APPLICATIONES MATHEMATICAE
39,4 (2012), pp. 445–461

Łukasz Stettner (Warszawa)

ASYMPTOTICS OF UTILITY FROM TERMINAL WEALTH FOR PARTIALLY OBSERVED PORTFOLIOS

Abstract. We study the asymptotical behaviour of expected utility from terminal wealth on a market in which asset prices depend on economic factors that are unobserved or observed with delay.

1. Introduction. In this paper we consider a discrete time market consisting of a bank account with interest rate for simplicity equal to 0 and a risky asset with price $S(n)$ at time n. We shall assume that the asset price, which is observed, depends on an unobserved or partially observed Markov process of economic factors (x_n) on a Polish space E with transition operator P, which is Feller, i.e., it transforms the set of continuous bounded functions into itself. Let $X^n = \sigma\{x_i, i \le n\}$ and $Y^n = \sigma\{S(i), i \le n\}$, and let $\mathcal{B}(E)$ denote the set of Borel subsets of E. We assume that

$$P\{x_{n+1} \in A \mid X^n, Y^n\} = P(x_n, A) \tag{1.1}$$

P-a.e. for $A \in \mathcal{B}(E)$. Let

$$r_n := \frac{S(n)}{S(n-1)}. \tag{1.2}$$

We shall assume the asset price $S(0)$ at time 0 is deterministic, and therefore $Y^n = \sigma\{r_i, i \le n\}$. The dynamics of r_n depends on economic factors in the following way:

$$P\{r_{n+1} \in B \mid X^{n+1}, Y^n\} = \int_B q(x_{n+1}, r_n, y)\, \nu(dy) \tag{1.3}$$

P-a.e. for $B \in \mathcal{B}((0,\infty))$, where $\nu \in \mathcal{P}((0,\infty))$, the set of probability measures on $(0,\infty)$, and $q > 0$ is a bounded continuous function. Since we would like to have formula (1.3) valid also for $n = 0$ we assume that we know

2010 *Mathematics Subject Classification*: Primary 93E20; Secondary 91B16.
Key words and phrases: utility, terminal wealth, asymptotics, partially observed economic factors.

DOI: 10.4064/am39-4-4

the initial value r_0 of the dynamics r_n. At each time n we choose, based on available observation, the portion b_n of capital invested in the risky asset. Since we do not admit shortselling or shortborrowing, our control parameter b_n at time n should take values from $\mathcal{S} = [0,1]$. Starting with initial wealth W_0 the wealth W_n at time n is given by the formula

$$W_n = W_{n-1}(1 - b_{n-1} + b_{n-1} r_n) \quad \text{for } n \geq 1. \tag{1.4}$$

Our problem is to determine the asymptotics of the optimal utility from terminal wealth, i.e. of $\sup_{(b_n)} E\{U(W_T)\}$, where U is one of the following utility functions: $U(W) = \ln W$, $U(W) = W^\alpha$ with $\alpha \in (0,1)$, or $U(W) = 1 - W^{-\alpha}$ with $\alpha > 0$. In other words we are looking for a λ such that for large $T > 0$,

$$\sup_{(b_n)} E\{U(W_T)\} \sim U(e^{\lambda T}), \tag{1.5}$$

where $\sim$ means the same order. This way the problem leads to a suitable infinite time horizon control problem, where λ plays the role of an optimal value of a suitable long run cost functional. This problem has been studied intensively with complete observation for the logarithmic utility function under the name of the Kelly capital growth investment criterion (see [6] and the references therein). A similar problem with transaction costs was considered in [4] and in [10], where additionally the power utility function was also studied. A risk sensitive portfolio problem with partial observation of economic factors was studied for a general discrete time model in [9].

In this paper we characterize the asymptotics of the logarithmic and negative power utility functions for a discrete time model with unobserved economic factors and then study the asymptotics of the logarithmic, negative and positive power utility functions in the case of delay in the observation of economic factors. Solutions to both asymptotical problems, i.e., the problems with partial or with delayed observation, seem to be new. The study of the growth optimal portfolio with unobserved economic factors strongly relies on the recent result of [11] on ergodicity of hidden Markov processes.

2. Ergodicity of hidden Markov processes. We shall assume that the portion $b_n \in \mathcal{S}$ of the capital invested in the asset at time n is adapted only to Y^n, i.e. we observe the asset prices and our information about the economic factors (x_n) comes from the observation of $(S(n))$. By (1.3) it is clear that the pair (r_n, x_n) forms a Markov process. Given the initial law μ of the Markov process (x_n) define the following sequence of random measures: $\pi_0(A) = \mu(A)$ and

$$\pi_{n+1}(A) = \frac{\int_A q(z, r_n, r_{n+1})\, P(\pi_n, dz)}{\int_E q(z, r_n, r_{n+1})\, P(\pi_n, dz)} =: \frac{N(r_n, r_{n+1}, \pi_n)(A)}{N(r_n, r_{n+1}, \pi_n)(E)} \tag{2.1}$$

for $A \in \mathcal{B}(E)$ where $P(\pi_n, A) = \int_E P(x, A)\, \pi_n(dx)$, defining moreover implicitly random measures $N(r_n, r_{n+1}, \pi_n)(\cdot) \in M(E)$, where $M(E)$ is the set of finite measures on E. One can easily prove (see e.g. [8, proof of Lemma 1.1.1])

Lemma 2.1. *The process π_n defined in* (2.1) *is a representation of the conditional probability*

$$\pi_n(A) = P\{x_n \in A \,|\, Y^n\} \tag{2.2}$$

for $A \in \mathcal{B}(E)$, P-a.e., and the pair (r_n, π_n) forms a Markov process with transition operator Π.

Notice that when studying the optimal asymptotics (1.5) of the utility from terminal wealth, it is important that the pair (r_n, π_n) is a non-controlled Markov process.

Let $\Lambda_n(\omega) = \prod_{i=0}^{n-1} q(x_{i+1}(\omega), r_i(\omega), r_{i+1}(\omega))$ and let P^0 be a probability measure such that the restrictions $P^0_{|n}$ and $P_{|n}$ of P^0 and P respectively to the σ-field $X^n \vee Y^n$ satisfy the formula $P_{|n}(d\omega) = \Lambda_n(\omega) P^0_{|n}(d\omega)$. We have (see [8, proof of Lemma 1.1.8])

Lemma 2.2. *Under P^0, (r_n) is i.i.d. with law ν independent of (x_n), and (x_n) is Markov with transition operator $P(x, dx')$.*

For $A \in \mathcal{B}(E)$ let

$$N_n(r, r_1, \ldots, r_n, \eta)(A) = N(r_{n-1}, r_n, N_{n-1}(r, r_1, \ldots, r_{n-1}, \eta))(A) \tag{2.3}$$

with $N_1 = N$ defined in (2.1).

