ with positive probability$\}$.

LEMMA 2.31. *Let X be an adapted càdlàg process. If X satisfies condition* (d), *then it satisfies condition* (AA).

Proof. Assume that X admits a simple arbitrage strategy and satisfies (d). As proved in Lemma 2.1, there is an investment strategy $G = (G^1,\ldots,G^d) \in \mathcal{S}_T$ of the form $G^j = g^j \chi_{(\tau_1,\tau_2]}$ such that $\sum_{j=1}^{d} g^j(X_{\tau_2}^j - X_{\tau_1}^j) \geq 0$ P-a.e. and $P(\sum_{j=1}^{d} g^j(X_{\tau_2}^j - X_{\tau_1}^j) > 0) > 0$. Note that, as in the proof of Lemma 2.29, the inequality $P(\sum_{j=1}^{d} g^j(X_{\tau_2}^j - X_{\tau_1}^j) > 0) > 0$ implies the existence of $k \in \{1,\ldots,d\}$ such that $P(\{g^k \neq 0\} \cap \{X_{\tau_2}^k \neq X_{\tau_1}^k\}) > 0$. Without loss of generality, we can assume $P(A) > 0$, where $A = \{g^k > 0\} \cap \{X_{\tau_2}^k \neq X_{\tau_1}^k\}$. It is clear that

$$P\Big(\bigcup_{(k_1,\ldots,k_d) \in \{0,1\}^d} \bigcap_{j \in J \setminus \{k\}} \{(-1)^{k_j} g^j \geq 0\} \Big) = 1.$$

Therefore we can assume that there are $(l_1, \ldots, l_d) \in \{0, 1\}^d$ such that

$$P\Big(A \cap \bigcap_{j \in J \setminus \{k\}} \{(-1)^{l_j} g^j \geq 0\}\Big) > 0.$$

Let $B = \{g^k > 0\} \cap \bigcap_{j \in J \setminus \{k\}} \{(-1)^{l_j} g^j \geq 0\}$. Then $B \in \mathcal{F}_{\tau_1}$ and $P(B \cap \{X_{\tau_2}^k \neq X_{\tau_1}^k\}) > 0$. It can be shown using the same arguments as in Lemma 2.29 that $k \in J$. From condition (d), we have

$$P\Big(\bigcap_{j \in J} \{(-1)^{l_j} (X_{\tau_2}^j - X_{\tau_1}^j) < 0\} \,\Big|\, \mathcal{F}_{\tau_1}\Big) > 0 \quad P\text{-a.e.}$$

Hence $E(\chi_B \chi_{\bigcap_{j \in J} \{(-1)^{l_j} (X_{\tau_2}^j - X_{\tau_1}^j) < 0\}} \mid \mathcal{F}_{\tau_1}) > 0$, which implies that

$$P\Big(B \cap \{X_{\tau_2}^k < X_{\tau_1}^k\} \cap \bigcap_{j \in J \setminus \{k\}} \{(-1)^{l_j} (X_{\tau_2}^j - X_{\tau_1}^j) < 0\}\Big) > 0.$$

Letting $C = B \cap \{X_{\tau_2}^k < X_{\tau_1}^k\} \cap \bigcap_{j \in J} \{(-1)^{l_j} (X_{\tau_2}^j - X_{\tau_1}^j) < 0\}$ we have $g^k(X_{\tau_2}^k - X_{\tau_1}^k) < 0$ a.s. on C and $g^j(X_{\tau_2}^j - X_{\tau_1}^j) \leq 0$ a.s. on C for $j \in J \setminus \{k\}$. Moreover, from the definition of J we infer that if $j \notin J$ then $P(X_{\tau_2}^j = X_{\tau_1}^j \mid \mathcal{F}_{\tau_1}) = 1$ almost surely, so $X_{\tau_2}^j = X_{\tau_1}^j$ a.s. and

$$\sum_{j=1}^{d} g^j(X_{\tau_2}^j - X_{\tau_1}^j) = \sum_{j \in J} g^j(X_{\tau_2}^j - X_{\tau_1}^j).$$

Finally, on C we have

$$\sum_{j=1}^{d} g^j(X_{\tau_2}^j - X_{\tau_1}^j) = g^k(X_{\tau_2}^k - X_{\tau_1}^k) + \sum_{\substack{j \in J \\ j \neq k}} g^j(X_{\tau_2}^j - X_{\tau_1}^j) < 0.$$

Hence $P(\sum_{j=1}^{d} g^j(X_{\tau_2}^j - X_{\tau_1}^j) < 0) \geq P(C) > 0$, again contradicting the fact that $\sum_{j=1}^{d} g^j(X_{\tau_2}^j - X_{\tau_1}^j) \geq 0$ almost surely. ∎

2.7. Embedded discrete time market characterization. The results in this section are based on the proof of the Dalang–Morton–Willinger theorem for a discrete-time model given in [9]. Directly from this theorem we obtain the following characterization for absence of arbitrage over simple strategies.

THEOREM 2.32. *The process X satisfies condition* (AA) *if and only if for any positive integer $n \geq 2$ and any sequence of stopping times $0 \leq \tau_1 \leq \cdots \leq \tau_n \equiv T$ there exists an equivalent martingale measure Q such that (X_{τ_n}) is a Q-martingale.*

The following theorem is an adaptation of Kabanov and Stricker's technique (see Theorem 1 in [9]) to the case of shortsale restrictions. Consider

the case $T = 1$. Let $X = (X_0, X_1)$ be an $(\mathcal{F}_0, \mathcal{F}_1)$-adapted $\mathbb{R}^d$-valued process and let $\Delta X^i = X_1^i - X_0^i$. Let $K = \{\xi : \xi = \sum_{i=1}^d N^i \Delta X^i, \ N \in \mathcal{P}\}$, where $\mathcal{P}$ is the set of predictable nonnegative processes and $A_1^+ = K - L_+^0$. By $\bar{A}_1^+$ we denote the closure of A_1^+ in probability.

