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EQUILIBRIA IN CONSTRAINED CONCAVE BIMATRIX GAMES

Abstract. We study a generalization of bimatrix games in which not all pairs of players' pure strategies are admissible. It is shown that under some additional convexity assumptions such games have equilibria of a very simple structure, consisting of two probability distributions with at most two-element supports. Next this result is used to get a theorem about the existence of Nash equilibria in bimatrix games with a possibility of payoffs equal to $-\infty$. The first of these results is a discrete counterpart of the Debreu Theorem about the existence of pure noncooperative equilibria in n-person constrained infinite games. The second one completes the classical theorem on the existence of Nash equilibria in bimatrix games. A wide discussion of the results is given.

1. Introduction. *Games with constraints* were introduced by Debreu [3]. His results were used in a model of a competitive economy by Arrow and Debreu [1]. Their main result generalizes Glicksberg's theorem [4] about the existence of pure Nash equilibria in n-person games. Debreu considered the case of n-person games in which the players are restricted in choosing their strategies in such a way that the "effective" set of pure strategies of each player depends on the decisions taken by the others. Thus, in such a game, some multistrategies (vectors describing the choices of all players) are not admissible, and the equilibria should be looked for only among admissible ones. Games with constraints were often applied in economic models, in particular the ones with a finite number of agents (see e.g. Ichiischi [5], Shafer and Sonnenschein [13] and Wieczorek [14]).

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In another approach, instead of a game with a set of "forbidden" multistrategies, a generalized model of classical noncooperative game is considered with some multistrategies determining players' payoffs to be equal to $-\infty$. One of our results shows a very close relationship between these two approaches to constrained bimatrix games.

The Debreu Theorem (also referred to as the Debreu–Nash Theorem, see Theorem 12.3 in [2]) will serve as the starting point for our considerations. We will formulate it in one of the equivalent forms suited to the way we present our results. First we need some terminology.

A (*noncooperative*) *n-person game with constraints* is a quadruple

$$\mathbb{G} = \langle N, \{X_i\}_{i \in N}, \{S_i\}_{i \in N}, \{F_i\}_{i \in N} \rangle, \tag{1}$$

where $N = \{1, \ldots, n\}$ is the set of players, and for each $i \in N$,

(a) X_i is the space of pure strategies of player i;
(b) S_i is a multifunction from the set $X_{-i} = \prod_{k \neq i} X_k$ to 2^{X_i} (the set of all subsets of X_i). $S_i(x_{-i})$ is the set of pure strategies admissible for player i, when the remaining players act according to x_{-i};
(c) $F_i : \prod_{k=1}^{n} X_k \to R$ is the payoff function of player i.

Such a game with constraints will also be called a *constrained game.*

A *multistrategy* $x^* = (x_1^*, \ldots, x_n^*) \in \prod_{i=1}^{n} X_i$ is a (*pure*) *equilibrium* in an n-person constrained game $\mathbb{G}$ of the form (1) if for all $i \in N$, $x_i^* \in S_i(x_{-i}^*)$, and

$$F_i(x^*) = \max_{y_i \in S(x_{-i}^*)} F_i(x_{-i}^*, y_i). \tag{2}$$

Now the Debreu Theorem can be written in the following equivalent form.

THEOREM A. *Let $\mathbb{G}$ be an n-person game with constraints, of the form* (1), *with $n \geq 2$. Assume that*

(a) *each pure strategy space X_i is a convex and compact subset of a Euclidean space $\mathbb{R}^{k_i}$,*
(b) *each payoff function F_i is continuous on $\prod_{i=1}^{n} X_i$ and concave with respect to the ith variable, and*
(c) *all the multifunctions S_i are continuous and have nonempty, convex values.*

Then the game $\mathbb{G}$ has a pure equilibrium.

The main purpose of the paper is to find a discrete counterpart of the Debreu Theorem for two-person games with constraints of the form (1), with finite sets X_1 and X_2 of players' pure strategies. We call such games *constrained bimatrix games.* To get discrete counterparts for the assumptions of Theorem A we need to extend the classical notions of convexity and concavity to finite sets and to functions with finite domains.

Our results are presented as Theorems 1 and 2 in Section 3, together with a wide discussion. The first result says that under some concavity assumptions, any constrained bimatrix game has an equilibrium of a very simple structure, in the form of two probability distributions with supports consisting of at most two "neighboring" pure strategies for each player. Theorem 1 can be seen as a discrete counterpart of the Debreu Theorem about the existence of pure noncooperative equilibria in n-person constrained infinite games.

The second result (Theorem 2) shows that, in fact, constrained bimatrix games are equivalent to classical bimatrix games they generate, where the players' payoffs for all nonadmissible multistrategies are defined to be $-\infty$ (the player loses "everything"). This shows that there exists a close relationship between equilibria in such constrained games and Nash equilibria in the bimatrix games generated by them. This result can be seen as completing the classical theorem on the existence of Nash equilibria in bimatrix games.

We also show (in Example 5 in Section 4) that a generalization of our results to n-person finite games is not possible even if the assumptions are strengthened. This suggests that a fully discrete counterpart of the Debreu Theorem in case $n > 2$ is an open problem. Section 5 is devoted to the proofs.

2. Preliminary definitions and results. We will mainly concentrate on constrained games of the form (1) where the pure strategy spaces X_i are finite. Such games will be called *finite constrained games*, and players' pure strategy spaces will be of the form

$$X_i = \{1, \ldots, k_i\}, \qquad 1 \le i \le n. \tag{3}$$

In general, there are no pure equilibria in such games, so we must extend our considerations to mixed equilibria.