From [1, Lemma 4] we obtain

Lemma 2.3. *The transition operator Π of the pair (r_n, π_n) and its iterations are respectively of the form*

$$\Pi F(r, \mu) = E_{r\mu}\{F(r_1, \pi_1)\} = E^0\{SF(r_1, N(r, r_1, \mu))\} \tag{2.4}$$

and

$$\Pi^n F(r, \mu) = E_{r\mu}\{F(r_n, \pi_n)\} = E^0\{SF(r_n, N_n(r, r_1, \ldots, r_n, \mu))\} \tag{2.5}$$

for any bounded Borel function $F : (0, \infty) \times \mathcal{P}(E) \to \mathbb{R}$, with $SF(r, \zeta) := \zeta(E) F(r, \zeta/\zeta(E))$ for $\zeta \in M(E)$.

We introduce the following assumption:

(A) there is a probability measure $\phi \in \mathcal{P}(E \times (0, \infty))$ such that

$$P\{(x_n, r_n) \in \cdot\} \to \phi(\cdot) \quad \text{in variation norm as } n \to \infty.$$

This assumption is clearly satisfied for a wide family of ergodic processes called positive aperiodic Harris processes (see Theorem 13.3.1 of [7]). From [11] we have

Theorem 2.4. *Under* (A) *there is a unique invariant measure* Φ *for the pair* (r_n,π_n), *and* Π^n *converges to* Φ *in variation norm as* $n\to\infty$.

Remark 2.5. If the convergence in (A) is replaced by convergence in the weak topology then we may have more invariant measures for the pair (r_n,π_n), as is shown in [13]. The long standing problem of filling the gap in the famous paper [5] has been partially solved in [12] and [11]. It is still an open problem to clarify what we should add to the weak convergence of $P\{(x_n,r_n)\in\cdot\}$ to $\phi(\cdot)$ as $n\to\infty$ to get a unique invariant measure Φ for the pair (r_n,π_n).

Using Theorem 2.4 we can characterize the optimal asymptotics of the logarithmic utility function. We have

Proposition 2.6. *Under* (A), *assuming that*

$$\text{(I)}\qquad L=\sup_{x\in E}\sup_{r\in(0,\infty)}\sup_{b\in[0,1]}\Big|\int_0^\infty\int_E \ln(1-b+by)q(x',r,y)\,P(x,dx')\,\nu(dy)\Big|<\infty$$

and that for all $\epsilon>0$, C *compact in* E *and* C_1 *compact in* $(0,\infty)$ *there exists* K *compact in* $(0,\infty)\times E$ *such that*

$$\sup_{x\in C}\sup_{r\in C_1}\sup_{b\in\mathcal{S}}\int_{K^c}|\ln(1-b+by)|q(x',r,y)\,P(x,dx')\,\nu(dy)<\epsilon, \tag{2.6}$$

we have

$$\lim_{T\to\infty}\sup_{(b_n)}\frac{1}{T}E\{\ln W_T\}=\lambda, \tag{2.7}$$

where

$$\lambda=\int_{(0,\infty)\times\mathcal{P}(E)} F(\tilde b(r,\eta),r,\eta)\,\Phi(dr,d\eta) \tag{2.8}$$

with

$$F(b,r,\eta)=\int_E\int_E\int_{(0,\infty)}\ln(1-b+by)q(x',r,y)\,\nu(dy)\,P(x,dx')\,\eta(dx)$$

and $\tilde b(r,\eta)=\operatorname{argmax}_{b\in\mathcal{S}}F(b,r,\eta)$. *Furthermore an optimal control is of the form* $b_i=\tilde b(r_i,\pi_i)$.

Proof. Note first that

$$E\{\ln W_T\}=\ln W_0+\sum_{i=0}^{T-1}E\{\ln(1-b_i+b_ir_{i+1})\}.$$

Furthermore

$$
\begin{aligned}
&E\{\ln(1-b_i+b_i r_{i+1})\}\\
&\qquad = E\Big\{\int_E \int_{(0,\infty)} \ln(1-b_i+b_i y) q(x',r_i,y)\,\nu(dy)\,P(x_i,dx')\Big\}\\
&\qquad = E\Big\{\int_E\int_E \int_{(0,\infty)} \ln(1-b_i+b_i y) q(x',r_i,y)\,\nu(dy)\,P(x,dx')\,\pi_i(dx)\Big\}\\
&\qquad = E\{F(b_i,r_i,\pi_i)\}
\end{aligned}
$$

and by (I) the function F is bounded. By (2.6) the condition (5.1) (see Appendix) is satisfied, and using Proposition 5.1 we obtain the existence of a continuous selector $\tilde{b}$. Consequently, b_i of the form $\tilde{b}(r_i,\pi_i)$ is an optimal control maximizing $E\{\ln(1-b_i+b_i r_{i+1})\}$ at time i. Since we practically maximize each term of $\sum_{i=0}^{T-1} E\{\ln(1-b_i+b_i r_{i+1})\}$, and by Theorem 2.4

$$E\{F(\tilde{b}(r_i,\pi_i),r_i,\pi_i)\} \to \int_{(0,\infty)\times\mathcal{P}(E)} F(\tilde{b}(r,\eta),r,\eta)\,\Phi(dr,d\eta)$$

as $i\to\infty$, we finally obtain (2.7), which completes the proof. ∎

3. Asymptotics of negative power utility functions. In this section we consider the case of the negative power utility function $U(W) = 1-W^{-\alpha}$, with $\alpha>0$. Maximizing the expected utility from terminal wealth in this case we minimize

$$(3.1)\qquad E\Big\{\prod_{i=0}^{T-1}(1-b_i+b_i r_{i+1})^{-\alpha}\Big\} = E\Big\{\prod_{i=0}^{T-1} e^{-\alpha G(r_{i+1},b_i)}\Big\}$$

with $G(r,b)=\ln(1-b+br)$. Consequently, we are looking for λ such that

$$(3.2)\qquad e^{-\alpha\lambda T} \sim \inf_{(b_i)} E\Big\{\prod_{i=0}^{T-1}(1-b_i+b_i r_{i+1})^{-\alpha}\Big\}.$$

The cost functional on the right hand side of (3.1) is close to the risk sensitive cost functional studied in [9]. We consider first the discounted risk sensitive control problem consisting in minimizing, for given $\beta\in(0,1)$ and $\gamma\in(0,1]$, the cost functional

$$(3.3)\quad J_{r,\mu}^{\beta\gamma}((b_n)) = E\Big[\prod_{i=0}^{\infty}(1-b_i+b_i r_{i+1})^{-\alpha\beta^i\gamma}\Big] = E\Big\{\prod_{i=0}^{\infty} e^{-\alpha\beta^i\gamma G(r_{i+1},b_i)}\Big\}.$$