THEOREM 2.33. *Assume* (AA^+) *for* $T = 1$. *Then the following conditions are equivalent:*

  (i) $A_1^+ \cap L_+^0 = \{0\}$;
  (ii) $A_1^+ \cap L_+^0 = \{0\}$ *and* $A_1^+ = \bar{A}_1^+$;
  (iii) $\bar{A}_1^+ \cap L_+^0 = \{0\}$;
  (iv) *there is a probability measure* $Q \sim P$ *with* $dQ/dP \in L^\infty$ *such that* X *is a* Q-*supermartingale.*

Proof. (i)⇒(ii). We have to show that A_1^+ is closed in probability. Suppose that $\sum_{i=1}^d N_n^i \Delta X^i - r_n \to \zeta$ P-a.e., where N_n^i is $\mathcal{F}_0$-measurable nonnegative and $r_n \in L_+^0$. Let $\Omega_1 = \{\liminf |N_n| < \infty\}$. On Ω_1 (by Lemma 2 in [9]) there is a random sequence $n_k(\omega)$ such that $N_{n_k}^i(\omega) \to N^i(\omega)$ for all ω. Then necessarily $r_{n_k} = \sum_{i=1}^d N_{n_k}^i \Delta X^i - \zeta_{n_k}$ tends a.s. to $\sum_{i=1}^d N^i \Delta X^i - \zeta = r \geq 0$ and therefore $\zeta \in A_1^+$ a.s. On $\Omega_2 = \{\liminf |N_n| = \infty\}$ we define $G_n = N_n/|N_n|$ and $h_n = r_n/|N_n|$. Note that $G_n \subset S^{d-1}$, where S^{d-1} denotes the unit sphere in $\mathbb{R}^d$, and we may (as in the proof of Theorem 1 in [9]) find $\mathcal{F}_0$-measurable nonnegative G_{n_k} such that $G_{n_k}(\omega)$ is a convergent subsequence of $G_n(\omega)$ for every ω. Since $\sum_{i=1}^d G_n^i \Delta X^i - h_n \to 0$ we obtain $\sum_{i=1}^d G^i \Delta X^i = h$, where $G^i(\omega) = \lim G_{n_k}(\omega)$, which (in view of (AA^+)) means that $\sum_{i=1}^d G^i \Delta X^i = 0$. Note that there exists a partition of Ω_2 into disjoint subsets $\Omega_2^i \in \mathcal{F}_0$ such that $G^i \neq 0$ on Ω_2^i. Let $\beta_n = \min_{G^i \neq 0} N_n^i/G^i$ on Ω_2^i. Now we can define the sequence $\tilde{N}_n^i = N_n^i - \beta_n G^i$ with the properties: $\sum_{i=1}^d \tilde{N}_n^i \Delta X^i = \sum_{i=1}^d N_n^i \Delta X^i$ and $\tilde{N}_n^i \geq 0$ for $i = 1, \ldots, d$. This procedure leads to elimination of nonzero components of the sequence N_n, so by iteration we can construct the desired sequence.

(ii)⇒(iii). Trivial.

(iii)⇒(iv) and (iv)⇒(i). We proceed in the same way as in the proof of Theorem 1 in [9]. ∎

REMARK 2.34. In the original paper the time horizon $T \geq 1$ was considered. Here we adapt their techniques only in the case $T = 1$. For $T > 1$ their procedure does not seem to work in the case of shortselling.

COROLLARY 2.35. (AA^+) *in discrete time is equivalent to existence of an equivalent supermartingale measure* Q.

Proof. To construct an equivalent supermartingale measure Q we use induction on T. For $T = 1$ Theorem 2.33 applies. Suppose that there is an equivalent probability measure Q^1, defined on $\mathcal{F}_T$, such that $dQ^1/dP \in L^\infty$

and $(X_t)_{t=1}^T$ is a Q^1-supermartingale, i.e. $X_1, \ldots, X_T \in L^1(\Omega, \mathcal{F}_T, Q^1)$ and $E_{Q^1}(X_{t+1} \mid \mathcal{F}_t) \leq X_t$ for $t = 1, \ldots, T - 1$. Theorem 2.33 applied to the probability space $(\Omega, \mathcal{F}_1, Q^1)$ gives us a probability measure $Q \sim Q^1$ with $dQ/dQ^1 \in L^\infty$ such that $E_Q(X_1 \mid \mathcal{F}_0) \leq X_0$. Define Q on $\mathcal{F}_T$ by

$$\frac{dQ}{dP} = \frac{dQ}{dQ^1} \frac{dQ^1}{dP}.$$

Then since dQ/dQ^1 is bounded and $\mathcal{F}_1$-measurable we have

$$E_Q(X_{t+1} \mid \mathcal{F}_t) \leq X_t \quad \text{for } t \geq 1. \ \blacksquare$$

We are now ready to state the main result of this section:

THEOREM 2.36. *The process X satisfies condition* (AA^+) *if and only if for any integer $n \geq 2$ and any sequence of stopping times $0 \leq \tau_1 \leq \cdots \leq \tau_n \equiv T$ there exists an equivalent supermartingale measure Q such that (X_{τ_n}) is a Q-supermartingale.*

3. Markets with proportional transaction costs

3.1. Problems and illustrative examples. Note that absence of arbitrage over one period and absence of arbitrage over a multiperiod are not equivalent in the case of proportional transaction costs. The following example gives a deterministic process that satisfies weak absence of arbitrage without shortselling $(\mathrm{AA}_{\mathrm{w}}^+)$ over any single period and has a strict arbitrage opportunity over the whole period.

EXAMPLE 3.1. Consider the deterministic model with $T = 2$, $S_0 = 1$, $S_1 = 2$ and $S_2 = 3$. Assume $\lambda, \mu \in (0, 1)$ are such that

(3.1) $$\frac{1 + \lambda}{1 - \mu} \in (2, 3).$$

We first show that S admits a strict arbitrage opportunity without shortselling. Consider the investment of buying $\frac{1}{1+\lambda}$ assets and borrowing -1 from the bank account at time $t = 0$ with no change of portfolio at time $t = 1$, so $(\Delta x_0, \Delta y_0) = (-1, \frac{1}{1+\lambda}) \in L^0(-G_0, \mathcal{F}_0)$ and $(\Delta x_1, \Delta y_1) = (\Delta x_2, \Delta y_2) = (0, 0)$. Then $(x_2, y_2) = (-1, \frac{1}{1+\lambda}) = \sum_{i=0}^2 (\Delta x_i, \Delta y_i) \in A_+^2$ and if $\frac{1}{1+\lambda} > \frac{1}{3(1-\mu)}$ then $(x_2, y_2) \in L^0(\mathrm{int}\, G_2, \mathcal{F}_2)$. But since $\frac{1+\lambda}{1-\mu} < 3$ by (3.1) we get the required inequality.

