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UNIQUENESS OF SOLUTIONS FOR SOME DEGENERATE NONLINEAR ELLIPTIC EQUATIONS

Abstract. We investigate the existence and uniqueness of solutions to the Dirichlet problem for a degenerate nonlinear elliptic equation

$$-\sum_{i,j=1}^{n} D_j(a_{ij}(x)D_iu(x))+b(x)u(x)+\operatorname{div}(\Phi(u(x))) = g(x)-\sum_{j=1}^{n} f_j(x) \quad \text{on } \Omega$$

in the setting of the space $H_0(\Omega)$.

1. Introduction. In this work we prove the existence of (weak) solutions in the space $H_0(\Omega)$ (see Definition 2.5) for the Dirichlet problem

$$\text{(P)} \qquad \begin{cases} Lu(x) + \operatorname{div}(\Phi(u(x))) = g(x) - \sum_{j=1}^{n} D_j f_j(x) & \text{on } \Omega, \\ u(x) = 0 & \text{on } \partial\Omega, \end{cases}$$

where L is the partial differential operator

$$Lu(x) = -\sum_{i,j=1}^{n} D_j(a_{ij}(x)D_iu(x)) + b(x)u(x) \tag{1.1}$$

with $D_j = \partial/\partial x_j$, where Ω is a bounded open set in $\mathbb{R}^n$ and we assume that Ω has a Lipschitz boundary $\partial\Omega$ with outward unit normal $\vec{\eta}(x) = (\eta_1(x),\ldots,\eta_n(x))$, the coefficients a_{ij} are measurable, real valued functions, the coefficient matrix $\mathcal{A} = (a_{ij}(x))$ is symmetric and satisfies the degenerate ellipticity condition

$$|\xi|^2\omega(x) \le \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \le |\xi|^2 v(x), \tag{1.2}$$

for all $\xi \in \mathbb{R}^n$ and a.e. $x \in \Omega$, ω and v are weight functions and $\Phi : \mathbb{R} \to \mathbb{R}^n$.

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By a *weight*, we shall mean a locally integrable function ω on $\mathbb{R}^n$ such that $\omega(x) > 0$ for a.e. $x \in \mathbb{R}^n$. Every weight ω gives rise to a measure on the measurable subsets of $\mathbb{R}^n$ through integration. This measure will also be denoted by ω. Thus, $\omega(E) = \int_E \omega(x)\,dx$ for measurable sets $E \subset \mathbb{R}^n$.

In general, the Sobolev spaces $W^{k,p}(\Omega)$ without weights occur as spaces of solutions for elliptic and parabolic partial differential equations. For degenerate partial differential equations, i.e., equations with various kinds of singularities in the coefficients, it is natural to look for solutions in weighted Sobolev spaces (see [1]–[4] and [7]). The type of the weight depends on the equation.

A class of weights which is particularly well understood is the class of A_p-weights (or Muckenhoupt class), introduced by B. Muckenhoupt [11]. These classes have found many applications in harmonic analysis (see [14] and [15]). Another reason for studying A_p-weights is the fact that powers of the distance to submanifolds of $\mathbb{R}^n$ often belong to A_p (see [9]). There are, in fact, many interesting examples of weights (see [7] for p-admissible weights).

Equations like (1.1) have been studied by many authors in the nondegenerate case (i.e. with $\omega(x) = v(x) \equiv 1$) (see e.g. [6], [8] and [12] and the references therein).

Let us briefly describe the content of the paper. In Section 2, we give necessary definitions and basic results. In Section 3, we prove the existence and uniqueness of weak solutions to problem (P).

The following theorem will be proved in Section 3.

THEOREM 1.1. *Let Ω be an open bounded set in $\mathbb{R}^n$ with a Lipschitz boundary $\partial\Omega$ and let ω and v be two weights. Suppose that*

(H1) $f_j/\omega \in L^p(\Omega, \omega)$ $(j = 1, \ldots, n)$ *with* $p > nr \geq 2$;
(H2) $g/v \in L^q(\Omega, v)$ *with* $1/q = 1/p + 1/nr$;
(H3) $(v, \omega) \in A_r$ *with* $1 < r < p' < nr$ (*where* $1/p + 1/p' = 1$);
(H4) $b(x) \geq 0$ *for a.e.* $x \in \Omega$ *and* $b/\omega \in L^\infty(\Omega)$;
(H5) $\Phi : \mathbb{R} \to \mathbb{R}^n$ $(\Phi = (\Phi_1, \ldots, \Phi_n))$, *with* $|\Phi(u)|/\omega \in L^2(\Omega, \omega)$ *if* $u \in H_0(\Omega)$ *and the functions* Φ_j *are continuous* $(j = 1, \ldots, n)$.

Then problem (P) *has a solution* $u \in H_0(\Omega)$. *Moreover,* $u \in L^\infty(\Omega)$ *with*

$$\|u\|_{L^\infty(\Omega)} \leq C\Big(\|g/v\|_{L^q(\Omega,v)} + \sum_{j=1}^n \|f_j/\omega\|_{L^p(\Omega,\omega)}\Big) \tag{1.3}$$

where C is a constant independent of u, g, f_j and Φ_j. If moreover

(H6) $|\Phi_j(u_1(x)) - \Phi_j(u_2(x))| \leq C_0 v(x)|u_1(x) - u_2(x)|$ *for all* $u_1, u_2 \in H_0(\Omega)$, *a.e.* $x \in \Omega$ *and C_0 is a positive constant,*

then problem (P) *has a unique solution.*

Remark 1.2. The estimate (1.3) is an important ingredient in the proof of the existence of a weak solution to problem (P). Under the assumption $\Phi_j = 0$ ($j = 1, \dots, n$) and $\omega = v \equiv 1$ (non-degenerate case), (1.3) is the usual L^∞-estimate of Stampacchia (see [8]).