Let

$$(3.4)\qquad w^\beta(r,\eta,\gamma) = \inf_{(b_n)} J_{r,\eta}^{\beta\gamma}((b_n)).$$

We have

Theorem 3.1. *Under the assumption* (I) *of Proposition* 2.6, *the function* w^β *defined in* (3.4) *is a solution to the Bellman equation*

$$w^\beta(r,\eta,\gamma)=\inf_{b\in\mathcal{S}}\int_0^\infty e^{-\alpha\gamma G(y,b)}Sw^\beta(y,N(r,y,\eta),\gamma\beta)\,\nu(dy), \tag{3.5}$$

with the operator S *defined in Lemma* 2.3, *and* w^β *takes values in the interval* $(0,1]$ *and is bounded away from* 0 (*i.e. there is an* $a>0$ *such that* $w^\beta\ge a$). *Moreover the mapping* $\mathcal{P}(E)\ni\eta\mapsto w^\beta(r,\eta,\gamma)$ *is concave.*

Proof. Define an operator T on bounded measurable functions w on $(0,\infty)\times\mathcal{P}(E)\times(0,1)$ by the formula

$$Tw(r,\eta,\gamma)=\inf_{b\in\mathcal{S}}\int_0^\infty e^{-\alpha\gamma G(y,b)}Sw(y,N(r,y,\eta),\gamma\beta)\,\nu(dy).$$

Letting $b=0$ it is clear that $T1(r,\eta,\gamma)\le 1$. The operator T is monotone and therefore the sequence $T^n1(r,\eta,\gamma)$ is nonincreasing and nonnegative. Moreover one can show that $T^n1(r,\eta,\gamma)$ is the optimal value of the cost functional $E\{\prod_{i=0}^n e^{-\alpha\beta^i\gamma G(r_{i+1},b_i)}\}$. Consequently, $T^n1(r,\eta,\gamma)$ has a limit $w^\beta(r,\eta,\gamma)$, and this limit is also a solution to (3.5).

We now show the concavity of the mapping $\mathcal{P}(E)\ni\eta\mapsto w^\beta(r,\eta,\gamma)$. Since

$$T1(r,\eta,\gamma)=\inf_{b\in\mathcal{S}}\int_0^\infty e^{-\alpha\gamma G(y,b)}N(r,y,\eta)(E)\,\nu(dy),$$

the mapping $\eta\mapsto T1(r,\eta,\gamma)$ is concave as the infimum of linear functions. By Lemma 2 of [2] the mapping $M(E)\ni\zeta\mapsto ST1(r,\zeta,\gamma)$ is also concave. Consequently, the mapping $\mathcal{P}(E)\ni\eta\mapsto ST^21(r,\eta,\gamma)$ is concave and by induction for each n the mapping $\mathcal{P}(E)\ni\eta\mapsto ST^n1(r,\eta,\gamma)$ is concave and the same is true for the limit function w^β.

To show that w^β is bounded away from 0 we use the Jensen inequality:

$$\begin{aligned}E\{e^{-\alpha\beta^i\gamma G(r_{i+1},b_i)}\mid Y^i\}&\ge\exp\{E\{-\alpha\beta^i\gamma G(r_{i+1},b_i)\mid Y^i\}\}\\&\ge\exp\Big\{-\alpha\beta^i\sup_{x\in E}\sup_{r\in(0,\infty)}\sup_{b\in[0,1]}\Big|\int_0^\infty\int_E G(y,b)q(x',r,y)\,P(x,dx')\,\nu(dy)\Big|\Big\}\end{aligned}$$

and by (I) we obtain

$$w^\beta(r,\eta,\gamma)\ge e^{-\alpha\gamma L\sum_{i=0}^\infty\beta^i}. \tag{3.6}$$

Consequently, w^β is bounded away from 0 and is an optimal value of the cost functional (3.3). ∎

We now impose the following assumptions:

$$\text{(B1)}\qquad \sup_{r,r'\in(0,\infty)}\sup_{x\in E}\sup_{y\in(0,\infty)}\frac{q(x,r,y)}{q(x,r',y)}:=\bar q<\infty,$$

$$\text{(B2)}\qquad \sup_{x,x'\in E}\sup_{A}\frac{P(x,A)}{P(x',A)}:=\bar p<\infty.$$

We have

PROPOSITION 3.2. *Under* (B1), (B2) *and* (I) *we have*

$$Sw^\beta(y,N(r,y,\eta),\gamma\beta)\ge\frac{1}{\bar p\bar q}Sw^\beta(y,N(r',y,\eta'),\gamma\beta)\tag{3.7}$$

for $r,y\in(0,\infty)$ *and* $\eta,\eta'\in\mathcal{P}(E)$. *Furthermore*

$$(\bar q(r',r)\bar p(\eta',\eta))^{-1}\le\frac{w^\beta(r,\eta,\gamma)}{w^\beta(r',\eta',\gamma)}\le\bar q(r,r')\bar p(\eta,\eta')\tag{3.8}$$

with

$$\bar p(\eta,\eta')=\sup_A\frac{P(\eta,A)}{P(\eta',A)}\quad\textit{and}\quad\bar q(r,r')=\sup_{x\in E,\,y\in(0,\infty)}\frac{q(x,r,y)}{q(x,r',y)}.$$

Proof. By (B1) and (B2) for $A\in\mathcal{B}(E)$, $r,r',y\in(0,\infty)$ and $\eta,\eta'\in\mathcal{P}(E)$ we have

$$\begin{aligned}N(r',y,\eta')(A)&=\int_A q(z,r',y)\,P(\eta',dz)\le\bar q\int_A q(z,r,y)\,P(\eta',dz)\\&\le\bar p\bar q\int_A q(z,r,y)\,P(\eta,dz)=\bar p\bar qN(r,y,\eta)(A).\end{aligned}$$

