APPLICATIONES MATHEMATICAE
40,2 (2013), pp. 237–258

I. K. Argyros (Lawton, OK)
S. K. Khattri (Stord)

INEXACT NEWTON METHOD UNDER WEAK AND CENTER-WEAK LIPSCHITZ CONDITIONS

Abstract. The paper develops semilocal convergence of **I**nexact **N**ewton **M**ethod (**INM**) for approximating solutions of nonlinear equations in Banach space setting. We employ weak Lipschitz and center-weak Lipschitz conditions to perform the error analysis. The results obtained compare favorably with earlier ones in at least the case of **N**ewton's **M**ethod (**NM**). Numerical examples, where our convergence criteria are satisfied but the earlier ones are not, are also explored.

1. Introduction. Let $\mathbf{X}$, $\mathbf{Y}$ be Banach spaces and $\mathcal{D}$ be a nonempty, convex and open subset in $\mathbf{X}$. Let $U(x,r)$ and $\overline{U}(x,r)$ stand, respectively, for the open and closed ball in $\mathbf{X}$ with center x and radius $r>0$. Denote by $\mathcal{L}(\mathbf{X},\mathbf{Y})$ the space of bounded linear operators from $\mathbf{X}$ into $\mathbf{Y}$. In the present paper, we are concerned with the problem of approximating a locally unique solution $x^\star$ of the equation

$$\mathcal{F}(x)=0 \tag{1.1}$$

where $\mathcal{F}$ is a Fréchet continuously differentiable operator defined on $\mathcal{D}$ with values in $\mathbf{Y}$.

Many problems from computational sciences and other disciplines can be brought into the form of equation (1.1) using mathematical modelling [1, 2, 3, 5, 6, 10]. The solution of these equations can rarely be found in closed form. That is why the solution methods for these equations are iterative. In particular, the practice of numerical analysis for finding such solutions is essentially connected to variants of Newton's method [2, 3, 4, 6, 7, 10, 11]. The study of convergence of iterative procedures is usually centered on two

2010 *Mathematics Subject Classification*: 65H10, 65G99, 47H99, 49H15.
Key words and phrases: inexact Newton method, Banach space, semilocal convergence, weak and center-weak Lipschitz condition, recurrent functions, Kantorovich hypotheses.

DOI: 10.4064/am40-2-6

types of analysis: semilocal and local convergence analysis. The semilocal convergence analysis is based on the information around an initial point, to give criteria ensuring the convergence of the iterative procedure; while the local one is, based on the information around a solution, to find estimates of the radii of convergence balls. There is a plethora of studies on the weakness and/or extension of the hypotheses made on the underlying operators; see for example [1–28].

Undoubtedly the most popular iterative method for generating a sequence approximating $x^\star$ is the Newton's method (**NM**), which is defined as

$$x_{n+1} = x_n - \mathcal{F}'(x_n)^{-1}\mathcal{F}(x_n) \quad \text{for } n = 0, 1, 2, \ldots, \tag{1.2}$$

where x_0 is an initial point. There are two difficulties with the implementation of (**NM**). The first is to evaluate $\mathcal{F}'$ and the second is to exactly solve the Newton equation

$$\mathcal{F}'(x_n)(x_{n+1} - x_n) = -\mathcal{F}(x_n) \quad \text{for } n = 0, 1, 2, \ldots. \tag{1.3}$$

It is well-known that evaluating $\mathcal{F}'$ and solving (1.3) may be computationally expensive [9, 10, 13–27, 28]. The computational cost can be reduced if the (**INM**)

$$x_{n+1} = x_n + s_n, \quad \mathcal{F}'(x_n)s_n = -\mathcal{F}(x_n) + r_n \quad \text{for } n = 0, 1, 2, \ldots, \tag{1.4}$$

is used. Here, $\{r_n\}$ is a null-sequence in the Banach space $\mathbf{Y}$. Clearly, the convergence behavior of (**INM**) depends on the residual controls $\{r_n\}$ and hypotheses on $\mathcal{F}'$. In particular, Lipschitz continuity conditions on $\mathcal{F}'$ have been used, together with residual controls of the form

$$\begin{aligned} \|r_n\| &\le \eta_n \|\mathcal{F}(x_n)\|, \\ \|\mathcal{F}'(x_0)^{-1} r_n\| &\le \eta_n \|\mathcal{F}'(x_0)^{-1}\mathcal{F}(x_n)\|, \\ \|\mathcal{F}'(x_0)^{-1} r_n\| &\le \eta_n \|\mathcal{F}'(x_0)^{-1}\mathcal{F}(x_n)\|^{1+\theta}, \\ \|P_n r_n\| &\le \theta_n \|P_n \mathcal{F}(x_n)\|^{1+\theta}, \end{aligned} \tag{1.5}$$

for some $\theta \in [0, 1]$ and all $n = 0, 1, 2, \ldots$. Here, $\{\eta_n\}$, $\{\theta_n\}$ are sequences in $[0, 1]$, $\{P_n\}$ is a sequence in $\mathcal{L}(\mathbf{Y}, \mathbf{X})$, and $\mathcal{F}'(x_0)^{-1}\mathcal{F}'$ satisfies a Lipschitz or Hölder condition on $U(x_0, r)$ [1–24].

In this work, motivated by the works of Argyros et al. [1–5, 7, 8], Shen et al. [22, 23] and Wang et al. [25, 26], we suppose that $\mathcal{F}$ has a continuous Fréchet derivative in $\overline{U}(x_0, r)$, $\mathcal{F}'(x_0)^{-1}\mathcal{F}'$ exists and $\mathcal{F}'(x_0)^{-1}\mathcal{F}'$ satisfies the weak Lipschitz condition

$$\|\mathcal{F}'(x_0)^{-1}(\mathcal{F}'(x) - \mathcal{F}'(x^\tau))\| \le \int_{\tau\rho(x)}^{\rho(x)} L(u)\, du \tag{1.6}$$

for each $x \in U(x_0, r)$. Here, $\rho(x) = \|x - x_0\|$, $x^\tau = x_0 + \tau(x - x_0)$, $\tau \in [0, 1]$ and L is a positive, integrable, nondecreasing function on $[0, r]$. Moreover, we suppose that the 3rd formula of (1.5) is satisfied for $\theta = 1$, that is,

$$\|\mathcal{F}'(x_0)^{-1} r_n\| \leq \eta_n \|\mathcal{F}'(x_0)^{-1}\mathcal{F}(x_n)\|^2 \quad \text{for } n = 0, 1, 2, \dots. \tag{1.7}$$

Moreover, we suppose that

$$\eta = \sup_{n \geq 0} \eta_n < 1. \tag{1.8}$$

In view of (1.6) there exists a positive, integrable and nondecreasing function on $[0, r]$ such that the *center-weak Lipschitz condition*

$$\|\mathcal{F}'(x_0)^{-1}(\mathcal{F}'(x) - \mathcal{F}'(x_0))\| \leq \int_0^{\rho(x)} L_0(u)\, du \tag{1.9}$$

holds for each $x \in U(x_0, r)$. Clearly,

$$L_0(u) \leq L(u) \tag{1.10}$$

for each $u \in [0, r]$ and L/L_0 can be arbitrarily large [4, 9, 6].

In this work, we use our idea of recurrent functions [8–6] to perform convergence analysis for the (**INM**). In the computation of $\|\mathcal{F}(x)^{-1}\mathcal{F}'(x_0)\|$ we use the condition (1.9) which is tighter than (1.6), and the Banach lemma on invertible operators [10], to obtain the perturbation bound

$$\|\mathcal{F}'(x)^{-1}\mathcal{F}'(x_0)\| \leq \Big(1 - \int_0^{\rho(x)} L_0(u)\, du\Big)^{-1} \quad \text{for } x \in U(x_0, r), \tag{1.11}$$

instead of using (1.6) to obtain

$$\|\mathcal{F}'(x)^{-1}\mathcal{F}'(x_0)\| \leq \Big(1 - \int_0^{\rho(x)} L(u)\, du\Big)^{-1} \quad \text{for } x \in U(x_0, r). \tag{1.12}$$

It turns out that using (1.12) instead of (1.11), in the case when $L_0(u) < L(u)$ for each $u \in [0, r]$, leads to tighter majorizing sequences for (**INM**). This observation in turn leads to the following advantages over the earlier works (for $\eta_n = 0$ for each $n = 0, 1, 2, \dots$ or not, and L being a constant or not (see also Section 4)):

- $\mathscr{A}_1$: Weaker sufficient convergence conditions.
- $\mathscr{A}_2$: Tighter error estimates on the distances $\|x_{n+1} - x_n\|$, $\|x_n - x^\star\|$ for each $n = 0, 1, 2, \dots$.
- $\mathscr{A}_3$: At least as precise information on the location of the solution.