A *mixed strategy* of player i in the game $\mathbb{G}$ is a probability distribution μ_i on the set X_i. Set $\mu = (\mu_1, \ldots, \mu_n)$ and $\mu_{-i} = (\mu_1, \ldots, \mu_{i-1}, \mu_{i+1}, \ldots, \mu_n)$. For an arbitrary vector μ of players' mixed strategies, we put

$$F_i(\mu) = \int_{x \in X_i} F_i(x)\, d\mu_1 \ldots d\mu_n, \qquad S_i(\mu_{-i}) = \bigcap_{x_{-i} \in \operatorname{supp}(\mu_{-i})} S_i(x_{-i}).$$

So $S_i(\mu_{-i})$ describes the set of all pure strategies of player i that are admissible for any pure multistrategy $x_{-i} \in \operatorname{supp}(\mu_{-i})$ of the remaining players. Hence, we have the following two natural definitions.

Definition 1. A (mixed) multistrategy $\mu = (\mu_1, \ldots, \mu_n)$ is *admissible* in a constrained game $\mathbb{G}$ if $\operatorname{supp} \mu_i \subset S_i(\mu_{-i})$ for all $i \in N$.

Definition 2. A (mixed) multistrategy $\mu^* = (\mu_1^*, \ldots, \mu_n^*)$ is called a (mixed) *equilibrium* in a constrained game $\mathbb{G}$ if it is admissible and $F_i(\mu^*) = \max_{y_i \in S(\mu^*_{-i})} F_i(\mu^*_{-i}, y_i)$ for each player $i \in N$.

Remark 1. Notice that if we take S_i in (1) satisfying $S_i(x_{-i}) = X_i$ for all x_{-i}, then the constrained game $\mathbb{G}$ becomes a (classical) noncooperative n-person game whose normal form can be simply written as

$$\Gamma = \langle N, \{X_i\}_{i\in N}, \{F_i\}_{i\in N}\rangle. \tag{4}$$

It is trivially seen that the set of equilibria in such a game coincides with the set of its Nash equilibria. It is also worth mentioning that according to the classical Glicksberg Theorem ([4]), if all the sets X_i are convex and compact in a Euclidean space $\mathbb{R}^k$, and each payoff function F_i is upper semicontinuous in the ith variable, then the game Γ has a Nash equilibrium in pure strategies.

Now we give three basic definitions of properties of finite games and strategies. Let Γ denote an n-person finite game described by (3) and (4).

Definition 3. A payoff function F_i of player i in the game Γ, $i \in N$, is called *concave* with respect to the ith variable if for $j = 1, \ldots, n$ there exist strictly increasing sequences $y^j = (y_1^j, y_2^j, \ldots, y_{k_j}^j)$ in $[0, 1]$, and a function $f_i(y_1, \ldots, y_n)$ from $[0, 1]^n$ to $\mathbb{R}$, concave with respect to y_i and such that $f_i(y_{x_1}^1, \ldots, y_{x_n}^n) = F_i(x_1, \ldots, x_n)$ for all $(x_1, \ldots, x_n) \in \prod_{i=1}^n X_i$.

Remark 2. Concavity of functions with a finite domain is a basic assumption in this paper, so it will be wider analyzed later. One thing worth mentioning right now is that one could think that the above definition would be more natural if $[0, 1]^n$ was replaced by $\mathrm{conv}(\prod_{i=1}^n X_i)$ and the sequences y_i^j were taken from the strategy space X_i. It appears, however, that such a modification is less general than Definition 3 and leads to a much smaller class of functions F_i concave with respect to x_i.

Definition 4. The game Γ is called *concave* if for each $i \in N$, the payoff function F_i is concave with respect to x_i.

Definition 5. A mixed strategy μ of player i in the game Γ is called *two-adjoining-pure* if it is of the form $\mu_i = \alpha\delta_a + (1 - \alpha)\delta_{a+1}$ for some $0 \le \alpha \le 1$ and $a \in X_i$, $1 \le i \le n$. (Here δ_t denotes the degenerate probability distribution concentrated at t.)

To end this section we will quote some basic results from [8], essential for our paper. Some new notation concerning only two-person finite games will be needed.

Let $A = [a_{ij}]_{p\times q}$ and $B = [b_{ij}]_{p\times q}$ $(p, q \ge 1)$ be payoff matrices of players 1 and 2 in a nonzero-sum two-person game Γ of the form (4) with

$$X_1 = \{1, \ldots, p\}, \qquad X_2 = \{1, \ldots, q\} \tag{5}$$

and

$$a_{ij} = F_1(i, j), \qquad b_{ij} = F_2(i, j) \quad \text{for } 1 \le i \le p,\ 1 \le j \le q. \tag{6}$$

Let $\Gamma(A, B)$ denote the bimatrix game described by (5) and (6). We will also use the notation $\Gamma(A, B)_{p\times q}$ for the $(p\times q)$-game $\Gamma(A, B)$, to emphasize the size of the payoff matrices in it.

For a given bimatrix game $\Gamma(A, B)$, it is rather difficult to check directly from Definition 3 whether the game is concave or not. It appears, however, that this can be easily verified with the help of the following proposition ([8, Theorem 4]).

Proposition 1. *A bimatrix game $\Gamma(A, B)_{p\times q}$ is concave if and only if there exist positive numbers $\theta_1, \ldots, \theta_{p-1}$ and $\tau_1, \ldots, \tau_{q-1}$ such that*

$$\theta_1(a_{2j} - a_{1j}) \ge \theta_2(a_{3j} - a_{2j}) \ge \cdots \ge \theta_{p-1}(a_{pj} - a_{p-1,j}) \quad \textit{for } j = 1, \ldots, q, \tag{7}$$

and

$$\tau_1(b_{i2} - b_{i1}) \ge \tau_2(b_{i3} - b_{i2}) \ge \cdots \ge \tau_{q-1}(b_{iq} - b_{i,q-1}) \quad \textit{for } i = 1, \ldots, p. \tag{8}$$

The second result we need is [8, Theorem 6]. We recall it in the form sufficient for our considerations.