Therefore $N(r,y,\eta)-(\bar p\bar q)^{-1}N(r',y,\eta')\in M(E)$ and whenever $\bar p\bar q>1$,

$$d(r,r',y,\eta,\eta'):=\frac{1}{1-(\bar p\bar q)^{-1}}N(r,y,\eta)-(\bar p\bar q)^{-1}N(r',y,\eta')$$

is a nontrivial measure from $M(E)$. When $\bar p\bar q=1$, since $\bar p\ge1$ and $\bar q\ge1$ we have a trivial case for which (3.7) is clearly satisfied. Therefore we can restrict ourselves to $\bar p\bar q>1$. Then by concavity of $\mathcal{P}(E)\ni\zeta\mapsto Sw^\beta(y,\zeta,\gamma\beta)$, which follows from Lemma 2 of [2] (or the proof of Theorem 3.1), we obtain

$$\begin{aligned}Sw^\beta(y,N(r,y,\eta),\gamma\beta)&\ge\frac{1}{\bar p\bar q}Sw^\beta(y,N(r',y,\eta'),\gamma\beta)\\&\quad+(1-(\bar p\bar q)^{-1})Sw^\beta(y,d(r,r',y,\eta,\eta'),\gamma\beta)\\&\ge\frac{1}{\bar p\bar q}Sw^\beta(y,N(r',y,\eta'),\gamma\beta),\end{aligned}$$

which completes the proof of (3.7). Moreover

$$N(r',y,\eta')\le\bar q(r',r)\bar p(\eta',\eta)N(r,y,\eta)$$

and by similar arguments to those above

$$Sw^{\beta}(y,N(r,y,\eta),\gamma\beta)\geq\frac{1}{\bar{p}(\eta',\eta)\bar{q}(r',r)}Sw^{\beta}(y,N(r',y,\eta'),\gamma\beta),$$

and therefore from (3.5) we obtain

$$\frac{w^{\beta}(r,\eta,\gamma)}{w^{\beta}(r',\eta',\gamma)}\geq\frac{1}{\bar{p}(\eta',\eta)\bar{q}(r',r)}.$$

Replacing r by r' and η by η' we immediately obtain the second part of (3.8). ∎

Remark 3.3. One could try to use the same method for the power utility function $U(W)=W^{\alpha}$ with $\alpha\in(0,1)$. However, then an analog of the function w^{β} is convex in η and consequently we do not obtain (3.7), which is crucial in the proof of Theorem 3.5, based on a vanishing discount argument.

In what follows we fix $\bar{\eta}\in\mathcal{P}(E)$ and $\bar{r}\in(0,\infty)$ and define

$$v^{\beta}(r,\eta,\gamma):=\frac{w^{\beta}(r,\eta,\gamma)}{w^{\beta}(\bar{r},\bar{\eta},\gamma)}\quad\text{and}\quad\kappa^{\beta}(\gamma):=\frac{w^{\beta}(\bar{r},\bar{\eta},\gamma)}{w^{\beta}(\bar{r},\bar{\eta},\gamma\beta)}.$$

Then from (3.5) we have

$$v^{\beta}(r,\eta,\gamma)\kappa^{\beta}(\gamma)=\inf_{b\in\mathcal{S}}\int_{0}^{\infty}e^{-\alpha\gamma G(y,b)}Sv^{\beta}(y,N(r,y,\eta),\gamma\beta)\,\nu(dy).\tag{3.9}$$

The following estimates will be important:

Corollary 3.4. *We have*

$$e^{-\alpha\gamma L}\leq\kappa^{\beta}(\gamma)\leq\bar{p}\bar{q}.\tag{3.10}$$

Proof. Since by Jensen's inequality

$$w^{\beta}(\bar{r},\bar{\eta},\gamma\beta)\leq w^{\beta}(\bar{r},\bar{\eta},\gamma)^{\beta},$$

using (3.6) we obtain

$$\kappa^{\beta}(\gamma)\geq w^{\beta}(\bar{r},\bar{\eta},\gamma)^{1-\beta}\geq e^{-\alpha\gamma L}.$$

From (3.8),

$$w^{\beta}\left(y,\frac{N(r,y,\eta)}{N(r,y,\eta)(E)},\gamma\beta\right)\leq\bar{p}\bar{q}w^{\beta}(y',\bar{\eta},\gamma\beta),$$

and therefore from (3.5),

$$\begin{aligned}w^{\beta}(\bar{r},\bar{\eta},\gamma)&\leq\bar{p}\bar{q}\inf_{b\in\mathcal{S}}\int_{0}^{\infty}\int_{E}e^{-\alpha\gamma G(y,b)}w^{\beta}(\bar{r},\bar{\eta},\gamma\beta)q(x',r,y)\,P(\bar{\eta},dx')\,\nu(dy)\\&\leq\bar{p}\bar{q}w^{\beta}(\bar{r},\bar{\eta},\gamma\beta),\end{aligned}$$

from which the second part of (3.10) follows. ∎

We impose a further assumption:

(C1) $\bar{p}(\eta,\eta') \to 0$ when $\eta' \Rightarrow \eta$ in the weak topology of $\mathcal{P}(E)$, and $\bar{q}(r,r') \to 0$ when $r' \to r$.

We have

THEOREM 3.5. *Under* (B1), (B2), (I) *and* (C1), *for every* $\gamma > 0$ *there is* $\lambda(\gamma)$ *and a continuous bounded function* $(r,\eta) \mapsto v(r,\eta,\gamma)$ *such that*

$$v(r,\eta,\gamma)e^{-\alpha\lambda(\gamma)} = \inf_{b\in\mathcal{S}} \int_0^\infty e^{-\alpha\gamma G(y,b)} Sv(y,N(r,y,\eta),\gamma)\,\nu(dy) \tag{3.11}$$

and

$$\lambda(\gamma) = \lim_{T\to\infty} \frac{-1}{\alpha\gamma T} \ln \inf_{(b_i)} E\Big\{\prod_{i=0}^{T-1}(1-b_i+b_i r_{i+1})^{-\alpha\gamma}\Big\},$$

so that $\lambda(1)$ *is the optimal asymptotics of the negative power utility function and the optimal control is of the form* $b_i = \tilde{b}(r_i,\pi_i,1)$ *for* $i = 0,1,\ldots,$ *where* $\tilde{b}$ *is a continuous selector for which the infimum on the right hand side of* (3.11) *with* $\gamma = 1$ *is attained.*

Proof. By (C1) and (3.8) the family $\{v^\beta(r,\eta,\gamma) : \beta \in (0,1),\ \gamma > 0\}$ is bounded and equicontinuous so that by Ascoli–Arzelà theorem and (3.10) we have the following convergences of suitable subsequences (with $\beta_n \to 1$) : $v^{\beta_n}(r,\eta,\gamma\beta_n^j) \to v_j(r,\eta)$ uniformly in r, η on compact sets and $\kappa^{\beta_n}(\gamma\beta_n^j) \to \kappa_j$ as $n \to \infty$. Choosing further subsequences (using again Proposition 3.2 and Corollary 3.4) we find a subsequence $j_m \to \infty$ such that $v_{j_m}(r,\eta) \to v(r,\eta,\gamma)$ uniformly in r and η on compact subsets and $\kappa_{j_m} \to e^{-\alpha\lambda(\gamma)}$. Consequently, from (3.9) we obtain (3.11). The mapping $(r,\eta) \mapsto v(r,\eta,\gamma)$ is continuous. Since $e^{-\alpha\gamma G(y,b)} = (1-b+by)^{-\alpha\gamma}$ is a strictly convex function of b, the function

$$b \mapsto \int_0^\infty e^{-\alpha\gamma G(y,b)} Sv(y,N(r,y,\eta),\gamma)\,\nu(dy)$$

is also strictly convex, and by a version of Proposition 5.1 for strictly convex functions we obtain the existence of a continuous (in r and η) selector $\tilde{b}(r,\eta,\lambda)$. The optimality of $\lambda(1)$ and of the control $\tilde{b}(r_i,\pi_i,1)$ follows from [2]. ■

