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ON THE EXISTENCE OF POSITIVE SOLUTIONS OF SECOND ORDER NEUTRAL DIFFERENCE EQUATIONS

Abstract. The neutral delay difference equations of second order with positive and negative coefficients

$$\text{(E)} \qquad \Delta^2(x_n + p_n x_{n-\tau}) + q_n x_{n-\sigma} - r_n x_{n-\lambda} = 0, \qquad n = 0, 1, 2, \ldots,$$

is studied, and a sufficient condition for the existence of a positive solution of this equation is obtained.

1. Introduction. Recently, there has been a lot of research activity concerning positive solutions of difference equations. See for example [4–6] and the references cited therein. Difference equations appear as natural descriptions of the observed evolution phenomena as well as in the study of discretization methods for differential equations. The application of the theory of difference equations is rapidly broadening to various fields such as numerical analysis, control theory, finite mathematics, and computer science; in particular, the connection between the theory of difference equations and computer science has become more important in recent years, because of the successful use of computers to solve difficult problems arising in practice. Furthermore, chaos and fractals are at the center of attention nowadays, and difference equations produce them [1–3].

The present paper deals with the neutral delay difference equations of second order with positive and negative coefficients

$$\text{(E)} \qquad \Delta^2(x_n + p_n x_{n-\tau}) + q_n x_{n-\sigma} - r_n x_{n-\lambda} = 0, \qquad n = 0, 1, 2, \ldots,$$

where τ is a positive integer; σ, λ are nonnegative integers; $\{p_n\}$ is a real sequence; $\{q_n\}, \{r_n\}$ are real positive sequences; Δ is the forward difference

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operator defined by

$$\Delta x_n = x_{n+1} - x_n \quad \text{and} \quad \Delta^2 = \Delta(\Delta).$$

Let $\mu = \max\{\tau, \sigma, \lambda\}$. Then by a solution of (E), we mean a real sequence $\{x_n\}$ which is defined for $n \geq -\mu$ and satisfies equation (E) for $n \geq n_0$. A solution $\{x_n\}$ of (E) is said to be *eventually positive* if $x_n > 0$ for all large n, and *eventually negative* if $x_n < 0$ for all large n. A solution of (E) is called *nonoscillatory* if it is eventually positive or negative. Otherwise it is called *oscillatory*.

2. Main results

THEOREM 1. *Suppose that*

$$\lim_{n\to\infty} p_n = p \in (0,1) \cup (1,\infty), \tag{1}$$

$$\sum_{n=n_0}^{\infty} nq_n < \infty, \qquad \sum_{n=n_0}^{\infty} nr_n < \infty. \tag{2}$$

If there exists a sufficiently large positive integer n_1 such that

$$aq_n - r_n \geq 0 \quad \text{for every } n \geq n_1 \text{ and any } a > 0, \tag{3}$$

then (E) *has a nonoscillatory solution.*

Proof. The proof will be divided into two cases, depending on the two ranges of the parameter p.

CASE 1: $p \in (0,1)$. By condition (1), we choose a number α such that $p < \alpha < 1$, and choose positive constants M_1 and M_2 such that

$$\alpha - M_2 < p < \frac{\alpha - M_1}{1 + M_2}$$

(this implies that $M_1 < M_2$). Let $\varepsilon_0 > 0$ be such that

$$\varepsilon_0 + \alpha < 1, \qquad \frac{\alpha - M_1}{1 + M_2} - \frac{M_2}{1 + M_2}\varepsilon_0 \geq p,$$

so that $\alpha - p - pM_2 - \varepsilon_0 M_2 \geq M_1$. By (1)–(3), there exists a sufficiently large $N \geq \max\{n_1, n_0 + \mu\}$ such that

$$0 < p - \varepsilon_0 < p_n < p + \varepsilon_0 \quad \text{for } n \geq N, \tag{4}$$

$$\sum_{s=N}^{\infty} s(q_s + r_s) < \alpha - p, \tag{5}$$

$$0 \leq \sum_{s=N}^{\infty} s[M_2 q_s - M_1 r_s] \leq p - \alpha + M_2, \tag{6}$$

$$\sum_{s=N}^{\infty} s(M_1 q_s - M_2 r_s) \geq 0. \tag{7}$$

Let B^N denote the Banach space of all bounded real sequences $x = \{x_n\}_{n=N-\mu}^{\infty}$ with the sup norm $\|x\| = \sup_{n\geq N-\mu} |x_n|$. Set

$$\Omega = \{x \in B^N : M_1 \leq x_n \leq M_2,\ n \geq N-\mu\}.$$

It is easy to see that Ω is a bounded, closed, and convex subset of B^N. Define a mapping $T : \Omega \to B^N$ as follows:

$$(Tx)_n = \begin{cases} \alpha - p - p_n x_{n-\tau} + (n-1) \displaystyle\sum_{s=n-1}^{\infty} (q_s x_{s-\sigma} - r_s x_{s-\lambda}) \\ \qquad\qquad + \displaystyle\sum_{s=N}^{n-2} s(q_s x_{s-\sigma} - r_s x_{s-\lambda}), \qquad n \geq N, \\ \alpha + p - p_N x_{N-\tau}, \qquad N-\mu \leq n \leq N. \end{cases}$$