Theorem B. *Every concave bimatrix game $\Gamma(A, B)$ has a Nash equilibrium (μ^*, ν^*) in two-adjoining-pure strategies.*

Remark 3. Theorem 6 from [8] gives the exact formulae for the two-adjoining-pure strategies μ^* and ν^* in Theorem B (described by conditions on elements of the payoff matrices A and B).

Remark 4. Theorem B can be seen as a discrete counterpart of the Glicksberg Theorem for two-person games. It will be used in the proof of our first main result (Theorem 1) in Section 4. The zero-sum version of Theorem B (with the exact formulae for μ^* and ν^*) was earlier proved in [11].

Remark 5. Matrix and bimatrix games with other properties weaker than concavity were also studied in the literature. Namely, in [10] and [7], one can find necessary and sufficient conditions for such games to have saddle points and pure Nash equilibria, respectively.

3. Two main results. In this section we formulate our two main results (Theorems 1 and 2) about the existence of equilibria in constrained bimatrix games. Their proofs will be given in Section 5.

Let $p, q \geq 1$ be arbitrary natural numbers. Similarly to the case of unconstrained bimatrix games, it is more convenient to replace the form (1) by one making use of payoff matrices. Namely, a *constrained bimatrix game* (of size $p \times q$) will be denoted by

$$\mathbb{G} = \mathbb{G}(A, B, E_A, E_B), \tag{9}$$

where $A = [a_{ij}]_{p\times q}$ and $B = [b_{ij}]_{p\times q}$ are two $(p \times q)$-matrices for players 1 and 2 respectively, and E_A and E_B are nonempty subsets of the set $\{(i,j) : i = 1,\dots,p,\ j = 1,\dots,q\}$ of pairs of players' pure strategies. Such a game is interpreted as a two-person game with constraints of the form (1) with $N = \{1,2\}$, $X_1 = \{1,\dots,p\}$, $X_2 = \{1,\dots,q\}$, $F_1(i,j) = a_{ij}$ and $F_2(i,j) = b_{ij}$ for $1 \leq i \leq p$, $1 \leq j \leq q$, and with multifunctions S_1 and S_2 defined by the equivalences: $i \in S_1(j) \Leftrightarrow (i,j) \in E_A$ and $j \in S_2(i) \Leftrightarrow (i,j) \in E_B$. So in the description of the constrained bimatrix game $\mathbb{G}$, A and B are the payoff matrices for players 1 and 2, respectively, and E_A and E_B describe admissible pairs of their pure strategies.

To formulate our main result about the existence of a mixed equilibrium in the constrained bimatrix game $\mathbb{G}$ of the form (9), we need to assume the following three conditions:

Z1 (*symmetry*) $E_A = E_B \neq \emptyset$.

Z2 (*biconvexity*)

(a) If $(i,j) \in E_A$ and $(l,j) \in E_A$ for some $i < l$ and j, then $(k,j) \in E_A$ for all k with $i \leq k \leq l$.

(b) If $(i,j) \in E_B$ and $(i,l) \in E_B$ for some i and $j < l$, then $(i,k) \in E_B$ for all k with $j \leq k \leq l$.

Z3 (*game concavity*) The bimatrix game $\Gamma(A,B)$ is concave.

Remark 6. Assumption Z1 does not have a counterpart in the Debreu Theorem. We will justify it in Example 1 in the next section. Assumption Z2 is a discrete counterpart of the assumption about the convexity of the images of the multifunctions S_i in the Debreu Theorem, and assumption Z3 (see also Proposition 1) is a discrete counterpart of the assumption of concavity of the payoff functions considered there.

Now we are ready to formulate our first main result.

Theorem 1. *If a constrained bimatrix game* $\mathbb{G}(A, B, E_A, E_B)$ *satisfies assumptions* Z1–Z3, *then it has an equilibrium consisting of two two-adjoining-pure strategies.*

Remark 7. One could ask if constrained bimatrix games can well represent noncooperative games, that is, the following question seems to be essential: can the players be "independent" if some of their joint multistrategies are excluded? A positive answer to this question is given in Theorem 2 below, our second main result. Namely, it appears that in fact, constrained bimatrix games are equivalent to classical bimatrix games with possibly $-\infty$ payoffs, constructed in such a way that the payoffs for all nonadmissible multistrategies are defined to be $-\infty$. Theorem 2 says that under an additional (not very restrictive) assumption, the set of equilibria in a constrained bimatrix game coincides with the set of Nash equilibria in the generated bimatrix game.

To formulate Theorem 2, we need to consider an additional (not very restrictive) assumption (Z4).

Let $\mathbb{G} = \mathbb{G}(A, B, E_A, E_B)$ be a constrained bimatrix game of size $p \times q$. A new generalized bimatrix game can be defined as $\Gamma_{\mathbb{G}} = \Gamma(A_{\mathbb{G}}, B_{\mathbb{G}})_{p\times q}$, with payoff matrices $A_{\mathbb{G}} = [a_{ij}]$ and $B_{\mathbb{G}} = [b_{ij}]$ given by

$$a_{ij} = \begin{cases} a_{ij} & \text{for } (i,j) \in E_A, \\ -\infty & \text{for } (i,j) \notin E_A, \end{cases} \qquad b_{ij} = \begin{cases} b_{ij} & \text{for } (i,j) \in E_B, \\ -\infty & \text{for } (i,j) \notin E_B. \end{cases}$$

(for the game $\Gamma_{\mathbb{G}}$ we use the convention $0 \cdot (-\infty) = 0$). Obviously, this procedure can be reversed, and every bimatrix game with possibly $-\infty$ payoffs generates some constrained bimatrix game.