4. Asymptotics of utility from terminal wealth for models with delayed observation. In this section we assume that economic factors are observed with a fixed delay k, i.e. at time $n+k$ we know the value of x_n but we do not know x_{n+i} for $i > 0$. Define the following sequence of measures for $A \in \mathcal{B}(E)$:

$$\tilde{\pi}^k_{n+k}(A) = \frac{N(r_{n+k-1}, r_{n+k}, \tilde{\pi}^{k-1}_{n+k-1})(A)}{N(r_{n+k-1}, r_{n+k}, \tilde{\pi}^{k-1}_{n+k-1})(E)} \tag{4.1}$$

with $\tilde{\pi}^0_n(A) = \delta_A(x_n)$, and the operator N defined in (2.1). Recalling that $X^n = \sigma\{x_i, i \le n\}$ and $Y^{n+k} = \sigma\{r_i, i \le n+k\}$ we have (see the proof of Lemma 1.1.1 in [8])

LEMMA 4.1. *For $A \in \mathcal{B}(E)$ we have*

$$\tilde{\pi}^k_{n+k}(A) = P\{x_{n+k} \in A \mid X^n, Y^{n+k}\} \tag{4.2}$$

P-a.e. and

$$\begin{aligned} \tilde{\pi}^k_{n+k}(A) &= \frac{N_k(r_n, r_{n+1}, \ldots, r_{n+k}, \delta_{x_n})(A)}{N_k(r_n, r_{n+1}, \ldots, r_{n+k}, \delta_{x_n})(E)} \\ &= \tilde{M}_k(r_n, r_{n+1}, \ldots, r_{n+k}, x_n)(A) \end{aligned} \tag{4.3}$$

with operator N_k defined in (2.3), *and δ_x the Dirac measure at x.*

For $A \in \mathcal{B}(E)$ let

$$\begin{aligned} \pi^{n,n+k}_{n+1}(A) &= \frac{\int_A N_{k-1}(r_{n+1}, r_{n+2}, \ldots, r_{n+k}, \delta_z)(E) q(z, r_n, r_{n+1})\, P(x_n, dz)}{N_k(r_n, r_{n+1}, \ldots, r_{n+k}, \delta_{x_n})(E)} \\ &=: M_k(r_n, r_{n+1}, \ldots, r_{n+k}, x_n)(A). \end{aligned} \tag{4.4}$$

We have (see again the proof of Lemma 1.1.1 in [8])

LEMMA 4.2. *For $A \in \mathcal{B}(E)$,*

$$\pi^{n,n+k}_{n+1}(A) = P\{x_{n+1} \in A \mid X^n, Y^{n+k}\} \tag{4.5}$$

P-a.e.

In the next lemma we markovianize our problem:

LEMMA 4.3. *The $(k+2)$-tuple $(x_n, r_n, r_{n+1}, \ldots, r_{n+k})$ forms a Markov process with transition operator $\mathbf{P}$ defined implicitly as follows:*

$$\begin{aligned} &E[f(x_{n+1}, r_{n+1}, r_{n+2}, \ldots, r_{n+k+1}) \mid X^n, Y^{n+k}] \\ &= \int_E \int_0^\infty f(z, r_{n+1}, \ldots, r_{n+k}, y) \int_E q(x', r_{n+k}, y) \\ &\quad \cdot P(\tilde{M}_{k-1}(r_{n+1}, \ldots, r_{n+k}, z), dx')\, \nu(dy)\, M_k(r_n, r_{n+1}, \ldots, r_{n+k}, x_n)(dz) \\ &= \int_E \int_0^\infty f(z, r_{n+1}, \ldots, r_{n+k}, y)\, \mathbf{P}(x_n, r_n, r_{n+1}, \ldots, r_{n+k}, dy, dz) \\ &= \mathbf{P} f(x_n, r_n, r_{n+1}, \ldots, r_{n+k}). \end{aligned} \tag{4.6}$$

Given nice ergodic properties of the processes (x_n) and (r_n), the kth iteration $\mathbf{P}^k$ of the operator $\mathbf{P}$ inherits ergodic properties. Namely, we have

PROPOSITION 4.4. *Under* (B1), (B2) *the operator* $\mathbf{P}$ *is uniformly ergodic, i.e., its iterations uniformly approximate a unique invariant measure* Ψ.

Proof. We show first that the operator $\mathbf{P}^{k+1}$ is uniformly ergodic. Note that for a bounded Borel measurable function f on $E \times (0,\infty)^{k+1}$ we have

$$\tag{4.7} \mathbf{P}^{k+1} f(x_n, r_n, r_{n+1}, \ldots, r_{n+k}) = E\{f(x_{n+k+1}, r_{n+k+1}, r_{n+k+2}, \ldots, r_{n+2k+1}) \mid X^n, Y^{n+k}\}.$$

Furthermore

$$\tag{4.8} \begin{aligned} &E\{f(x_{n+k+1}, r_{n+k+1}, r_{n+k+2}, \ldots, r_{n+2k+1}) \mid X^{n+k}, Y^{n+k}\} \\ &= \int_E \int_0^\infty \ldots \int_E \int_0^\infty f(x^{(k+1)}, y^{(k+1)}, y^{(k)}, \ldots, y^{(1)}, y) q(x, y^{(1)}, y) \\ &\quad \cdot \nu(dy) P(x^{(1)}, dx) q(x^{(1)}, y^{(2)}, y^{(1)}) \nu(dy^{(1)}) P(x^{(2)}, dx^{(1)}) \\ &\quad \ldots q(x^{(k+1)}, r_{n+k}, y^{(k+1)}) \nu(dy^{(k+1)}) P(x_{n+k}, dx^{(k+1)}) \end{aligned}$$

and therefore

$$\tag{4.9} \begin{aligned} &E\{f(x_{n+k+1}, r_{n+k+1}, r_{n+k+2}, \ldots, r_{n+2k+1}) \mid X^{n}, Y^{n+k}\} \\ &= \int_E \int_0^\infty \ldots \int_E \int_0^\infty f(x^{(k+1)}, y^{(k+1)}, y^{(k)}, \ldots, y^{(1)}, y) q(x, y^{(1)}, y) \\ &\quad \cdot \nu(dy) P(x^{(1)}, dx) q(x^{(1)}, y^{(2)}, y^{(1)}) \nu(dy^{(1)}) P(x^{(2)}, dx^{(1)}) \\ &\quad \ldots q(x^{(k+1)}, r_{n+k}, y^{(k+1)}) \nu(dy^{(k+1)}) P(\tilde{\pi}^k_{n+k}, dx^{(k+1)}). \end{aligned}$$

