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INTEGRAL REPRESENTATIONS OF RISK FUNCTIONS FOR BASKET DERIVATIVES

Abstract. The risk minimizing problem $\mathbf{E}[l((H - X_T^{x,\pi})^+)] \overset{\pi}{\to} \min$ in the multidimensional Black–Scholes framework is studied. Specific formulas for the minimal risk function and the cost reduction function for basket derivatives are shown. Explicit integral representations for the risk functions for $l(x) = x$ and $l(x) = x^p$, with $p > 1$ for digital, quantos, outperformance and spread options are derived.

1. Introduction. The paper is devoted to the stochastic control problem arising in the risk analysis of financial markets. Let H be a random variable representing future random payoff which is traded on the market. Denote by $p(H)$ its price determined by the no arbitrage method. If the initial capital x of the writer exceeds $p(H)$ then he is able to hedge H perfectly, i.e. he can follow some trading strategy π such that the wealth process at the final time is greater than H, i.e.

$$P(X_T^{x,\pi} \geq H) = 1.$$

If $x < p(H)$ then the above equality fails for each π and as a consequence shortfall risk appears. The aim of the trader is to find a strategy which is optimal in some sense. Let $l : [0, \infty) \to [0, \infty)$ be a loss function which describes the attitude of the trader to hedging losses. The goal is to minimize

$$\mathbf{E}[l((H - X_T^{x,\pi})^+)]. \tag{1.1}$$

This problem was studied with various model settings in many papers. The ones mentioned below do not form a complete list. Existence of the optimal strategy for the case when $l(x) = x$ in a complete market with the stock prices modeled by diffusion processes was shown in [3]. These results were generalized to incomplete markets in [2] where existence of solution with the

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use of dual methods was shown. Existence of the optimal strategy in a general semimartingale model was shown in [9]. In [5] two aspects of the problem are studied. The first is, as above, to find the minimal value of (1.1) which will be denoted here by $\Phi_1^l(x)$ and called the *minimal risk function*. The second aspect is to minimize the initial costs when (1.1) is smaller than or equal to v. The corresponding cost minimizing function is denoted by $\Phi_2^l(v)$. Proposition 3.1 and Theorem 3.2 of [5] provide a description of the solution to the first problem in a general semimartingale framework and Section 7 in [5] shows its relation to the solution of the second problem. In fact these results can be treated as a general method for finding $\Phi_1^l(x)$, $\Phi_2^l(v)$, but they do not provide explicit formulas in the general situation. Then using regularity of the one-dimensional Black–Scholes model both problems have been solved explicitly for a call option (see Section 6 of [5]).

In this paper we examine a multidimensional Black–Scholes model and extend the results of [5] towards more direct formulas for the functions Φ_1^l and Φ_2^l. First we treat the case when l is linear, i.e. $l(x) = x$. Using the general results from [5] and the fact that the density of the martingale measure is regular, we show that

$$\Phi_1^l = \Psi_1 \circ \Psi_2^{-1}, \qquad \Phi_2^l = \Psi_2 \circ \Psi_1^{-1},$$

where Ψ_1, Ψ_2 are certain deterministic functions (for precise formulation see Theorem 3.6). This shows in particular that Φ_2^l is the inverse of Φ_1^l. We show similar results for a strictly convex loss function l. As an immediate consequence of Theorem 3.2 in [5] we obtain the following characterization of the risk minimizing function:

$$\Phi_1^l = \Psi_1^l \circ (\Psi_2^l)^{-1},$$

where again Ψ_1^l, Ψ_2^l are certain deterministic functions. The analogous result for Φ_2^l requires an auxiliary result, Proposition 3.8. Finally, in Theorem 3.9 we show that

$$\Phi_2^l = \Psi_2^l \circ (\Psi_1^l)^{-1}.$$

The risk functions $\Phi_1, \Phi_2, \Phi_1^l, \Phi_2^l$ are thus determined provided that the auxiliary functions $\Psi_1, \Psi_2, \Psi_1^l, \Psi_2^l$ are given. We present concrete integral forms of these functions for some widely traded derivatives like digital option, quantos, outperformance and spread options. The cases when l is a linear loss function or $l(x) = x^p$, $p > 1$, are treated as well.

Let us stress that both functions Φ_1^l, Φ_2^l reflect the interplay between hedging risk and trading costs and thus they serve as an important tool for risk management. Although the model under consideration is a particular case of a general framework studied in [5], it is commonly used in practice due to its tractability (see [6, p. 104]). Thus more explicit computing methods for finding Φ_1^l and Φ_2^l seem to be important for practitioners. The results presented in this paper extend the results from [1], where analogous integral representations for the quantile hedging problem for basket derivatives have been shown.

The paper is organized as follows. In Section 2 we describe the model settings and formulate the problem precisely. Section 3 contains the main results which consist of two parts concerning a linear and a convex loss function respectively. Section 4 is devoted to presenting an explicit integral form for the risk functions in the two-dimensional model when $l(x) = x$ and $l(x) = x^p$ with $p > 1$.

2. Problem formulation. Let $(\Omega, \mathcal{F}, \mathcal{F}_t, t \in [0,T], P)$ be a filtered probability space supporting a d-dimensional Wiener process $W = (W^1, \ldots, W^d)$ with a positive definite correlation matrix of the form

$$Q = \begin{bmatrix} 1 & \rho_{1,2} & \rho_{1,3} & \cdots & \rho_{1,d} \\ \rho_{2,1} & 1 & \rho_{2,3} & \cdots & \rho_{2,d} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \rho_{d,1} & \rho_{d,2} & \rho_{d,3} & \cdots & 1 \end{bmatrix},$$

where

$$\rho_{i,j} = \mathrm{cor}\{W_1^i, W_1^j\}, \quad i,j = 1,\ldots,d.$$

The process W as above will be called a *Q-Wiener process*. The multidimensional Black–Scholes model is specified by the dynamics of d stocks,

$$dS_t^i = S_t^i(\alpha_i dt + \sigma_i dW_t^i), \quad i = 1,\ldots,d,\ t \in [0,T],$$

and evolution of the money market account

$$dB_t = rB_t dt, \quad t \in [0,T].$$

Above, $\alpha_i \in \mathbb{R}$, $\sigma_i > 0$, $i = 1,\ldots,d$, and r stands for a constant interest rate. It is known that such a market is complete and that the unique martingale measure $\tilde{P}$ is given by the density

$$\text{(2.1)} \qquad \frac{d\tilde{P}}{dP} = \tilde{Z}_T := e^{-(Q^{-1}[\frac{\alpha - r\mathbf{1}_d}{\sigma}], W_T) - 1/2|Q^{-1/2}[\frac{\alpha - r\mathbf{1}_d}{\sigma}]|^2 T}, \quad t \in [0,T],$$

with the notation

$$Q^{-1}\left[\frac{\alpha - r\mathbf{1}_d}{\sigma}\right] := Q^{-1}\begin{bmatrix} \frac{\alpha_1 - r}{\sigma_1} \\ \vdots \\ \frac{\alpha_d - r}{\sigma_d} \end{bmatrix}, \quad t \in [0,T]$$

(for more details see, for instance, [1]). Moreover,

$$\widetilde{W}_t := W_t + \frac{\alpha - r\mathbf{1}_d}{\sigma}\, t, \quad t \in [0,T],$$

is a Q-Wiener process under $\widetilde{P}$. The dynamics of the prices under the measure $\widetilde{P}$ can be written as

$$dS_t^i = S_t^i(rdt + \sigma_i d\widetilde{W}_t^i), \quad i = 1, \dots, d, \, t \in [0, T].$$

The wealth process corresponding to the initial endowment x and the trading strategy π is given by

$$X_0^{x,\pi} = x, \quad X_t^{x,\pi} := \pi_t^0 B_t + \sum_{i=1}^d \pi_t^i S_t^i, \quad t \in [0, T].$$

Each strategy is assumed to be admissible, i.e. $X_t^{x,\pi} \geq 0$ for each $t \in [0, T]$ almost surely, and self-financing, i.e.

$$dX_t^{x,\pi} = \pi_t^0 dB_t + \sum_{i=1}^d \pi_t^i dS_t^i, \quad t \in [0, T].$$

A contingent claim is represented by an $\mathcal{F}_T$-measurable random variable H which is assumed to be nonnegative, i.e. $H \geq 0$ and $\mathbf{E}[e^{-rT}\tilde{Z}_T H] < \infty$. As the market is complete, the price of H defined by

$$p(H) := \inf\{x : P(X_T^{x,\pi} \geq H) = 1 \text{ for some } \pi\}$$

is given by $p(H) = \tilde{\mathbf{E}}[e^{-rT}H] = \mathbf{E}[e^{-rT}\tilde{Z}_T H]$.

The trader's attitude towards risk is measured by

$$\mathbf{E}[l((H - X_T^{x,\pi})^+)],$$

where $l : [0, \infty) \to [0, \infty)$ is a loss function which is assumed to be increasing with $l(0) = 0$. It is clear that if $x \geq p(H)$ then the risk equals zero for the replicating strategy. In the opposite case the risk is strictly positive and the question under consideration is to find a strategy such that

$$\mathbf{E}[l((H - X_T^{x,\pi})^+)] \underset{\pi}{\to} \min.$$

We will refer to the corresponding function $\Phi_1 : [0, \infty) \to [0, \mathbf{E}[l(H)]]$ given by

$$\Phi_1^l(x) := \min_{\pi} \mathbf{E}[l((H - X_T^{x,\pi})^+)] \tag{2.2}$$

as the *minimal risk function*. The strategy $\hat{\pi}$ such that $\mathbf{E}[l((H - X_T^{x,\hat{\pi}})^+)] = \Phi_1^l(x)$ will be called the *risk minimizing strategy for* x. If $x \geq p(H)$ then $\Phi_1^l(x) = 0$, and $\Phi_1^l(x) > 0$ otherwise.

We also consider the cost reduction problem. Let $v \geq 0$ be a fixed number describing the level of shortfall risk accepted by the trader. We are searching for a minimal initial cost such that there exists a strategy with risk not exceeding v, i.e.

$$x \to \min; \quad \mathbf{E}[l((H - X_T^{x,\pi})^+)] \leq v \text{ for some } \pi.$$

The *cost reduction function* $\Phi_2^l : [0,\infty) \to [0, p(H)]$ is thus defined by

$$\Phi_2^l(v) := \min\{x : \mathbf{E}[l((H - X_T^{x,\pi})^+)] \le v \text{ for some } \pi\}. \tag{2.3}$$

The strategy $\hat{\pi}$ such that $\mathbf{E}[l((X_T^{\Phi_2^l(v),\hat{\pi}} - H)^+)] \le v$ will be called the *cost minimizing strategy for* v. Notice that $\Phi_2^l(0) = p(H)$.

3. Main results

3.1. Linear loss function. In this section we examine the case when $l(x) = x$. Denote for simplicity the corresponding functions Φ_1^l, Φ_2^l by Φ_1, Φ_2 respectively.

Let us start with two auxiliary results.

LEMMA 3.1. *Let $X, Y \ge 0$ be random variables such that $\mathbf{E}X < \infty$. Then the function $g : [0,\infty) \to [0,\infty)$ given by*

$$g(c) := \mathbf{E}[X\mathbf{1}_{\{Y \ge c\}}]$$

is:

(a) *left continuous on $(0,\infty)$ with right limits on $[0,\infty)$,*
(b) *right continuous on $[0,\infty)$ if the cumulative distribution function of Y is continuous,*
(c) *strictly decreasing if for any $0 \le a < b < \infty$,*

$$P(X > 0, Y \in [a, b)) > 0. \tag{3.1}$$

Proof. The function g is decreasing and thus it has right and left limits. If $X = 0$ then the assertion follows trivially. In the opposite case let us consider an auxiliary probability measure $\hat{P}$ defined by

$$\frac{d\hat{P}}{dP} = \frac{X}{\mathbf{E}[X]},$$

which is absolutely continuous with respect to P, i.e. $\hat{P} \ll P$.