We now prove $(\mathrm{AA}_{\mathrm{w}}^+)$ over the single period $0 \leftrightarrow 1$. Let $(x_1, y_1) = (\Delta x_0, \Delta y_0) + (\Delta x_1, \Delta y_1)$ be a position such that $(x_1, y_1) \in A_+^1 \cap L^0(G_1, \mathcal{F}_1)$. We have to show that $(x_1, y_1) \in L^0(\partial G_1, \mathcal{F}_1)$. Shortsale restrictions imply that $y_1 \geq 0$ and $\Delta y_0 \geq 0$, hence as $(x_1, y_1) \in L^0(G_1, \mathcal{F}_1)$ and $(\Delta x_0, \Delta y_0) \in L^0(-G_0, \mathcal{F}_0)$ we get $y_1 \geq \frac{-x_1}{2(1-\mu)}$ and $\Delta y_0 \leq \frac{-\Delta x_0}{1+\lambda}$. Moreover (3.1) implies

that $\frac{-\Delta x_0}{1+\lambda} \leq \frac{-\Delta x_0}{2(1-\mu)}$ (note that $\Delta x_0 \leq 0$), so we have $\Delta y_0 \leq \frac{-\Delta x_0}{2(1-\mu)}$. Then

$$\frac{-x_1}{2(1-\mu)} \leq y_1 \leq \frac{-\Delta x_0}{2(1-\mu)} + \Delta y_1,$$

from which it follows that $\Delta y_1 \geq \frac{-\Delta x_1}{2(1-\mu)}$. On the other hand we know that $(\Delta x_1, \Delta y_1) \in L^0(-G_1, \mathcal{F}_1)$, so $\Delta y_1 \leq \frac{-\Delta x_1}{2(1-\mu)}$. Consequently, $\Delta y_1 = \frac{-\Delta x_1}{2(1-\mu)}$ and $y_1 = \frac{-x_1}{2(1-\mu)}$, which implies that $(x_1, y_1) \in L^0(\partial G_1, \mathcal{F}_1)$.

We finish by proving (AA_w^+) over the single period $1 \leftrightarrow 2$. We assume that at time 1 we make an investment starting from $(0, 0)$ borrowing $-\Delta x_1$ from the bank account and buying Δy_1 assets. Take any $(x_2, y_2) = (\Delta x_1, \Delta y_1) + (\Delta x_2, \Delta y_2) \in A_1^2 \cap L^0(G_2, \mathcal{F}_2)$ such that $\Delta y_1 \geq 0$ and $y_2 \geq 0$. Then $y_2 \geq \frac{-x_2}{3(1-\mu)}$ and $\Delta y_1 \leq \frac{-\Delta x_1}{2(1+\lambda)}$. By using (3.1) we have $\frac{-\Delta x_1}{2(1+\lambda)} \leq \frac{-\Delta x_1}{3(1-\mu)}$ (we know that $\Delta x_1 \leq 0$), hence

$$\frac{-x_2}{3(1-\mu)} \leq y_2 \leq \frac{-\Delta x_1}{3(1-\mu)} + \Delta y_2,$$

which means $\Delta y_2 \geq \frac{-\Delta x_2}{3(1-\mu)}$. But $(\Delta x_2, \Delta y_2) \in L^0(-G_2, \mathcal{F}_2)$, so $\Delta y_2 \leq \frac{-\Delta x_2}{3(1-\mu)}$. Consequently, $\Delta y_2 = \frac{-\Delta x_2}{3(1-\mu)}$ and $y_2 = \frac{-x_2}{3(1-\mu)}$, which means that $(x_2, y_2) \in L^0(\partial G_2, \mathcal{F}_2)$ (i.e. (AA_w^+) holds over $1 \leftrightarrow 2$).

Another example shows that the study of arbitrage using the strategies without shortselling cannot be restricted to one period (as in the case without transaction costs).

EXAMPLE 3.2. Let $T = 2$ and $S_0 = 1$, $S_1 = S_0(1 + \xi_1)$, $S_2 = S_1(1 + \xi_2)$, where $\xi_1, \xi_2 > -1$ almost surely. Assume that

(i) $1 + \xi_i < \frac{1+\lambda}{1-\mu}$ with positive probability,

(ii) $\frac{1+\lambda}{1-\mu}(1 - \delta) \leq 1 + \xi_i$ P-a.e., where $\delta > 0$ is such that $(1 - \delta)^2 > \frac{1-\mu}{1+\lambda}$.

We claim that there is no strict arbitrage over periods $0 \leftrightarrow 1$ and $1 \leftrightarrow 2$. It is sufficient to show that the position of the form $\left(-1, \frac{1}{1+\lambda}\right) \in A_1^+$ is not an arbitrage opportunity. In fact, if at $t = 0$ we have $x_0 = -1$, $y_0 = \frac{1}{1+\lambda}$, then liquidating it at time $t = 1$ we find that $-1 + \frac{1-\mu}{1+\lambda}(1 + \xi_1) \geq 0$ P-a.s. whenever $1 + \xi_1 < \frac{1+\lambda}{1-\mu}$, which is not satisfied (by (i)). Consider the whole period; let again $x_0 = -1$, $y_0 = \frac{1}{1+\lambda}$; then liquidating the position at time $t = 2$ (with no change of portfolio at time $t = 1$) we find that $-1 + \frac{1-\mu}{1+\lambda}(1 + \xi_1)(1 + \xi_2) > 0$ is satisfied when $(1 + \xi_1)(1 + \xi_2) > \frac{1+\lambda}{1-\mu}$, which holds when $(1 - \delta)^2 > \frac{1-\mu}{1+\lambda}$.

3.2. Sticky type conditions. Let $Y = (Y_t)_{t \in [0,T]} \in \mathcal{B}_T$, where

$$Y_t = \sum_{i=1}^{n-2} y_i \chi_{(\tau_i, \tau_{i+1}]}(t) + y_{n-1}\chi_{(\tau_{n-1}, \tau_n)}(t) + y_n \chi_{\{T\}}(t)$$

for some integer $n \geq 2$ and a sequence of $\mathbb{F}$-stopping times $0 \leq \tau_1 \leq \cdots \leq \tau_n \equiv T$. The value process generated by the simple strategy Y is given by

$$(3.2) \quad W_T^Y = \sum_{j=1}^{d} \Big[\sum_{i=1}^{n-1} y_i^j (X_{\tau_{i+1}}^j - X_{\tau_i}^j) - \lambda^j \sum_{i=1}^{n} X_{\tau_i}^j (y_i^j - y_{i-1}^j)^+$$

$$- \mu^j \sum_{i=1}^{n} X_{\tau_i}^j (y_i^j - y_{i-1}^j)^- - \lambda^j (y_n^j)^- X_{\tau_n}^j - \mu^j (y_n^j)^+ X_{\tau_n}^j \Big],$$

where $y_0^j = 0$ for all $j = 1, \ldots, d$. The term $\sum_{j=1}^{d} [\lambda^j \sum_{i=1}^{n} X_{\tau_i}^j (y_i^j - y_{i-1}^j)^+ + \mu^j \sum_{i=1}^{n} X_{\tau_i}^j (y_i^j - y_{i-1}^j)^-]$ corresponds to the cost of trading. The liquidation cost at the end of trading equals $\sum_{j=1}^{d} [\lambda^j (y_n^j)^- X_{\tau_n}^j + \mu^j (y_n^j)^+ X_{\tau_n}^j]$. Observe that $W_T^Y = R_T(x_n, y_n^1, \ldots, y_n^d)$, so we can now reformulate the definition of strict absence of arbitrage. We say that we have a strict arbitrage opportunity with respect to the class of simple strategies (resp. simple strategies without shortselling) if there exists a simple investment strategy $Y \in \mathcal{B}_T$ (resp. $Y \in \mathcal{B}_T^+$) with the properties

$$(3.3) \qquad W_T^Y \geq 0 \ P\text{-a.e.} \quad \text{and} \quad P(W_T^Y > 0) > 0.$$

We start with the case $d = 1$.