Z4 (*full section*) There exists $j \in \{1, \ldots, q\}$ such that $(i,j) \in E_B$ for each $i \in \{1, \ldots, p\}$, or there exists $k \in \{1, \ldots, p\}$ such that $(k,l) \in E_A$ for each $l \in \{1, \ldots, q\}$.

THEOREM 2. *Assume that a constrained game* $\mathbb{G} = \mathbb{G}(A, B, E_A, E_B)$ *satisfies assumptions* Z1–Z3. *Then the generalized bimatrix game* $\Gamma_{\mathbb{G}}$ *has a Nash equilibrium in two-adjoining-pure strategies with finite payoffs for both players. Moreover, each equilibrium in* $\mathbb{G}$ *is a Nash equilibrium in* $\Gamma_{\mathbb{G}}$.

If additionally assumption Z4 *holds, then the set of all equilibria in* $\mathbb{G}$ *coincides with the set of all Nash equilibria in* $\Gamma_{\mathbb{G}}$.

REMARK 8. By Definition 2 one can easily see that any equilibrium in the constrained game $\mathbb{G}$ generates finite payoffs for both players (matrices A and B in (9) have, by assumption, finite values). On the other hand, if assumption Z4 is not satisfied, then any pair of completely mixed strategies is a Nash equilibrium in the generalized bimatrix game $\Gamma_{\mathbb{G}}$ with payoffs $(-\infty, -\infty)$. Therefore when assumption Z4 does not hold, the set of equilibria in $\mathbb{G}$ is smaller than the set of Nash equilibria in $\Gamma_{\mathbb{G}}$. It is worth mentioning that bimatrix games with possibly $-\infty$ payoffs may have no Nash equilibria at all, neither with finite nor infinite payoffs (Example 4 in the next section).

REMARK 9. It is easily seen that for $n > 2$ any n-person finite constrained game (described by (1) and (3)) can be equivalently written in the form

$$\mathbb{G}_n = \mathbb{G}(A_1, \ldots, A_n, E_{A_1}, \ldots, E_{A_n}),$$

a natural extension of (9), where $A_1, \dots, A_n$ are payoff "multimatrices" of size $k_1 \times \dots \times k_n$ each, and $E_{A_1}, \dots, E_{A_n}$ are subsets of the set of pure multistrategies. Obviously, for such a game, the definition of three conditions Z1–Z3 can be extended in an obvious way to n-person finite constrained games. By Definition 2.2, condition Z3 allows us to view each finite action set X_i as a subset of $[0,1]$, and conversely, every pure strategy in $[0,1]$ is a convex combination of two-adjoining-pure strategies in X_i. Therefore, one could think that under Z1–Z3 for n-person finite constrained game, the game $\mathbb{G}_n$ generates (after applying some multilinear interpolation arguments) a "continuous" n-person game with constraints of the form (1) with all $X_i = [0,1]$, satisfying the assumptions of the Debreu Theorem. Hence, the question arises if Theorem 1 (and even its generalization with $n \geq 2$) are implied by the Debreu Theorem. Unfortunately, the answer is negative. A counterexample is given in Example 5 where a certain 3-person finite constrained game is discussed which satisfies assumptions Z1–Z3 for a 3-player game, but has no equilibria. Thus the generalization of Theorem 1 to a greater number of players is not true. These facts strongly suggest that Theorem 1 cannot be derived from the Debreu Theorem, and its direct generalization to all $n > 2$ is not true.

To give the last remark we need to introduce the following notation. Let $p, q \geq 2$ and let $\Gamma(A,B)$ be a $(p \times q)$-bimatrix game. The game $\Gamma(A_1, B_1)$ is said to be a *subgame* of $\Gamma(A,B)$ if the matrices A_1 and B_1 can be obtained by removing some rows and/or columns from A and B (the same for A and B).

Now let $\Gamma^{ij}_{kl} = \Gamma(A^{ij}_{kl}, B^{ij}_{kl})$, $1 \leq i \leq k \leq p$, $1 \leq j \leq l \leq q$, where for any matrix $W = [w_{sr}]$ of size $p \times q$ we put

$$W^{ij}_{kl} = \begin{bmatrix} w_{ij} & w_{i,j+1} & \dots & w_{il} \\ w_{i+1,j} & w_{i+1,j+1} & \dots & w_{i+1,l} \\ \vdots & \vdots & & \vdots \\ w_{kj} & w_{k,j+1} & \dots & w_{kl} \end{bmatrix}.$$

(Obviously, each Γ^{ij}_{kl} is a subgame of $\Gamma(A,B)$.)

Remark 10. An essential question is how to find the equilibrium described in Theorem 1. The proof (given in Section 5) implies the following procedure, which finds it in at most four steps (1)–(4):

(1) first check if there is a pure equilibrium in the constrained bimatrix game $\mathbb{G} = \mathbb{G}(A, B, E_A, E_B)$. If not,

(2) check if there is an admissible (2×2)-subgame of the form $\Gamma^{sr}_{s+1,r+1}$ in $\mathbb{G}$ which has no pure Nash equilibria. If it exists, the Nash equilibrium for $\Gamma^{sr}_{s+1,r+1}$ coincides with an equilibrium in $\mathbb{G}$. If not then for the bimatrix game $\Gamma = \Gamma(A,B)$ defined by assumption Z3,