2. Definitions and basic results

Definition 2.1. Let ω be a locally integrable nonnegative function in $\mathbb{R}^n$ and assume that $0 < \omega(x) < \infty$ almost everywhere. We say that ω belongs to the *Muckenhoupt class* A_p, $1 < p < \infty$, or that ω is an A_p-*weight*, if there is a constant $C = C_{p,\omega}$ such that

$$\left(\frac{1}{|B|}\int_B \omega(x)\,dx\right)\left(\frac{1}{|B|}\int_B \omega^{1/(1-p)}(x)\,dx\right)^{p-1} \le C_{p,\omega}$$

for all balls $B \subset \mathbb{R}^n$, where $|\cdot|$ denotes the n-dimensional Lebesgue measure in $\mathbb{R}^n$ (see [5], [7], [15] or [16] for more information about A_p-weights).

The union of all Muckenhoupt classes is denoted by

$$A_\infty = \bigcup_{p>1} A_p.$$

The weight ω satisfies the *doubling condition* if there exists a positive constant C such that

$$\omega(B(x,2r)) \le C\omega(B(x,r))$$

for every ball $B = B(x,r) \subset \mathbb{R}^n$, where $\omega(B) = \int_B \omega(x)\,dx$. If $\omega \in A_p$, then ω is doubling (see [7, Corollary 15.7]).

As an example the function $\omega(x) = |x|^\alpha$, $x \in \mathbb{R}^n$, is in A_p if and only if $-n < \alpha < n(p-1)$ (see [15, Corollary 4.4, Chapter IX]). If $\varphi \in \mathrm{BMO}(\mathbb{R}^n)$ then $\omega(x) = e^{\alpha\varphi(x)} \in A_2$ for some $\alpha > 0$ (see [14]).

Definition 2.2. Let ω be a weight, and let $\Omega \subset \mathbb{R}^n$ be open. For $0 < p < \infty$, we define $L^p(\Omega,\omega)$ as the set of measurable functions f on Ω such that

$$\|f\|_{L^p(\Omega,\omega)} = \left(\int_\Omega |f(x)|^p \omega(x)\,dx\right)^{1/p} < \infty.$$

Remark 2.3. If $\omega \in A_p$, $1 < p < \infty$, then since $\omega^{-1/(p-1)}$ is locally integrable, we have $L^p(\Omega,\omega) \subset L^1_{\mathrm{loc}}(\Omega)$ for every open set Ω (see [16, Remark 1.2.4]). It thus makes sense to talk about weak derivatives of functions in $L^p(\Omega,\omega)$. We also know that the dual space of $L^p(\Omega,\omega)$ is $L^{p'}(\Omega,\omega^{1-p'})$.

Definition 2.4. Let $\Omega \subset \mathbb{R}^n$ be open, $1 < p < \infty$, and let ω be an A_p-weight, $1 < p < \infty$. We define the weighted Sobolev space $W^{1,p}(\Omega,\omega)$ as the set of functions $u \in L^p(\Omega,\omega)$ with weak derivatives $D_j u \in L^p(\Omega,\omega)$

for $j = 1, \ldots, n$. The norm of u in $W^{1,p}(\Omega, \omega)$ is defined by

$$(2.1) \qquad \|u\|_{W^{1,p}(\Omega,\omega)} = \Big(\int_\Omega |u(x)|^p \omega(x)\, dx + \sum_{j=1}^n \int_\Omega |D_j u(x)|^p \omega(x)\, dx \Big)^{1/p}.$$

The space $W_0^{1,p}(\Omega, \omega)$ is the closure of $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{W_0^{1,p}(\Omega,\omega)} = \Big(\sum_{j=1}^n \int_\Omega |D_j u(x)|^p \omega(x)\, dx \Big)^{1/p}.$$

The dual space of $W_0^{1,p}(\Omega, \omega)$ is $W^{-1,p'}(\Omega, \omega)$ (see [3]), where

$$W^{-1,p'}(\Omega, \omega) = \{T = f_0 - \operatorname{div} f : f = (f_1, \ldots, f_n),\ f_j/\omega \in L^{p'}(\Omega, \omega),\ j = 0, \ldots, n\}.$$

It is evident that the weights ω which satisfy $0 < c_1 \le \omega(x) \le c_2$ for $x \in \Omega$ (c_1 and c_2 positive constants) give nothing new (the space $W_0^{1,p}(\Omega, \omega)$ is then identical with the classical Sobolev space $W_0^{1,p}(\Omega)$). Consequently, we shall be interested above all in weight functions ω which either vanish somewhere in $\bar{\Omega}$ or increase to infinity (or both). For more information about weighted Sobolev spaces see [7], [9], [15] and [16].