DEFINITION 3.3. We say that an adapted càdlàg process X satisfies *condition* (S) with respect to the filtration $\mathbb{F}$ if for any stopping time $\tau \leq T$ and any $\epsilon > 0$ we have

$$P \left(\sup_{\tau \leq t \leq T} \left| \ln \frac{X_t}{X_\tau} \right| < \epsilon \ \Big| \ \mathcal{F}_\tau \right) > 0 \qquad P\text{-a.e.}$$

In the next lemma we show that condition (S) is sufficient for weak absence of arbitrage at time T. This result is based on Proposition 1 in [13].

PROPOSITION 3.4. *If the adapted càdlàg process X satisfies condition* (S), *then there is no strict arbitrage at time T (i.e.* (AA$_w$) *holds).*

Proof. Assume $Y = (Y_t)_{t \in [0,T]} \in \mathcal{B}_T$ is an arbitrage strategy. Then the value process W^Y satisfies $W_T^Y \geq 0$ P-a.e. and $P\{W_T^Y > 0\} > 0$. Let $\sigma = \min\{i \in \{1, \ldots, n\} : y_i \neq 0\}$. Since $P\{W_T^Y > 0\} > 0$, the event $A = \{\tau_\sigma < T\}$ has positive probability. Observe that on the set $A \in \mathcal{F}_\sigma$ we can write

$$W_T^Y = \sum_{i=\sigma}^{n} y_i (X_{\tau_{i+1}} - X_{\tau_i}) - \lambda \sum_{i=\sigma}^{n+1} X_{\tau_i} (y_i - y_{i-1})^+ - \mu \sum_{i=\sigma}^{n+1} X_{\tau_i} (y_i - y_{i-1})^-,$$

where $\tau_{n+1} \equiv T$, $y_0 = 0$ and $y_{n+1} = 0$. If we denote $\tilde{X}_{\tau_i} = X_{\tau_i} - X_{\tau_\sigma}$, then

$$\sum_{i=\sigma}^{n} y_i(X_{\tau_{i+1}} - X_{\tau_i}) = \sum_{i=\sigma}^{n} y_i \tilde{X}_{\tau_{i+1}} - \sum_{i=\sigma}^{n} y_i \tilde{X}_{\tau_i} = \sum_{i=\sigma+1}^{n+1} y_{i-1} \tilde{X}_{\tau_i} - \sum_{i=\sigma}^{n+1} y_i \tilde{X}_{\tau_i}$$

$$= \sum_{i=\sigma}^{n+1} (y_{i-1} - y_i) \tilde{X}_{\tau_i}.$$

Hence

$$W_T^Y = \sum_{i=\sigma}^{n+1} \left[(y_{i-1} - y_i) \tilde{X}_{\tau_i} - \lambda \sum_{i=\sigma}^{n+1} X_{\tau_i}(y_i - y_{i-1})^+ - \mu \sum_{i=\sigma}^{n+1} X_{\tau_i}(y_i - y_{i-1})^- \right].$$

Note that

$$(y_{i-1} - y_i)\tilde{X}_{\tau_i} - \lambda X_{\tau_i}(y_i - y_{i-1})^+ - \mu X_{\tau_i}(y_i - y_{i-1})^-$$
$$= -(y_i - y_{i-1})^+[(1 + \lambda)X_{\tau_i} - X_{\tau_\sigma}] + (y_i - y_{i-1})^-[(1 - \mu)X_{\tau_i} - X_{\tau_\sigma}]$$

for all $i \in \{1, \ldots, n+1\}$. It follows that

$$W_T^Y = \sum_{i=\sigma}^{n+1} (y_i - y_{i-1})^-[(1 - \mu)X_{\tau_i} - X_{\tau_\sigma}] - \sum_{i=\sigma}^{n+1} (y_i - y_{i-1})^+[(1 + \lambda)X_{\tau_i} - X_{\tau_\sigma}].$$

Since X satisfies condition (S), the event

$$B^\epsilon = A \cap \left\{ \sup_{\tau_\sigma \leq t \leq T} \left| \ln \frac{X_t}{X_{\tau_\sigma}} \right| < \epsilon \right\}$$

has positive probability for any $\epsilon > 0$. Observe that on B^ϵ,

$$\frac{1}{1 + \lambda} < \frac{X_{\tau_i}}{X_{\tau_\sigma}} < \frac{1}{1 - \mu} \qquad \text{for all } i \in \{\sigma, \ldots, n+1\}$$

whenever $\epsilon < \min\{\ln(1 + \lambda), \ln\left(\frac{1}{1-\mu}\right)\}$. Moreover, on A there is $i \in \{\sigma, \ldots, n+1\}$ such that $(y_i - y_{i-1})^- \neq 0$ or $(y_i - y_{i-1})^+ \neq 0$ (by the definition of A and σ). Therefore $W_T^Y < 0$ on B^ϵ, which contradicts the assumption $P\{W_T^Y \geq 0\} = 1$. ∎

REMARK 3.5. Under (S) we also have $(\mathrm{AA}_\mathrm{w}^+)$ (clearly, if there is a strict arbitrage opportunity without shortselling, then there is a strict arbitrage).

REMARK 3.6. Under (b) we also have $(\mathrm{AA}_\mathrm{w}^+)$ (absence of arbitrage without transaction costs implies weak absence of arbitrage: observe that

$$W_T^Y \leq \sum_{i=1}^{n-1} y_i(X_{\tau_{i+1}} - X_{\tau_i}) = (\tilde{Y} \cdot X)_T,$$

so if $Y \in \mathcal{B}_T^+$ satisfies (3.3), then the simple strategy $\tilde{Y} = \sum_{i=1}^{n-1} y_i \chi_{(\tau_i, \tau_{i+1}]}(t)$ is an arbitrage opportunity without shortselling).

The following lemma has important consequences for characterization of weak absence of arbitrage under shortsale restrictions.