The rest of the paper is organized as follows. Section 2 develops the semilocal convergence analysis I for (**INM**) by employing the traditional majorant principle in combination with the Kantorovich theory for (**INM**) and nonlinear equations. Section 3 contains the semilocal convergence analysis II of (**INM**) using recurrent functions [8–6]. In the concluding Section 4, we present special cases and numerical examples where the claims $\mathscr{A}_1$–$\mathscr{A}_3$ are verified.

2. Semilocal convergence I. For convenience, let us define some majorizing functions. Let $b, c, d > 0$ and $\theta \in [0, 1)$. Define functions f, g_0 and g on $[0, R]$ by

$$f(t) = b - (1-\theta)t + dt^2 + c\int_0^t L(u)(t-u)\,du, \tag{2.1}$$

$$g_0(t) = b - t + c\int_0^t L_0(u)(t-u)\,du, \tag{2.2}$$

$$g(t) = b - t + c\int_0^t L(u)(t-u)\,du. \tag{2.3}$$

We have

$$f'(t) = -(1-\theta) + 2dt + c\int_0^t L(u)\,du, \tag{2.4}$$

$$f''(t) = 2d + cL(t) > 0, \tag{2.5}$$

$$g_0'(t) = -1 + c\int_0^t L_0(u)\,du, \tag{2.6}$$

$$g'(t) = -1 + c\int_0^t L(u)\,du, \tag{2.7}$$

$$f'(t) = g'(t) + \theta + 2dt, \tag{2.8}$$

$$g_0(t) \le g(t), \tag{2.9}$$

$$g_0'(t) \le g'(t). \tag{2.10}$$

Define

$$r := \sup\Big\{r \in (0, R) : c\int_0^r L(u)\,du + 2dr \le 1-\theta\Big\}, \tag{2.11}$$

$$p := (1-\theta)r - dr^2 - c\int_0^r L(u)(r-u)\,du, \tag{2.12}$$

$$\mu := c\int_0^R L(u)\,du + 2dR. \tag{2.13}$$

It follows that

$$r = \begin{cases} R, & \mu < 1-\theta, \\ \overline{r}, & \mu \geq 1-\theta, \end{cases} \tag{2.14}$$

where $\overline{r} \in [0, R]$ is such that

$$c\int_0^{\overline{r}} L(u)\,du + 2d\overline{r} = 1-\theta.$$

Hence,

$$p \begin{cases} \geq c\int_0^{r} L(u)u\,du + dr^2, & \mu < 1-\theta, \\ = c\int_0^{r} L(u)u\,du + dr^2, & \mu \geq 1-\theta. \end{cases} \tag{2.15}$$

As in [8], let us define the scalar sequences $\{s_n\}$ and $\{t_n\}$ for each $\theta \in [0,1)$ by

$$s_0 = 0, \qquad s_{n+1} = s_n - \frac{f(s_n)}{g_0'(s_n)}, \tag{2.16}$$

$$t_0 = 0, \qquad t_{n+1} = t_n - \frac{f(t_n)}{g'(t_n)}. \tag{2.17}$$

Note that if $d = 0$ and $c = 1$ (i.e. if $r_n = 0$ for all $n = 0, 1, 2, \ldots$) the functions and sequences defined above reduce to the corresponding ones in [8]. If equality holds in (1.10), then $g_0(t) = g(t)$ for each $t \in [0, R]$. Otherwise (i.e. if $g_0(t) < g(t)$ for each $t \in [0, R]$) then we shall show in Lemma 2.2 that the scalar sequence $\{s_n\}$ is at least as tight as $\{t_n\}$. But first we need a crucial result on majorizing sequences for (**INM**). The proof is similar to the corresponding Lemma 2.1 in [8] where $d = 0$ and $c = 1$. However there are some crucial differences.

LEMMA 2.1. *Suppose*

$$b \leq p. \tag{2.18}$$

Then

- ($\mathscr{G}_2$) *A function f is strictly decreasing on $[0, r]$ and has exactly one zero $s^\star \in [0, r]$ for each $\theta \in [0, 1)$ such that*

$$b < s^\star. \tag{2.19}$$

- ($\mathscr{G}_2$) *The sequence $\{s_n\}$ defined by* (2.16) *is strictly increasing and converges to $s^\star$.*

Proof. ($\mathscr{G}_1$) It follows from (2.5) and (2.11) that f' is strictly increasing on $[0,r]$. In particular, $f'(0)<0$ and $f'(r)\le 0$. That is, f is strictly decreasing on $[0,r]$. Moreover, using (2.1) we find that $f(0)=b>0$ and $f(r)\le 0$. Hence, the graph of f crosses the interval $(0,r]$ only once at a zero denoted by s. Furthermore,

$$f(b)=\theta b+db^2+c\int_0^b L(u)(b-u)\,du>0, \tag{2.20}$$

which shows (2.19).

($\mathscr{G}_2$) We use induction on n. It follows from (2.16) and (2.19) that

$$0=s_0<s_1=b<s^\star. \tag{2.21}$$

Assume that

$$s_{k-1}<s_k<s^\star \quad \text{for each } k\le n.$$

We have $g_0''(t)=cL_0(t)$. Hence, $-g_0'$ is strictly decreasing on $[0,r]$. In view of (2.8), (2.10), (2.21) and the definition of r we get

$$-g_0'(s_k)>-g_0'(s^\star)\ge -g_0'(r)\ge -g'(r)=-f'(r)+\theta+2dr\ge 0.$$

We also have $f(s_k)>0$ by ($\mathscr{G}_1$). Hence,

$$s_{k+1}=s_k-\frac{f(s_k)}{g_0'(s_k)}>s_k, \tag{2.22}$$

which completes the induction. Let us define a function q on $[0,s^\star]$ by

$$q(t)=t-\frac{f(t)}{g_0'(t)}. \tag{2.23}$$

We have

$$g_0'(t)<0 \quad \text{for each } t\in[0,s^\star] \tag{2.24}$$

except if $\theta=0$ and $t=s^\star=r$. As in [8], we use the L'Hôpital rule to obtain

$$\frac{f(s^\star)}{g_0'(s^\star)}=\lim_{t\to s^{\star-}}\frac{f(t)}{g_0'(t)}=0. \tag{2.25}$$

Hence, the function q is well defined and continuous on $[0,s^\star]$. It then follows from ($\mathscr{G}_1$), (2.5) and (2.24) that

$$q'(t)=1-\frac{f'(t)g_0'(t)-f(t)g''(t)}{(g_0'(t))^2}=\frac{-g_0'(t)(\theta+2dt)+f(t)g_0''(t)}{(g_0'(t))^2}>0 \tag{2.26}$$

for each $t\in[0,s^\star]$. Hence, q is strictly increasing on $[0,s^\star]$. Using (2.21), (2.22) and (2.26) we obtain

$$s_k<s_{k+1}=q(s_k)<q(s^\star)=s^\star, \tag{2.27}$$

which completes the induction. That is, $\{s_n\}$ is increasing, bounded from above by $s^\star$ and so converges to its least upper bound $\overline{s}^\star \in [0, s^\star]$ with $f(\overline{s}^\star) = 0$. But by $(\mathscr{G}_1)$, $\overline{s}^\star = s^\star$. ∎

Next, we compare the sequences $\{s_n\}$ and $\{t_n\}$.