(3) look for a $(k \times 2)$-subgame of the form $\Gamma^{sr}_{s+k-1,r+1}$ admissible in $\mathbb{G}$ $(k \geq 3)$, which has no pure equilibria and satisfies $b_{lr} = b_{l,r+1}$ for all $l, s < l < s+k-1$. If such a subgame exists, then it has an Nash equilibrium of the form $(\delta_{s+1},\, \gamma\delta_r + (1-\gamma)\delta_{r+1})$ for some $0 < \gamma < 1$; this Nash equilibrium coincides with an equilibrium in $\mathbb{G}$. If there is no such subgame $\Gamma^{sr}_{s+k-1,r+1}$,

(4) look for a $(2 \times k)$-subgame of the form $\Gamma^{sr}_{s+1,r+k-1}$ in Γ and admissible in $\mathbb{G}$ $(k \geq 3)$ which has no pure equilibria and satisfies $a_{sl} = a_{s+1,l}$ for all $l, r < l < r+k-1$. Then it has a Nash equilibrium of the form $(\lambda\delta_s + (1-\lambda)\delta_{s+1},\, \delta_{r+1})$ for some $0 < \lambda < 1$, which coincides with an equilibrium in $\mathbb{G}$.

4. Counterexamples. In this section we present five examples. In the first three examples we discuss assumptions Z1–Z3 of Theorem 1, showing that none of them can be omitted. We will consider three constrained bimatrix games and check the existence of equilibria consisting of two two-adjoining-pure strategies. To increase the clarity of the examples, all "nonadmissible payoffs" will be additionally marked by "$\star$" in the payoff matrices. We start by discussing assumption Z1.

Example 1. Consider the constrained bimatrix game $\mathbb{G}(A, B, E_A, E_B)$, where

$$A = \begin{bmatrix} 0\star & 3 & 0 \\ 1 & 0 & 1 \\ 2 & -3\star & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0\star & 1 & 2 \\ 3 & 0 & -3\star \\ 0 & 1 & 0 \end{bmatrix}.$$

In this case, $E_A = \{1,2,3\} \times \{1,2,3\} \setminus \{(1,1),(3,2)\}$ and $E_B = \{1,2,3\} \times \{1,2,3\} \setminus \{(1,1),(2,3)\}$. One can easily check that the game $\mathbb{G}$ satisfies assumptions Z2–Z3, but not Z1. A direct analysis shows that this game has no equilibria, neither in pure nor in mixed strategies.

The next example shows that assumption Z2 cannot be omitted in Theorem 1 either.

Example 2. Consider the constrained bimatrix game $\mathbb{G}(A, B, E_A, E_B)$, where

$$A = \begin{bmatrix} 3\star & 2 & 1 \\ 2 & 0 & -2\star \\ 1 & 2\star & 2 \end{bmatrix}, \quad B = \begin{bmatrix} -1\star & 0 & 1 \\ 0 & 2 & 4\star \\ 1 & 1\star & 0 \end{bmatrix},$$

and $E_A = E_B = \{1,2,3\} \times \{1,2,3\} \setminus \{(1,1),(2,3),(3,2)\}$. It is easy to check that this game satisfies assumptions Z1 and Z3 (but not Z2), and that it has no equilibrium with a pure strategy for one of the players. Nor are there equilibria in mixed strategies, since they are not admissible, as a consequence of the form of the sets E_A and E_B.

The third example shows that assumption Z3 is also necessary in Theorem 1.

EXAMPLE 3. Consider the constrained bimatrix game $\mathbb{G}(A, B, E_A, E_B)$, where

$$A = \begin{bmatrix} 0 & 1 & 2\star \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 2\star \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and $E_A = E_B = \{1,2,3\} \times \{1,2,3\} \setminus \{(1,3)\}$. This game satisfies assumptions Z1 and Z2, but not Z3. It is not difficult to check that it has no equilibrium. Therefore, assumption Z3 about concavity of the game cannot be omitted in Theorem 1.

Now we quote an interesting example (in the context of Remark 8) of a bimatrix game with some payoffs $-\infty$, having no Nash equilibria.This example belongs to Rosenthal [12].

EXAMPLE 4. Consider the generalized bimatrix game $\Gamma(A, B)$, where

$$A = \begin{bmatrix} -\infty & 3 & 0 \\ 1 & 0 & 1 \\ 2 & -\infty & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -\infty & 1 & 2 \\ 3 & 0 & -\infty \\ 0 & 1 & 0 \end{bmatrix}.$$

It is not difficult to check that in this game there is no Nash equilibrium (with finite or infinite payoffs). Note that the corresponding constrained bimatrix game does not satisfy the assumption of Theorem 2 (conditions Z1 and Z2 do not hold).

One could hope that Theorem 1 can be generalized to the case of n-person finite games with $n \geq 2$. The last example given below shows that this is impossible.

EXAMPLE 5. Consider the 3-person constrained game in which each of the players has exactly two pure strategies, described graphically in Fig. 1:

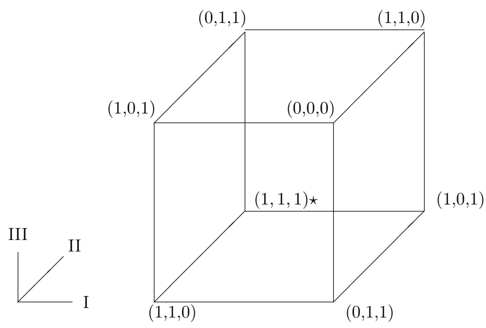

Fig. 1

Player 1 chooses one of his two pure strategies along line I, player 2 along line II, and player 3 along line III. Triplets of numbers in the figure denote payoffs for players 1, 2 and 3, respectively, associated with triples of pure strategies, except the one "forbidden" common multistrategy, denoted by "$\star$". Therefore this 3-person constrained game can be described (see Remark 9) by $\mathbb{G}_3 = \mathbb{G}(A_1, A_2, A_3, E_{A_1}, E_{A_2}, E_{A_3})$ and we easily deduce that $\mathbb{G}_3$ satisfies the extension of conditions Z1, Z2 and Z3 to 3-person games. It is also easy to verify that this game has no pure equilibrium.