(a) For any $c > 0$ we have

$$\bigcap_n \{c - 1/n \le Y < c\} = \emptyset,$$

and thus

$$|g(c - 1/n) - g(c)| = \mathbf{E}(X\mathbf{1}_{\{c-1/n \le Y < c\}}) = \mathbf{E}[X]\hat{P}(c - 1/n \le Y < c) \underset{n}{\to} 0.$$

(b) For any $c \ge 0$ we have

$$\bigcap_n \{c \le Y < c + 1/n\} = \{Y = c\},$$

and thus

$$\begin{aligned}|g(c)-g(c+1/n)| &= \mathbf{E}(X\mathbf{1}_{\{c\le Y<c+1/n\}})\\ &= \mathbf{E}[X]\hat{P}(c\le Y<c+1/n) \underset{n}{\to} \mathbf{E}[X]\hat{P}(Y=c)=0,\end{aligned}$$

as $\hat{P} \ll P$ and $P(Y=c)=0$.

(c) Let us notice that (3.1) is equivalent to the condition

$$P(X>\varepsilon, Y\in[a,b))>0 \quad \text{for some } \varepsilon>0,$$

and thus for $0\le a<b<\infty$ we have

$$\begin{aligned}|g(a)-g(b)| &= \mathbf{E}(X\mathbf{1}_{\{a\le Y<b\}})\\ &= \mathbf{E}(X\mathbf{1}_{\{a\le Y<b\}}\mathbf{1}_{\{X=0\}}) + \mathbf{E}(X\mathbf{1}_{\{a\le Y<b\}}\mathbf{1}_{\{X>0\}})\\ &\ge \mathbf{E}(X\mathbf{1}_{\{a\le Y<b\}}\mathbf{1}_{\{X>\varepsilon\}}) \ge \varepsilon P(X>\varepsilon, a\le Y<b)>0. \ \blacksquare\end{aligned}$$

REMARK 3.2. Let us notice that condition (3.1) implies that Y has a strictly increasing cumulative distribution function.

COROLLARY 3.3. *If the cumulative distribution function of Y is continuous then the function g in Lemma 3.1 is continuous on $(0,\infty)$ and right continuous at 0.*

PROPOSITION 3.4. *Let (Z_1,Z_2) be a random vector with nondegenerate normal distribution on the plane. Let f,h be functions such that*

$$f:\mathbb{R}^2\to(0,\infty), \qquad h:\mathbb{R}\to(0,\infty) \text{ is strictly monotone.}$$

Let $\alpha,\beta,\gamma,\delta\in\mathbb{R}$ be such that the vectors $(\alpha,\beta),(\gamma,\delta)$ are not parallel, $(\alpha,\beta)\nparallel(\gamma,\delta)$. Let

$$X := f(Z_1,Z_2)\mathbf{1}_{\{\alpha Z_1+\beta Z_2>k\}}, \quad Y := h(\gamma Z_1+\delta Z_2),$$

where k is some constant. Then the function $g(c) := \mathbf{E}[X\mathbf{1}_{\{Y\ge c\}}]$ is strictly decreasing on $[0,\infty)$.

Proof. We will show that (3.1) in Lemma 3.1 holds. We have

$$P(X>0,\, Y\in[a,b)) = P\big(\alpha Z_1+\beta Z_2>k,\, h^{-1}(a)\le\gamma Z_1+\delta Z_2<h^{-1}(b)\big)$$

whennever h is strictly increasing. The probability above is positive because the set

$$\{(x,y):\alpha x+\beta y>k,\, h^{-1}(a)\le\gamma x+\delta y<h^{-1}(b)\}$$

is of positive Lebesgue measure and (Z_1,Z_2) has a nondegenerate distribution. $\blacksquare$

We will need two auxiliary functions defined by

$$\Psi_1(c) := \mathbf{E}(H\mathbf{1}_{A_c}), \tag{3.2}$$

$$\Psi_2(c) := \tilde{\mathbf{E}}(H\mathbf{1}_{A_c}), \tag{3.3}$$

where

$$A_c := \{\tilde{Z}_T^{-1} \geq c\}, \quad c \geq 0,$$

and $\tilde{Z}_T$ is given by (2.1). Let us notice that since Q is nonsingular, the random variable

$$\tilde{Z}_T^{-1} := e^{(Q^{-1}[\frac{\alpha - r\mathbf{1}_d}{\sigma}], W_T) + \frac{1}{2}|Q^{-1/2}[\frac{\alpha - r\mathbf{1}_d}{\sigma}]|^2 T} \tag{3.4}$$

has a continuous cumulative distribution function with respect to P and $\tilde{P}$. Thus it follows from Corollary 3.3 that the functions Ψ_1, Ψ_2 are continuous for any $H \geq 0$. It is clear that both are decreasing. For some special contingent claims these functions are strictly decreasing. Indeed, using Proposition 3.4 one can show that this is the case for the following payoffs.

EXAMPLE 3.5. The functions Ψ_1, Ψ_2 are strictly decreasing if

(a) H is a *digital option*, i.e.

$$H = K\mathbf{1}_{\{S_T^1 \geq S_T^2\}} \quad \text{and} \quad (\sigma_1, -\sigma_2) \nparallel Q^{-1}\left[\frac{\alpha - r\mathbf{1}_d}{\sigma}\right],$$

(b) H is a *quanto domestic option*, i.e.

$$H = S_T^2(S_T^1 - K)^+ \quad \text{and} \quad (\sigma_1, 0) \nparallel Q^{-1}\left[\frac{\alpha - r\mathbf{1}_d}{\sigma}\right],$$

(c) H is a *quanto foreign option*, i.e.

$$H = \left(S_T^1 - \frac{K}{S_T^2}\right)^+ \quad \text{and} \quad (\sigma_1, \sigma_2) \nparallel Q^{-1}\left[\frac{\alpha - r\mathbf{1}_d}{\sigma}\right].$$

Below we present the description of the risk functions Φ_1, Φ_2.

THEOREM 3.6.

(a) *Let $c = c(x)$ be a solution of the equation*

$$\Psi_2(c) = e^{rT}x, \qquad x \in [0, p(H)). \tag{3.5}$$

Then

$$\Phi_1(x) = \begin{cases} \Psi_1(0) - \Psi_1(c) & \text{for } x \in [0, p(H)), \\ 0 & \text{for } x \geq p(H). \end{cases}$$

Moreover, the replicating strategy for the payoff $H\mathbf{1}_{A_{c(x)}}$ is a risk minimizing strategy for x.

(b) *Let $c = c(v)$ be a solution of the equation*

$$\Psi_1(c) = \Psi_1(0) - v, \qquad v \in [0, \mathbf{E}[H]). \tag{3.6}$$

Then

$$\Phi_2(v) = \begin{cases} e^{-rT}\Psi_2(c) & \text{for } v \in [0, \mathbf{E}[H]), \\ 0 & \text{for } v \geq \mathbf{E}[H]. \end{cases}$$

Moreover, the replicating strategy for the payoff $H\mathbf{1}_{A_{c(v)}}$ is a cost minimizing strategy for v.

Proof. First let us notice that the equations (3.5), (3.6) actually have solutions. Indeed, this follows from the fact that Ψ_1, Ψ_2 are continuous and decreasing with images $[0, \mathbf{E}[H]]$, $[0, e^{rT}p(H)]$ respectively.

For any admissible strategy (x, π) let us defin the success function

$$\varphi_{x,\pi} := \mathbf{1}_{\{X_T^{x,\pi} \geq H\}} + \frac{X_T^{x,\pi}}{H}\mathbf{1}_{\{X_T^{x,\pi} < H\}}.$$

One can check the identity

$$(H - X_T^{x,\pi})^+ = H - X_T^{x,\pi} \wedge H = H - H\varphi_{x,\pi},$$

which implies that

$$\mathbf{E}[(H - X_T^{x,\pi})^+] = \mathbf{E}[H] - \mathbf{E}[H\varphi_{x,\pi}]. \tag{3.7}$$

(a) In view of (3.7) the problem (2.2) of finding $\Phi_1(x)$ is equivalent to that of finding a strategy π satisfying

$$\mathbf{E}[H\varphi_{x,\pi}] \underset{\pi}{\rightarrow} \max .$$

If $x \geq p(H)$ then $\varphi_{x,\pi} = 1$ for the replicating strategy and $\Phi_1(x) = 0$, so consider the case $0 \leq x < p(H)$. Let us formulate an auxiliary problem of determining $\varphi \in \mathcal{R}$ solving

$$\begin{cases} \mathbf{E}[H\varphi] \rightarrow \max, \\ \tilde{\mathbf{E}}[e^{-rT}H\varphi] \leq x, \end{cases} \tag{3.8}$$

where

$$\mathcal{R} := \{\varphi : 0 \leq \varphi \leq 1 \text{ and } \varphi \text{ is } \mathcal{F}_T\text{-measurable}\}. \tag{3.9}$$

It is clear that if $\hat{\varphi}$ such that $\tilde{\mathbf{E}}[e^{-rT}H\hat{\varphi}] = x$ is a solution of (3.8) then the replicating strategy $\tilde{\pi}$ for the payoff $H\hat{\varphi}$ is a risk minimizing strategy for x and

$$\Phi_1(x) = \mathbf{E}[(H - X_T^{x,\tilde{\pi}})^+] = \mathbf{E}[H] - \mathbf{E}[H\hat{\varphi}]. \tag{3.10}$$

Now let us focus on determining a solution $\hat{\varphi}$ of (3.8). To this end introduce two probability measures P_1, P_2 with densities

$$\frac{dP_1}{dP} = \frac{H}{\mathbf{E}[H]}, \qquad \frac{dP_2}{dP} = \frac{e^{-rT}\tilde{Z}_T H}{\mathbf{E}[e^{-rT}\tilde{Z}_T H]}.$$

Then (3.8) reads

$$\begin{cases} \mathbf{E}^{P_1}[\varphi] \rightarrow \max, \\ \mathbf{E}^{P_2}[\varphi] \leq x/p(H), \end{cases} \tag{3.11}$$

which is a standard problem in the theory of statistical tests. One should try to search for a solution in the class of 0-1 valued functions of the form $\mathbf{1}_{A_c}$,

$c \geq 0$, where

$$A_c := \left\{ \frac{dP_1}{dP_2} \geq c \right\} = \left\{ \frac{dP_1}{dP} \frac{dP}{dP_2} \geq c \right\} = \left\{ \frac{H}{\mathbf{E}[H]} \frac{\mathbf{E}[\tilde{Z}_T H]}{\tilde{Z}_T H} \geq c \right\}$$
$$= \left\{ \tilde{Z}_T^{-1} \geq c \frac{\mathbf{E}[H]}{\mathbf{E}[\tilde{Z}_T H]} \right\}.$$

For simplicity we can reparametrize A_c by denoting $c\mathbf{E}[H]/\mathbf{E}[\tilde{Z}_T H]$ above just by c. Then $A_c = \{\tilde{Z}_T^{-1} \geq c\}$. It is known by the Neyman–Pearson lemma that if there exists $c = c(x)$ such that

$$\mathbf{E}^{P_2}[\mathbf{1}_{A_c}] = P_2(A_c) = x/p(H), \tag{3.12}$$

then the solution of (3.11), or equivalently (3.8), is given by $\hat{\varphi} = \mathbf{1}_{A_{c(x)}}$. But (3.12) is equivalent to

$$\Psi_2(c) = e^{rT} x,$$

and the existence of the required constant c follows from (3.5). Finally, coming back to (3.10) and using the definition of Ψ_1, we obtain

$$\Phi_1(x) = \mathbf{E}[H] - \mathbf{E}[H\hat{\varphi}] = \mathbf{E}[H] - \mathbf{E}[H\mathbf{1}_{A_c}] = \Psi_1(0) - \Psi_1(c).$$

(b) If $v \geq \mathbf{E}[H]$ then the cost minimizing strategy is trivial, i.e. ($x = 0$, $\pi = 0$), and thus $\Phi_2(v) = 0$. Let us focus on the case when $v \in [0, \mathbf{E}[H])$. In view of (3.7) the cost minimizing strategy is the one which solves the problem

$$\begin{cases} \mathbf{E}[H\varphi_{x,\pi}] \geq \mathbf{E}[H] - v, \\ \tilde{\mathbf{E}}[e^{-rT} H\varphi_{x,\pi}] \to \min. \end{cases}$$