LEMMA 2.2. *Suppose that condition* (2.18) *holds. Then, for each* $n = 0, 1, 2, \ldots,$

$$s_n \le t_n, \tag{2.28}$$

$$s_{n+1} - s_n \le t_{n+1} - t_n. \tag{2.29}$$

Moreover, if $L_0(t) < L(t)$ *for each* $t \in [0, s^\star]$ *then inequalities* (2.28) *and* (2.29) *are strict.*

Proof. Clearly, under condition (2.18) assertions $(\mathscr{G}_1')$ and $(\mathscr{G}_2)$ of Lemma 2.1 hold with the sequence $\{t_n\}$ replaced by $\{s_n\}$. We shall show estimates (2.28) and (2.29) by induction. Using (1.10), (2.16) and (2.17) we get

$$s_0 = t_0, \qquad s_1 = t_1 = b,$$
$$s_1 = s_0 - \frac{f(s_0)}{g_0'(s_0)} \le t_0 - \frac{f(t_0)}{g'(t_0)} = t_1$$

and so

$$s_1 - s_0 = -\frac{f(s_0)}{g_0'(s_0)} = -\frac{f(t_0)}{g'(t_0)} = t_1 - t_0.$$

Hence, (2.28) and (2.29) hold for $n = 0$. Let us assume that estimates (2.28) and (2.29) hold for each integer $k \le n$. Then

$$s_{k+1} = s_k - \frac{f(s_k)}{g_0'(s_k)} \le t_k - \frac{f(t_k)}{g'(t_k)} = t_{k+1}$$

and

$$s_{k+1} - s_k = -\frac{f(s_k)}{g_0'(s_k)} \le -\frac{f(t_k)}{g'(t_k)} = t_{k+1} - t_k. \ ∎$$

Let us denote

$$a = \|\mathcal{F}'(x_0)^{-1}\mathcal{F}(x_0)\|, \quad b = (1+\sqrt{\eta})a, \tag{2.30}$$

$$c = 1 + \sqrt{\eta}, \qquad d = \frac{\eta(1+\sqrt{\eta})(1+\int_0^R L(u)\,du)^2}{(1-\sqrt{\eta})^2}. \tag{2.31}$$

We need the following auxiliary result connecting (**INM**) with the majorizing sequence (2.16).

LEMMA 2.3. *Suppose the sequence* $\{x_n\}$ *generated by* (**INM**) *is well defined for each* $n = 0, 1, 2, \ldots.$ *Furthermore, let* $\mathcal{F}$ *satisfy the weak Lipschitz condition on* $U(x_0, s^\star)$, $b \le p$, *and for some integer* $k \ge 1$,

$$\sqrt{\eta}\,\|\mathcal{F}'(x_0)^{-1}\mathcal{F}(x_{n-1})\| \le 1 \quad \textit{and} \quad \|x_n - x_{n-1}\| \le s_n - s_{n-1} \tag{2.32}$$

are satisfied for each $n = 1, \dots, k$. *Then*

$$(1+\sqrt{\eta})\|\mathcal{F}'(x_0)^{-1}\mathcal{F}(x_k)\| \le f(s_k), \tag{2.33}$$

$$\sqrt{\eta}\,\|\mathcal{F}'(x_0)^{-1}\mathcal{F}(x_k)\| \le 1. \tag{2.34}$$

Proof. Suppose that (2.32) holds for each $1 \le n \le k$. We write $x_{k-1}^r = x_{k-1} + \tau(x_k - x_{k-1})$, where $\tau \in [0,1]$. Using (1.4), we obtain

$$\begin{aligned}\mathcal{F}(x_k) &= \mathcal{F}(x_k) - \mathcal{F}(x_{k-1}) - \mathcal{F}'(x_{k-1})(x_k - x_{k-1}) + r_{k-1}\\ &= \int_0^1 [\mathcal{F}'(x_{k-1}^r) - F'(x_{k-1})]\,d\tau\,(x_k - x_{k-1}) + r_{k-1}.\end{aligned}$$

Hence,

$$\begin{aligned}&\|\mathcal{F}'(x_0)^{-1}\mathcal{F}(x_k)\| \\ &\le \left\|\mathcal{F}'(x_0)^{-1}\int_0^1 [\mathcal{F}'(x_{k-1}^r) - \mathcal{F}'(x_{k-1})]\,d\tau\,(x_k - x_{k-1})\right\| + \|\mathcal{F}'(x_0)^{-1}r_{k-1}\| \\ &= A + B.\end{aligned} \tag{2.35}$$

To estimate A, by (2.32), we observe that

$$\begin{aligned}\|x_{k-1}^r - x_0\| &= \|x_{k-1} + r(x_k - x_{k-1}) - x_0\|\\ &\le \sum_{n=1}^{m-1} \|x_n - x_{n-1}\| + r\|x_k - x_{k-1}\|\\ &\le s_{k-1} + \tau(s_k - s_{k-1}) = (1-\tau)s_{k-1} + rs_k < s^\star.\end{aligned} \tag{2.36}$$

In particular,

$$\|x_{k-1} - x_0\| \le s_{k-1} < s^\star, \qquad \|x_k - x_0\| \le s_k < s^\star,$$

and so

$$A \le \int_0^{\|x_k - x_{k-1}\|} (\|x_k - x_{k-1}\| - u)L(\|x_{k-1} - x_0\| + u)\,du. \tag{2.37}$$

Now, we estimate B. We notice that (1.7) and (2.32) yield

$$\begin{aligned}&\|\mathcal{F}'(x_0)^{-1}\mathcal{F}'(x_{k-1})(x_k - x_{k-1})\|\\ &\qquad \ge \|\mathcal{F}'(x_0)^{-1}\mathcal{F}(x_{k-1})\| - \|\mathcal{F}'(x_0)^{-1}r_{k-1}\|\\ &\qquad \ge \|\mathcal{F}'(x_0)^{-1}\mathcal{F}(x_{k-1})\| - \eta\|\mathcal{F}'(x_0)^{-1}\mathcal{F}(x_{k-1})\|^2\\ &\qquad \ge (1-\sqrt{\eta})\|\mathcal{F}'(x_0)^{-1}\mathcal{F}(x_{k-1})\|.\end{aligned} \tag{2.38}$$

Since

$$\|\mathcal{F}'(x_0)^{-1}\mathcal{F}'(x_k)\| = \|I + \mathcal{F}'(x_0)^{-1}[\mathcal{F}'(x_k) - \mathcal{F}'(x_0)]\| \leq 1 + \int_0^{\rho(x_k)} L(u)\,du \leq 1 + \int_0^{R} L(u)\,du,$$

we have

$$\|\mathcal{F}'(x_0)^{-1}\mathcal{F}(x_{k-1})\| \leq \frac{\|\mathcal{F}'(x_0)^{-1}\mathcal{F}'(x_{k-1})\|\,\|x_k - x_{k-1}\|}{1-\sqrt{\eta}} \leq \frac{1 + \int_0^R L(u)\,du}{1-\sqrt{\eta}}\|x_k - x_{k-1}\|. \tag{2.39}$$

Combining the preceding relations with (1.7), we get

$$B \leq \eta\|\mathcal{F}'(x_0)^{-1}\mathcal{F}(x_{k-1})\|^2 \leq \eta\frac{(1 + \int_0^R L(u)\,du)^2}{(1-\sqrt{\eta})^2}\|x_k - x_{k-1}\|^2. \tag{2.40}$$