Moreover, it cannot have an equilibrium in completely mixed strategies either. There is also no equilibrium where only one of the players uses a mixed strategy, since under fixed strategies of the other two players, the third player will always choose exactly one of his pure strategies (since one of them is dominated). Therefore we should only check equilibria in which one of the players uses a pure strategy, and two others use mixed strategies. But it is not difficult to verify that, also in this case, one of these two players has a strictly better strategy, no matter what the second player does. Therefore, in fact, this game has no equilibrium. This demonstrates that Theorem 1 cannot be directly generalized to n-person constrained finite games.

5. Proofs of the theorems. For $p, q \geq 1$, let a constrained bimatrix game $\mathbb{G} = \mathbb{G}(A, B, E_A, E_B)$ of size $p \times q$ with two matrices $A = [a_{ij}]_{p\times q}$ and $B = [b_{ij}]_{p\times q}$ satisfy assumptions Z1–Z3. By assumption Z1 ($E_A = E_B$), we can simplify the description of $\mathbb{G}$ to

$$\mathbb{G} = \mathbb{G}(A, B, E) \tag{10}$$

(instead of (9)), putting $E = E_A = E_B$. We will also use the notation $\mathbb{G}_{p\times q}$ to emphasize that $\mathbb{G}$ is of size $p \times q$.

Let naturals i, k, j and l satisfy $1 \leq i \leq k \leq p$ and $1 \leq j \leq l \leq q$. The matrices A^{ij}_{kl} and B^{ij}_{kl} (see notation of W^{ij}_{kl} before Remark 8 in Section 3) are of the same size $(k-i+1)\times(l-j+1)$, so we can consider a new constrained bimatrix game

$$\mathbb{G}^{ij}_{kl} = \mathbb{G}(A^{ij}_{kl}, B^{ij}_{kl}, E^{ij}_{kl}),$$

where $E^{ij}_{kl} = E \cap (\{i, i+1, \ldots, k\} \times \{j, j+1, \ldots, l\})$. It will be called a (*constrained*) *subgame* of $\mathbb{G}$.

In general, constrained subgames of $\mathbb{G}$ arise when we "remove" some fixed rows or/and columns in the matrices A and B, and appropriate pairs (i, j) from the set E. Obviously, an arbitrary constrained subgame $\mathbb{G}' = \mathbb{G}'(A', B', E')$ of $\mathbb{G}$ with $E' \neq \emptyset$ still satisfies assumptions Z1–Z3. The justification for Z1 and Z2 is trivial, and Z3 follows easily from Definition 3.

Proof of Theorem 1. One can easily see from Definition 2 that the set of equilibria of the constrained game $\mathbb{G}$ (of the form (10)) is independent of

a_{ij} and b_{ij} for $(i,j) \notin E$. By assumption Z3 and Proposition 1, inequalities (7) and (8) hold for some positive numbers $\theta_1, \ldots, \theta_{p-1}$ and $\tau_1, \ldots, \tau_{q-1}$. Now, considering (7) for each j separately and (8) for each i separately, we easily conclude with the help of assumption Z2 that the values a_{ij} and b_{ij} for $(i,j) \notin E$ can be changed in such a way that

$$a_{ij} < \min_{(k,l)\in E} a_{kl} \quad \text{and} \quad b_{ij} < \min_{(k,l)\in E} b_{kl} \quad \text{for } (i,j) \notin E, \tag{11}$$

and inequalities (7) and (8) remain true. Therefore, for the rest of the proof we can additionally assume (without loss of generality) that the entries of the matrices A and B satisfy (11).

Further, one can easily deduce from (7) and (8) that for every j, $1 \le j \le q$, there are $1 \le k \le l \le p$ such that

$$a_{1j} < \cdots < a_{kj} = \cdots = a_{lj} > \cdots > a_{pj}, \tag{12}$$

and for every i, $1 \le i \le p$, there are $1 \le t \le u \le q$ such that

$$b_{i1} < \cdots < b_{it} = \cdots = b_{iu} > \cdots > b_{iq}. \tag{13}$$

Now suppose that there is an $i, 1 \le i \le p$, such that $(i,j) \notin E$ for $j = 1, \ldots, q$, and let $\mathbb{G}' = \mathbb{G}(A', B', E')$ be the constrained subgame of $\mathbb{G}$ obtained from $\mathbb{G}$ by removing the ith rows in A and B, with $E' = E$. One can easily conclude from Definition 2 that if (μ, ν) is an equilibrium in $\mathbb{G}$ then $i \notin \operatorname{supp}(\mu)$, and hence the sets of equilibria in $\mathbb{G}$ and $\mathbb{G}'$ coincide. We get the same conclusion when there is an $j, 1 \le i \le q$, such that $(i,j) \notin E$ for $i = 1, \ldots, p$ and the subgame $\mathbb{G}'$ of $\mathbb{G}$ is obtained from $\mathbb{G}$ by removing the jth columns in A and B. Therefore, we can additionally assume that

$$\forall i \in \{1, \ldots, p\}\ \exists j \in \{1, \ldots, q\} \quad \text{such that} \quad (i,j) \in E, \tag{14}$$

$$\forall j \in \{1, \ldots, q\}\ \exists i \in \{1, \ldots, p\} \quad \text{such that} \quad (i,j) \in E. \tag{15}$$

Now we give four lemmata needed for the proof of Theorem 1. The first two are easy consequences of (11)–(13). The details are omitted.