We are thus looking for a solution $\hat{\varphi} \in \mathcal{R}$ of the problem

$$\begin{cases} \mathbf{E}[H\varphi] \geq \mathbf{E}[H] - v, \\ \tilde{\mathbf{E}}[e^{-rT} H\varphi] \to \min. \end{cases} \tag{3.13}$$

If (3.13) has a solution satisfying $\mathbf{E}[H\hat{\varphi}] = \mathbf{E}[H] - v$ then the cost minimizing strategy is the one which replicates $H\hat{\varphi}$, and the cost minimizing function equals

$$\Phi_2(r) = e^{-rT} \tilde{\mathbf{E}}[H\hat{\varphi}]. \tag{3.14}$$

Let us focus on determining the solution $\hat{\varphi}$ of (3.13). Using the notation from part (a) we can reformulate (3.13) as

$$\begin{cases} \mathbf{E}^{P_1}[\varphi] \geq \dfrac{\mathbf{E}[H] - v}{\mathbf{E}[H]}, \\ \mathbf{E}^{P_2}[\varphi] \to \min. \end{cases} \tag{3.15}$$

It can be shown in the same way as in the proof of the Neyman–Pearson lemma that the solution should be searched among the 0-1 valued functions of the form $\mathbf{1}_{B_c}$, $c \geq 0$, where

$$B_c := \left\{ \frac{dP_2}{dP_1} \leq c \right\} = \left\{ \frac{dP_2}{dP} \frac{dP}{dP_1} \leq c \right\} = \left\{ \tilde{Z}_T^{-1} \geq \frac{1}{c} \frac{\mathbf{E}[H]}{\mathbf{E}[\tilde{Z}_T H]} \right\}.$$

Denoting, for simplicity, the constant $\frac{1}{c} \frac{\mathbf{E}[H]}{\mathbf{E}[\tilde{Z}_T H]}$ above by c, we have

$$B_c = \{\tilde{Z}_T^{-1} \geq c\}.$$

If there exists a constant $c = c(v)$ satisfying

$$\mathbf{E}^{P_1}[\mathbf{1}_{B_c}] = P_1(B_c) = \frac{\mathbf{E}[H] - v}{\mathbf{E}[H]} \tag{3.16}$$

then $\hat{\varphi} = \mathbf{1}_{B_c}$ is a solution of (3.15) or, equivalently, (3.13). Let us notice that (3.16) can be written as

$$\Psi_1(c) = \Psi_1(0) - v$$

and existence of the required constant $c(v)$ follows from (3.6). Coming back to (3.14) we obtain

$$\Phi_2(v) = e^{-rT} \tilde{\mathbf{E}}[H \mathbf{1}_{B_c}] = e^{-rT} \Psi_2(c). \ \blacksquare$$

3.2. Convex loss function. In this section we study the case when $l : [0, \infty) \to [0, \infty)$ is an increasing, strictly convex function such that $l(0) = 0$. We assume that $l \in C^2(0, \infty)$ and that l' is strictly increasing with $l'(0+) = 0$, $l'(\infty) = \infty$. The inverse of the first derivative will be denoted by I, i.e.

$$I = (l')^{-1}.$$

Moreover, the contingent claim H is assumed to satisfy $\mathbf{E}[l(H)] < \infty$. The functions Φ_1^l, Φ_2^l can be characterized in terms of the functions

$$\Psi_1^l(c) := \mathbf{E}[l((1 - \varphi_c)H)], \tag{3.17}$$

$$\Psi_2^l(c) := \tilde{\mathbf{E}}[H \varphi_c], \tag{3.18}$$

where φ_c is defined by

$$\varphi_c := \left\{ 1 - \left(\frac{I(c\tilde{Z}_T)}{H} \wedge 1 \right) \right\} \mathbf{1}_{\{H > 0\}}, \qquad c \geq 0. \tag{3.19}$$

It was shown in [5, Theorem 5.1] that the problem of determining Φ_1^l is equivalent to finding a solution $\tilde{\varphi}$ of the problem

$$\begin{cases} \mathbf{E}[l((1 - \varphi)H)] \xrightarrow[\varphi \in \mathcal{R}]{} \min, \\ \tilde{\mathbf{E}}[e^{-rT} H \varphi] \leq x, \end{cases} \tag{3.20}$$

where $\mathcal{R}$ is defined in (3.9). Then $\Phi_1^l(x) = \mathbf{E}[l((1 - \tilde{\varphi})H)]$ and the risk minimizing strategy is the one which replicates $H\tilde{\varphi}$. Moreover, since the function Ψ_2^l is continuous with image $[0, e^{rT} p(H)]$ (see the proof of Theorem 5.1 in [5]), it follows that for any $x \in [0, e^{rT} p(H)]$ there exists a constant

c such that $\Psi_2^l(c) = \tilde{\mathbf{E}}[H\varphi_c] = e^{rT}x$. This φ_c solves the auxiliary problem (3.20) and thus

$$\Phi_1^l(x) = \mathbf{E}[l((1-\varphi_c)H)],$$

and the minimal risk strategy is that replicating the payoff $H\varphi_c$ (see Theorem 3.2 in [5]). Thus the results from [5] can be expressed in our notation as follows.

THEOREM 3.7. *Let $c = c(x)$ be a solution of the equation*

$$\Psi_2^l(c) = e^{rT}x, \qquad x \in [0, p(H)).$$

Then

$$\Phi_1^l(x) = \begin{cases} \Psi_1^l(c) & \text{for } x \in [0, p(H)), \\ 0 & \text{for } x \geq p(H). \end{cases}$$

Although Theorem 3.7 is only a reformulation of Theorem 3.2 in [5], it provides an effective method for practical applications if one is able to derive the functions Ψ_1^l, Ψ_2^l for concrete derivatives.

We will show that the function Φ_2^l can be characterized in terms of the functions Ψ_1^l, Ψ_2^l as well. It is easy to show that the cost reduction problem is equivalent to that of finding $\varphi \in \mathcal{R}$ such that

$$\begin{cases} \mathbf{E}[l((1-\varphi)H)] \leq v, \\ \tilde{\mathbf{E}}[e^{-rT}H\varphi] \to \min. \end{cases} \tag{3.21}$$

Let us notice that (3.21) cannot be solved with the same method as (3.20). In (3.20) the constraints are linear and thus the solution could be found via the Neyman–Pearson approach to the variational problem (see the proof of Theorem 5.1 in [5] and p. 210 in [8]). The constraints in (3.21) are no longer linear and the method above fails. Below we present the proof based on Lagrange multipliers.

PROPOSITION 3.8. *Let l'' be increasing and let H additionally satisfy $\mathbf{E}[l'(H)H] < \infty$ and $\mathbf{E}[l''(H)H^2] < \infty$. Then the random variable*

$$\tilde{\varphi} := \left\{1 - \left(\frac{I(c\tilde{Z}_T)}{H} \wedge 1\right)\right\}\mathbf{1}_{\{H>0\}}$$

with a constant c such that $\mathbf{E}[l((1-\tilde{\varphi})H)] = v$ is a solution of the problem (3.21).

Proof. First, if $\varphi \in \mathcal{R}$ is a solution to (3.21) then $\mathbf{E}[l((1-\varphi)H)] = v$. Indeed, assume to the contrary that φ is a solution to (3.21) with $\mathbf{E}[l((1-\varphi)H)] < v$ and consider the family of random variables $\varphi_\alpha := \varphi \wedge \alpha$, $\alpha \in [0,1]$. Then the function $\alpha \mapsto \mathbf{E}[l((1-\varphi_\alpha)H)]$ is continuously decreasing from $\mathbf{E}[l(H)]$ to 0. Thus there exists $\tilde{\alpha} \in [0,1]$ such that $\mathbf{E}[l((1-\varphi_{\tilde{\alpha}})H)] = v$. Then $\varphi_{\tilde{\alpha}} \leq \varphi$ and thus $\tilde{\mathbf{E}}[H\varphi_{\tilde{\alpha}}] < \tilde{\mathbf{E}}[H\varphi]$, which is a contradiction.

Let $\varphi \neq \tilde{\varphi}$ be any element of $\mathcal{R}$ such that $\mathbf{E}[l((1-\varphi)H)] = v$. We need to show that $\tilde{\mathbf{E}}[H\tilde{\varphi}] \leq \tilde{\mathbf{E}}[H\varphi]$. Let us define

$$\varphi_\varepsilon := (1-\varepsilon)\tilde{\varphi} + \varepsilon\varphi, \qquad \varepsilon \in [0,1],$$

and

$$F_\varphi(\varepsilon) := \tilde{\mathbf{E}}(H\varphi_\varepsilon) = \mathbf{E}(\tilde{Z}_T H \varphi_\varepsilon).$$

We need to show that $F_\varphi(0) \leq F_\varphi(1)$. We will show that F_φ has a minimum at 0. Let us define the auxiliary function

$$G_\varphi(\varepsilon) := \mathbf{E}[l((1-\varphi_\varepsilon)H)],$$

and notice that due to the convexity of l we have $G_\varphi(\varepsilon) \leq v$ for each $\varepsilon \in [0,1]$. Thus the problem of minimizing F_ε on $[0,1]$ is equivalent to

$$\begin{cases} F_\varphi(\varepsilon) \to \min, \\ G_\varphi(\varepsilon) \leq v, \\ \varepsilon \geq 0, \\ 1-\varepsilon \geq 0. \end{cases} \tag{3.22}$$

In view of the assumptions on l and H, both F_φ, G_φ are smooth, with

$$\begin{aligned} F'_\varphi(\varepsilon) &\equiv \mathbf{E}[\tilde{Z}_T(\varphi-\tilde{\varphi})H], \\ G'_\varphi(\varepsilon) &= \mathbf{E}[l'((1-\varphi_\varepsilon)H)\cdot(\tilde{\varphi}-\varphi)H], \\ G''_\varphi(\varepsilon) &= \mathbf{E}[l''((1-\varphi_\varepsilon)H)\cdot(\tilde{\varphi}-\varphi)^2 H^2], \end{aligned}$$

and thus the Lagrange function for (3.22) is of the form

$$L(\varepsilon,\lambda_1,\lambda_2,\lambda_3) = F_\varphi(\varepsilon) - \lambda_1(v - G_\varphi(\varepsilon)) - \lambda_2\varepsilon - \lambda_3(1-\varepsilon).$$

As the function F_φ is linear, it attains its minimal value at 0 or 1. We will show that the first and the second order differential conditions are satisfied for $\varepsilon = 0$.