Accordingly, from (2.35), (2.37), (2.40) and Lemma 2.2, we get in turn

$$(1+\sqrt{\eta})\|\mathcal{F}'(x_0)^{-1}\mathcal{F}(x_k)\| \leq (1+\sqrt{\eta})(A+B) \tag{2.41}$$

$$\begin{aligned}
&\leq (1+\sqrt{\eta}) \int_0^{\|x_k - x_{k-1}\|} (\|x_k - x_{k-1}\| - u) L(\|x_{k-1} - x_0\| + u)\,du \\
&\quad + \frac{\eta(1+\sqrt{\eta})\big(1 + \int_0^R L(u)\,du\big)^2}{(1-\sqrt{\eta})^2}\|x_k - x_{k-1}\|^2 \\
&= c \int_0^{\|x_k - x_{k-1}\|} (\|x_k - x_{k-1}\| - u) L(\|x_{k-1} - x_0\| + u)\,du + d\|x_k - x_{k-1}\|^2 \\
&= \bigg(\frac{c}{\|x_k - x_{k-1}\|^2} \int_0^{\|x_k - x_{k-1}\|} (\|x_k - x_{k-1}\| - u) \\
&\quad \times L(\|x_{k-1} - x_0\| + u)\,du + d\bigg)\|x_k - x_{k-1}\|^2 \\
&\leq \bigg(\frac{c}{(s_k - s_{k-1})^2} \int_0^{s_k - s_{k-1}} (s_k - s_{k-1} - u) L(s_{k-1} + u)\,du + d\bigg)(s_k - s_{k-1})^2 \\
&= c \int_0^{s_k - s_{k-1}} (s_k - s_{k-1} - u) L(s_{k-1} + u)\,du + d[s_k^2 - s_{k-1}^2 - 2s_{k-1}(s_k - s_{k-1})] \\
&= f(s_k) - f(s_{k-1}) - f'(s_{k-1})(s_k - s_{k-1})
\end{aligned}$$

$$
\begin{aligned}
&\le f(s_k) - f(s_{k-1}) - (g'(s_{k-1}) + \theta + 2ds_{k-1})(s_k - s_{k-1}) \\
&\le f(s_k) - f(s_{k-1}) - g_0'(s_{k-1})(s_k - s_{k-1}) \\
&\quad + (g_0'(s_{k-1}) - g'(s_{k-1}))(s_k - s_{k-1}) - (\theta + 2ds_{k-1})(s_k - s_{k-1}) \\
&\le f(s_k)
\end{aligned}
$$

by (2.16), (2.8), (2.9) and (2.32). Moreover, since f is decreasing on $[0, s^\star]$, one obtains

$$(1 + \sqrt{\eta})\|\mathcal{F}'(x_0)^{-1}\mathcal{F}(x_k)\| \le f(s_k) \le f(s_0) = b. \tag{2.42}$$

Therefore, we deduce that

$$\sqrt{\eta}\,\|\mathcal{F}'(x_0)^{-1}\mathcal{F}(x_k)\| \le \frac{\sqrt{\eta}}{1+\sqrt{\eta}}\, b = \sqrt{\eta}\,\|\mathcal{F}'(x_0)^{-1}\mathcal{F}(x_0)\| \le 1. \tag{2.43}$$

Hence, we conclude that (2.33) and (2.34) hold. ∎

Moreover, we need the following version of the Banach lemma on invertible operators.

LEMMA 2.4 ([8]). *Suppose that $\mathcal{F}'$ satisfies the center-Lipschitz condition* (1.9). *Let r satisfy $\int_0^r L_0(u)\,du \le 1$. Then $F'(x)^{-1} \in \mathcal{L}(\mathbf{Y}, \mathbf{X})$ for each $x \in U(x_0, r)$ and*

$$\|\mathcal{F}'(x)^{-1}\mathcal{F}'(x_0)\| \le \Big(1 - \int_0^{\rho(x)} L_0(u)\,du\Big)^{-1}. \tag{2.44}$$

Furthermore, we need a standard result concerning the behavior of a certain function.

LEMMA 2.5 ([24–26]). *Suppose $0 \le R_0 < R$. Define a function φ on $[0, R - R_0)$ by*

$$f(t) = \frac{1}{t^2}\int_0^t L(R_0 + u)(t - u)\,du. \tag{2.45}$$

Then f is increasing on $[0, R - R_0]$.

We can now show the following semilocal convergence result for (**INM**).

THEOREM 2.6. *Suppose that*

$$b \le \min\{1/\sqrt{\eta}, b_\lambda\}, \qquad \overline{U}(x_0, s^\star) \subseteq U(x_0, R)$$

and $\mathcal{F}'(x_0)^{-1}\mathcal{F}'$ satisfies the weak Lipschitz condition (1.9) *on $U(x_0, s^\star)$. Then the sequence $\{x_n\}$ generated by the* (**INM**) (1.4) *converges to a solution $x^\star$ of equation* (1.1). *Moreover,*

$$\|x_n - x^\star\| \le s^\star - s_n, \qquad n = 0, 1, 2, \ldots. \tag{2.46}$$

Proof. First we show by induction that (2.32) holds for each $n = 1, 2, \ldots$. For $n = 1$, by the above condition and (2.30), the first inequality in (2.32) is satisfied, while the second can be proved as follows. First,

$$\begin{aligned}\|x_1 - x_0\| &\le \|\mathcal{F}'(x_0)^{-1}\mathcal{F}(x_0)\| + \|\mathcal{F}'(x_0)^{-1}r_0\| \\ &\le a + \eta a^2 \le a + a\sqrt{\eta} = (1+\sqrt{\eta})a = b = s_1 - s_0.\end{aligned}$$

Suppose that (2.32) holds for all $n \le k$. Then from Lemma 2.4,

$$(1+\sqrt{\eta})\|\mathcal{F}'(x_0)^{-1}\mathcal{F}(x_k)\| \le f(s_k), \qquad \sqrt{\eta}\,\|\mathcal{F}'(x_0)^{-1}\mathcal{F}(x_k)\| \le 1.$$

Then, by (2.33), (2.37), the weak Lipschitz condition (1.9) and Lemma 2.3, we obtain

(2.47)

$$\begin{aligned}\|x_{k+1} - x_k\| &\le \|\mathcal{F}'(x_k)^{-1}\mathcal{F}'(x_0)\|(\|\mathcal{F}'(x_0)^{-1}\mathcal{F}(x_k)\| + \|\mathcal{F}'(x_0)^{-1}r_k\|) \\ &\le \frac{1}{1 - \int_0^{\rho(x_k)} L(u)\,du}(\|\mathcal{F}'(x_0)^{-1}\mathcal{F}(x_k)\| + \eta\|\mathcal{F}'(x_k)^{-1}\mathcal{F}(x_k)\|^2) \\ &\le \frac{1+\sqrt{\eta}}{1 - c\int_0^{\rho(x_k)} L(u)\,du}\|\mathcal{F}'(x_0)^{-1}\mathcal{F}(x_k)\| \\ &\le -\frac{f(s_k)}{g_0'(s_k)} = s_{k+1} - s_k.\end{aligned}$$

Thus (2.32) is valid for $n = k+1$. Hence, it holds for each $n \ge 1$. Hence, for $n \ge 0$ and $k \ge 0$ we have

$$\begin{aligned}\|x_{k+n} - x_n\| &\le \sum_{i=1}^{k} \|x_{i+n} - x_{i+n-1}\| \\ &\le \sum_{i=1}^{k} (s_{i+n} - s_{i+n-1}) = s_{k+n} - s_n.\end{aligned} \tag{2.48}$$

The sequence $\{s_n\}$ converges to $s^\star$ according to Lemma 2.1. Hence, it follows from (2.48) that $\{x_n\}$ is a Cauchy sequence in the Banach space $\mathbf{X}$ and so it converges to some $x^\star \in \overline{U}(x_0, s^\star)$. Letting $k \to \infty$ in (2.48) yields

$$\|x_n - x^\star\| \le s^\star - s_n, \qquad n = 0, 1, 2, \ldots. \ \blacksquare$$

REMARK 2.7. (i) If $c = 1$ and $d = 0$, Theorem 2.6 reduces to Theorem 3.6 in [8]. Moreover, if $\eta_n = 0$ for each $n = 0, 2, 3, \ldots$ then (**INM**) reduces to (**NM**). If $g_0(t) = g(t)$, the results further reduce to the corresponding ones developed by Shen et al. [21–23].