Lemma 1. *Let $(i,j) \in E$, $(k,j) \notin E$ and $(k+1,j) \notin E$. Then*

(a) $a_{ij} > a_{kj} > a_{k+1,j}$ *if* $i < k$,
(b) $a_{kj} < a_{k+1,j} < a_{ij}$ *if* $i > k+1$.

Lemma 2. *Let $(i,j) \in E$, $(i,k) \notin E$ and $(i,k+1) \notin E$. Then*

(a) $b_{ij} > b_{ik} > b_{i,k+1}$ *if* $j < k$,
(b) $b_{ik} < b_{i,k+1} < b_{ij}$ *if* $j > k+1$.

Lemma 3. *Let $\mathbb{G}^{ij}_{kl} = \mathbb{G}(A^{ij}_{kl}, B^{ij}_{kl}, E^{ij}_{kl})$ be a subgame of $\mathbb{G}$ of the form* (10) *and let $E^{ij}_{kl} \ne \emptyset$. If $\mathbb{G}$ satisfies assumptions* Z1–Z3 *then so does $\mathbb{G}^{ij}_{kl}$.*

Proof. Obviously, $\mathbb{G}^{ij}_{kl}$ satisfies Z1 and Z2. The fact that it also satisfies Z3 immediately follows from Proposition 1.

LEMMA 4. *Let $\mathbb{G}$ be a constrained bimatrix game of the form* (10) *satisfying assumptions* Z1–Z3, *and* (11), (14) *and* (15). *Assume that the bimatrix game $\Gamma = \Gamma(A, B)$ has a (2×2)-subgame of the form $\Gamma^{sr}_{s+1,r+1}$ without pure Nash equilibria, such that $\{(s, r), (s+1, r), (s, r+1), (s+1, r+1)\} \cap E = \emptyset$. Then one of the following two cases must hold:*

(a) *The constrained subgame $\mathbb{G}^{11}_{s,r-1}$ of $\mathbb{G}$ satisfies assumptions* Z1–Z3 *and each equilibrium in $\mathbb{G}^{11}_{s,r-1}$ (if any) is an equilibrium in $\mathbb{G}$.*

(b) *Constrained subgame $\mathbb{G}^{11}_{s-1,r}$ of $\mathbb{G}$ satisfies assumptions* Z1–Z3 *and each equilibrium in $\mathbb{G}^{11}_{s-1,r}$ (if any) is an equilibrium in $\mathbb{G}$.*

Proof. Let $\Gamma^{sr}_{s+1,r+1}$ satisfy the assumption. By (14) and (15), there exist i, j, k and l such that four pairs of pure strategies, (i, r), $(k, r + 1)$, (s, j) and $(s + 1, l)$, belong to E. Since $\Gamma^{sr}_{s+1,r+1}$ has no pure Nash equilibrium, it follows that either

$$(16) \quad a_{sr} < a_{s+1,r}, \quad a_{s,r+1} > a_{s+1,r+1}, \quad b_{sr} > b_{s,r+1}, \quad b_{s+1,r} < b_{s+1,r+1},$$

or

$$(17) \quad a_{sr} > a_{s+1,r}, \quad a_{s,r+1} < a_{s+1,r+1}, \quad b_{sr} < b_{s,r+1}, \quad b_{s+1,r} > b_{s+1,r+1}.$$

First suppose that (16) holds. Then by assumption:

(C1) $(i, r) \in E$, $(s, r) \notin E$, $(s + 1, r) \notin E$ and $a_{sr} < a_{s+1,r}$,
(C2) $(k, r+1) \in E$, $(s, r+1) \notin E$, $(s+1, r+1) \notin E$ and $a_{s,r+1} > a_{s+1,r+1}$,
(C3) $(s, j) \in E$, $(s, r) \notin E$, $(s, r + 1) \notin E$ and $b_{s,r} > b_{s,r+1}$,
(C4) $(s+1, l) \in E$, $(s+1, r) \notin E$, $(s+1, r+1) \notin E$ and $b_{s+1,r} < b_{s+1,r+1}$.

Now applying Lemma 1 to conditions (C1) and (C2), and Lemma 2 to (C3) and (C4), we get the inequalities

$$i > s + 1, \ k < s, \quad \text{and} \quad j < r, \ l > r + 1.$$

Cosider now the constrained subgame $\mathbb{G}^{11}_{s,r-1} = \mathbb{G}(A^{11}_{s,r-1}, B^{11}_{s,r-1}, E^{11}_{s,r-1})$ of $\mathbb{G}$. Since $1 \le j < r$ and $(s, j) \in E$, the subgame is well-defined and $(s, j) \in E^{11}_{s,r-1}$, whence $E^{11}_{s,r-1} \neq \emptyset$. Therefore, by Lemma 3, $\mathbb{G}^{11}_{s,r-1}$ satisfies assumptions Z1–Z3.

We also have $(s, r) \notin E$, $(i, r) \in E$ and $i > s + 1$. Therefore assumption Z2 implies

$$(18) \qquad (t, r) \notin E \quad \forall t \le s.$$

In a similar way we show that

$$(19) \qquad (s + 1, t) \notin E \quad \forall t \le r.$$

With the help of (18), (19) and assumption Z2 for the set E, it is not difficult to verify that each equilibrium of $\mathbb{G}^{11}_{s,r-1}$ (if any) is also an equilibrium in $\mathbb{G}$.

The proof of the lemma when (17) holds is analogous and leads to case (b). The details are omitted. ∎

Now we return to the proof of Theorem 1. Obviously the conclusion is true for any game $\mathbb{G}_{1\times 1}$. Let $p \geq 1$ and $q \geq$ be natural numbers. Assume now that Theorem 1 holds for all constrained games of the form $\mathbb{G}_{k\times l}$ with $1 \leq k \leq p$, $1 \leq l \leq q$, $(k,l) \neq (p,q)$. We will show that this implies the validity of the theorem for every game of the form $\mathbb{G} = \mathbb{G}_{p\times q}$. This, by induction, will complete the proof.