Consequently, under (B1) and (B2) we have

$$\tag{4.10} \begin{aligned} &\sup_{A \in \mathcal{B}(E\times(0,\infty)^{k+1})} \sup_{(x,r,r^1,\ldots,r^k),(\bar{x},\bar{r},\bar{r}^1,\ldots,\bar{r}^k) \in E\times(0,\infty)^{k+1}} \\ &\qquad (\mathbf{P}(x, r, r^1, \ldots, r^k, A) - \mathbf{P}(\bar{x}, \bar{r}, \bar{r}^1, \ldots, \bar{r}^k, A)) < 1, \end{aligned}$$

which by part (b) of Section 5.5 in [3] implies uniform ergodicity of $\mathbf{P}^{k+1}$, i.e. the iterations of the operator $\mathbf{P}^{k+1}$ approximate uniformly an invariant measure Ψ. Therefore denoting by $[x]$ the integer part of x we have

$$\mathbf{P}^n f = \mathbf{P}^{n-[\frac{n}{k+1}](k+1)} (\mathbf{P}^{k+1})^{[\frac{n}{k+1}]} f \to \mathbf{P}^{n-[\frac{n}{k+1}](k+1)} \Psi(f) = \Psi(f)$$

as $n \to \infty$, which shows uniqueness of the invariant measure Ψ and completes the proof. ■

We can now summarize our results for the logarithmic utility function.

Proposition 4.5. *Under* (B1), (B2) *and* (I), *for the logarithmic utility function the optimal growth rate* λ *is of the form*

$$\lambda = \int\limits_{E\times(0,\infty)^{k+1}} \int\limits_0^\infty \ln\big(1-\tilde b(x,r,r^{(1)},\ldots,r^{(k)}) + \tilde b(x,r,r^{(1)},\ldots,r^{(k)})y\big)\,\mathbf{P}(x,r,r(1),\ldots,r^{(k)},dy,E) \cdot \Psi(dx,dr,dr^{(1)},\ldots,dr^{(k)}), \tag{4.11}$$

where Ψ *is the unique invariant measure of the operator* $\mathbf{P}$, *and* $\tilde b$ *is the unique continuous selector for which the following supremum is attained:*

$$\sup_{b\in\mathcal{S}} \int\limits_0^\infty \ln(1-b+by)\,\mathbf{P}(x_i,r_i,r_{i+1},\ldots,r_{i+k},dy,E). \tag{4.12}$$

Furthermore the optimal control is of the form $b_{i+k} = \tilde b(x_i,r_i,r_{i+1},\ldots,r_{i+k})$.

Proof. We have P-a.e.

$$\begin{aligned}\sup_{b\in\mathcal{S}} E[\ln(1-b+br_{n+k+1}) \,|\, X^n, Y^{n+k}] &= \sup_{b\in\mathcal{S}} \int\limits_0^\infty \ln(1-b+by)N(r_{n+k},y,\tilde\pi^k_{n+k})(E)\,\nu(dy)\\ &= \sup_{b\in\mathcal{S}} \int\limits_0^\infty \ln(1-b+by)\,\mathbf{P}(x_n,r_n,r_{n+1},\ldots,r_{n+k},dy,E).\end{aligned} \tag{4.13}$$

By Proposition 5.1 there exists a continuous selector $\tilde b$ for which the supremum in (4.12) is attained. By (I) the last term of (4.13) is a bounded function of $(x_n,r_n,r_{n+1},\ldots,r_{n+k})$, so that by uniform ergodicity of $\mathbf{P}$ (using Proposition 4.4) we obtain the form (4.11) of λ. The form of the optimal control immediately follows from (4.12) and (4.13). ∎

We now consider the cases with multiplicative functionals, i.e. first when $U(W) = W^\alpha$ with $\alpha\in(0,1)$ and then $U(W) = 1-W^{-\alpha}$ with $\alpha>0$. We use the vanishing discount approach in a similar way to Section 3. Consider first the discounted control problem for $\beta\in(0,1)$ and $\gamma\in(0,1]$,

$$\begin{aligned}J^{\beta\gamma}_{x,r,r_1,\ldots,r_k}((b_n)) &= E\Big[\prod_{i=0}^\infty (1-b_{k+i}+b_{k+i}r_{k+i+1})^{\alpha\beta^i\gamma}\Big]\\ &= E\Big\{\prod_{i=0}^\infty e^{\alpha\beta^i\gamma G(r_{k+i+1},b_{k+i})}\Big\}\end{aligned} \tag{4.14}$$

with $G(r,b) = \ln(1-b+br)$. Let

$$w^\beta(x,r,r_1,\ldots,r_k,\gamma) = \sup_{(b_n)} J^{\beta\gamma}_{x,r,r_1,\ldots,r_k}((b_n)). \tag{4.15}$$

We shall assume that

(I1) $L_1 := \sup_{(x,r,r_1,\ldots,r_k)\in E\times(0,\infty)^{k+1}} \sup_{b\in\mathcal{S}} E_{x,r,r_1,\ldots,r_k}\{(1-b+br_{k+1})^{\alpha}\} < \infty.$

We have

PROPOSITION 4.6. *Under the assumption* (I1) *function* w^β *defined in* (4.15) *is a bounded solution, with values in* $(1,\infty)$, *to the Bellman equation*

$$(4.16)\qquad w^\beta(x,r,r_1,\ldots,r_k,\gamma) = \sup_{b\in\mathcal{S}} \int_0^\infty e^{\alpha\gamma G(y,b)} \cdot w^\beta(x',r_1,\ldots,r_k,y,\gamma\beta) q(x',r_{n+k},y)\, P(\tilde{\pi}^k_{n+k},dx')\,\nu(dy)$$

with $\tilde{\pi}^k_{n+k}$ *defined in* (4.1).