(ii) Condition (2.18) is sufficient for the convergence of the majorizing sequences $\{s_n\}$ and $\{t_n\}$. At this point we may wonder if (2.18) can be dropped and replaced by a possibly weaker sufficient convergence condition. Let us define the scalar sequences $\{\alpha_n\}$ and $\{\varphi_n\}$ by

(2.49)

$$\alpha_0 = 0, \qquad \alpha_1 = b,$$

$$\alpha_2 = \alpha_1 - \frac{\left[\frac{c}{(\alpha_1-\alpha_0)^2}\int_0^{\alpha_1-\alpha_0}(\alpha_1-\alpha_0-u)L_0(\alpha_0+u)\,du + d\right](\alpha_1-\alpha_0)^2}{g_0'(\alpha_1)},$$

$$\alpha_{n+1} = \alpha_n - \frac{\left[\frac{c}{(\alpha_n-\alpha_{n-1})^2}\int_0^{\alpha_n-\alpha_{n-1}}(\alpha_n-\alpha_{n-1}-u)L(\alpha_{n-1}+u)\,du + d\right](\alpha_n-\alpha_{n-1})^2}{g_0'(\alpha_n)},$$

and

(2.50)

$$\varphi_0 = 0, \qquad \varphi_1 = b,$$

$$\varphi_{n+1} = \varphi_n - \frac{\left[\frac{c}{(\varphi_n-\varphi_{n-1})^2}\int_0^{\varphi_n-\varphi_{n-1}}(\varphi_n-\varphi_{n-1}-u)L(\varphi_{n-1}+u)\,du + d\right](\varphi_n-\varphi_{n-1})^2}{g_0'(\varphi_n)}.$$

Then, according to (2.41) and (2.47), the sequences $\{\alpha_n\}$ and $\{\varphi_n\}$ are also majorizing sequences for $\{x_n\}$. Moreover, in view of (1.10), the sequence $\{\alpha_n\}$ is tighter than $\{\varphi_n\}$, which is tighter than $\{s_n\}$ (and $\{t_n\}$). Clearly the sequences $\{\alpha_n\}$ and $\{\varphi_n\}$ converge under (2.18) and can replace $\{s_n\}$ in Theorem 2.6. Moreover a simple inductive argument (see also the proof of Lemma 2.2) shows that

$$\alpha_n \le \varphi_n \le s_n, \tag{2.51}$$

$$\alpha_{n+1} - \alpha_n \le \varphi_{n+1} - \varphi_n \le s_{n+1} - s_n, \tag{2.52}$$

$$\alpha^\star = \lim_{n\to\infty} \alpha_n \le \varphi^\star = \lim_{n\to\infty} \varphi_n \le s^\star. \tag{2.53}$$

In the next section, we shall present sufficient convergence conditions for $\{\alpha_n\}$ and $\{\varphi_n\}$ which are different and can be weaker than (2.18).

3. Semilocal convergence of (INM). Using our new idea of recurrent functions, we provide a semilocal convergence analysis for (**INM**). The concept of recurrent functions has already been used to produce finer convergence analysis for iterative methods using invertible operators or outer or generalized inverses [8].

Let us first define some sequences and functions.

Definition 3.1. Let $b > 0$, $c > 0$ and $d > 0$. Let us define the sequence $\{\phi_n\}$, the functions f_n, ϵ_n, μ_n on $[0, 1)$ and ξ on $I = [0, 1]^2 \times [1, 1/(1-t)]^2$ $(t \in [0, 1))$ by

$$\begin{aligned} &\phi_0 = 0, \qquad \phi_1 = b, \\ &\phi_{n+1} = \phi_n + \frac{\delta_n + d(\phi_n - \phi_{n-1})}{1 - \overline{\delta}_n}(\phi_n - \phi_{n-1}) \quad (n \ge 1). \end{aligned} \tag{3.1}$$

(3.2) $$f_n(t) = c\int_0^1 \int_{w_{n-1}(t)}^{z_{n-1}(t)} L(u)\,du\,d\theta + t^{n-1}b - \lambda + t\int_0^{w_n(t)} L_0(u)\,du$$

(3.3) $$\epsilon_n(t) = c\int_0^1 \Big(\int_{w_n(t)}^{z_n(t)} L(u)\,du - \int_{w_{n-1}(t)}^{z_{n-1}(t)} L(u)\,du\Big)\,d\theta + d(\lambda^n - \lambda^{n-1})b + d\int_{w_n(t)}^{z_{n+1}(t)} L_0(u)\,du,$$

(3.4) $$\mu_n(t) = c\int_0^1 \Big(\int_{w_{n+1}(t)}^{z_{n+1}(t)} L(u)\,du + \int_{w_{n-1}(t)}^{z_{n-1}(t)} L(u)\,du - 2\int_{w_n(t)}^{z_n(t)} L(u)\,du\Big) + d(\lambda^{n+1} - \lambda^{n-1} - 2\lambda^{n-1})b + t\Big(\int_{w_{n+1}(t)}^{z_{n+1}(t)} L_0(u)\,du - \int_{w_n(t)}^{z_n(t)} L_0(u)\,du\Big),$$

(3.5) $$\xi(\theta, t, \lambda_1, \lambda_2) = \int_0^1 \Big(\int_{(\lambda_1+\lambda_2+\lambda_2 t)b}^{(\lambda_1+\lambda_2+\lambda_2 t+\theta t^2)b} L(u)\,du + \int_{\lambda_1 b}^{(\lambda_1+\theta\lambda_2)b} L(u)\,du - 2\int_{(\lambda_1+\lambda_2)b}^{(\lambda_1+\lambda_2+\theta\lambda_2 t)b} bL(u)\,du\Big)\,d\theta + d(\lambda^{n+1} - \lambda^{n-1} - 2\lambda^{n-1})b + t\Big(\int_{(\lambda_1+\lambda_2+\lambda_2 t)b}^{(\lambda_1+\lambda_2 t+t^2\lambda_2)b} L_0(u)\,du - \int_{(\lambda_1+\lambda_2)b}^{(\lambda_1+\lambda_2 t)b} L_0(u)\,du\Big),$$

where

(3.6) $$\delta_n = \int_0^1 \int_{\phi_{n-1}}^{\phi_{n-1}+\theta(\phi_n-\phi_{n-1})} L(u)\,du\,d\theta,$$

(3.7) $$\overline{\delta}_n = \int_0^{\phi_n} L_0(u)\,du,$$

(3.8) $$z_n(t) = \left(\frac{1-t^n}{1-t} + \theta t^n\right)b,$$

(3.9) $$w_n(t) = \frac{1-t^n}{1-t}b,$$

(3.10) $$\gamma = db - t.$$

The function f_∞ is defined on $[0, 1)$ by

(3.11) $$f_\infty(t) = \lim_{n\to\infty} f_n(t).$$

Remark 3.2. Using (3.2) and (3.11), we obtain

$$f_\infty(t) = t \int_0^{b/(1-t)} L_0(u)\,du + \gamma. \tag{3.12}$$

From (3.2)–(3.10), we deduce the following identities:

$$f_{n+1}(t) = f_n(t) + \epsilon_n(t), \tag{3.13}$$

$$\epsilon_{n+1}(t) = \epsilon_n(t) + \mu_n(t), \tag{3.14}$$

$$\mu_n(t) = \xi(\theta, t, \lambda_1) = (1 + t + \cdots + t^{n-2})b, \qquad \lambda_2 = t^{n-1}b. \tag{3.15}$$

We need the following result on majorizing sequences for (**INM**).