LEMMA 3.7. *Let $\xi_0, \xi_1, \ldots, \xi_n$ be a sequence of nonnegative random variables on a probability space $(\Omega, \mathcal{F}, P)$, and $\eta_0, \eta_1, \ldots, \eta_n$ be a sequence of nonnegative random variables with $\eta_n = 0$. Then*

$$\bigcap_{\substack{i,j \in \{0,\ldots,n\} \\ i<j}} \{(1-\mu)\xi_j < (1+\lambda)\xi_i\} \subset \{W_{\xi_0,\xi_1,\ldots,\xi_n}(\eta_0, \eta_1 \ldots, \eta_n) < 0\},$$

where $\lambda, \mu \in (0,1)$ and

$$W_{\xi_0,\xi_1,\ldots,\xi_n}(\eta_0, \ldots, \eta_n) = \sum_{i=1}^{n} \eta_{i-1}(\xi_i - \xi_{i-1}) - \lambda\eta_0\xi_0 - \sum_{i=1}^{n-1} \lambda(\eta_i - \eta_{i-1})^+\xi_i$$
$$- \sum_{i=1}^{n-1} \mu(\eta_i - \eta_{i-1})^-\xi_i - \mu\eta_{n-1}\xi_n$$

for $n \geq 2$ and

$$W_{\xi_0,\xi_1}(\eta_0, \eta_1) = \eta_0(\xi_1 - \xi_0) - \lambda\eta_0\xi_0 - \mu\eta_0\xi_1.$$

Proof. We use induction on n. First we prove the statement for $n = 1$. Since $\eta_0 \geq 0$ we have

$$W_{\xi_0,\xi_1}(\eta_0, \eta_1) = \eta_0[(1-\mu)\xi_1 - (1+\lambda)\xi_0] < 0$$

on the set $\{(1-\mu)\xi_1 < (1+\lambda)\xi_0\}$. Now suppose that the assertion holds true for $k-1$. Observe that

$$W_{\xi_0,\xi_1,\ldots,\xi_k}(\eta_0, \eta_1, \ldots, \eta_k) = \eta_0[(1+\lambda)\xi_1\chi_{\{\eta_1>\eta_0\}} + (1-\mu)\xi_1\chi_{\{\eta_1<\eta_0\}}]$$
$$- \eta_0(1+\lambda)\xi_0 + \sum_{i=1}^{k-2} \eta_i[(1+\lambda)\xi_{i+1}\chi_{\{\eta_{i+1}>\eta_i\}} + (1-\mu)\xi_{i+1}\chi_{\{\eta_{i+1}<\eta_i\}}]$$
$$- \sum_{i=1}^{k-2} \eta_i[(1+\lambda)\xi_i\chi_{\{\eta_i>\eta_{i-1}\}} + (1-\mu)\xi_i\chi_{\{\eta_i<\eta_{i-1}\}}]$$
$$+ \eta_{k-1}[(1-\mu)\xi_k - (1+\lambda)\xi_{k-1}\chi_{\{\eta_{k-1}>\eta_{k-2}\}} - (1-\mu)\xi_{k-1}\chi_{\{\eta_{k-1}<\eta_{k-2}\}}].$$

We consider first the process $W_{\xi_0,\xi_1,\ldots,\xi_k}(\eta_0, \eta_1, \ldots, \eta_k)\chi_{\{\eta_0=\eta_1\}}$. Clearly

$$W_{\xi_0,\xi_1,\ldots,\xi_k}(\eta_0, \eta_1, \ldots, \eta_k)\chi_{\{\eta_0=\eta_1\}} = W_{\xi_0,\xi_2,\ldots,\xi_k}(\eta_1, \ldots, \eta_k)\chi_{\{\eta_0=\eta_1\}}$$

and

$$\bigcap_{\substack{i,j \in \{0,\ldots,k\} \\ i<j}} \{(1-\mu)\xi_j < (1+\lambda)\xi_i\} \subset \bigcap_{\substack{i,j \in \{0,\ldots,n\}\setminus\{1\} \\ i<j}} \{(1-\mu)\xi_j < (1+\lambda)\xi_i\},$$

so by the inductive hypothesis we have

$$(3.4) \qquad \bigcap_{\substack{i,j \in \{0,\ldots,k\} \\ i<j}} \{(1-\mu)\xi_j < (1+\lambda)\xi_i\}$$

$$\subset \{W_{\xi_0,\xi_1,\ldots,\xi_k}(\eta_0,\eta_1,\ldots,\eta_k)\chi_{\{\eta_0=\eta_1\}} < 0\}.$$

The next step is to consider the process

$$W_{\xi_0,\xi_1,\ldots,\xi_k}(\eta_0,\eta_1,\ldots,\eta_k)\chi_{\{\eta_1<\eta_0\}} = \chi_{\{\eta_1<\eta_0\}}\Big\{\eta_0[(1-\mu)\xi_1 - (1+\lambda)\xi_0]$$

$$+ \eta_1[(1+\lambda)\xi_2\chi_{\{\eta_2>\eta_1\}} + (1-\mu)\xi_2\chi_{\{\eta_2<\eta_1\}}]$$

$$- \eta_1(1-\mu)\xi_1 + \sum_{i=2}^{k-2}\eta_i[(1+\lambda)\xi_{i+1}\chi_{\{\eta_{i+1}>\eta_i\}} + (1-\mu)\xi_{i+1}\chi_{\{\eta_{i+1}<\eta_i\}}]$$

$$- \sum_{i=2}^{k-2}\eta_i[(1+\lambda)\xi_i\chi_{\{\eta_i>\eta_{i-1}\}} + (1-\mu)\xi_i\chi_{\{\eta_i<\eta_{i-1}\}}]$$

$$+ \eta_{k-1}[(1-\mu)\xi_k - (1+\lambda)\xi_{k-1}\chi_{\{\eta_{k-1}>\eta_{k-2}\}} - (1-\mu)\xi_{k-1}\chi_{\{\eta_{k-1}<\eta_{k-2}\}}]\Big\}.$$