The first order conditions are

$$\begin{aligned} &L'_\varepsilon(\varepsilon,\lambda_1,\lambda_2,\lambda_3) \\ &\quad = \mathbf{E}[\tilde{Z}_T(\varphi-\tilde{\varphi})H] + \lambda_1\mathbf{E}[l'((1-\varphi_\varepsilon)H)\cdot(\tilde{\varphi}-\varphi)H] - \lambda_2 + \lambda_3 = 0. \end{aligned} \tag{3.23}$$

$$\lambda_1,\lambda_2,\lambda_3 \geq 0, \quad \lambda_1(v - G_\varphi(\varepsilon)) = 0, \quad \lambda_2\varepsilon = 0, \quad \lambda_3(1-\varepsilon) = 0. \tag{3.24}$$

By the definition of $\tilde{\varphi}$ we have

$$\tilde{\varphi} = 1 - I(c\tilde{Z}_T)/H \quad \text{and} \quad c\tilde{Z}_T = l'((1-\tilde{\varphi})H) \quad \text{on } A,$$
$$\tilde{\varphi} = 0 \quad \text{on } A^{\mathsf{C}},$$

where $A := \{c\tilde{Z}_T < l'(H)\}$ and A^{C} stands for the complement of A. For $\varepsilon = 0$ it follows from (3.24) that $\lambda_3 = 0$ and the equation (3.23) is of the form

$$\begin{aligned}(3.25)\qquad &\mathbf{E}[\tilde{Z}_T(\varphi-\tilde{\varphi})H\mathbf{1}_A]+\mathbf{E}[\tilde{Z}_T(\varphi-\tilde{\varphi})H\mathbf{1}_{A^{\complement}}]+c\lambda_1\mathbf{E}[\tilde{Z}_T(\tilde{\varphi}-\varphi)H\mathbf{1}_A]\\ &\qquad+\lambda_1\mathbf{E}[l'((1-\tilde{\varphi})H)(\tilde{\varphi}-\varphi)H\mathbf{1}_{A^{\complement}}\\ &=(1-c\lambda_1)\mathbf{E}[\tilde{Z}_T(\varphi-\tilde{\varphi})H\mathbf{1}_A]+\mathbf{E}[\tilde{Z}_T\varphi H\mathbf{1}_{A^{\complement}}]-\lambda_1\mathbf{E}[l'(H)\varphi H\mathbf{1}_{A^{\complement}}]=\lambda_2.\end{aligned}$$

The left side of (3.25) satisfies the estimate

$$\begin{aligned}&(1-c\lambda_1)\mathbf{E}[\tilde{Z}_T(\varphi-\tilde{\varphi})H\mathbf{1}_A]+\mathbf{E}[\tilde{Z}_T\varphi H\mathbf{1}_{A^{\complement}}]-\lambda_1\mathbf{E}[l'(H)\varphi H\mathbf{1}_{A^{\complement}}]\\ &\quad\ge(1-c\lambda_1)\mathbf{E}[\tilde{Z}_T(\varphi-\tilde{\varphi})H\mathbf{1}_A]+\mathbf{E}[\tilde{Z}_T\varphi H\mathbf{1}_{A^{\complement}}]-\lambda_1 c\mathbf{E}[\tilde{Z}_T\varphi H\mathbf{1}_{A^{\complement}}]\\ &\quad\ge(1-c\lambda_1)\mathbf{E}[\tilde{Z}_T(\varphi-\tilde{\varphi})H\mathbf{1}_A+\tilde{Z}_T\varphi H\mathbf{1}_{A^{\complement}}].\end{aligned}$$

If $\mathbf{E}[\tilde{Z}_T(\varphi-\tilde{\varphi})H\mathbf{1}_A+\tilde{Z}_T\varphi H\mathbf{1}_{A^{\complement}}]>0$ then we take λ_1 such that $1-c\lambda_1>0$; in the opposite case we take λ_1 such that $1-c\lambda_1<0$. In both cases λ_2 given by (3.25) is nonnegative.

The second order condition for $\varepsilon=0$ is

$$L''_\varepsilon(\varepsilon,\lambda_1,\lambda_2,\lambda_3)=\lambda_1\mathbf{E}[l''((1-\tilde{\varphi})H)\cdot(\tilde{\varphi}-\varphi)^2H^2]\ge 0,$$

and thus the solution of (3.22) is $\varepsilon=0$. ∎

Proposition 3.8 and the definitions of Ψ_1^l,Ψ_2^l lead us to the following result.

THEOREM 3.9. *Assume that l'' is increasing and H satisfies $\mathbf{E}[l'(H)H]<\infty$ and $\mathbf{E}[l''(H)H^2]<\infty$. Let $c=c(v)$ be a solution of the equation*

$$\Psi_1^l(c)=v,\quad v\in[0,\mathbf{E}[l(H)]).$$

Then

$$\Phi_2^l(v)=\begin{cases}e^{-rT}\Psi_2^l(c) & \text{for } v\in[0,\mathbf{E}[l(H)]),\\ 0 & \text{for } v\ge\mathbf{E}[l(H)].\end{cases}$$

4. Two-dimensional model. In this section we determine explicit integral formulas for the functions Ψ_1^l, Ψ_2^l for several popular options in the case $d=2$. Some of the results can be generalized to higher dimensions.

First let us introduce some notation relating to the multidimensional normal distribution. The fact that an $\mathbb{R}^d$-valued random vector X has a normal distribution with mean $m\in\mathbb{R}^d$ and covariance matrix Σ will be denoted by $X\sim N_d(m,\Sigma)$ or $\mathcal{L}(X)=N_d(m,\Sigma)$. We denote by f_X the density of X. If $d=1$ then the subscript is omitted and $N(m,\sigma)$ denotes the normal distribution with mean m and variance σ. If $X\sim N_d(m,\Sigma)$ and A is a $k\times d$ matrix, then

$$AX\sim N_k(Am,A\Sigma A^T);\tag{4.1}$$

in particular if $a\in\mathbb{R}^d$ then

$$a^TX\sim N(a^Tm,a^T\Sigma a).\tag{4.2}$$

Let X be a random vector taking values in $\mathbb{R}^d$ and fix an integer $0 < k < d$. Let us divide X into two vectors $X^{(1)}$ and $X^{(2)}$ of lengths k, $d-k$ respectively, i.e.

$$X^{(1)} = (X_1, \ldots, X_k)^T, \qquad X^{(2)} = (X_{k+1}, \ldots, X_d)^T.$$

Analogously, divide the mean vector m and the covariance matrix Σ:

$$m = \begin{pmatrix} m^{(1)} \\ m^{(2)} \end{pmatrix}, \qquad \Sigma = \begin{bmatrix} \Sigma^{(11)} & \Sigma^{(12)} \\ \Sigma^{(21)} & \Sigma^{(22)} \end{bmatrix},$$

so that $\mathbf{E}X^{(1)} = m^{(1)}$, $\mathbf{E}X^{(2)} = m^{(2)}$, $\operatorname{Cov} X^{(1)} = \Sigma^{(11)}$, $\operatorname{Cov} X^{(2)} = \Sigma^{(22)}$, $\operatorname{Cov}(X^{(1)}, X^{(2)}) = \Sigma^{(12)} = {\Sigma^{(21)}}^T$. Denote by $\mathcal{L}(X^{(1)} \mid X^{(2)} = x^{(2)})$ the conditional distribution of $X^{(1)}$ given $X^{(2)} = x^{(2)} \in \mathbb{R}^{d-k}$. If $\Sigma^{(22)}$ is nonsingular then

$$\mathcal{L}(X^{(1)} \mid X^{(2)} = x^{(2)}) = N_k(m^{(1)}(x^{(2)}), \Sigma^{(11)}(x^{(2)})), \tag{4.3}$$

where

$$\begin{aligned} m^{(1)}(x^{(2)}) &= m^{(1)} + \Sigma^{(12)} {\Sigma^{(22)}}^{-1} (x^{(2)} - m^{(2)}), \\ \Sigma^{(11)}(x^{(2)}) &= \Sigma^{(11)} - \Sigma^{(12)} {\Sigma^{(22)}}^{-1} \Sigma^{(21)}. \end{aligned}$$

Actually the conditional variance $\Sigma^{(11)}(x^{(2)})$ does not depend on $x^{(2)}$ but we keep the notation for consistency. The conditional density will be denoted by $f_{X^{(1)}|X^{(2)}=x^{(2)}}(x^{(1)})$, where $x^{(1)} \in \mathbb{R}^k$. In particular if (X, Y) is a two-dimensional normal vector with parameters

$$m = \begin{pmatrix} m_1 \\ m_2 \end{pmatrix}, \qquad \Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix},$$

then

$$\mathcal{L}(X \mid Y = y) = N(m_1(y), \sigma_1(y)),$$

where

$$m_1(y) := m_1 + \frac{\sigma_{12}}{\sigma_{22}}(y - m_2), \quad \sigma_1(y) := \sigma_{11} - \frac{\sigma_{12}^2}{\sigma_{22}}.$$

If X is a random vector then its distribution with respect to the measure $\widetilde{P}$ will be denoted by $\tilde{\mathcal{L}}(X)$ and its density by $\tilde{f}_X$. Also, $\tilde{f}_{X^{(1)}|X^{(2)}=x^{(2)}}(x^{(1)})$ stands for the conditional density with respect to $\widetilde{P}$.

In the case $d = 2$ the correlation matrix is of the form

$$Q = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix},$$

and thus

$$Q^{-1} = \frac{1}{\rho^2-1}\begin{bmatrix} -1 & \rho \\ \rho & -1 \end{bmatrix}, \qquad Q^{-1/2} = \frac{1}{2}\begin{bmatrix} \frac{1}{\sqrt{1+\rho}} + \frac{1}{\sqrt{1-\rho}} & \frac{1}{\sqrt{1+\rho}} - \frac{1}{\sqrt{1-\rho}} \\ \frac{1}{\sqrt{1+\rho}} - \frac{1}{\sqrt{1-\rho}} & \frac{1}{\sqrt{1+\rho}} + \frac{1}{\sqrt{1-\rho}} \end{bmatrix}.$$

Hence the density of the martingale measure (2.1) can be written as

$$\tilde{Z}_T = e^{-A_1 W_T^1 - A_2 W_T^2 - BT} = e^{-A_1 \widetilde{W}_T^1 - A_2 \widetilde{W}_T^2 - \tilde{B}T}, \tag{4.4}$$

where

$$A_1 := \frac{1}{\rho^2-1}\left(-\frac{\alpha_1-r}{\sigma_1} + \rho\frac{\alpha_2-r}{\sigma_2}\right)$$

$$A_2 := \frac{1}{\rho^2-1}\left(\rho\frac{\alpha_1-r}{\sigma_1} - \frac{\alpha_2-r}{\sigma_2}\right)$$

$$B := \frac{1}{8}\left(\left(\left(\frac{1}{\sqrt{1+\rho}} + \frac{1}{\sqrt{1-\rho}}\right)\frac{\alpha_1-r}{\sigma_1} + \left(\frac{1}{\sqrt{1+\rho}} - \frac{1}{\sqrt{1-\rho}}\right)\frac{\alpha_2-r}{\sigma_2}\right)^2 + \left(\left(\frac{1}{\sqrt{1+\rho}} - \frac{1}{\sqrt{1-\rho}}\right)\frac{\alpha_1-r}{\sigma_1} + \left(\frac{1}{\sqrt{1+\rho}} + \frac{1}{\sqrt{1-\rho}}\right)\frac{\alpha_2-r}{\sigma_2}\right)^2\right)$$

$$\tilde{B} := B - A_1\frac{\alpha_1-r}{\sigma_1} - A_2\frac{\alpha_2-r}{\sigma_2}.$$

In the following subsections we will use the universal constants $A_1, A_2, B, \tilde{B}$ appearing in (4.4) as well as $a_1, a_2, b, \tilde{a}_1, \tilde{a}_2, \tilde{b}$ introduced below.

Fix numbers $K > 0$, $c \geq 0$. One can check the following:

$$\{S_T^1 \geq K\} = \{W_T^1 \geq a_1\} = \{\widetilde{W}_T^1 \geq \tilde{a}_1\}, \tag{4.5}$$

$$\{S_T^2 \geq K\} = \{W_T^2 \geq a_2\} = \{\widetilde{W}_T^2 \geq \tilde{a}_2\}, \tag{4.6}$$

$$\{S_T^1 \geq S_T^2\} = \{\sigma_1 W_T^1 - \sigma_2 W_T^2 \geq b\} = \{\sigma_1 \widetilde{W}_T^1 - \sigma_2 \widetilde{W}_T^2 \geq \tilde{b}\}, \tag{4.7}$$

$$\begin{aligned}\{\tilde{Z}_T^{-1} \geq c\} &= \{A_1 W_T^1 + A_2 W_T^2 \geq \ln c - BT\} \\ &= \{A_1 \widetilde{W}_T^1 + A_2 \widetilde{W}_T^2 \geq \ln c - \tilde{B}T\},\end{aligned} \tag{4.8}$$

where

$$a_1 := \frac{1}{\sigma_1}\left(\ln\frac{K}{S_0^1} - \left(\alpha_1 - \frac{1}{2}\sigma_1^2\right)T\right), \quad \tilde{a}_1 := \frac{1}{\sigma_1}\left(\ln\frac{K}{S_0^1} - \left(r - \frac{1}{2}\sigma_1^2\right)T\right),$$

$$a_2 := \frac{1}{\sigma_2}\left(\ln\frac{K}{S_0^2} - \left(\alpha_2 - \frac{1}{2}\sigma_2^2\right)T\right), \quad \tilde{a}_2 := \frac{1}{\sigma_2}\left(\ln\frac{K}{S_0^2} - \left(r - \frac{1}{2}\sigma_2^2\right)T\right),$$

$$b := \ln\frac{S_0^2}{S_0^1} + \left(\alpha_2 - \alpha_1 - \frac{1}{2}(\sigma_2^2 - \sigma_1^2)\right)T, \quad \tilde{b} := \ln\frac{S_0^2}{S_0^1} - \frac{1}{2}(\sigma_2^2 - \sigma_1^2)T.$$

In all the formulas below it is understood that $\ln 0 = -\infty$ and Φ stands for the distribution function of $N(0,1)$.