Lemma 3.3. *Let the sequence* $\{\phi_n\}$ *and functions* f_n, ϵ_n, μ_n *and* ξ *be as in Definition* 3.1. *Furthermore assume that there exists* $\alpha \in (0,1)$ *such that*

$$b\int_0^1\int_0^{\theta b} L(u)\,du\,d\theta + db \le \alpha\Big(1 - \int_0^b L_0(u)\,du\Big), \tag{3.16}$$

$$\gamma = db - \alpha < 0, \tag{3.17}$$

$$\xi(\theta, \lambda_1, \lambda_2, \lambda_3) \ge 0 \qquad \text{on } I, \tag{3.18}$$

$$\epsilon_1(\alpha) \ge 0, \tag{3.19}$$

$$f_\infty(\alpha) \le 0. \tag{3.20}$$

Then the sequence $\{\phi_n\}$ *given by* (3.1) *is nondecreasing, bounded from above by*

$$\phi^{\star\star} = \frac{b}{1-\alpha} \tag{3.21}$$

and converges to its least upper bound $\phi^\star$ *such that*

$$\phi^\star \in [0, \phi^{\star\star}]. \tag{3.22}$$

Moreover, for all $n \ge 0$,

$$0 \le \phi_{n+1} - \phi_n \le \alpha(\phi_n - \phi_{n-1}) \le \alpha^n b, \tag{3.23}$$

$$0 \le \phi^\star - \phi_n \le \frac{\alpha^n b}{1-\alpha}. \tag{3.24}$$

Proof. Estimate (3.23) holds if

$$\delta_n + d(\phi_n - \phi_{n-1}) \le \alpha(1 - \overline{\delta}_n) \tag{3.25}$$

for all $n \ge 1$. It follows from (3.1), (3.16) and (3.17) that (3.25) holds for $n = 1$. We also notice that (3.23) holds for $n = 1$ and

$$\phi_n \le \frac{1-\alpha^n}{1-\alpha} b < \phi^{\star\star}. \tag{3.26}$$

By the induction hypotheses and (3.26), the estimate (3.25) is true if

$$\delta_k + d(\phi_k - \phi_{k-1})b + \alpha\overline{\delta}_k + d(\phi_k - \phi_{k-1}) - \delta \le 0 \tag{3.27}$$

or

$$(3.28)\quad \int_0^1 \int_{w_{k-1}(\alpha)}^{z_{k-1}(\alpha)} L(u)\,du\,d\theta + d\lambda^{k-1}b + \alpha \int_0^{w_k(\alpha)} L_0(u)\,du + d(\phi_k - \phi_{k-1}) - \alpha \le 0$$

for all $k \le n$. Estimate (3.28) motivates introducing the function f_k given by (3.2) and we will show instead of (3.28) that

$$(3.29)\qquad f_k(\alpha) \le 0.$$

From (3.13)–(3.15) (for $t = \alpha$) and (3.19) we have

$$(3.30)\qquad f_{k+1}(\alpha) \ge f_k(\alpha).$$

In view of (3.11), (3.12) and (3.30), the estimate (3.29) will hold if (3.20) holds since

$$(3.31)\qquad f_k(\alpha) \le f_\infty(\alpha),$$

and this completes the induction. It follows from (3.23) and (3.26) that the sequence $\{\phi_n\}$ is nondecreasing, bounded from above by $\phi^{\star\star}$ given by (3.21) and so converges to $\phi^\star$. Finally, (3.24) follows from (3.23) by using standard majorization techniques [10]. ■

Using the recurrent function approach, we now show the following semilocal convergence result for (**INM**).

THEOREM 3.4. *Suppose that $\mathcal{F}'$ satisfies* (3.4), (3.6) *on $U(x_0, \phi^\star) \subseteq \mathcal{D}$, and the hypotheses of Lemma* 3.3 *as well as*

$$(3.32)\qquad \overline{U}(x_0, \phi^\star) \subseteq \mathcal{D}.$$

Then the sequence $\{x_n\}$ generated by (**INM**) *is well defined, remains in $\overline{U}(x_0, \phi^\star)$ for all $n \ge 0$ and converges to a zero $x^\star$ of $\mathcal{F}'(\cdot)\mathcal{F}(\cdot)$ in $\overline{U}(x_0, \phi^\star)$. Moreover,*

$$(3.33)\qquad \|x_{n+1} - x_n\| \le \phi_{n+1} - \phi_n,$$

$$(3.34)\qquad \|x_n - x^\star\| \le \phi^\star - \phi_n.$$

Proof. As in Theorem 2.6, we deduce the estimates (3.33) and (3.34) (with ϕ_k replacing s_k), which in view of (3.1) lead to

$$(3.35)\qquad \|x_{k+1} - x_k\| \le \phi_{n+1} - \phi_n.$$

Estimates (3.18), (3.19), (3.35) and Lemma 3.5 imply that $\{x_k\}$ is a Cauchy sequence in $\mathbb{R}^m$ and hence converges to some $x^\star \in \overline{U}(x_0, \phi^\star)$. ■

REMARK 3.5. (i) The point $\phi^{\star\star}$ given by (3.21) can replace $\phi^\star$ in hypothesis (3.32).

(ii) The hypotheses of Lemma 3.2 involve only computations with the initial data. These hypotheses differ from (3.13) given in Theorem 2.6. In practice, one has to test which of the two are satisfied. If both are satisfied, we use the more precise error bounds given in Theorem 3.4.

In the next section, we show that the conditions of Theorem 3.4 can be weaker than those of Theorem 2.6.

4. Special cases and numerical examples. We compare the Kantorovich type conditions introduced in Section 2 with the corresponding conditions in Section 3.

Special case (Newton's method). Let $L(u) = l > 0$, $L_0(u) = l_0 > 0$, $\eta = 0$, $\theta = 0$, $c = 1$. Then the sequences introduced in Sections 2 and 3 reduce to

$$s_0 = 0, \quad s_{n+1} = s_n - \frac{f(s_n)}{g_0'(s_n)} \quad \text{for } n = 0, 1, 2, \ldots, \tag{4.1}$$

$$t_0 = 0, \quad t_{n+1} = t_n - \frac{f(t_n)}{g'(t_n)} \quad \text{for } n = 0, 1, 2, \ldots, \tag{4.2}$$

$$\alpha_0 = 0, \quad \alpha_1 = b, \quad \alpha_2 = \alpha_1 + \frac{l_0(\alpha_1 - \alpha_0)^2}{2(1 - l_0\alpha_1)}, \tag{4.3}$$

$$\alpha_{n+2} = \alpha_{n+1} + \frac{l(\alpha_{n+1} - \alpha_n)^2}{2(1 - l_0\alpha_{n+1})} \quad \text{for } n = 1, 2, \ldots, \tag{4.4}$$

$$\phi_0 = 0, \quad \phi_1 = b, \quad \phi_{n+2} = \phi_{n+1} + \frac{l(\phi_{n+1} - \phi_n)^2}{2(1 - l_0\phi_{n+1})} \quad \text{for } n = 0, 1, 2, \ldots, \tag{4.5}$$

$$f_n(t) = \frac{l}{2}bt^n + l_0 b(1 + t + \cdots + t^n)t - t, \quad f_\infty(t) = t\left(\frac{l_0 b}{1 - t} - 1\right), \tag{4.6}$$

$$\epsilon_n(t) = \frac{1}{2}(2l_0t^2 + lt - l)t^n b, \quad \mu_n(t) = \frac{1}{2}b(2l_0t^2 + lt - l)(t_{n+1} - t_n), \tag{4.7}$$

$$\delta_n(t) = \frac{l}{2}(t_{n+1} - t_n)^2, \quad \overline{\delta}_n(t) = l_0 t_n, \tag{4.8}$$

$$z_n(t) = (1 + t + \cdots + t^{n-1} + \theta t^n)b, \quad w_n(t) = (1 + t + \cdots + t^{n-1})b, \tag{4.9}$$

$$\alpha = \frac{2l}{l + \sqrt{l^2 + 8l_0 l}}. \tag{4.10}$$

Then the sequences $\{s_n\}$ and $\{t_n\}$ converge if the hypotheses of Theorem 2.5 are satisfied, that is, if the Kantorovich criterion [6]

$$h_1 = 2l\eta \leq 1 \tag{4.11}$$

is satisfied, and then we have

$$s^\star = t^\star = \frac{1 - \sqrt{1 - 2l\eta}}{l}. \tag{4.12}$$

The sequence $\{\phi_n\}$ converges if the hypotheses of Theorem 3.4 are satisfied [9, 6], i.e. if

$$h_2 = \bar{l}\eta \le 1, \tag{4.13}$$

where

$$\bar{l} = \frac{1}{4}(l + 4l_0 + \sqrt{l^2 + 8l_0 l}), \tag{4.14}$$

and then we have

$$\phi^\star \le \overline{\phi}^\star = \frac{\eta}{1-\alpha}. \tag{4.15}$$

Finally, the sequence $\{\alpha_n\}$ converges [9] if

$$h_3 = \bar{\bar{l}}\eta \le 1, \tag{4.16}$$

where

$$\bar{\bar{l}} = \frac{1}{4}\Big(4l_0 + \sqrt{l_0 l} + \sqrt{l_0 l + 8l_0^2}\Big), \tag{4.17}$$

and we have

$$\alpha^\star \le \overline{\alpha}^\star = \left(1 + \frac{l_0\eta}{2(1-\alpha)(1-l_0\eta)}\right)\eta. \tag{4.18}$$