Definition 2.5. Let $\Omega \subset \mathbb{R}^n$ be open. The space $H(\Omega)$ is defined to be the completion of $C^\infty(\bar{\Omega})$ with respect to the norm

$$(2.2) \qquad \|u\|_{H(\Omega)} = \Big(\int_\Omega u^2 v\, dx + \int_\Omega \langle \mathcal{A} \nabla u, \nabla u \rangle\, dx \Big)^{1/2}$$

where $\mathcal{A} = (a_{ij}(x))$ is the coefficient matrix of the operator L defined in (1.1), $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $\mathbb{R}^n$, and the symbol ∇ indicates the gradient. The space $H_0(\Omega)$ is defined to be the completion of $C_0^\infty(\Omega)$ with respect to the norm

$$(2.3) \qquad \|u\|_{H_0(\Omega)} = \Big(\int_\Omega \langle \mathcal{A} \nabla u, \nabla u \rangle\, dx \Big)^{1/2}.$$

The spaces $H(\Omega)$ and $H_0(\Omega)$ are Hilbert spaces. For more information about them see [2].

Remark 2.6. Using condition (1.2) we obtain

$$\int_\Omega |\nabla u|^2 \omega\, dx \le \int_\Omega \langle \mathcal{A} \nabla u, \nabla u \rangle\, dx \le \int_\Omega |\nabla u|^2 v\, dx,$$

and $W_0^{1,2}(\Omega, v) \subset H_0(\Omega) \subset W_0^{1,2}(\Omega, \omega)$, $\|\cdot\|_{W_0^{1,2}(\Omega,\omega)} \le \|\cdot\|_{H_0(\Omega)} \le \|\cdot\|_{W_0^{1,2}(\Omega,v)}$.

Definition 2.7. We shall say that a pair of weights (v,ω) satisfies the *condition* A_r, $1<r<\infty$, if there is a constant $C>0$ such that

$$\left(\frac{1}{|B|}\int_B v(x)\,dx\right)\left(\frac{1}{|B|}\int_B \omega^{1/(1-r)}(x)\,dx\right)^{r-1}\le C$$

for all balls $B\subset\mathbb{R}^n$. The smallest such C will be called the A_r-*constant* for the pair (v,ω).

Remark 2.8. If $(v,\omega)\in A_r$ and $\omega\le v$ then $v\in A_r$ and $\omega\in A_r$.

In this work we use the following six results.

Theorem 2.9 (The Weighted Sobolev Inequality). *Let Ω be an open bounded set in $\mathbb{R}^n$ $(n\ge 2)$ and $\omega\in A_p$ $(1<p<\infty)$. There exist positive constants C_Ω and δ such that for all $u\in W_0^{1,p}(\Omega,\omega)$ and all θ satisfying $1\le\theta\le n/(n-1)+\delta$,*

$$\|u\|_{L^{p\theta}(\Omega,\omega)}\le C_\Omega\|\nabla u\|_{L^p(\Omega,\omega)}. \tag{2.4}$$

Proof. For $u\in C_0^\infty(\Omega)$ the inequality is proved in [3, Theorem 1.3]. To extend the estimate (2.4) to arbitrary $u\in W_0^{1,p}(\Omega,\omega)$, we let $\{u_m\}$ be a sequence of $C_0^\infty(\Omega)$ functions tending to u in $W_0^{1,p}(\Omega,\omega)$. Applying (2.4) to the differences $u_{m_1}-u_{m_2}$, we see that $\{u_m\}$ will be a Cauchy sequence in $L^{\theta p}(\Omega,\omega)$. Consequently, the limit functions u will lie in the desired spaces and satisfy (2.4). ∎

Theorem 2.10 (The Hardy Inequality; see [10, Theorem 15.8]). *Let $1<r<p_1<nr$, $1/p_2=1/p_1-1/nr$ and $(v,\omega)\in A_r$. Then there exists a constant $C_\Omega>0$ such that*

$$\left(\int_\Omega |u(x)|^{p_2}v\,dx\right)^{1/p_2}\le C_\Omega\left(\int_\Omega |\nabla u(x)|^{p_1}\omega\,dx\right)^{1/p_1}$$

for every $u\in C_0^1(\Omega)$.

The following lemma is due to Stampacchia (see [13, Lemme 4.1]).

Lemma 2.11. *Let α, β, C be positive real constants, where $\beta>1$. Let $\phi:[0,\infty)\to\mathbb{R}_+$ be a decreasing function such that*

$$\phi(h)\le\frac{C}{(h-k)^\alpha}[\phi(k)]^\beta\quad\textit{for all } h>k.$$

Then $\phi(d)=0$, where $d^\alpha=C[\phi(0)]^{\beta-1}2^{\alpha\beta/(\beta-1)}$.

Lemma 2.12 (see [7, Theorem 15.5]). *If $\omega\in A_p$, then*

$$\left(\frac{|E|}{|B|}\right)^p\le C_{p,\omega}\frac{\omega(E)}{\omega(B)}$$

whenever B is a ball in $\mathbb{R}^n$ and E is a measurable subset of B.

By Lemma 2.12, if $\omega(E) = 0$ then $|E| = 0$.

Lemma 2.13 (see [10, Lemma 15.5]). *Let* $(v, \omega) \in A_r$. *Then* $(v, \omega) \in A_p$ *for every* $p \in (r, \infty)$.

Theorem 2.14. *If* $\omega \in A_2$ *then the embedding* $W_0^{1,2}(\Omega, \omega) \hookrightarrow L^2(\Omega, \omega)$ *is compact.*

Proof. The proof follows the lines of the proof of [4, Theorem 4.6]. ∎

Remark 2.15. (a) Since $p > nr \geq 2$ and $r < p' < 2$, if $\omega \in A_r$ then $\omega \in A_p$ and $\omega \in A_2$ (by Lemma 2.13 and Remark 2.8) and we also have $L^p(\Omega, \omega) \subset L^2(\Omega, \omega) \subset L^{p'}(\Omega, \omega)$ (since $\omega(\Omega) < \infty$) and $\|\cdot\|_{L^{p'}(\Omega,\omega)} \leq C_1\|\cdot\|_{L^2(\Omega,\omega)} \leq C_2\|\cdot\|_{L^p(\Omega,\omega)}$.