For each derivative we calculate the risk functions for the cases when $l(x) = x$ and when $l(x) = x^p/p$, $p > 1$. In the latter case we use the notation

$\Psi_1^p = \Psi_1^l, \Psi_2^p = \Psi_2^l$. For $l(x) = x^p/p$ we have $I(x) = x^{1/(p-1)}$ and in view of (3.19),

$$\Psi_1^p(c) = \frac{1}{p}\mathbf{E}[H^p \mathbf{1}_{A_c^{\complement}}] + \frac{1}{p}\mathbf{E}[(c\tilde{Z}_T)^{\frac{p}{p-1}} \mathbf{1}_{A_c}], \tag{4.9}$$

$$\Psi_2^p(c) = \tilde{\mathbf{E}}[(H - (c\tilde{Z}_T)^{\frac{1}{p-1}})\mathbf{1}_{A_c}], \tag{4.10}$$

where

$$A_c := \{c\tilde{Z}_T \le H^{p-1}\}, \tag{4.11}$$

and $A_c^{\complement}$ stands for the complement of A_c.

4.1. Digital option. Digital option is a contract with payoff function of the form

$$H = K \cdot \mathbf{1}_{\{S_T^1 \ge S_T^2\}}, \qquad K > 0.$$

Let (X, Y), $(\tilde{X}, \tilde{Y})$ be random vectors defined by $X := \sigma_1 W_T^1 - \sigma_2 W_T^2$, $Y := A_1 W_T^1 + A_2 W_T^2$, $\tilde{X} := \sigma_1 \widetilde{W}_T^1 - \sigma_2 \widetilde{W}_T^2$, $\tilde{Y} := A_1 \widetilde{W}_T^1 + A_2 \widetilde{W}_T^2$. They are normally distributed under P, resp. $\tilde{P}$ and their parameters are given by (4.1).

Linear loss function. Using (4.7) and (4.8) we obtain

$$\begin{aligned}\Psi_1(c) &= K\mathbf{E}(\mathbf{1}_{\{S_T^1 \ge S_T^2\}} \mathbf{1}_{\{\tilde{Z}_T^{-1} \ge c\}})\\ &= KP(\sigma_1 W_T^1 - \sigma_2 W_T^2 \ge b, A_1 W_T^1 + A_2 W_T^2 \ge \ln c - BT),\end{aligned}$$

and thus

$$\Psi_1(c) = K \int_b^\infty \int_{\ln c - BT}^\infty f_{X,Y}(x, y)\, dy\, dx.$$

An analogous computation yields

$$\begin{aligned}\Psi_2(c) &= K\tilde{P}(\sigma_1 \widetilde{W}_T^1 - \sigma_2 \widetilde{W}_T^2 \ge \tilde{b},\ A_1 \widetilde{W}_T^1 + A_2 \widetilde{W}_T^2 \ge \ln c - \tilde{B}T)\\ &= K \int_{\tilde{b}}^\infty \int_{\ln c - \tilde{B}T}^\infty \tilde{f}_{\tilde{X},\tilde{Y}}(x, y)\, dy\, dx.\end{aligned}$$

Power loss function. In view of (4.7) and (4.4) we have

$$\begin{aligned}A_c &:= \{c\tilde{Z}_T \le H^{p-1}\} = \{c\tilde{Z}_T \le K^{p-1}\mathbf{1}_{\{\sigma_1 W_T^1 - \sigma_2 W_T^2 \ge b\}}\}\\ &= \{\sigma_1 W_T^1 - \sigma_2 W_T^2 \ge b,\ c\tilde{Z}_T \le K^{p-1}\}\\ &= \{\sigma_1 W_T^1 - \sigma_2 W_T^2 \ge b,\ A_1 W_T^1 + A_2 W_T^2 \ge \ln(K^{p-1}/c) - BT\}\\ &= \{\sigma_1 \widetilde{W}_T^1 - \sigma_2 \widetilde{W}_T^2 \ge \tilde{b},\ A_1 \widetilde{W}_T^1 + A_2 \widetilde{W}_T^2 \ge \ln(K^{p-1}/c) - \tilde{B}T\},\end{aligned} \tag{4.12}$$

and thus

$$\Psi_1^p(c) = \frac{1}{p}\mathbf{E}[K^p \mathbf{1}_{\{\sigma_1 W_T^1 - \sigma_2 W_T^2 \geq b\}} \mathbf{1}_{A_c^\complement}] + \frac{1}{p} c^{\frac{p}{p-1}} \mathbf{E}[\tilde{Z}_T^{\frac{p}{p-1}} \mathbf{1}_{A_c}],$$
$$\Psi_2^p(c) = \tilde{\mathbf{E}}[K \mathbf{1}_{\{\sigma_1 \widetilde{W}_T^1 - \sigma_2 \widetilde{W}_T^2 \geq \tilde{b}\}} \mathbf{1}_{A_c}] - c^{\frac{1}{p-1}} \tilde{\mathbf{E}}[\tilde{Z}_T^{\frac{1}{p-1}} \mathbf{1}_{A_c}].$$

In view of (4.12) we have

$$\begin{aligned}\Psi_1^p(c) &= \frac{K^p}{p} P(\sigma_1 W_T^1 - \sigma_2 W_T^2 \geq b,\ A_1 W_T^1 + A_2 W_T^2 < \ln(K^{p-1}/c) - BT) \\ &\quad + \frac{1}{p} c^{\frac{p}{p-1}} \mathbf{E}[\tilde{Z}_T^{\frac{p}{p-1}} \mathbf{1}_{A_c}] \\ &= \frac{K^p}{p} \int_b^\infty \int_{-\infty}^{\ln(K^{p-1}/c)-BT} f_{X,Y}(x,y)\, dy\, dx \\ &\quad + \frac{1}{p} c^{\frac{p}{p-1}} \int_b^\infty \int_{\ln(K^{p-1}/c)-BT}^\infty e^{-\frac{p(y+BT)}{p-1}} f_{X,Y}(x,y)\, dy\, dx,\end{aligned}$$

and

$$\begin{aligned}\Psi_2^p(c) &= K\tilde{P}(A_c) - c^{\frac{1}{p-1}} \tilde{\mathbf{E}}[e^{-A_1 \widetilde{W}_T^1 - A_2 \widetilde{W}_T^2 - \tilde{B}T} \mathbf{1}_{A_c}] \\ &= K \int_{\tilde{b}}^\infty \int_{\ln(K^{p-1}/c)-\tilde{B}T}^\infty \tilde{f}_{\tilde{X},\tilde{Y}}(x,y)\, dy\, dx \\ &\quad - c^{\frac{1}{p-1}} \int_{\tilde{b}}^\infty \int_{\ln(K^{p-1}/c)-\tilde{B}T}^\infty e^{-\frac{y+\tilde{B}T}{p-1}} \tilde{f}_{\tilde{X},\tilde{Y}}(x,y)\, dy\, dx.\end{aligned}$$

4.2. Quantos

4.2.1. *Quanto domestic.* The contingent claim is of the form

$$H = S_T^2 (S_T^1 - K)^+, \quad K > 0.$$

Linear loss function. Using (4.5) we obtain

$$\begin{aligned}\Psi_1(c) &= \mathbf{E}[S_T^2 (S_T^1 - K)^+ \mathbf{1}_{\{\tilde{Z}_T^{-1} \geq c\}}] \\ &= \mathbf{E}[S_T^2 (S_T^1 - K) \mathbf{1}_{\{\tilde{Z}_T^{-1} \geq c\}} \mid S_T^1 > K] P(S_T^1 > K) \\ &= \mathbf{E}[S_T^2 (S_T^1 - K) \mathbf{1}_{\{A_1 W_T^1 + A_2 W_T^2 \geq \ln c - BT\}} \mid W_T^1 > a_1] P(W_T^1 > a_1) \\ &= \int_{a_1}^\infty \mathbf{E}\big[S_0^2 e^{(\alpha_2 - \frac{1}{2}\sigma_2^2)T + \sigma_2 W_T^2} (S_0^1 e^{(\alpha_1 - \frac{1}{2}\sigma_1^2)T + \sigma_1 W_T^1} - K) \\ &\qquad\qquad \cdot \mathbf{1}_{\{W_T^2 \geq \frac{\ln c - BT - A_1 W_T^1}{A_2}\}} \big| W_T^1 = x\big] f_{W_T^1}(x) dx\end{aligned}$$

$$= S_0^2 e^{(\alpha_2-\frac{1}{2}\sigma_2^2)T} \int_{a_1}^{\infty} \int_{\frac{\ln c - BT - A_1 x}{A_2}}^{\infty} (S_0^1 e^{(\alpha_1-\frac{1}{2}\sigma_1^2)T+\sigma_1 x} - K) e^{\sigma_2 y} \cdot f_{W_T^2|W_T^1=x}(y) f_{W_T^1}(x)\, dy\, dx,$$

and

$$\begin{aligned}
\Psi_2(c) &= \tilde{\mathbf{E}}[S_T^2(S_T^1-K)^+ \mathbf{1}_{\{\tilde{Z}_T^{-1}\geq c\}}] \\
&= \tilde{\mathbf{E}}[S_T^2(S_T^1-K)\mathbf{1}_{\{\tilde{Z}_T^{-1}\geq c\}} \mid S_T^1 > K]\tilde{P}(S_T^1 > K) \\
&= \tilde{\mathbf{E}}[S_T^2(S_T^1-K)\mathbf{1}_{\{A_1\widetilde{W}_T^1 + A_2\widetilde{W}_T^2 \geq \ln c - \tilde{B}T\}} \mid \widetilde{W}_T^1 > \tilde{a}_1]\tilde{P}(\widetilde{W}_T^1 > \tilde{a}_1) \\
&= \int_{\tilde{a}_1}^{\infty} \tilde{\mathbf{E}}\big[S_0^2 e^{(r-\frac{1}{2}\sigma_2^2)T+\sigma_2\widetilde{W}_T^2}(S_0^1 e^{(r-\frac{1}{2}\sigma_1^2)T+\sigma_1\widetilde{W}_T^1} - K) \\
&\qquad \cdot \mathbf{1}_{\{\widetilde{W}_T^2 \geq \frac{\ln c - \tilde{B}T - A_1\widetilde{W}_T^1}{A_2}\}} \mid \widetilde{W}_T^1 = x\big]\tilde{f}_{\widetilde{W}_T^1}(x)\, dx \\
&= S_0^2 e^{(r-\frac{1}{2}\sigma_2^2)T} \int_{\tilde{a}_1}^{\infty} \int_{\frac{\ln c - \tilde{B}T - A_1 x}{A_2}}^{\infty} (S_0^1 e^{(r-\frac{1}{2}\sigma_1^2)T+\sigma_1 x} - K)e^{\sigma_2 y} \\
&\qquad \cdot \tilde{f}_{\widetilde{W}_T^2|\widetilde{W}_T^1=x}(y)\, dy\, \tilde{f}_{\widetilde{W}_T^1}(x)\, dx.
\end{aligned}$$