For the rest of the proof let us fix $\mathbb{G} = \mathbb{G}_{p\times q}$ of the form (10). In view of the reasoning given before Lemma 1, we can assume that (11), (14) and (15) hold. Let $\Gamma = \Gamma(A,B)$ be the bimatrix game associated with $\mathbb{G}$. Obviously, by assumption Z3, Γ is a concave bimatrix game. Therefore Theorem B (in Section 2) implies that Γ has a Nash equilibrium (μ^*,ν^*) of the form $\mu^* = \lambda\delta_s + (1-\lambda)\delta_{s+1}$, $\nu^* = \gamma\delta_r + (1-\gamma)\delta_{r+1}$, for some $1 \leq s \leq p$, $1 \leq r \leq q$ and $0 \leq \lambda,\gamma \leq 1$. Using this fact, we will show (in four cases) that $\mathbb{G}$ also has an equilibrium of the same form.

Case 1: (μ^*,ν^*) is a pure Nash equilibrium in Γ. Then $(\mu^*,\nu^*) = (\delta_s,\delta_r)$ for some s and r. Thus either $(s,r) \in E$ and (μ^*,ν^*) is an equilibrium in $\mathbb{G}$, or $(s,r) \notin E$, which is impossible because of (11).

Case 2: $(\mu^*,\nu^*) = (\lambda\delta_s + (1-\lambda)\delta_{s+1}, \gamma\delta_r + (1-\gamma)\delta_{r+1})$ is a mixed Nash equilibrium in Γ, for some $0 < \lambda,\gamma < 1$, and the subgame $\Gamma^{sr}_{s+1,r+1}$ has no pure Nash equilibria.

First assume that all the four pairs of pure strategies in $\Gamma^{sr}_{s+1,r+1}$ belong to E. Then obviously, the pair (μ^*,ν^*) of mixed strategies is an equilibrium in $\mathbb{G}_{t\times u}$.

Next, assume that among the pairs of pure strategies in $\Gamma^{sr}_{s+1,r+1}$, there are both pairs in E and pairs off this set. Now, a simple analysis of all possible configurations shows with the help of (11) that $\Gamma^{sr}_{s+1,r+1}$ has a pure Nash equilibrium. But this is impossible by assumption. Therefore, we can assume that all the pairs of pure strategies in $\Gamma^{sr}_{s+1,r+1}$ are outside of E.

The induction assumption implies that both $\mathbb{G}^{11}_{s,r-1}$ and $\mathbb{G}^{11}_{s-1,r}$ have equilibria consisting of two-adjoining-pure strategies. Hence, by Lemma 4, the proof in Case 2 is complete.

Case 3: $(\mu^*,\nu^*) = (\delta_s, \gamma\delta_r + (1-\gamma)\delta_{r+1})$ is a mixed Nash equilibrium in Γ, for some $0 < \gamma < 1$. From the properties of Nash equilibrium we know that in that case we have

$$b_{s,r} = b_{s,r+1} \quad \text{and} \quad b_{s,j} \leq b_{s,r} \quad \text{for } j = 1,\ldots,q. \tag{20}$$

First suppose that both pairs of pure strategies, (s,r) and $(s,r+1)$, belong to the set E of admissible strategies in $\mathbb{G}$. Then obviously, the pair (μ^*,ν^*) of mixed strategies is an equilibrium in $\mathbb{G}$.

Assume next that only one pair of pure strategies, (s, r) or $(s, r+1)$, belongs to E in $\mathbb{G}_{t\times u}$. Then it follows from (11) that $b^*_{s,r} \neq b^*_{s,r+1}$, which is impossible in the present case. Therefore $(s, r) \notin E$ and $(s, r+1) \notin E$. But this, in view of (11) and (20), is also impossible, completing the proof in Case 3.

Case 4: $(\mu^*, \nu^*) = (\lambda\delta_s + (1-\lambda)\delta_{s+1}, \delta_r)$ is a mixed Nash equilibrium in Γ, for some $0 < \lambda < 1$. The proof in this case is analogous to that of Case 3. The details are omitted.

This completes the proof that $\mathbb{G}_{p\times q}$ has an equilibrium in two-adjoining-pure strategies. Thus, by induction, Theorem 1 has been proved for any constrained bimatrix game. ■

Proof of Theorem 2. It follows from Theorem 1 that the constrained bimatrix game $\mathbb{G}(A, B, E)$ has an equilibrium (μ^*, ν^*) in two-adjoining-pure strategies, with finite payoffs for both players (since the payoff matrices A and B have finite payoffs). Hence, we conclude immediately that (μ^*, ν^*) is a Nash equilibrium in the bimatrix game $\Gamma_{\mathbb{G}} = \Gamma(A_{\mathbb{G}}, B_{\mathbb{G}})$, since the matrices $A_{\mathbb{G}}$ and $B_{\mathbb{G}}$ have all entries equal to $-\infty$ outside E.

Now assume that additionally assumption Z4 is satisfied. Then $A_{\mathbb{G}}$ and $B_{\mathbb{G}}$ have a row or a column without $-\infty$. If (μ^*, ν^*) is any equilibrium of $\Gamma_{\mathbb{G}}$, then, obviously, this equilibrium generates finite payoffs for both players. Hence it is not difficult to deduce that the cartesian product of the supports of the probability distributions μ^* and ν^* must be contained in E, since otherwise both payoffs would be equal to $-\infty$. Therefore (μ^*, ν^*) is also an equilibrium in $\mathbb{G}(A, B, E_A, E_B)$. This completes the proof of Theorem 2. ■

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