Note that

$$\bar{\bar{l}} \le \bar{l} \le l. \tag{4.19}$$

If $l_0 = l$ double equality holds in (4.19). Otherwise (i.e. if $l_0 < l$) then inequalities in (4.19) are strict. Consequently, it follows from (4.11), (4.13), (4.16) and (4.19) that

$$h_1 \le 1 \;\Rightarrow\; h_2 \le 1 \;\Rightarrow\; h_3 \le 1 \tag{4.20}$$

and

$$h_3/h_1 \to 0, \quad h_3/h_2 \to 0, \quad h_2/h_1 \to 1/4 \quad \text{as } l/l_0 \to \infty. \tag{4.21}$$

Next we present examples where $l_0 < l$.

EXAMPLE 4.1. Let $\mathbf{X} = \mathbf{Y} = \mathbb{R}$, $x_0 = 1$ and $\mathcal{D} = [\zeta, 2-\zeta]$ for $\zeta \in (0, 0.5)$. Let us consider the nonlinear map

$$\mathcal{F}(x) = x^3 - \zeta.$$

Then we get

$$\eta = \frac{1-\zeta}{3}, \quad l_0 = 3-\zeta, \quad l = 2(2-\zeta).$$

From the Kantorovich hypothesis (4.11), we get

$$h_1 > 1 \quad \text{for all } \zeta \in (0, 0.5).$$

Hence, there is no guarantee that the sequence $\{x_n\}$ converges to $\sqrt[3]{\zeta}$. But from (4.13) and (4.16), we observe that

$$h_2 \le 1 \quad \text{for all } \zeta \in [0.450339002, 0.5),$$
$$h_3 \le 1 \quad \text{for all } \zeta \in [0.4271907643, 0.5).$$

Hence, our approach extends the applicability of (**NM**).

EXAMPLE 4.2. Let $x_0 = 0$. Define $F(x) = d_0 x + d_1 + d_2 \sin e^{d_3 x}$, where d_i, $i = 0, 1, 2, 3$, are given parameters. It can be easily seen that for d_3 large and d_2 sufficiently small, l_0/l can be arbitrarily small.

EXAMPLE 4.3. Let $\mathbf{X} = \mathbf{Y} = \mathcal{C}[0, 1]$ be equipped with the max-norm. Consider the following nonlinear boundary value problem [6]

$$\begin{cases} u'' = -u^3 - \Psi u^2, \\ u(0) = 0, \quad u(1) = 1. \end{cases}$$

It is well known that this problem can be formulated as the integral equation

$$u(s) = s + \int_0^1 G(s,t)(u^3(t) + \Psi u^2(t))\, dt, \tag{4.22}$$

where G is the Green's function

$$G(s,t) = \begin{cases} t(1-s), & t \le s, \\ s(1-t), & s < t. \end{cases}$$

We observe that

$$\max_{0 \le s \le 1} \int_0^1 |G(s,t)|\, dt = \frac{1}{8}.$$

Then problem (4.22) is in the form (1.1), where $\mathcal{F} : \mathcal{D} \to \mathbf{Y}$ is defined by

$$[\mathcal{F}(x)](s) = x(s) - s - \int_0^1 G(s,t)(x^3(t) + \Psi x^2(t))\, dt.$$

Set $u_0(s) = s$ and $\mathcal{D} = U(u_0, R_0)$. It is easy to verify that $U(u_0, R_0) \subset U(0, R_0 + 1)$, since $\|u_0\| = 1$. If $2\Psi < 5$, the operator $\mathcal{F}'$ satisfies the conditions (4.13) and (3.16) with

$$\eta = \frac{1 + \Psi}{5 - 2\Psi}, \qquad l = \frac{\Psi + 6R_0 + 3}{4}, \qquad l_0 = \frac{2\Psi + 3R_0 + 6}{8}.$$

Note that $l_0 < l$.

EXAMPLE 4.4. Let $\mathbf{X} = \mathbf{Y} = \mathcal{C}[0, 1]$ with the max-norm $\|.\|$. Let $\theta \in [0, 1]$ be a given parameter. Consider the "cubic" Chandrasekhar integral equation

$$u(s) = u^3(s) + \lambda u(s) \int_0^1 q(s,t)u(t)\,dt + y(s) - \theta \tag{4.23}$$

[1–10]. Here the kernel $q(s,t)$ is a continuous function on $[0,1] \times [0,1]$. The parameter λ in (4.23) is a real number called the "albedo" for scattering, $y(s)$ is a given continuous function defined on $[0,1]$ and $x(s)$ is the unknown function sought in $\mathcal{C}[0,1]$. Equations of the form (4.23) arise in the kinetic theory of gases [6]. For simplicity, we choose $u_0(s) = y(s) = 1$ and $q(s,t) = s/(s+t)$ for all $s \in [0,1]$ and $t \in [0,1]$, with $s + t \neq 0$. If we let $\mathcal{D} = U(u_0, 1-\theta)$ and define the operator $\mathcal{F}$ on $\mathcal{D}$ by

$$\mathcal{F}(x)(s) = x^3(s) - x(s) + \lambda x(s) \int_0^1 q(s,t)x(t)\,dt + y(s) - \theta \tag{4.24}$$

for all $s \in [0,1]$, then every zero of $\mathcal{F}$ satisfies equation (4.23). We have

$$\max_{0\le s\le 1} \left| \int_0^1 \frac{s}{s+t}\,dt \right| = \ln 2.$$

Therefore, if we set $\xi = \|\mathcal{F}'(u_0)^{-1}\|$, we obtain

$$\eta = \xi(|\lambda| \ln 2 + 1 - \theta),$$
$$l = 2\xi(|\lambda| \ln 2 + 3(2-\theta)), \qquad l_0 = \xi(2|\lambda| \ln 2 + 3(3-\theta)).$$

It follows that if conditions (4.13) and (4.16) hold, then problem (4.23) has a unique solution near u_0. These assumptions are weaker than the one given before using the Newton–Kantorovich hypothesis (4.11), since we notice that $l = l_0 + 3\xi(1-\theta)$, thus $l_0 \le l$ for all $\theta \in [0,1]$.