Power loss function. The set (4.11) is of the form

$$\begin{aligned}
A_c &= \left\{ \frac{(ce^{-A_1W_T^1 - A_2W_T^2 - BT})^{\frac{1}{p-1}}}{S_0^2 e^{(\alpha_2-\frac{1}{2}\sigma_2^2)T+\sigma_2 W_T^2}} \leq (S_T^1 - K)^+ \right\} \\
&= \left\{ \frac{c^{\frac{1}{p-1}}}{S_0^2} e^{-\frac{A_1}{p-1}W_T^1 - (\frac{A_2}{p-1}+\sigma_2)W_T^2 - (B+\alpha_2-\frac{1}{2}\sigma_2^2)T} \leq S_T^1 - K, S_T^1 \geq K) \right\}.
\end{aligned}$$

For simplicity we assume that $\frac{A_2}{p-1} + \sigma_2 > 0$. In the opposite case one has to modify the form of the set A_c and thus also the integration limits in the formulas below. We obtain

$$A_c = \{W_T^2 \geq w(W_T^1),\, W_T^1 \geq a_1\} = \{\widetilde{W}_T^2 \geq \tilde{w}(\widetilde{W}_T^1),\, \widetilde{W}_T^1 \geq \tilde{a}_1\},$$

where

$$w(x) := \frac{\frac{A_1}{p-1}x + \ln\big(\frac{S_0^2(S_0^1 e^{(\alpha_1-\sigma_1^2)T+\sigma_1 x} - K)}{c^{1/(p-1)}}\big) + (B + \alpha_2 - \frac{1}{2}\sigma_2^2)T}{-\big(\frac{A_2}{p-1} + \sigma_2\big)},$$

$$\tilde{w}(x) := \frac{\frac{A_1}{p-1}x + \ln\big(\frac{S_0^2(S_0^1 e^{(r-\sigma_1^2)T+\sigma_1 x} - K)}{c^{1/(p-1)}}\big) + (\tilde{B} + \alpha_2 - \frac{1}{2}\sigma_2^2)T}{-\big(\frac{A_2}{p-1} + \sigma_2\big)}.$$

In view of the above, (4.9), (4.10) and using conditional densities we obtain

$$\begin{aligned}\Psi_1^p(c) = \frac{(S_0^2)^p e^{(\alpha_2-\frac{1}{2}\sigma_2^2)pT}}{p}\Big(\int_{a_1}^{\infty}\int_{-\infty}^{\infty} e^{p\sigma_2 y}(S_0^1 e^{(\alpha_1-\frac{1}{2}\sigma_1^2)T+\sigma_2 x} - K)^p \\ \cdot f_{W_T^2|W_T^1=x}(y) f_{W_T^1}(x)\,dy\,dx \\ - \int_{a_1}^{\infty}\int_{w(x)}^{\infty} e^{p\sigma_2 y}(S_0^1 e^{(\alpha_1-\frac{1}{2}\sigma_1^2)T+\sigma_2 x} - K)^p f_{W_T^2|W_T^1=x}(y) f_{W_T^1}(x)\,dy\,dx\Big) \\ + \frac{c^{\frac{p}{p-1}} e^{-\frac{BTp}{p-1}}}{p}\int_{a_1}^{\infty}\int_{w(x)}^{\infty} e^{-(\frac{A_1 p}{p-1}x+\frac{A_2 p}{p-1}y)} f_{W_T^2|W_T^1=x}(y) f_{W_T^1}(x)\,dy\,dx,\end{aligned}$$

and

$$\begin{aligned}\Psi_2^p(c) = S_0^2 e^{(r-\frac{1}{2}\sigma_2^2)T}\int_{\tilde a_1}^{\infty}\int_{\tilde w(x)}^{\infty} e^{\sigma_2 y}(S_0^1 e^{(r-\frac{1}{2}\sigma_1^2)T+\sigma_2 x} - K) \\ \cdot \tilde f_{\widetilde W_T^2|\widetilde W_T^1=x}(y)\tilde f_{\widetilde W_T^1}(x)\,dy\,dx \\ - c^{\frac{1}{p-1}} e^{-\frac{BT}{p-1}}\int_{\tilde a_1}^{\infty}\int_{\tilde w(x)}^{\infty} e^{-\frac{1}{p-1}(A_1 x+A_2 y)}\tilde f_{\widetilde W_T^2|\widetilde W_T^1=x}(y)\tilde f_{\widetilde W_T^1}(x)\,dy\,dx.\end{aligned}$$

4.2.2. *Quanto foreign.* The payoff is of the form

$$H = (S_T^1 - K/S_T^2)^+, \quad K > 0.$$

Linear loss function. First let us notice that

$$\{S_T^1 - K/S_T^2 \ge 0\} = \{\sigma_1 W_T^1 + \sigma_2 W_T^2 \ge d\} = \{\sigma_1 \widetilde W_T^1 + \sigma_2 \widetilde W_T^2 \ge \tilde d\}, \tag{4.13}$$

where

$$\begin{aligned} d &:= \ln\frac{K}{S_0^1 S_0^2} - \Big(\alpha_1 + \alpha_2 - \frac{1}{2}(\sigma_1^2+\sigma_2^2)\Big)T, \\ \tilde d &:= \ln\frac{K}{S_0^1 S_0^2} - \Big(2r - \frac{1}{2}(\sigma_1^2+\sigma_2^2)\Big)T. \end{aligned} \tag{4.14}$$

We have

$$\begin{aligned}\Psi_1(c) &= \mathbf{E}\bigg[\Big(S_T^1 - \frac{K}{S_T^2}\Big)^+ \mathbf{1}_{\{\tilde Z_T^{-1} \ge c\}}\bigg] \\ &= \mathbf{E}\bigg[\Big(S_T^1 - \frac{K}{S_T^2}\Big)\mathbf{1}_{\{W_T^2 \ge \frac{\ln c - BT - A_1 W_T^1}{A_2}\}}\ \bigg|\ \sigma_1 W_T^1 + \sigma_2 W_T^2 \ge d\bigg] \\ &\quad \cdot P(\sigma_1 W_T^1 + \sigma_2 W_T^2 \ge d).\end{aligned}$$

Denoting $Z := \sigma_1 W_T^1 + \sigma_2 W_T^2$ and taking into account the conditional distribution $\mathcal{L}(W_T^1, W_T^2 \mid Z)$ we obtain

$$\Psi_1(c) = \int_d^\infty \int_{-\infty}^\infty \int_{\frac{\ln c - BT - A_1 x}{A_2}}^\infty (S_0^1 e^{(\alpha_1 - \frac{1}{2}\sigma_1^2)T + \sigma_1 x} - K S_0^2 e^{(-\alpha_2 + \frac{1}{2}\sigma_2^2)T - \sigma_2 y}) \cdot f_{(W_T^1, W_T^2)|Z=z}(x,y)\, dy\, dx\, f_Z(z)\, dz.$$

Using the same argument under the measure $\tilde{P}$ with $\tilde{Z} := \sigma_1 \widetilde{W}_T^1 + \sigma_2 \widetilde{W}_T^2$ yields

$$\begin{aligned}\Psi_2(c) &= \tilde{\mathbf{E}}\Big[\Big(S_T^1 - \frac{K}{S_T^2}\Big)\mathbf{1}_{\{\widetilde{W}_T^2 \geq \frac{\ln c - \tilde{B}T - A_1 \widetilde{W}_T^1}{A_2}\}} \,\Big|\, \sigma_1 \widetilde{W}_T^1 + \sigma_2 \widetilde{W}_T^2 \geq \tilde{d}\Big] \cdot \tilde{P}(\sigma_1 \widetilde{W}_T^1 + \sigma_2 \widetilde{W}_T^2 \geq \tilde{d}) \\ &= \int_{\tilde{d}}^\infty \int_{-\infty}^\infty \int_{\frac{\ln c - \tilde{B}T - A_1 x}{A_2}}^\infty (S_0^1 e^{(r - \frac{1}{2}\sigma_1^2)T + \sigma_1 x} - K S_0^2 e^{(-r + \frac{1}{2}\sigma_2^2)T - \sigma_2 y}) \cdot \tilde{f}_{(\widetilde{W}_T^1, \widetilde{W}_T^2)|\tilde{Z}=z}(x,y)\, dy\, dx\, \tilde{f}_{\tilde{Z}}(z)\, dz.\end{aligned}$$

Power loss function. Using (4.13) one can check the following:

$$\begin{aligned} A_c &:= \Big\{c\tilde{Z}_T \leq \Big(\Big(S_T^1 - \frac{K}{S_T^2}\Big)^+\Big)^{p-1},\ S_T^1 - \frac{K}{S_T^2} > 0\Big\} \\ &= \Big\{c\tilde{Z}_T \leq \Big(\Big(S_T^1 - \frac{K}{S_T^2}\Big)^+\Big)^{p-1},\ \sigma_1 W_T^1 + \sigma_2 W_T^2 > d\Big\} \\ &= \Big\{\frac{A_1}{p-1} W_T^1 + \Big(\frac{A_2}{p-1} - \sigma_2\Big) W_T^2 \geq v(\sigma_1 W_T^1 + \sigma_2 W_T^2),\ \sigma_1 W_T^1 + \sigma_2 W_T^2 > d\Big\} \qquad (4.15) \\ &= \Big\{\frac{A_1}{p-1} \widetilde{W}_T^1 + \Big(\frac{A_2}{p-1} - \sigma_2\Big) \widetilde{W}_T^2 \geq \tilde{v}(\sigma_1 \widetilde{W}_T^1 + \sigma_2 \widetilde{W}_T^2),\ \sigma_1 \widetilde{W}_T^1 + \sigma_2 \widetilde{W}_T^2 > \tilde{d}\Big\}, \qquad (4.16) \end{aligned}$$

where $d, \tilde{d}$ are given by (4.14) and

$$v(x) = \ln\Big\{\frac{S_0^1 S_0^2 e^{(\alpha_1 + \alpha_2 - \frac{1}{2}(\sigma_1^2 + \sigma_2^2))T + x} - K}{c^{\frac{1}{p-1}} S_0^2 e^{(\alpha_2 - \frac{1}{2}\sigma_2^2 - \frac{B}{p-1})T}}\Big\},$$

$$\tilde{v}(x) = \ln\Big\{\frac{S_0^1 S_0^2 e^{(2r - \frac{1}{2}(\sigma_1^2 + \sigma_2^2))T + x} - K}{c^{\frac{1}{p-1}} S_0^2 e^{(r - \frac{1}{2}\sigma_2^2 - \frac{\tilde{B}}{p-1})T}}\Big\}.$$

To calculate Ψ_1^l, Ψ_2^l we use the conditional distributions $\mathcal{L}(X \mid Y)$, $\mathcal{L}(\tilde{X} \mid \tilde{Y})$, where $X := \frac{A_1}{p-1}W_T^1 + \left(\frac{A_2}{p-1} - \sigma_2\right)W_T^2$, $Y := \sigma_1 W_T^1 + \sigma_2 W_T^2$, $\tilde{Y} := \frac{A_1}{p-1}\widetilde{W}_T^1 + \left(\frac{A_2}{p-1} - \sigma_2\right)\widetilde{W}_T^2$, $\tilde{Y} := \sigma_1 \widetilde{W}_T^1 + \sigma_2 \widetilde{W}_T^2$. Denote by k_1, k_2, k_3, k_4 constants satisfying $W_T^1 = k_1 X + k_2 Y$, $W_T^2 = k_3 X + k_4 Y$, $\widetilde{W}_T^1 = k_1 \widetilde{X} + k_2 \widetilde{Y}$, $\widetilde{W}_T^2 = k_3 \widetilde{X} + k_4 \widetilde{Y}$. Then we have

$$\begin{aligned}\Psi_1^p(c) &= \frac{1}{p}\int_d^\infty \int_{-\infty}^{v(y)} \left(S_0^1 e^{(\alpha_1 - \frac{1}{2}\sigma_1^2)T + \sigma_1(k_1 x + k_2 y)} - \frac{K}{S_0^1 e^{(\alpha_2 - \frac{1}{2}\sigma_2^2)T + \sigma_1(k_3 x + k_4 y)}} \right)^p \\ &\qquad\qquad \cdot f_{X|Y=y}(x) f_Y(y)\,dx\,dy \\ &\quad + \frac{1}{p} c^{\frac{p}{p-1}} e^{-\frac{pBT}{p-1}} \int_d^\infty \int_{v(y)}^\infty e^{-\frac{pA_1}{p-1}(k_1 x + k_2 y) - \frac{pA_2}{p-1}(k_3 x + k_4 y)} f_{X|Y=y}(x) f_Y(y)\,dx\,dy,\end{aligned}$$

and

$$\begin{aligned}\Psi_2^p(c) &= \int_{\tilde{d}}^\infty \int_{\tilde{v}(y)}^\infty \left(S_0^1 e^{(r - \frac{1}{2}\sigma_1^2)T + \sigma_1(k_1 x + k_2 y)} - \frac{K}{S_0^1 e^{(r - \frac{1}{2}\sigma_2^2)T + \sigma_1(k_3 x + k_4 y)}} \right) \\ &\qquad\qquad \cdot \tilde{f}_{\tilde{X}|\tilde{Y}=y}(x) \tilde{f}_{\tilde{Y}}(y)\,dx\,dy \\ &\quad - c^{\frac{1}{p-1}} e^{-\frac{\tilde{B}T}{p-1}} \int_{\tilde{d}}^\infty \int_{\tilde{v}(y)}^\infty e^{-\frac{A_1}{p-1}(k_1 x + k_2 y) - \frac{A_2}{p-1}(k_3 x + k_4 y)} \tilde{f}_{\tilde{X}|\tilde{Y}=y}(x) \tilde{f}_{\tilde{Y}}(y)\,dx\,dy.\end{aligned}$$