(b) Since $1/q = 1/p + 1/nr$ we have $1/q' = 1/p' - 1/nr$. By (H3) we have $1 < r < p' < nr$ and using Theorem 2.10 and (1.2) we obtain

$$\begin{aligned}\|\varphi\|_{L^{q'}(\Omega,v)} &\leq C_\Omega \|\nabla\varphi\|_{L^{p'}(\Omega,\omega)} \leq C_\Omega C_1 \|\nabla\varphi\|_{L^2(\Omega,\omega)} \\ &\leq C_\Omega C_1 \Big(\int_\Omega \langle \mathcal{A}\nabla\varphi, \nabla\varphi\rangle\, dx\Big)^{1/2} = C_3\|\varphi\|_{H_0(\Omega)}.\end{aligned}$$

(c) Since $\omega \in A_2$, by Theorem 2.9 (with $\theta = 1$) and (1.2) we obtain

$$\|u\|_{L^2(\Omega,\omega)} \leq C_\Omega \|\nabla u\|_{L^2(\Omega,\omega)} \leq C_\Omega \|u\|_{H_0(\Omega)}.$$

Definition 2.16. We say that an element $u \in H_0(\Omega)$ is a (*weak*) *solution of problem* (P) if

$$\begin{aligned}\int_\Omega a_{ij}(x) D_i u(x) D_j\varphi(x)\, dx + \int_\Omega b(x) u(x)\varphi(x)\, dx - \int_\Omega \Phi_j(u(x)) D_j\varphi(x)\, dx \\ = \int_\Omega g(x)\varphi(x)\, dx + \sum_{j=1}^n \int_\Omega f_j(x) D_j\varphi(x)\, dx\end{aligned} \tag{2.5}$$

for all $\varphi \in H_0(\Omega)$.

Remark 2.17. By (1.2), (H1)–(H5), Theorem 2.9 (with $\theta = 1$), and Remark 2.15(a), (b) we have

(i) $$\Big|\int_\Omega a_{i,j}(x) D_i u D_j\varphi\, dx\Big| \leq \|u\|_{H_0(\Omega)} \|\varphi\|_{H_0(\Omega)};$$

(ii) $$\Big|\int_\Omega \Phi_j(u(x)) D_j\varphi\, dx\Big| \leq \|\, |\Phi(u)|/\omega\|_{L^2(\Omega,\omega)} \|\varphi\|_{H_0(\Omega)};$$

(iii) $$\Big|\int_\Omega b(x) u\varphi\, dx\Big| \leq C_\Omega^2 \|b/\omega\|_{L^\infty(\Omega)} \|u\|_{H_0(\Omega)} \|\varphi\|_{H_0(\Omega)};$$

$$\text{(iv)}\quad \Big|\int_\Omega g\varphi\,dx+\sum_{j=1}^n\int_\Omega f_jD_j\varphi\,dx\Big|\le\int_\Omega\frac{|g|}{v}|\varphi|v\,dx+\sum_{j=1}^n\int_\Omega\frac{|f_j|}{\omega}|D_j\varphi|\omega$$
$$\le\|g/v\|_{L^q(\Omega,v)}\|\varphi\|_{L^{q'}(\Omega,v)}+\sum_{j=1}^n\|f_j/\omega\|_{L^p(\Omega,\omega)}\|\nabla\varphi\|_{L^{p'}(\Omega,\omega)}$$
$$\le\Big(C_\Omega C_1\|g/v\|_{L^q(\Omega,v)}+C_1\sum_{j=1}^n\|f_j/\omega\|_{L^p(\Omega,\omega)}\Big)\|\varphi\|_{H_0(\Omega)}$$
$$\le C_4\Big(\|g/v\|_{L^q(\Omega,v)}+\sum_{j=1}^n\|f_j/\omega\|_{L^p(\Omega,\omega)}\Big)\|\varphi\|_{H_0(\Omega)},$$

where $C_4=\max\{C_1,C_1C_\Omega\}$.

3. Proof of Theorem 1.1

STEP 1: *Proof of* (1.3). Assuming problem (P) has a solution $u\in H_0(\Omega)$, set $\Omega(k)=\{x\in\Omega:|u(x)|>k\}$ for $k\ge 0$. We choose for φ in (2.5) the function

$$\tilde\varphi=(u-k)^++(u+k)^- \tag{3.1}$$

where $(u-k)^+=\max\{u-k,0\}$ and $(u+k)^-=\min\{u+k,0\}$. The functions $(u-k)^+$, $(u+k)^-$ and $\tilde\varphi$ are in $H_0(\Omega)$, and

$$D_i(u-k)^+=\chi_{\{u>k\}}D_iu\quad\text{and}\quad D_i(u+k)^-=\chi_{\{u<-k\}}D_iu,$$

where χ_E denotes the characteristic function of a measurable set $E\subset\mathbb{R}^n$. Moreover, if we set $\psi_j(s)=\int_0^s\Phi_j(t+k)\,dt$, by the divergence theorem we have

$$\int_\Omega\Phi_j(u)D_j(u-k)^+\,dx=\int_\Omega D_j\psi_j((u-k)^+)\,dx=\int_{\partial\Omega}\psi_j((u-k)^+)\eta_j\,d\sigma(x)=0, \tag{3.2}$$

since $\psi_j(0)=0$ and $(u-k)^+=0$ on $\partial\Omega$. Analogously, we deduce that $\int_\Omega\Phi_j(u)D_j(u+k)^-\,dx=0$. Moreover, on $u>k>0$, u is positive and on $u<-k<0$, u is negative. So we have

$$\int_\Omega b(x)u(x)\tilde\varphi(x)\,dx=\int_\Omega b(x)u(x)(u(x)-k)^+\,dx+\int_\Omega b(x)u(x)(u(x)+k)^-\,dx\ge 0. \tag{3.3}$$