Note that

$$\eta_0[(1-\mu)\xi_1 - (1+\lambda)\xi_0] < \eta_1[(1-\mu)\xi_1 - (1+\lambda)\xi_0]$$

on the set $\{(1-\mu)\xi_1 < (1+\lambda)\xi_0\} \cap \{\eta_1 < \eta_0\}$, from which it follows that

$$W_{\xi_0,\xi_1,\ldots,\xi_k}(\eta_0,\eta_1,\ldots,\eta_k)\chi_{\{\eta_1<\eta_0\}} < \chi_{\{\eta_1<\eta_0\}}\Big\{-\eta_1(1+\lambda)\xi_0$$

$$+ \eta_1[(1+\lambda)\xi_2\chi_{\{\eta_2>\eta_1\}} + (1-\mu)\xi_2\chi_{\{\eta_2<\eta_1\}}]$$

$$+ \sum_{i=2}^{k-2}\eta_i[(1+\lambda)\xi_{i+1}\chi_{\{\eta_{i+1}>\eta_i\}} + (1-\mu)\xi_{i+1}\chi_{\{\eta_{i+1}<\eta_i\}}]$$

$$- \sum_{i=2}^{k-2}\eta_i[(1+\lambda)\xi_i\chi_{\{\eta_i>\eta_{i-1}\}} + (1-\mu)\xi_i\chi_{\{\eta_i<\eta_{i-1}\}}]$$

$$+ \eta_{k-1}[(1-\mu)\xi_k - (1+\lambda)\xi_{k-1}\chi_{\{\eta_{k-1}>\eta_{k-2}\}} - (1-\mu)\xi_{k-1}\chi_{\{\eta_{k-1}<\eta_{k-2}\}}]\Big\}.$$

Observe that the right hand side represents $W_{\xi_0,\xi_2,\ldots,\xi_k}(\eta_1,\ldots,\eta_k)$ on the set $\{\eta_1 < \eta_0\}$ and hence

$$\bigcap_{\substack{i,j \in \{0,\ldots,k\} \\ i<j}} \{(1-\mu)\xi_j < (1+\lambda)\xi_i\}$$

$$\subset \{W_{\xi_0,\xi_1,\ldots,\xi_k}(\eta_0,\eta_1,\ldots,\eta_k)\chi_{\{\eta_1<\eta_0\}} < W_{\xi_0,\xi_2,\ldots,\xi_k}(\eta_1,\ldots,\eta_k)\chi_{\{\eta_1<\eta_0\}}\}.$$

By the induction hypothesis we have

$$\bigcap_{\substack{i,j\in\{0,\dots,k\}\setminus\{1\}\\i<j}}\big\{(1-\mu)\xi_j<(1+\lambda)\xi_i\big\}\subset\big\{W_{\xi_0,\xi_2,\dots,\xi_k}(\eta_1,\dots,\eta_k)<0\big\},$$

which means that

$$(3.5)\qquad \bigcap_{\substack{i,j\in\{1,\dots,k\}\\i<j}}\big\{(1-\mu)\xi_j<(1+\lambda)\xi_i\big\}$$

$$\subset\big\{W_{\xi_0,\xi_1,\dots,\xi_k}(\eta_0,\eta_1,\dots,\eta_k)\chi_{\{\eta_1<\eta_0\}}<0\big\}.$$

Now we turn to the process $W_{\xi_0,\xi_1,\dots,\xi_k}(\eta_0,\eta_1,\dots,\eta_k)\chi_{\{\eta_1>\eta_0\}}$. Let $l=\max\{s:\eta_1<\cdots<\eta_{s-1}\}$. We remark that l is well defined because we assumed that $\eta_0,\eta_1,\dots,\eta_k$ are nonnegative random variables and $\eta_k=0$. Clearly,

$$W_{\xi_0,\xi_1,\dots,\xi_k}(\eta_0,\eta_1,\dots,\eta_k)\chi_{\{\eta_1>\eta_0\}}$$

$$=\chi_{\{\eta_1>\eta_0\}}\Big\{\sum_{i=0}^{\tilde{l}-2}\eta_i[(1+\lambda)\xi_{i+1}-(1+\lambda)\xi_i]$$

$$+\eta_{\tilde{l}-1}[(1-\mu)\xi_{\tilde{l}}-(1+\lambda)\xi_{\tilde{l}-1}]$$

$$+\eta_{\tilde{l}}[(1+\lambda)\xi_{\tilde{l}+1}\chi_{\{\eta_{\tilde{l}+1}>\eta_{\tilde{l}}\}}+(1-\mu)\xi_{\tilde{l}+1}\chi_{\{\eta_{\tilde{l}+1}<\eta_{\tilde{l}}\}}-(1-\mu)\xi_{\tilde{l}}]$$

$$+\sum_{i=\tilde{l}+1}^{k-2}\eta_i[(1+\lambda)\xi_{i+1}\chi_{\{\eta_{i+1}>\eta_i\}}+(1-\mu)\xi_{i+1}\chi_{\{\eta_{i+1}<\eta_i\}}]$$

$$-\sum_{i=\tilde{l}+1}^{k-2}\eta_i[(1+\lambda)\xi_i\chi_{\{\eta_i>\eta_{i-1}\}}+(1-\mu)\xi_i\chi_{\{\eta_i<\eta_{i-1}\}}]$$

$$+\eta_{k-1}[(1-\mu)\xi_k-(1+\lambda)\xi_{k-1}\chi_{\{\eta_{k-1}>\eta_{k-2}\}}-(1-\mu)\xi_{k-1}\chi_{\{\eta_{k-1}<\eta_{k-2}\}}]\Big\}$$

on the set $\{l=\tilde{l}\}$ for some $\tilde{l}\in\{1,\dots,k\}$. Since

$$\eta_{\tilde{l}-1}[(1-\mu)\xi_{\tilde{l}}-(1+\lambda)\xi_{\tilde{l}-1}]<\eta_{\tilde{l}}[(1-\mu)\xi_{\tilde{l}}-(1+\lambda)\xi_{\tilde{l}-1}]$$

on the set $\{l=\tilde{l}\}\cap\{(1-\mu)\xi_{\tilde{l}}<(1+\lambda)\xi_{\tilde{l}-1}\}$ we have

$$W_{\xi_0,\xi_1,\dots,\xi_k}(\eta_0,\eta_1,\dots,\eta_k)\chi_{\{\eta_1>\eta_0\}}\chi_{\{l=\tilde{l}\}}$$

$$<\chi_{\{\eta_1>\eta_0\}}\chi_{\{l=\tilde{l}\}}\Big\{\sum_{i=0}^{\tilde{l}-2}\eta_i[(1+\lambda)\xi_{i+1}-(1+\lambda)\xi_i]$$

$$+\eta_{\tilde{l}}[(1+\lambda)\xi_{\tilde{l}+1}\chi_{\{\eta_{\tilde{l}+1}>\eta_{\tilde{l}}\}}+(1-\mu)\xi_{\tilde{l}+1}\chi_{\{\eta_{\tilde{l}+1}<\eta_{\tilde{l}}\}}-(1+\lambda)\xi_{\tilde{l}-1}]$$

$$+ \sum_{i=\tilde{l}+1}^{k-2} \eta_i[(1+\lambda)\xi_{i+1}\chi_{\{\eta_{i+1}>\eta_i\}} + (1-\mu)\xi_{i+1}\chi_{\{\eta_{i+1}<\eta_i\}}]$$

$$- \sum_{i=\tilde{l}+1}^{k-2} \eta_i[(1+\lambda)\xi_i\chi_{\{\eta_i>\eta_{i-1}\}} + (1-\mu)\xi_i\chi_{\{\eta_i<\eta_{i-1}\}}]$$

$$+ \eta_{k-1}[(1-\mu)\xi_k - (1+\lambda)\xi_{k-1}\chi_{\{\eta_{k-1}>\eta_{k-2}\}} - (1-\mu)\xi_{k-1}\chi_{\{\eta_{k-1}<\eta_{k-2}\}}]\Big\}.$$