4.3. Outperformance option. The problem is studied for

$$H = (\max\{S_T^1, S_T^2\} - K)^+, \quad K > 0.$$

Linear loss function. By (4.5)–(4.7) we get

$$\begin{aligned}\Psi_1(c) &= \mathbf{E}[(S_T^1 - K)\mathbf{1}_{\{\tilde{Z}_T^{-1} \ge c\}} \mid S_T^1 \ge K,\ S_T^1 \ge S_T^2] P(S_T^1 \ge K, S_T^1 \ge S_T^2) \\ &\quad + \mathbf{E}[(S_T^2 - K)\mathbf{1}_{\{\tilde{Z}_T^{-1} \ge c\}} \mid S_T^2 \ge K,\ S_T^1 < S_T^2] P(S_T^2 \ge K, S_T^1 < S_T^2) \\ &= \mathbf{E}[(S_T^1 - K)\mathbf{1}_{\{\tilde{Z}_T^{-1} \ge c\}} \mid W_T^1 \ge a_1,\ \sigma_1 W_T^1 - \sigma_2 W_T^2 \ge b] \\ &\qquad\qquad \cdot P(W_T^1 \ge a_1, \sigma_1 W_T^1 - \sigma_2 W_T^2 \ge b) \\ &\quad + \mathbf{E}[(S_T^2 - K)\mathbf{1}_{\{\tilde{Z}_T^{-1} \ge c\}} \mid W_T^2 \ge a_2,\ \sigma_1 W_T^1 - \sigma_2 W_T^2 < b] \\ &\qquad\qquad \cdot P(W_T^2 \ge a_2,\ \sigma_1 W_T^1 - \sigma_2 W_T^2 < b),\end{aligned}$$

and further

$$\begin{aligned}\Psi_1(c) &= \int_{a_1}^\infty \int_b^\infty (S_0^1 e^{(\alpha_1 - \frac{1}{2}\sigma_1^2)T + \sigma_1 x} - K)\mathbf{1}_{\{A_1 x + A_2 \frac{\sigma_1 x - z}{\sigma_2} \ge \ln c - BT\}} \\ &\qquad\qquad \cdot f_{W_T^1, \sigma_1 W_T^1 - \sigma_2 W_T^2}(x, z)\,dz\,dx\end{aligned}$$

$$+\int_{a_2}^{\infty}\int_{-\infty}^{b}(S_0^2 e^{(\alpha_2-\frac{1}{2}\sigma_2^2)T+\sigma_2 y}-K)\mathbf{1}_{\{A_1\frac{z+\sigma_2 y}{\sigma_1}+A_2 y\geq \ln c-BT\}} \cdot f_{W_T^2,\sigma_1 W_T^1-\sigma_2 W_T^2}(y,z)\,dz\,dy.$$

Similarly,

$$\begin{aligned}\Psi_2(c) &= \tilde{\mathbf{E}}[(S_T^1-K)\mathbf{1}_{\{\tilde{Z}_T^{-1}\geq c\}}\,|\,S_T^1\geq K,\ S_T^1\geq S_T^2]\tilde{P}(S_T^1\geq K, S_T^1\geq S_T^2)\\ &\quad+\tilde{\mathbf{E}}[(S_T^2-K)\mathbf{1}_{\{\tilde{Z}_T^{-1}\geq c\}}\,|\,S_T^2\geq K,\ S_T^1< S_T^2]\tilde{P}(S_T^2\geq K, S_T^1< S_T^2)\\ &= \tilde{\mathbf{E}}[(S_T^1-K)\mathbf{1}_{\{\tilde{Z}_T^{-1}\geq c\}}\,|\,\widetilde{W}_T^1\geq \tilde{a}_1, \sigma_1\widetilde{W}_T^1-\sigma_2\widetilde{W}_T^2\geq \tilde{b}]\\ &\qquad\cdot \tilde{P}(\widetilde{W}_T^1\geq \tilde{a}_1, \sigma_1\widetilde{W}_T^1-\sigma_2\widetilde{W}_T^2\geq \tilde{b})\\ &\quad+\tilde{\mathbf{E}}[(S_T^2-K)\mathbf{1}_{\{\tilde{Z}_T^{-1}\geq c\}}\,|\,\widetilde{W}_T^2\geq \tilde{a}_2, \sigma_1\widetilde{W}_T^1-\sigma_2\widetilde{W}_T^2< \tilde{b}]\\ &\qquad\cdot \tilde{P}(\widetilde{W}_T^2\geq \tilde{a}_2, \sigma_1\widetilde{W}_T^1-\sigma_2\widetilde{W}_T^2< \tilde{b}),\end{aligned}$$

which leads to

$$\begin{aligned}\Psi_2(c) = &\int_{\tilde{a}_1}^{\infty}\int_{\tilde{b}}^{\infty}(S_0^1 e^{(r-\frac{1}{2}\sigma_1^2)T+\sigma_1 x}-K)\mathbf{1}_{\{A_1 x+A_2\frac{\sigma_1 x-z}{\sigma_2}\geq \ln c-\tilde{B}T\}}\\ &\qquad\cdot \tilde{f}_{\widetilde{W}_T^1,\sigma_1\widetilde{W}_T^1-\sigma_2\widetilde{W}_T^2}(x,z)\,dz\,dx\\ &+\int_{\tilde{a}_2}^{\infty}\int_{-\infty}^{\tilde{b}}(S_0^2 e^{(r-\frac{1}{2}\sigma_2^2)T+\sigma_2 y}-K)\mathbf{1}_{\{A_1\frac{z+\sigma_2 y}{\sigma_1}+A_2 y\geq \ln c-\tilde{B}T\}}\\ &\qquad\cdot \tilde{f}_{\widetilde{W}_T^2,\sigma_1\widetilde{W}_T^1-\sigma_2\widetilde{W}_T^2}(y,z)\,dz\,dy.\end{aligned}$$

Power loss function. Taking into account (4.5)–(4.7) we can write

$$\begin{aligned}A_c &= \{c\tilde{Z}_T\leq (S_T^1\vee S_T^2-K)^{p-1},\ S_T^1\vee S_T^2-K>0\}\\ &= \{c\tilde{Z}_T\leq (S_T^1-K)^{p-1},\ S_T^1>K,\ S_T^1\geq S_T^2\}\\ &\quad\cup\{c\tilde{Z}_T\leq (S_T^2-K)^{p-1},\ S_T^2>K,\ S_T^1\leq S_T^2\}.\end{aligned}$$

We consider the case when $A_1>0, A_2>0$:

$$\begin{aligned}A_c = &\Big\{W_T^2\geq -\Big(A_1W_T^1+BT+\ln\Big(\frac{1}{c}(S_T^1-K)^{p-1}\Big)\Big),\ W_T^1>a_1,\\ &\qquad \sigma_1W_T^1-\sigma_2W_T^2\geq b\Big\}\\ &\cup\Big\{W_T^1\geq -\Big(A_2W_T^2+BT+\ln\Big(\frac{1}{c}(S_T^2-K)^{p-1}\Big)\Big),\\ &\qquad W_T^2>a_2,\ \sigma_1W_T^1-\sigma_2W_T^2\leq b\Big\}\end{aligned}\tag{4.17}$$

$$= \left\{W_T^2 \geq v_1(W_T^1),\, W_T^1 > a_1,\, W_T^2 \leq \frac{\sigma_1 W_T^1 - b}{\sigma_2}\right\}$$
$$\cup \left\{W_T^1 \geq v_2(W_T^2),\, W_T^2 > a_2,\, W_T^1 \leq \frac{\sigma_2 W_T^2 - b}{\sigma_1}\right\}$$
$$= \left\{\widetilde{W}_T^2 \geq \tilde{v}_1(\widetilde{W}_T^1),\, \widetilde{W}_T^1 > \tilde{a}_1,\, \widetilde{W}_T^2 \leq \frac{\sigma_1 \widetilde{W}_T^1 - \tilde{b}}{\sigma_2}\right\}$$
$$\cup \left\{\widetilde{W}_T^1 \geq \tilde{v}_2(\widetilde{W}_T^2),\, \widetilde{W}_T^2 > \tilde{a}_2,\, \widetilde{W}_T^1 \leq \frac{\sigma_2 \widetilde{W}_T^2 - \tilde{b}}{\sigma_1}\right\},$$

where

$$v_1(x) = -\frac{1}{A_2}\left(A_1 x + BT + \ln\left(\frac{1}{c}(S_0^1 e^{(\alpha_1 - \frac{1}{2}\sigma_1^2)T + \sigma_1 x} - K)^p\right)\right),$$
$$v_2(x) = -\frac{1}{A_1}\left(A_2 x + BT + \ln\left(\frac{1}{c}(S_0^2 e^{(\alpha_2 - \frac{1}{2}\sigma_2^2)T + \sigma_2 x} - K)^p\right)\right),$$
$$\tilde{v}_1(x) = -\frac{1}{A_2}\left(A_1 x + \tilde{B}T + \ln\left(\frac{1}{c}(S_0^1 e^{(r - \frac{1}{2}\sigma_1^2)T + \sigma_1 x} - K)^p\right)\right),$$
$$\tilde{v}_2(x) = -\frac{1}{A_1}\left(A_2 x + \tilde{B}T + \ln\left(\frac{1}{c}(S_0^2 e^{(r - \frac{1}{2}\sigma_2^2)T + \sigma_2 x} - K)^p\right)\right).$$