Proof. By Jensen's inequality

$$(4.17)\qquad E[(1-b_{k+i}+b_{k+i}r_{k+i+1})^{\alpha\beta^i\gamma}] \le (E[(1-b_{k+i}+b_{k+i}r_{k+i+1})^{\alpha}])^{\beta^i\gamma} \le L_1^{\beta^i\gamma}.$$

Define the operator T by the right hand side of (4.16) on bounded functions w on $E\times(0,\infty)^{k+1}\times(0,1]$:

$$Tw(x,r,r_1,\ldots,r_k,\gamma) = \sup_{b\in\mathcal{S}} \int_0^\infty e^{\alpha\gamma G(y,b)} \cdot w(x',r_1,\ldots,r_k,y,\gamma\beta) q(x',r_{n+k},y)\, P(\tilde{\pi}^k_{n+k},dx')\,\nu(dy).$$

Clearly letting $b=0$ we have $T1\ge 1$ and therefore T^n1 is nondecreasing and bounded by (4.17). Furthermore T^n1 is an optimal value of the cost functional

$$E\Big\{\prod_{i=0}^{n-1} e^{\alpha\beta^i\gamma G(r_{k+i+1},b_{k+i})}\Big\}$$

and the limit w^β of the sequence T^n1 is a solution to (4.16). ∎

Now we fix $\bar{x}\in E$, $\bar{r},\bar{r}_1,\ldots,\bar{r}_k\in(0,\infty)$ and define

$$v^\beta(x,r,r_1,\ldots,r_k,\gamma) := \frac{w^\beta(x,r,r_1,\ldots,r_k,\gamma)}{w^\beta(\bar{x},\bar{r},\bar{r}_1,\ldots,\bar{r}_k,\gamma)},$$

$$\kappa^\beta(\gamma) := \frac{w^\beta(\bar{x},\bar{r},\bar{r}_1,\ldots,\bar{r}_k,\gamma)}{w^\beta(\bar{x},\bar{r},\bar{r}_1,\ldots,\bar{r}_k,\gamma\beta)}.$$

From (4.16) we then have

$$(4.18)\qquad v^\beta(x,r,r_1,\ldots,r_k,\gamma)\kappa^\beta(\gamma)=\sup_{b\in\mathcal{S}}\int_0^\infty e^{-\alpha\gamma G(y,b)} \cdot v^\beta(x',r_1,\ldots,r_k,y,\gamma\beta)q(x',r_{n+k},y)\,P(\tilde{\pi}^k_{n+k},dx')\,\nu(dy)$$

and letting $\beta\to 1$ and choosing subsequences we obtain

THEOREM 4.7. *Under* (B1), (B2), (C1) *and* (I1) *there is* λ *and a continuous bounded function* v *such that*

$$(4.19)\qquad v(x,r,r_1,\ldots,r_k)e^{\alpha\lambda} =\sup_{b\in\mathcal{S}}\int_0^\infty\int_E e^{\alpha G(y,b)}v(z,r_1,\ldots,r_k,y)\,\mathbf{P}(x,r,r_1,\ldots,r_k,dy,dz)$$

and

$$\lambda=\lim_{T\to\infty}\frac{1}{\alpha T}\sup_{(b_n)}\ln E\Big[\prod_{i=0}^{T-1}(1-b_{k+i}+b_{k+i}r_{k+i+1})^\alpha\Big],$$

i.e. λ *is the optimal asymptotics of the power utility function. Furthermore the optimal strategy is of the form* $b_{i+k}=\tilde{b}(x_i,r_i,r_{i+1},\ldots,r_{i+k})$, *where* $\tilde{b}$ *is a continuous function for which the supremum on the right hand side of* (4.19) *is attained.*

Proof. Note that under (C1) and (I1), $w^\beta=T^{k+1}w^\beta$ with T defined in the proof of Proposition 4.6 is a continuous function. Consequently, v^β is continuous as well. By (B1) and (B2), v^β is uniformly (in β) bounded from below and from above by a positive constant. By (4.17), $\kappa^\beta(\gamma)\ge L_1^{(1-\beta)\frac{\gamma}{1-\beta}} = L_1^\gamma$. Since v^β is bounded (from below and from above by a positive constant), by (4.18) there is a convergent subsequence $\beta_n\to 1$ such that for fixed $\gamma\in(0,1]$ the sequence $\kappa^{\beta_n}(\gamma)$ converges to $0<\kappa(\gamma)<\infty$. By (C1) the functions v^β are equicontinuous in β so that using the Ascoli–Arzelà theorem there is a subsequence of (β_n) (not relabelled) and a continuous function v such that $v^{\beta_n}(x,r,r_1,\ldots,r_k,\gamma)\to v(x,r,r_1,\ldots,r_k,\gamma)$ as $n\to\infty$ uniformly in $(x,r,r_1,\ldots,r_k)$ from compact subsets of $E\times(0,\infty)^{k+1}$. Letting $\gamma=1$ we obtain (4.19) with $\lambda=\frac{1}{\alpha}\ln\kappa(1)$. The existence of a continuous selector $\tilde{b}$ follows from Proposition 5.1 below. The optimality of λ and the form of the optimal strategy $\tilde{b}$ follow directly from (4.19) and the boundedness of v. ∎

In the case of the negative power utility function we first minimize the discounted cost functional for $\beta\in(0,1)$ and $\gamma\in(0,1]$,

$$(4.20)\qquad \bar{J}^{\beta\gamma}_{x,r,r_1,\ldots,r_k}((b_n)) = E\Big[\prod_{i=0}^{\infty}(1-b_{k+i}+b_{k+i}r_{k+i+1})^{-\alpha\beta^i\gamma}\Big] = E\Big\{\prod_{i=0}^{\infty} e^{-\alpha\beta^i\gamma G(r_{k+i+1},b_{k+i})}\Big\},$$

and characterize

$$(4.21)\qquad \bar{w}^{\beta}(x,r,r_1,\ldots,r_k,\gamma) = \inf_{(b_n)} \bar{J}^{\beta\gamma}_{x,r,r_1,\ldots,r_k}((b_n)).$$

By analogy to Theorem 3.1, Proposition 4.6 and Theorem 4.7 one can prove

Proposition 4.8. *Under the assumption* (I) *the function* $\bar{w}^\beta$ *defined in* (4.15) *is a bounded solution, with values in* $(0,1]$, *to the Bellman equation*

$$(4.22)\qquad \bar{w}^{\beta}(x,r,r_1,\ldots,r_k,\gamma) = \inf_{b\in\mathcal{S}}\int_0^\infty e^{-\alpha\gamma G(y,b)} \cdot \bar{w}^{\beta}(x',r_1,\ldots,r_k,y,\gamma\beta)q(x',r_{n+k},y)\,P(\tilde{\pi}^k_{n+k},dx')\,\nu(dy)$$

with $\tilde{\pi}^k_{n+k}$ *defined in* (4.1).