Using (3.1)–(3.3) and (1.2), we obtain

$$\begin{aligned}\int_\Omega |\nabla\tilde{\varphi}|^2\omega\,dx &\le \int_\Omega a_{ij}D_i\tilde{\varphi}D_j\tilde{\varphi}\,dx\\ &\le \int_\Omega a_{ij}D_iuD_j\tilde{\varphi}\,dx + \int_\Omega bu\tilde{\varphi}\,dx - \int_\Omega \Phi_j(u)D_j\tilde{\varphi}\,dx\\ &= \int_\Omega g\tilde{\varphi}\,dx + \sum_{j=1}^n \int_\Omega f_jD_j\tilde{\varphi}\,dx.\end{aligned}$$

Hence, by the Hölder inequality, we obtain

(3.4)
$$\|\nabla\tilde{\varphi}\|^2_{L^2(\Omega,\omega)} \le \bigg(\bigg\|\frac{g}{v}\bigg\|_{L^q(\Omega,v)}\|\tilde{\varphi}\|_{L^{q'}(\Omega,v)} + \sum_{j=1}^n\bigg\|\frac{f_j}{\omega}\bigg\|_{L^p(\Omega,\omega)}\|\nabla\tilde{\varphi}\|_{L^{p'}(\Omega,\omega)}\bigg).$$

Since $p > nr \ge 2$ and $1 < r < p' < 2$, if $(v,\omega)\in A_r$ and $\omega \le v$ then $\omega\in A_r$ (see Remark 2.8), $\omega\in A_p$ and $\omega\in A_2$ (see Lemma 2.13), and since $1/q' = 1/p' - 1/nr$, by Theorem 2.10 we have

$$\|\tilde{\varphi}\|_{L^{q'}(\Omega,v)} \le C_\Omega\|\nabla\tilde{\varphi}\|_{L^{p'}(\Omega,\omega)}. \tag{3.5}$$

Hence, by (3.4), we get

$$\|\nabla\tilde{\varphi}\|^2_{L^2(\Omega,\omega)} \le C_5\bigg(\bigg\|\frac{g}{v}\bigg\|_{L^q(\Omega,v)} + \sum_{j=1}^n\bigg\|\frac{f_j}{\omega}\bigg\|_{L^p(\Omega,\omega)}\bigg)\|\nabla\tilde{\varphi}\|_{L^{p'}(\Omega,\omega)}. \tag{3.6}$$

Now let us remark that $\tilde{\varphi} = 0$ outside $\Omega(k)$, so by Hölder's inequality we obtain

$$\begin{aligned}\|\nabla\tilde{\varphi}\|^{p'}_{L^{p'}(\Omega,\omega)} &= \int_\Omega |\nabla\tilde{\varphi}|^{p'}\omega\,dx = \int_{\Omega(k)}|\nabla\tilde{\varphi}|^{p'}\omega\,dx\\ &\le \Big(\int_{\Omega(k)}|\nabla\tilde{\varphi}|^2\omega\,dx\Big)^{p'/2}\Big(\int_{\Omega(k)}\omega\,dx\Big)^{(2-p')/2}\\ &= \|\nabla\tilde{\varphi}\|^{p'}_{L^2(\Omega,\omega)}[\omega(\Omega(k))]^{(2-p')/2}.\end{aligned} \tag{3.7}$$

Hence, we obtain

$$\|\nabla\tilde{\varphi}\|^2_{L^{p'}(\Omega,\omega)} \le \|\nabla\tilde{\varphi}\|^2_{L^2(\Omega,\omega)}[\omega(\Omega(k))]^{(2-p')/p'}. \tag{3.8}$$

From (3.6) and (3.8), we then deduce

$$\|\nabla\tilde{\varphi}\|_{L^{p'}(\Omega,\omega)} \le C_5\bigg(\bigg\|\frac{g}{v}\bigg\|_{L^q(\Omega,v)} + \sum_{j=1}^n\bigg\|\frac{f_j}{\omega}\bigg\|_{L^p(\Omega,\omega)}\bigg)[\omega(\Omega(k))]^{(2-p')/p'}. \tag{3.9}$$

If $h > k$ then $\Omega(h)\subset\Omega(k)$, $\tilde{\varphi} = \pm(|u|-k)$ on $\Omega(k)$ and $|\tilde{\varphi}| \ge h-k$ on

$\Omega(h)$ for $h > k$. We obtain (using $\omega \le v$)

$$(h-k)[\omega(\Omega(h))]^{1/q'} \le (h-k)[v(\Omega(h))]^{1/q'} \le \Big(\int_{\Omega(h)} |\tilde{\varphi}|^{q'} v\,dx\Big)^{1/q'} \le \Big(\int_{\Omega(k)} |\tilde{\varphi}|^{q'} v\,dx\Big)^{1/q'} = \|\tilde{\varphi}\|_{L^{q'}(\Omega,v)}. \tag{3.10}$$