Using the representation (4.17) and adopting the convention that the integral over the empty set is zero, we obtain

$$\Psi_1^p(c) = \int_{a_1}^{\infty} \int_{-\infty}^{v_1(x) \wedge \frac{\sigma_1 x - b}{\sigma_2}} (S_0^1 e^{(\alpha_1 - \frac{1}{2}\sigma_1^2)T + \sigma_1 x} - K)^p f_{W_T^2 | W_T^1 = x}(y)\, dy\, f_{W_T^1}(x)\, dx$$
$$+ \frac{1}{p} c^{\frac{p}{p-1}} e^{-\frac{pB}{p-1}} \int_{a_1}^{\infty} \int_{v_1(x)}^{\frac{\sigma_1 x - b}{\sigma_2}} e^{-\frac{pA_1}{p-1}x - \frac{pA_2}{p-1}y} f_{W_T^2 | W_T^1 = x}(y)\, dy\, f_{W_T^1}(x)\, dx$$
$$+ \int_{a_2}^{\infty} \int_{-\infty}^{v_2(x) \wedge \frac{\sigma_2 x - b}{\sigma_1}} (S_0^2 e^{(\alpha_2 - \frac{1}{2}\sigma_2^2)T + \sigma_2 x} - K)^p f_{W_T^1 | W_T^2 = x}(y)\, dy\, f_{W_T^2}(x)\, dx$$
$$+ \frac{1}{p} c^{\frac{p}{p-1}} e^{-\frac{pB}{p-1}} \int_{a_2}^{\infty} \int_{v_2(x)}^{\frac{\sigma_2 x - b}{\sigma_1}} e^{-\frac{pA_1}{p-1}x - \frac{pA_2}{p-1}y} f_{W_T^1 | W_T^2 = x}(y)\, dy\, f_{W_T^2}(x)\, dx,$$

$$\Psi_2^p(c) = \int_{\tilde{a}_1}^{\infty} \int_{\tilde{v}_1(x)}^{\frac{\sigma_1 x - \tilde{b}}{\sigma_2}} (S_0^1 e^{(r - \frac{1}{2}\sigma_1^2)T + \sigma_1 x} - K) \tilde{f}_{\widetilde{W}_T^2 | \widetilde{W}_T^1 = x}(y)\, dy\, \tilde{f}_{\widetilde{W}_T^1}(x)\, dx$$

$$- c^{\frac{1}{p-1}} e^{-\frac{\tilde{B}}{p-1}T} \int\limits_{\tilde{a}_1}^{\infty} \int\limits_{\tilde{v}_1(x)}^{\frac{\sigma_1 x - \tilde{b}}{\sigma_2}} e^{-\frac{A_1}{p-1}x - \frac{A_2}{p-1}y} \tilde{f}_{\widetilde{W}_T^2 | \widetilde{W}_T^1 = x}(y)\, dy\, \tilde{f}_{\widetilde{W}_T^1}(x)\, dx$$

$$+ \int\limits_{\tilde{a}_2}^{\infty} \int\limits_{\tilde{v}_2(x)}^{\frac{\sigma_2 x - \tilde{b}}{\sigma_1}} (S_0^2 e^{(r - \frac{1}{2}\sigma_2^2)T + \sigma_2 x} - K) \tilde{f}_{\widetilde{W}_T^1 | \widetilde{W}_T^2 = x}(y)\, dy\, \tilde{f}_{\widetilde{W}_T^2}(x)\, dx$$

$$- c^{\frac{1}{p-1}} e^{-\frac{\tilde{B}}{p-1}T} \int\limits_{\tilde{a}_2}^{\infty} \int\limits_{\tilde{v}_2(x)}^{\frac{\sigma_1 x - \tilde{b}}{\sigma_1}} e^{-\frac{A_1}{p-1}x - \frac{A_2}{p-1}y} \tilde{f}_{\widetilde{W}_T^1 | \widetilde{W}_T^2 = x}(y)\, dy\, \tilde{f}_{\widetilde{W}_T^2}(x)\, dx.$$

4.4. Spread option. The payoff is of the form

$$H = (S_T^1 - S_T^2 - K)^+, \quad K > 0.$$

One can check the following:

$$\{S_T^1 \geq S_T^2 + K\} = \{W_T^1 \geq d(W_T^2)\} = \{\widetilde{W}_T^1 \geq \tilde{d}(\widetilde{W}_T^2)\},$$

where

$$d(y) := \frac{1}{\sigma_1} \ln \frac{S_0^2 e^{(\alpha_2 - \frac{1}{2}\sigma_2^2)T + \sigma_2 y} + K}{S_0^1 e^{(\alpha_1 - \frac{1}{2}\sigma_1^2)T}}, \quad \tilde{d}(y) := \frac{1}{\sigma_1} \ln \frac{S_0^2 e^{(r - \frac{1}{2}\sigma_2^2)T + \sigma_2 y} + K}{S_0^1 e^{(r - \frac{1}{2}\sigma_1^2)T}}.$$

Linear loss function. We have

$$\begin{aligned} \Psi_1(c) &= \mathbf{E}[(S_T^1 - S_T^2 - K)^+ \mathbf{1}_{\{\tilde{Z}_T^{-1} \geq c\}}] \\ &= \int\limits_{-\infty}^{\infty} \mathbf{E}\big[(S_T^1 - S_T^2 - K)^+ \mathbf{1}_{\{\tilde{Z}_T^{-1} \geq c\}} \,|\, W_T^2 = y\big] f_{W_T^2}(y)\, dy \\ &= \int\limits_{-\infty}^{\infty} \int\limits_{d(y)}^{\infty} (S_0^1 e^{(\alpha_1 - \frac{1}{2}\sigma_1^2)T + \sigma_1 x} - S_0^2 e^{(\alpha_2 - \frac{1}{2}\sigma_2^2)T + \sigma_2 y} - K) \\ &\qquad \cdot \mathbf{1}_{\{A_1 x + A_2 y \geq \ln c - BT\}} f_{W_T^1 | W_T^2 = y}(x)\, dx\, f_{W_T^2}(y)\, dy \end{aligned}$$

and

$$\begin{aligned} \Psi_2(c) &= \tilde{\mathbf{E}}[(S_T^1 - S_T^2 - K)^+ \mathbf{1}_{\{\tilde{Z}_T^{-1} \geq c\}}] \\ &= \int\limits_{-\infty}^{\infty} \tilde{\mathbf{E}}[(S_T^1 - S_T^2 - K)^+ \mathbf{1}_{\{\tilde{Z}_T^{-1} \geq c\}} \,|\, \widetilde{W}_T^2 = y] \tilde{f}_{\widetilde{W}_T^2}(y)\, dy \\ &= \int\limits_{-\infty}^{\infty} \int\limits_{\tilde{d}(y)}^{\infty} (S_0^1 e^{(r - \frac{1}{2}\sigma_1^2)T + \sigma_1 x} - S_0^2 e^{(r - \frac{1}{2}\sigma_2^2)T + \sigma_2 y} - K) \\ &\qquad \cdot \mathbf{1}_{\{A_1 x + A_2 y \geq \ln c - \tilde{B}T\}} \tilde{f}_{\widetilde{W}_T^1 | \widetilde{W}_T^2 = y}(x)\, dx\, f_{W_T^2}(y)\, dy. \end{aligned}$$

Power loss function. We have

$$
\begin{aligned}
(4.18)\qquad A_c &:= \{c\tilde{Z}_T \le (S_T^1 - S_T^2 - K)^{p-1}, S_T^1 - S_T^2 - K > 0\} \\
&= \{c^{\frac{1}{p-1}} e^{-\frac{A_1}{p-1}W_T^1 - \frac{A_2}{p-1}W_T^2 - \frac{B}{p-1}T} \le S_0^1 e^{(\alpha_1 - \frac{1}{2}\sigma_1^2)T + \sigma_1 W_T^1} \\
&\qquad - S_0^2 e^{(\alpha_2 - \frac{1}{2}\sigma_2^2)T + \sigma_2 W_T^2} - K,\, W_T^1 \ge d(W_T^2)\} \\
&= \{W_T^1 \in \mathcal{A}(W_T^2)\} = \{\widetilde{W}_T^1 \in \tilde{\mathcal{A}}(\widetilde{W}_T^2)\},
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{A}(y) &:= \{x : c^{\frac{1}{p-1}} e^{-\frac{A_1}{p-1}x - \frac{A_2}{p-1}y - \frac{B}{p-1}T} \\
&\qquad \le S_0^1 e^{(\alpha_1 - \frac{1}{2}\sigma_1^2)T + \sigma_1 x} - S_0^2 e^{(\alpha_2 - \frac{1}{2}\sigma_2^2)T + \sigma_2 y} - K,\, x \ge d(y)\}, \\
\tilde{\mathcal{A}}(y) &:= \{x : c^{\frac{1}{p-1}} e^{-\frac{A_1}{p-1}x - \frac{A_2}{p-1}y - \frac{\tilde{B}}{p-1}T} \\
&\qquad \le S_0^1 e^{(r - \frac{1}{2}\sigma_1^2)T + \sigma_1 x} - S_0^2 e^{(r - \frac{1}{2}\sigma_2^2)T + \sigma_2 y} - K,\, x \ge \tilde{d}(y)\}.
\end{aligned}
$$

Let us notice that the set $A_c^{\complement} \cap \{H > 0\}$ is of the form

$$
(4.19)\qquad A_c^{\complement} \cap \{H > 0\} = \{W_T^1 \in \mathcal{B}(W_T^2)\},
$$

where

$$
\begin{aligned}
\mathcal{B}(y) &:= \{x : c^{\frac{1}{p-1}} e^{-\frac{A_1}{p-1}x - \frac{A_2}{p-1}y - \frac{B}{p-1}T} \\
&\qquad > S_0^1 e^{(\alpha_1 - \frac{1}{2}\sigma_1^2)T + \sigma_1 x} - S_0^2 e^{(\alpha_2 - \frac{1}{2}\sigma_2^2)T + \sigma_2 y} - K,\, x \ge d(y)\}.
\end{aligned}
$$

Taking into account (4.18) and (4.19) we obtain

$$
\begin{aligned}
\Psi_1^p(c) &= \frac{1}{p} \int_{-\infty}^{\infty} \int_{\mathcal{B}(y)} (S_0^1 e^{(\alpha_1 - \frac{1}{2}\sigma_1^2)T + \sigma_1 x} - S_0^2 e^{(\alpha_2 - \frac{1}{2}\sigma_2^2)T + \sigma_2 y} - K)^p \\
&\qquad \cdot f_{W_T^1 | W_T^2 = y}(x)\, dx\, f_{W_T^2}(y)\, dy \\
&\quad + \frac{1}{p} c^{\frac{p}{p-1}} e^{-\frac{pBT}{p-1}} \int_{-\infty}^{\infty} \int_{\mathcal{A}(y)} \big(e^{-\frac{pA_1}{p-1}x - \frac{pA_2}{p-1}y}\big) f_{W_T^1 | W_T^2 = y}(x)\, dx\, f_{W_T^2}(y)\, dy, \\
\Psi_2^p(c) &= \int_{-\infty}^{\infty} \int_{\tilde{\mathcal{A}}(y)} (S_0^1 e^{(r - \frac{1}{2}\sigma_1^2)T + \sigma_1 x} - S_0^2 e^{(r - \frac{1}{2}\sigma_2^2)T + \sigma_2 y} - K) \\
&\qquad \cdot \tilde{f}_{\widetilde{W}_T^1 | \widetilde{W}_T^2 = y}(x)\, dx\, \tilde{f}_{\widetilde{W}_T^2}(y)\, dy \\
&\quad + c^{\frac{1}{p-1}} e^{-\frac{\tilde{B}T}{p-1}} \int_{-\infty}^{\infty} \int_{\tilde{\mathcal{A}}(y)} \big(e^{-\frac{A_1}{p-1}x - \frac{A_2}{p-1}y}\big) \tilde{f}_{\widetilde{W}_T^1 | \widetilde{W}_T^2 = y}(x)\, dx\, \tilde{f}_{\widetilde{W}_T^2}(y)\, dy.
\end{aligned}
$$

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References

[1] M. Barski, *Quantile hedging for basket derivatives*, Appl. Math. (Warsaw) 39 (2012), 103–127.
[2] J. Cvitanić, *Minimizing expected loss of hedging in incomplete and constrained markets*, SIAM J. Control Optim. 38 (2000), 1050–1066.
[3] J. Cvitanić and I. Karatzas, *On dynamic measures of risk*, Finance Stoch. 3 (1999), 451–482.
[4] H. Föllmer and P. Leukert, *Quantile hedging*, Finance Stoch. 3 (1999), 251–273.
[5] H. Föllmer and P. Leukert, *Efficient hedging: cost versus shortfall risk*, Finance Stoch. 4 (2000), 117–146.
[6] P. Glasserman, *Monte Carlo Methods in Financial Engineering*, Springer, 2003.
[7] I. Karatzas, *Lectures on the Mathematics of Finance*, CRM Monogr. Ser. 8, Amer. Math. Soc., 1997.
[8] S. Karlin, *Mathematical Methods and Theory in Games, Programming and Economics*, Addison-Wesley, 1959.
[9] H. Pham, *Minimizing shortfall risk and applications to finance and insurance problems*, Ann. Appl. Probab. 12 (2002), 143–172.

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