This means that

$$W_{\xi_0,\xi_1,\dots,\xi_k}(\eta_0,\eta_1,\dots,\eta_k)\chi_{\{\eta_1>\eta_0\}}\chi_{\{l=\tilde{l}\}}$$
$$< W_{\xi_0,\dots,\xi_{\tilde{l}-1},\xi_{\tilde{l}+1},\dots,\xi_k}(\eta_0,\dots,\eta_{\tilde{l}-2},\eta_{\tilde{l}},\dots,\eta_k)\chi_{\{\eta_1>\eta_0\}}\chi_{\{l=\tilde{l}\}}$$

on $\bigcap_{i,j\in\{0,\dots,k\},\, i<j}\{(1-\mu)\xi_j < (1+\lambda)\xi_i\}$ and by induction we have

$$(3.6) \qquad \bigcap_{\substack{i,j\in\{0,\dots,k\}\\ i<j}} \{(1-\mu)\xi_j < (1+\lambda)\xi_i\}$$
$$\subset \{W_{\xi_0,\xi_1,\dots,\xi_k}(\eta_0,\eta_1,\dots,\eta_k)\chi_{\{\eta_1>\eta_0\}}\chi_{\{l=\tilde{l}\}} < 0\}.$$

Summing up, we have shown (by (3.4)–(3.6)) that

$$\bigcap_{\substack{i,j\in\{0,\dots,k\}\\ i<j}} \{(1-\mu)\xi_j < (1+\lambda)\xi_i\} \subset \{W_{\xi_0,\xi_1,\dots,\xi_k}(\eta_0,\eta_1,\dots,\eta_k) < 0\},$$

which proves the lemma. ∎

DEFINITION 3.8. We say that a process X satisfies *condition* (D) with respect to the filtration $\mathbb{F}$ if for any stopping times $0 \le \tau_1 \le \cdots \le \tau_n \equiv T$ we have

$$P\Big(\bigcap_{i<j}\{(1-\mu)X_{\tau_j} < (1+\lambda)X_{\tau_i}\}\Big) > 0.$$

THEOREM 3.9. *If X is a process satisfying condition* (D)*, then it satisfies* $(\mathrm{AA}_\mathrm{w}^+)$ *(weak absence of arbitrage without shortselling).*

Proof. Apply Lemma 3.7 to the sequence of nonnegative random variables

$$X_{\tau_1}, X_{\tau_2}, \dots X_{\tau_{n-1}}, X_{\tau_n}, X_{\tau_{n+1}}$$

where $\tau_n \equiv \tau_{n+1} \equiv T$, and the sequence of nonnegative random variables

$$y_1, \dots, y_{n+1},$$

where $y_{n+1} = 0$. ∎

We now define the multidimensional extension of condition (S) and show that it is sufficient for weak absence of arbitrage for a multi-asset model (see Proposition 1 in [13]).

DEFINITION 3.10. We say that an adapted càdlàg process X satisfies *condition* (S^d) with respect to the filtration $\mathbb{F}$ if for any stopping time $\tau \leq T$ and any $\epsilon > 0$ we have

$$P\Big(\bigcap_{j=1}^{d} \Big\{ \sup_{\tau \leq t \leq T} \Big| \ln \frac{X_t^j}{X_\tau^j} \Big| < \epsilon \Big\} \, \Big| \, \mathcal{F}_\tau \Big) > 0 \quad P\text{-a.e.}$$

PROPOSITION 3.11. *If the adapted càdlàg process X satisfies condition* (S^d), *then it satisfies* (AA_w) (*no strict arbitrage*).

Proof. The idea of the proof is the same as for the case $d = 1$. Suppose to the contrary that there is a strategy $Y = (Y_t)_{t \in [0,T]} \in \mathcal{B}_T$ such that $W_T^Y \geq 0$ P-a.e. and $P\{W_T^Y > 0\} > 0$. Let $\sigma = \min\{i \in \{1, \ldots, n\} : y_i^j \neq 0, \ j = 1, \ldots, d\}$. Since $P\{W_T^Y > 0\} > 0$, the event $A = \{\tau_\sigma < T\}$ has positive probability. Note that on the set $A \in \mathcal{F}_{\tau_\sigma}$ we can write

$$W_T^Y = \sum_{j=1}^{d} \Big(\sum_{i=\sigma}^{n} y_i^j (X_{\tau_{i+1}}^j - X_{\tau_i}^j)$$
$$- \lambda^j \sum_{i=\sigma}^{n+1} X_{\tau_i}^j (y_i^j - y_{i-1}^j)^+ - \mu^j \sum_{i=\sigma}^{n+1} X_{\tau_i}^j (y_i^j - y_{i-1}^j)^- \Big),$$

where $\tau_{n+1} \equiv T$, $y_0^j = 0$ and $y_{n+1}^j = 0$ for $j = 1, \ldots, d$. As observed in the proof of Lemma 3.4, the value process can be expressed by

$$W_T^Y = \sum_{j=1}^{d} \Big[\sum_{i=\sigma}^{n+1} (y_i^j - y_{i-1}^j)^- [(1 - \mu^j) X_{\tau_i}^j - X_{\tau_\sigma}^j]$$
$$- \sum_{i=\sigma}^{n+1} (y_i^j - y_{i-1}^j)^+ [(1 + \lambda^j) X_{\tau_i}^j - X_{\tau_\sigma}^j] \Big].$$

Since X satisfies condition (S^d), the event

$$B^\epsilon = A \cap \bigcap_{j=1}^{d} \Big\{ \sup_{\tau_\sigma \leq t \leq T} \Big| \ln \frac{X_t^j}{X_{\tau_\sigma}^j} \Big| < \epsilon \Big\}$$

has positive probability for any $\epsilon > 0$. Note that on B^ϵ, $(1 - \mu^j) X_{\tau_i}^j < X_{\tau_\sigma}^j$ and $(1 + \lambda^j) X_{\tau_i}^j > X_{\tau_\sigma}^j$ for all $i \in \{\sigma, \ldots, n+1\}$, $j \in \{1, \ldots, d\}$ whenever $\epsilon < \min\{\ln(1 + \lambda^j), \ln\big(\frac{1}{1-\mu^j}\big)\}$. Moreover, on A there are $i \in \{\sigma, \ldots, n+1\}$ and $j \in \{1, \ldots, d\}$ such that $(y_i^j - y_{i-1}^j)^- \neq 0$ or $(y_i^j - y_{i-1}^j)^+ \neq 0$ (by the definition of A and σ). Therefore $W_T^Y < 0$ on B^ϵ, which contradicts the assumption $P\{W_T^Y \geq 0\} = 1$. ■

As an extension of Theorem 3.9 to the multidimensional case, we give a sufficient condition for weak absence of arbitrage with shortsale restrictions.