Example 4.5. In this example, we examine influence of the nonzero residual vectors on the convergence of (**INM**) (1.4) when solving the Hammerstein type nonlinear integral equation

$$x(s) = 1 + \int_0^1 G(s,t)x(t)^2\,dt, \qquad t, s \in [0,1], \tag{4.25}$$

where $x \in \mathcal{C}[0,1]$ and the kernel is given as

$$G(s,t) = \begin{cases} (1-s)t, & t \le s, \\ s(1-t), & s \le t. \end{cases}$$

Here, we approximate the integral with the Gauss–Legendre quadrature. It will result in the nonlinear equation

$$x(s) = 1 + \sum_{i=1}^{m} w_i\, G(s,t_i)x(t_i)^2.$$

If we denote the approximation of $x(s)$ at the nodes t_j by x_j, then the above is equivalent to the following system of nonlinear equations:

$$\mathcal{F}(x) := x(s_j) - I - \sum_{i=1}^{m} b_{i,j}x(t_i)^2 = 0, \qquad i,j = 1,\dots,m, \tag{4.26}$$

where

$$b_{i,j} = \begin{cases} w_i(1-s_j)t_i, & i \le j, \\ w_i(1-t_i)s_j, & j \le i. \end{cases}$$

Now we employ (**INM**), given by (1.4), to solve the system (4.26). The integral in (4.25) is discretized by 8-point Gauss–Legendre quadrature. We consider two cases. In the first case the residual is fixed, $\|r_n\| = 10^{-10}$, while in the second case the residual depends upon the nonlinear iteration as follows: $\|r_n\| = 10^{-n}$. To solve the linear system, we use the Gauss–Seidel method. The iterations of the Gauss–Seidel method terminate when the residual stop criterion is satisfied, which in the first case is $\|r_n\| \le 10^{-10}$, and in the second case it is $\|r_n\| \le 10^{-n}$, where n is the number of (**INM**) iteration. For the first ten nonlinear iterations the results are reported in Figure 1. We observe that when the residual is $\|r_n\| = 10^{-n}$, we obtain $\|\mathcal{F}(x_{10})\| = 6.5 \cdot 10^{-11}$ after 54 Gauss–Seidel iterations; on the other hand, if the residual is $\|r_n\| = 10^{-10}$, we obtain $\|\mathcal{F}(x_{10})\| = 4.8 \cdot 10^{-11}$ after 71 Gauss–Seidel iterations. For computations, we use MATLAB 7.12.0, on a 64-bit Linux operating system, with double-precision floating point data type.

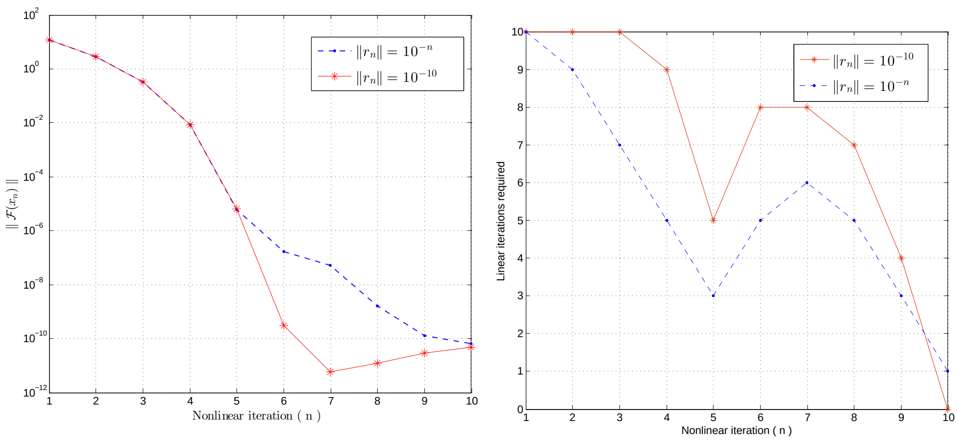

Fig. 1. Total number of iterations of linear solver: 71 if $\|r_n\| = 10^{-10}$, and 54 if $\|r_n\| = 10^{-n}$.

Acknowledgements. We appreciate reviewers' comments and suggestions which have enriched our work.

References

[1] I. K. Argyros, *A new convergence theorem for the inexact Newton methods based on assumptions involving the second Fréchet derivative*, Comput. Math. Appl. 37 (1999), 109–115.

[2] I. K. Argyros, *On the semilocal convergence of inexact Newton methods in Banach spaces*, J. Comput. Appl. Math. 228 (2009), 434–443.

[3] I. K. Argyros, *Relation between forcing sequences and inexact Newton iterates in Banach spaces*, Computing 63 (1999), 134–144.

[4] I. K. Argyros, *A unifying local-semilocal convergence analysis and applications for two point Newton-like methods in Banach space*, J. Math. Anal. Appl. 298 (2004), 374–397.

[5] I. K. Argyros, *Local convergence of inexact Newton-like-iterative methods and applications*, Comput. Math. Appl. 39 (2000), 69–75.

[6] I. K. Argyros, Y. J. Cho and S. Hilout, *Numerical Methods for Equations and its Applications*, CRC Press/Taylor and Francis, New York, 2012.

[7] I. K. Argyros and S. Hilout, *Inexact Newton methods and recurrent functions*, Appl. Math. (Warsaw) 37 (2010), 113–126.

[8] I. K. Argyros and S. Hilout, *Extending the applicability of the Gauss–Newton method under average Lipschitz-type conditions*, Numer. Algorithms 58 (2011), 23–52.

[9] I. K. Argyros and S. Hilout, *Weaker conditions for the convergence of Newton's method*, J. Complexity 28 (2012), 364–387.

[10] R. S. Dembo, S. C. Eisenstat and T. Steihaug, *Inexact Newton methods*, SIAM J. Numer. Anal. 19 (1982), 400–408.

[11] J. A. Ezquerro and M. A. Hernández, *Generalized differentiability conditions for Newton's method*, IMA J. Numer. Anal. 22 (2002), 187–205.

[12] J. A. Ezquerro and M. A. Hernández, *On an application of Newton's method to nonlinear operators with w-conditioned second derivative*, BIT 42 (2002), 519–530.

[13] M. G. Gasparo and B. Morini, *Inexact methods: forcing terms and conditioning*, J. Optim. Theory Appl. 107 (2000), 573–589.

[14] X. Guo, *On semilocal convergence of inexact Newton methods*, J. Comput. Math. 25 (2007), 231–242.

[15] J. M. Gutiérrez, *A new semilocal convergence theorem for Newton's method*, J. Comput. Appl. Math. 79 (1997), 131–145.

[16] J. M. Gutiérrez and M. A. Hernández, *Newton's method under weak Kantorovich conditions*, IMA J. Numer. Anal. 20 (2000), 521–532.

[17] M. A. Hernández, *The Newton method for operators with Hölder continuous first derivative*, J. Optim. Theory Appl. 109 (2001), 631–648.

[18] J. M. Martínez and L. Qi, *Inexact Newton method for solving nonsmooth equations*, J. Comput. Appl. Math. 60 (1995), 127–145.

[19] I. Moret, *A Kantorovich type theorem for inexact Newton methods*, Numer. Funct. Anal. Optim. 10 (1989), 351–365.

[20] B. Morini, *Convergence behaviour of inexact Newton methods*, Math. Comput. 68 (1999), 1605–1613.

[21] W. P. Shen and C. Li, *Convergence criterion of inexact methods for operators with Hölder continuous derivatives*, Taiwan. J. Math. 12 (2008), 1865–1882.

[22] W. P. Shen and C. Li, *Kantorovich-type convergence criterion for inexact Newton methods*, Appl. Numer. Math. 59 (2009), 1599–1611.

[23] W. P. Shen and C. Li, *Smale's α-theory for inexact Newton methods under the γ-condition*, J. Math. Anal. Appl. 369 (2010), 29–42.

[24] X. H. Wang, *Convergence of Newton's method and inverse function theorem in Banach space*, Math. Comp. 68 (1999), 169–186.

[25] X. H. Wang, *Convergence of Newton's method and uniqueness of the solution of equations in Banach space*, IMA J. Numer. Anal. 20 (2000), 123–134.

[26] X. H. Wang and C. Li, *Convergence of Newton's method and uniqueness of the solution of equations in Banach spaces, II*, Acta Math. Sinica (English Ser.) 19 (2003), 405–412.

[27] M. Wu, *A new semi-local convergence theorem for the inexact Newton methods*, Appl. Math. Comp. 200 (2008), 80–86.

[28] T. J. Ypma, *Local convergence of inexact Newton methods*, SIAM J. Numer. Anal. 21 (1984), 583–590.

I. K. Argyros
Department of Mathematical Sciences
Cameron University
Lawton, OK 73505-6377, U.S.A.
E-mail: iargyros@cameron.edu

S. K. Khattri
Department of Engineering
Stord/Haugesund University College
N-414 Stord, Norway
E-mail: sanjay.khattri@hsh.no

Received on 5.11.2012;
revised version on 16.4.2013 (2157)