Using (3.5) and (3.9) we get

$$(h-k)[\omega(\Omega(h))]^{1/q'} \le \|\tilde{\varphi}\|_{L^{q'}(\Omega,v)} \le C_\Omega \|\nabla\tilde{\varphi}\|_{L^{p'}(\Omega,\omega)} \le C_5 C_\Omega \bigg(\Big\|\frac{g}{v}\Big\|_{L^q(\Omega,v)} + \sum_{j=1}^n \Big\|\frac{f_j}{\omega}\Big\|_{L^p(\Omega,\omega)}\bigg)[\omega(\Omega(k))]^{(2-p')/p'}. \tag{3.11}$$

Hence

$$\omega(\Omega(h)) \le \bigg[C_6 \frac{\|g/v\|_{L^q(\Omega,v)} + \sum_{j=1}^n \|f_j/\omega\|_{L^p(\Omega,\omega)}}{h-k}\bigg]^{q'} \times [\omega(\Omega(k))]^{(2-p')q'/p'}. \tag{3.12}$$

Since $p > nr \ge 2$ and $1/q' = 1/p' - 1/nr$, we see that $\beta = (2-p')q'/p' = (nrp - 2nr)/(nrp - nr - p) > (nrp - nr - p)/(nrp - nr - p) = 1$ (since $p > nr \ge 2$ we have $nrp - nr - p > 0$). By Lemma 2.11 applied to $\phi(h) = \omega(\Omega(h))$ we have $\phi(d) = \omega(\Omega(d)) = 0$ where

$$d = C_7 \bigg(\Big\|\frac{g}{v}\Big\|_{L^q(\Omega,v)} + \sum_{j=1}^n \Big\|\frac{f_j}{\omega}\Big\|_{L^p(\Omega,\omega)}\bigg)[\varphi(0)]^{\beta-1} 2^{\beta/(\beta-1)}.$$

(Note that $\phi(0) = \omega(\Omega(0)) \le \omega(\Omega) < \infty$.) By Lemma 2.12, if $\omega(\Omega(d)) = 0$ then $|\Omega(d)| = 0$. Therefore

$$\|u\|_{L^\infty(\Omega)} \le C_8 \bigg(\Big\|\frac{g}{v}\Big\|_{L^q(\Omega,v)} + \sum_{j=1}^n \Big\|\frac{f_j}{\omega}\Big\|_{L^p(\Omega,\omega)}\bigg).$$

Step 2: *Proof of existence of a solution.* Let us denote

$$M = C_8 \bigg(\Big\|\frac{g}{v}\Big\|_{L^q(\Omega,v)} + \sum_{j=1}^n \Big\|\frac{f_j}{\omega}\Big\|_{L^p(\Omega,\omega)}\bigg),$$

and define, for all $j = 1, \dots, n$ ($t \in \mathbb{R}$),

$$\tilde{\Phi}_j(t) = \begin{cases} \Phi_j(-M) & \text{if } t < -M, \\ \Phi_j(t) & \text{if } |t| \le M, \\ \Phi_j(M) & \text{if } t > M. \end{cases}$$

By (H5), the $\tilde{\Phi}_j$ are bounded. For each $\vartheta \in L^2(\Omega, \omega)$, there exists a unique solution u to the problem

$$
\text{(P1)} \qquad \begin{cases} u \in H_0(\Omega), \\ \int\limits_\Omega a_{ij} D_i u D_j \varphi\, dx + \int\limits_\Omega b(x) u \varphi\, dx \\ \qquad = \int\limits_\Omega \tilde{\Phi}_j(\vartheta) D_j \varphi\, dx + \int\limits_\Omega g\varphi\, dx + \sum_{j=1}^n \int\limits_\Omega f_j D_j \varphi\, dx, \quad \forall \varphi \in H_0(\Omega). \end{cases}
$$

In fact, we define $\mathcal{B} : H_0(\Omega) \times H_0(\Omega) \to \mathbb{R}$ and $\mathcal{T} : H_0(\Omega) \to \mathbb{R}$ by

$$
\mathcal{B}(u, \varphi) = \int\limits_\Omega a_{ij} D_i u D_j \varphi\, dx + \int\limits_\Omega b u \varphi\, dx,
$$

$$
\mathcal{T}(\varphi) = \int\limits_\Omega \tilde{\Phi}_j(\vartheta) D_j \varphi\, dx + \int\limits_\Omega g\varphi\, dx + \sum_{j=1}^n \int\limits_\Omega f_j D_j \varphi\, dx.
$$

Then $\mathcal{B}$ is continuous (by (1.2), (H4), $\omega \in A_2$ and Theorem 2.9 with $\theta = 1$):

$$
\begin{aligned}
|\mathcal{B}(u, \varphi)| &\le \int\limits_\Omega |\langle \mathcal{A}\nabla u, \nabla\varphi\rangle|\, dx + \int\limits_\Omega \frac{b(x)}{\omega} |u|\, |\varphi| \omega\, dx \\
&\le (1 + C_\Omega^2 \|b/\omega\|_{L^\infty(\Omega)}) \|u\|_{H_0(\Omega)} \|\varphi\|_{H_0(\Omega)},
\end{aligned}
$$

and $\mathcal{B}$ is coercive (using (H4)),

$$
\mathcal{B}(u, u) = \int\limits_\Omega a_{ij}(x) D_i u D_j u\, dx + \int\limits_\Omega b(x) u^2\, dx \ge \int\limits_\Omega \langle \mathcal{A}\nabla u, \nabla u\rangle\, dx = \|u\|^2_{H_0(\Omega)}.
$$