DEFINITION 3.12. We say that a process X satisfies *condition* (D^d) with respect to the filtration $\mathbb{F}$ if for any stopping times $0 \leq \tau_1 \leq \cdots \leq \tau_n \equiv T$ we have

$$P\Big(\bigcap_{j=1}^d \bigcap_{i<k} \{(1 - \mu^j)X_{\tau_k}^j < (1 + \lambda^j)X_{\tau_i}^j\} \Big) > 0.$$

THEOREM 3.13. *If X is a process satisfying condition* (D^d), *then it satisfies* (AA_w^+) *(weak absence of arbitrage without shortselling).*

Proof. Assume $Y = (Y_t)_{t \in [0,T]} \in \mathcal{B}_T^+$ is an arbitrage strategy. Then the value process generated by Y with shortsales restrictions, given by

$$W_T^Y = \sum_{j=1}^d \Big[\sum_{i=1}^n y_i^j (X_{\tau_{i+1}}^j - X_{\tau_i}^j) - \lambda^j y_1^j X_{\tau_1}^j + \lambda^j \sum_{i=2}^n X_{\tau_i}^j (y_i^j - y_{i-1}^j)^+ $$
$$ - \mu^j \sum_{i=2}^n X_{\tau_i}^j (y_i^j - y_{i-1}^j)^- - \mu^j y_n^j X_{\tau_n}^j \Big],$$

where $\tau_n = \tau_{n+1} \equiv T$, satisfies $W_T^Y \geq 0$ P-a.e. and $P\{W_T^Y > 0\} > 0$. Note that

$$W_T^Y = \sum_{j=1}^d W_{X_{\tau_1}^j, \ldots, X_{\tau_n}^j, X_{\tau_{n+1}}^j}(y_1^j, \ldots, y_{n+1}^j),$$

and by Lemma 3.7,

$$\bigcap_{j=1}^d \bigcap_{\substack{i,k \in \{1,\ldots,n\} \\ i<k}} \{(1 - \mu^j)X_{\tau_k}^j < (1 + \lambda^j)X_{\tau_i}^j\}$$
$$\subset \bigcap_{j=1}^d \{W_{X_{\tau_1}^j, \ldots, X_{\tau_n}^j, X_{\tau_{n+1}}^j}(y_1^j, \ldots, y_{n+1}^j) < 0\}.$$

As X satisfies (D^d), we have $P(\bigcap_{j=1}^d \{W_{X_{\tau_1}^j, \ldots, X_{\tau_{n+1}}^j}(y_1^j, \ldots, y_{n+1}^j) < 0\}) > 0$. Hence the set $\{W_T^Y < 0\}$ has positive probability, contrary to $W_T^Y \geq 0$ almost surely. ∎

3.3. Embedded discrete time market approach. The positive dual cone of a cone $G \subset \mathbb{R}^{d+1}$ is defined as

$$G^* = \{v \in \mathbb{R}^{d+1} : \langle v, w \rangle \geq 0 \text{ for all } w \in G\}.$$

Denote by $\mathcal{M}_0^T(G^* \setminus \{0\})$ the set of martingales $Z = (Z_t)_{t=0}^T$ such that $Z_t \in L^0(G_t^* \setminus \{0\}, \mathcal{F}_t)$ for all $t = 0, \ldots, T$.

For a discrete time model, [5] and [12] give criteria for $(\mathrm{AA_w})$ in the two-dimensional case.

THEOREM 3.14. *Let* $d = 1$. *Then* $(\mathrm{AA_w})$ *holds iff*
$$\mathcal{M}_0^T(G^* \setminus \{0\}) \neq \emptyset.$$

Let us recall the result of [10] about the characterization of $(\mathrm{AA_s})$ in a discrete time model.

THEOREM 3.15. $(\mathrm{AA_s})$ *holds iff there exists* $Z \in \mathcal{M}_0^T(G^* \setminus \{0\})$ *such that* $Z_T \in L^0(\mathrm{int}\, G_T^*, \mathcal{F}_T)$.

REMARK 3.16. Note that in the case of shortsale restrictions similar criteria do not seem to be true. It is therefore an open problem to find a proper equivalent characterization for absence of arbitrage on the market without shortselling and with proportional transaction costs.

Let $0 \leq \tau_1 \leq \cdots \leq \tau_n \equiv T$ be a sequence of stopping times and denote by $\mathcal{M}_{\{\tau_1, \ldots, \tau_n\}}(G^* \setminus \{0\})$ the set of martingales $Z = (Z_{\tau_n})$ such that $Z_{\tau_i} \in L^0(G_{\tau_i}^* \setminus \{0\}, \mathcal{F}_{\tau_i})$ for all $i = 1, \ldots, n$. Using Theorems 3.14 and 3.15 we obtain

THEOREM 3.17. *Let* $d = 1$. *Then* $A^T \cap L^0(\mathbb{R}^2, \mathcal{F}_T) = \{0\}$ *if and only if* $\mathcal{M}_{\{\tau_1, \ldots, \tau_n\}}(G^* \setminus \{0\}) \neq \emptyset$ *for any integer* $n \geq 2$ *and for any sequence of stopping times* $0 \leq \tau_1 \leq \cdots \leq \tau_n \equiv T$.

THEOREM 3.18. $A^T \cap L^0(G_T, \mathcal{F}_T) = \{0\}$ *if and only if for any integer* $n \geq 2$ *and for any sequence of stopping times* $0 \leq \tau_1 \leq \cdots \leq \tau_n \equiv T$ *there exists* $Z \in \mathcal{M}_{\{\tau_1, \ldots, \tau_n\}}(G^* \setminus \{0\})$ *such that* $Z_T \in L^0(\mathrm{int}\, G_T^*, \mathcal{F}_T)$.

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Agnieszka Rygiel
Higher Vocational School in Tarnow
33-100 Tarnów, Poland
E-mail: a_rygiel@pwsztar.edu.pl

Łukasz Stettner
Institute of Mathematics
Polish Academy of Sciences
00-956 Warszawa, Poland
E-mail: L.Stettner@impan.pl