Theorem 4.9. *Under* (B1), (B2), (C1) *and* (I) *there is* λ *and a continuous bounded function* $\bar{v}$ *such that*

$$(4.23)\qquad \bar{v}(x,r,r_1,\ldots,r_k)e^{-\alpha\lambda} = \inf_{b\in\mathcal{S}}\int_0^\infty\int_E e^{-\alpha G(y,b)}\bar{v}(z,r_1,\ldots,r_k,y)\,\mathbf{P}(x,r,r_1,\ldots,r_k,dy,dz)$$

and

$$\lambda = \lim_{T\to\infty}\frac{-1}{\alpha T}\inf_{(b_n)}\ln E\Big[\prod_{i=0}^{T-1}(1-b_{k+i}+b_{k+i}r_{k+i+1})^{-\alpha}\Big],$$

i.e. λ *is the optimal asymptotics of the negative power utility function. Furthermore the optimal strategy is of the form* $b_{i+k}=\tilde{b}(x_i,r_i,r_{i+1},\ldots,r_{i+k})$, *where* $\tilde{b}$ *is a continuous function for which the infimum on the right hand side of* (4.23) *is attained.*

5. Appendix. We formulate here a continuous selection theorem which is frequently used in this paper.

Proposition 5.1. *Let* G *be a Polish space,* $(0,\infty)\ni u\mapsto f(u)\in\mathbb{R}$ *be strictly concave and* $G\ni z\mapsto Q(z,\cdot)\in\mathcal{P}((0,\infty))$ *be weakly continuous. Suppose that the following tightness property is satisfied: for all* $\epsilon>0$ *and* C *compact in* G *there exists* K *compact in* $(0,\infty)$ *such that*

$$(5.1)\qquad \sup_{z\in C}\sup_{b\in\mathcal{S}}\int_{K^c}|f(1-b+by)|\,Q(z,dy)<\epsilon.$$

Then $g(z) = \sup_{b\in\mathcal{S}} F(z,b)$ *is continuous, where* $F(z,b) = \int_0^\infty f(1-b+by)\,Q(z,dy)$*, and there is a unique continuous function* $z \mapsto b(z) \in \mathcal{S}$ *such that* $g(z) = F(z,b(z))$.

Proof. Let $z_n \to z$ and $b_n \to b$ as $n \to \infty$. For a given $\epsilon > 0$ and $C = \{z, z_1, z_2, \ldots\}$ we choose a compact set $K \subset (0,\infty)$ for which (5.1) holds. Then

$$\begin{aligned}|F(z_n,b_n) - F(z,b)| &\le 2\epsilon + \int_K |f(1-b+by) - f(1-b_n+b_n y)|\,Q(z_n,dy)\\ &\quad + \Big|\int_K f(1-b+by)\,(Q(z_n,dy) - Q(z,dy))\Big|\\ &= 2\epsilon + a_n^1 + a_n^2,\end{aligned}$$

and since f as a concave function is continuous, $a_n^1 \to 0$ as $n \to \infty$. Choosing K such that its boundary ∂K is a continuity point of the measure $Q(z,\cdot)$, i.e. $Q(z,\partial K) = 0$, we also obtain $a_n^2 \to 0$ as $n \to \infty$. Consequently, the mapping $(z,b) \mapsto F(z,b)$ is continuous, and clearly $z \mapsto g(z)$ is continuous. Since f is strictly concave, for each $z \in G$ the mapping $b \mapsto F(z,b)$ is also strictly concave, so that there is only one $b(z) \in \mathcal{S}$ for which $g(z) = F(z,b(z))$. Since $z_n \to z$ as $n \to \infty$, we have $g(z) = F(z,b(z)) = F(z,\tilde{b})$, where $\tilde{b}$ is any limit of a converging subsequence of $(b(z_n))$. By strict concavity $b(z) = \tilde{b}$ and therefore $b(z_n) \to b(z)$. ∎

REMARK 5.2. A suitable version of Proposition 5.1 is true for a strictly convex function f with sup replaced by inf; we obtain it immediately using the fact that $-f$ is then a strictly concave function.

Acknowledgements. This research was partly supported by MNiSzW (grant NN 201 371836). The author would like to thank Jan Palczewski for helpful comments and suggestions.

References

[1] G. Di Masi and Ł. Stettner, *Ergodicity of filtering process by vanishing discount approach*, Systems Control Lett. 57 (2008), 150–157.

[2] G. B. Di Masi and Ł. Stettner, *Risk sensitive control of discrete time partially observed Markov processes with infinite horizon*, Stoch. Stoch. Rep. 67 (1999), 309–322.

[3] J. L. Doob, *Stochastic Processes*, Wiley, 1953.

[4] T. Duncan, B. Pasik-Duncan and Ł. Stettner, *Growth optimal portfolio selection under proportional transaction costs with obligatory diversification*, Appl. Math. Optim. 63 (2011), 107–132.

[5] H. Kunita, *Asymptotic behavior of the nonlinear filtering errors of Markov processes*, J. Multivariate Anal. 1 (1971), 365–393.

[6] L. C. Maclean, E. O. Thor and W. T. Ziemba, *The Kelly Capital Growth Investment Criterion*, World Sci., 2011.
[7] S. P. Meyn and R. L. Tweedie, *Markov Chains and Stochastics Stability*, Springer, 1993.
[8] W. Runggaldier and Ł. Stettner, *Approximations of Discrete Time Partially Observed Control Problems*, Appl. Math. Monogr. CNR, Giardini, Pisa, 1994.
[9] Ł. Stettner, *Risk sensitive portfolio optimization with completely and partially observed factors*, IEEE Trans. Automat. Control 49 (2004), 457–464.
[10] Ł. Stettner, *Asymptotics of HARA utility from terminal wealth under proportional transaction costs with decision lag or execution delay and obligatory diversification*, in: Advanced Mathematical Methods for Finance, G. Di Nunno and B. Øksendal (eds.), Springer, 2011, 509–536.
[11] X. T. Tong and R. van Handel, *Ergodicity and stability of the conditional distributions of nondegenerate Markov chains*, Ann. Appl. Probab. 22 (2012), 1495–1540.
[12] R. van Handel, *The stability of conditional Markov processes and Markov chains in random environments*, Ann. Probab. 37 (2009), 1876–1925.
[13] R. van Handel, *A nasty filtering problem*, arXiv:1009.0507.

Łukasz Stettner
Institute of Mathematics
Polish Academy of Sciences
Śniadeckich 8
00-956 Warszawa, Poland
and
Academy of Finance
Warszawa, Poland
E-mail: stettner@impan.pl

Received on 26.1.2012;
revised version on 30.4.2012 (2120)