Moreover, since the $\tilde{\Phi}_j$ are bounded ($|\tilde{\Phi}_j| \le \tilde{C}$, $j = 1, \ldots, n$), $\mathcal{T}$ is continuous and

$$
\begin{aligned}
&|\mathcal{T}(\varphi)| \\
&\quad \le \bigg(\tilde{C}\Big(\int\limits_\Omega \omega^{-1}\, dx\Big)^{1/2} + C_\Omega C_1 \bigg\|\frac{g}{v}\bigg\|_{L^q(\Omega, v)} + C_1 \sum_{j=1}^n \bigg\|\frac{f_j}{\omega}\bigg\|_{L^p(\Omega,\omega)}\bigg) \|\varphi\|_{H_0(\Omega)}.
\end{aligned}
$$

Hence, by the Lax–Milgram theorem there is a unique solution $u \in H_0(\Omega)$ to $\mathcal{B}(u, \varphi) = \mathcal{T}(\varphi)$ for all $\varphi \in H_0(\Omega)$ (that is, u is the unique solution of problem (P1)).

Therefore let us consider the mapping $T : L^2(\Omega, \omega) \to L^2(\Omega, \omega)$, defined for $\vartheta \in L^2(\Omega, \omega)$ by $T(\vartheta) = u$ where u is the solution of problem (P1). By taking $\varphi = u$ in (P1), we obtain

$$
\begin{aligned}
&\int\limits_\Omega a_{ij}(x) D_i u D_j u\, dx + \int\limits_\Omega b(x) u^2\, dx \\
&\qquad\qquad = \int\limits_\Omega \tilde{\Phi}_j(\vartheta) D_j u\, dx + \int\limits_\Omega g u\, dx + \sum_{j=1}^n \int\limits_\Omega f_j D_j u\, dx.
\end{aligned}
$$

Using (1.2) and (H4) we have

$$\|u\|^2_{H_0(\Omega)} = \int_\Omega \langle \mathcal{A}\nabla u, \nabla u\rangle\, dx \le \int_\Omega a_{ij} D_i u D_j u\, dx + \int_\Omega bu^2\, dx, \tag{3.13}$$

and using the fact that the $\tilde{\Phi}_j$ are bounded (i.e., $|\tilde{\Phi}_j| \le \tilde{C}$), $\omega \in A_2$ ($\omega^{-1} \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$), we obtain

$$\Big|\int_\Omega \tilde{\Phi}_j(\vartheta) D_i u\, dx + \int_\Omega gu\, dx + \sum_{j=1}^n \int_\Omega f_j D_j u\, dx\Big| \le \bigg(\tilde{C}\Big(\int_\Omega \omega^{-1}\, dx\Big)^{1/2} + C_\Omega C_1 \Big\|\frac{g}{v}\Big\|_{L^q(\Omega,v)} + C_1 \sum_{j=1}^n \Big\|\frac{f_j}{\omega}\Big\|_{L^p(\Omega,\omega)}\bigg)\|u\|_{H_0(\Omega)}. \tag{3.14}$$

By (3.13) and (3.14) we obtain

$$\|u\|_{H_0(\Omega)} \le \bigg(\tilde{C}\Big(\int_\Omega \omega^{-1}\, dx\Big)^{1/2} + C_\Omega C_1 \Big\|\frac{g}{v}\Big\|_{L^q(\Omega,v)} + C_1 \sum_{j=1}^n \Big\|\frac{f_j}{\omega}\Big\|_{L^p(\Omega,\omega)}\bigg) = C_9, \tag{3.15}$$

where C_9 is a constant which does not depend on $\vartheta \in L^2(\Omega,\omega)$.

By combining this with Remark 2.15(c), we get

$$\|T(\vartheta)\|_{L^2(\Omega,\omega)} = \|u\|_{L^2(\Omega,\omega)} \le C_\Omega \|u\|_{H_0(\Omega)} \le C_\Omega C_9 = C_{10}. \tag{3.16}$$

Let us denote by $B = B(0, C_{10})$ the ball in $L^2(\Omega,\omega)$ of center 0 and radius C_{10}. From (3.16) (and Remark 2.15(c))) we have $T(B) \subset B$. Since $W^{1,2}_0(\Omega,\omega)$ is compactly embedded in $L^2(\Omega,\omega)$ (by Theorem 2.14) and $H_0(\Omega) \subset W^{1,2}_0(\Omega,\omega)$, it follows that $T(B)$ is precompact in B.

To prove that T is continuous, let $\{\vartheta_m\}$ be a sequence in $L^2(\Omega,\omega)$ such that $\vartheta_m \to \vartheta$. By (3.16) and Theorem 2.14, it is enough to show that $T(\vartheta)$ is the only limit point of the sequence $T(\vartheta_m)$. Let us assume that a subsequence $T(\vartheta_{m_k})$ tends to u as $k \to \infty$. One can extract a subsequence (still denoted by ϑ_{m_k}) such that

$$\begin{aligned} &\vartheta_{m_k} \to v && \omega\text{-a.e. in } \Omega,\\ &u_{m_k} = T(\vartheta_{m_k}) \rightharpoonup u && \text{in } H_0(\Omega). \end{aligned} \tag{3.17}$$

By (3.17) and the Lebesgue dominated convergence theorem we have $\tilde{\Phi}_j(\vartheta_{m_k}) \to \tilde{\Phi}_j(\vartheta)$ in $L^2(\Omega,\omega)$. Now, passing to the limit in

$$\int_\Omega a_{ij} D_i u_{m_k} D_i \varphi\, dx + \int_\Omega b u_{m_k} \varphi\, dx = \int_\Omega \tilde{\Phi}_j(\vartheta_{m_k}) D_j \varphi\, dx + \int_\Omega g\varphi\, dx + \sum_{j=1}^n \int_\Omega f_j D_j \varphi\, dx$$

we deduce from (3.17) that

$$\int_\Omega a_{ij} D_i u D_i \varphi \, dx + \int_\Omega b u \varphi \, dx = \int_\Omega \tilde{\Phi}_j(\vartheta) D_j \varphi \, dx + \int_\Omega g \varphi \, dx + \sum_{j=1}^n \int_\Omega f_j D_j \varphi \, dx$$

for all $\varphi \in H_0(\Omega)$.

Hence, $u = T(\vartheta)$ and T is continuous. Therefore, by the Schauder fixed point theorem (see [5, Theorem 10.1]), T has a fixed point $u \in B$. Such a fixed point is a solution to problem (P) with $\tilde{\Phi}_j$ instead of Φ_j; but from Step 1 we have $\|u\|_{L^\infty(\Omega)} \le M$ and thus $\tilde{\Phi}_j(u) = \Phi_j(u)$.

STEP 3: *Uniqueness.* If u_1 and u_2 are two solutions to problem (P), then

$$\int_\Omega a_{ij} D_i u_l D_j \varphi \, dx + \int_\Omega b u_l \varphi \, dx - \int_\Omega \langle \Phi(u_l), \nabla \varphi \rangle \, dx = \int_\Omega g \varphi \, dx + \sum_{j=1}^n \int_\Omega f_j D_j \varphi \, dx \qquad (l = 1, 2)$$

for all $\varphi \in H_0(\Omega)$. We obtain

$$\int_\Omega a_{ij} \langle \nabla u_1 - \nabla u_2, \nabla \varphi \rangle \, dx + \int_\Omega b(u_1 - u_2) \varphi \, dx - \int_\Omega \langle \Phi(u_1) - \Phi(u_2), \nabla \varphi \rangle \, dx = 0$$

for all $\varphi \in H_0(\Omega)$. Then, since $L = -\sum_{i,j=1}^n D_j(a_{ij} D_i) + b$, using integration by parts we obtain

$$\int_\Omega [(u_1 - u_2) L\varphi - \langle \Phi(u_1) - \Phi(u_2), \nabla \varphi \rangle] \, dx = 0. \tag{3.18}$$

We set

$$G_j = \begin{cases} \dfrac{\Phi_j(u_1) - \Phi_j(u_2)}{u_1 - u_2} & \text{if } u_1 \neq u_2, \\ 0 & \text{if } u_1 = u_2. \end{cases}$$

By (H6) we have $G_j / v \in L^\infty(\Omega)$ $(j = 1, \dots, n)$. By (3.18) we obtain

$$\int_\Omega (u_1 - u_2)(L\varphi - G_j D_j \varphi) \, dx = 0, \qquad \forall \varphi \in H_0(\Omega). \tag{3.19}$$

But, similarly to problem (P1), there exists a unique $\varphi \in H_0(\Omega)$ satisfying the equations $L\varphi - G_j D_j \varphi = (u_1 - u_2) v$, and for such a φ, (3.19) becomes

$$\int_\Omega (u_1 - u_2)^2 v \, dx = 0.$$

Therefore $u_1 = u_2$. ■

Example. Let $\Omega = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}$. Consider the weights

$$\omega(x,y) = \lambda_1 (x^2+y^2)^{-1/2}, \qquad v(x,y) = \lambda_2 (x^2+y^2)^{-1/2} \qquad (0 < \lambda_1 < \lambda_2)$$

$((v,\omega) \in A_r$, $r = 5/4$, $p' = 3/2$, $p = 3$, $q = 15/11)$, and the functions

$$\Phi : \mathbb{R} \to \mathbb{R}^2, \qquad \Phi(t) = (\cos(t), \sin(t)),$$

$$g(x,y) = \frac{\arctan(1/(x^2+y^2))}{(x^2+y^2)^{1/2}}, \qquad b(x,y) = e^{-(x^2+y^2)},$$

$$f_1(x,y) = \frac{\cos(1/(x^2+y^2))}{(x^2+y^2)^{1/3}}, \qquad f_2(x,y) = \frac{\sin(1/(x^2+y^2))}{(x^2+y^2)^{1/3}}.$$

Consider the partial differential operator

$$Lu(x,y) = -\frac{\partial}{\partial x}\left(\lambda_1 (x^2+y^2)^{-1/2} \frac{\partial u}{\partial x}\right) - \frac{\partial}{\partial y}\left(\lambda_2 (x^2+y^2)^{-1/2} \frac{\partial u}{\partial y}\right) + b(x,y)u.$$

By Theorem 1.1, the problem

$$\begin{cases} Lu(x,y) + \operatorname{div}(\Phi(u(x,y))) = g(x,y) - \dfrac{\partial f_1}{\partial x} - \dfrac{\partial f_2}{\partial y} & \text{on } \Omega, \\ u(x) = 0 \quad \text{on } \partial\Omega, \end{cases}$$

has a unique solution $u \in H_0(\Omega)$.

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