Clearly, T is continuous. For every $x \in \Omega$ and $n \geq N$, using (3) and (6) we get

$$\begin{aligned} (Tx)_n &= \alpha - p - p_n x_{n-\tau} + (n-1) \sum_{s=n-1}^{\infty} (q_s x_{s-\sigma} - r_s x_{s-\lambda}) \\ &\quad + \sum_{s=N}^{n-2} s(q_s x_{s-\sigma} - r_s x_{s-\lambda}) \\ &\leq \alpha - p + (n-1) \sum_{s=n-1}^{\infty} (M_2 q_s - M_1 r_s) + \sum_{s=N}^{n-2} s(M_2 q_s - M_1 r_s) \\ &\leq \alpha - p + \sum_{s=N}^{\infty} s(M_2 q_s - M_1 r_s) \leq M_2. \end{aligned}$$

Furthermore, in view of (3) and (7) we have

$$\begin{aligned} (Tx)_n &= \alpha - p - p_n x_{n-\tau} + (n-1) \sum_{s=n-1}^{\infty} (q_s x_{s-\sigma} - r_s x_{s-\lambda}) \\ &\quad + \sum_{s=N}^{n-2} s(q_s x_{s-\sigma} - r_s x_{s-\lambda}) \\ &\geq \alpha - p - p_n M_2 + (n-1) \sum_{s=n-1}^{\infty} (M_1 q_s - M_2 r_s) + \sum_{s=N}^{n-2} s(M_1 q_s - M_2 r_s) \\ &> \alpha - p - p_n M_2 > \alpha - p - p_n M_2 - \varepsilon_0 M_2 \geq M_1. \end{aligned}$$

Thus, we proved that $T\Omega \subseteq \Omega$.

Now, for $x^1, x^2 \in \Omega$ and $n \geq N$ we have

$$\begin{aligned}
|(Tx^1)_n - (Tx^2)_n| \\
&\leq p_n|x^1_{n-\tau} - x^2_{n-\tau}| + (n-1)\sum_{s=n-1}^{\infty} q_s|x^1_{n-\sigma} - x^2_{n-\sigma}| \\
&\quad + (n-1)\sum_{s=n-1}^{\infty} r_s|x^1_{s-\lambda} - x^2_{s-\lambda}| \\
&\quad + \sum_{s=N}^{n-2} sq_s|x^1_{n-\sigma} - x^2_{n-\sigma}| + \sum_{s=N}^{n-2} sr_s|x^1_{n-\lambda} - x^2_{n-\lambda}| \\
&\leq p_n\|x^1 - x^2\| + \|x^1 - x^2\|\Big[\sum_{s=n-1}^{\infty} s(q_s + r_s) + \sum_{s=N}^{n-2} s(q_s + r_s)\Big] \\
&< \|x^1 - x^2\|\Big\{p + \varepsilon_0 + \sum_{s=N}^{\infty} s(q_s + r_s)\Big\}.
\end{aligned}$$

This immediately implies that

$$\|Tx^1 - Tx^2\| \leq q_1\|x^1 - x^2\|,$$

where

$$q_1 = p + \varepsilon_0 + \sum_{s=N}^{\infty} s(q_s - r_s) < p + \varepsilon_0 + \alpha - p = \varepsilon_0 + \alpha < 1.$$

Hence, T is a contraction mapping. By the contraction principle, T has a unique fixed point x, which is obviously a positive solution of (E). This completes the proof in Case 1.

Case 2: $p \in (1, \infty)$. We choose a number α such that $1 < \alpha < p$, and choose positive constants N_1 and N_2 such that $1 < N_2 < \alpha(1 - N_1) < \alpha$. Take $\varepsilon_0 > 0$ such that

$$\varepsilon_0 < p - 1, \qquad \varepsilon_0 < \frac{p(p-\alpha)(1-N_1)}{(1-N_1)\alpha + 2p}, \qquad \varepsilon_0 \leq \frac{N_1 p}{2p(N_2 - 1) + N_1}.$$

By (1)–(3), there exists a sufficiently large $N \geq \max\{n_1, n_0 + \mu\}$ such that

(8) $$0 < p - \varepsilon_0 < p_n < p + \varepsilon_0 \quad \text{for } n \geq N,$$

(9) $$\sum_{s=N}^{\infty} s(q_s + r_s) < p - \varepsilon_0 - 1,$$

(10) $$0 \leq \sum_{s=N}^{\infty} s[N_2 q_s - N_1 r_s] \leq pN_2 - p,$$

$$\sum_{s=N}^{\infty} s(N_1 q_s - N_2 r_s) \geq 0. \tag{11}$$

Let B^N denote the same Banach space as in Case 1 and set

$$\Omega = \{x \in B^N : N_1 \leq x_n \leq N_2,\ n \geq N - \mu\}.$$

It is easy to see that Ω is a bounded, closed, and convex subset of B^N. Define a mapping $T : \Omega \to B^N$ as follows:

$$(Tx)_n = \begin{cases} \dfrac{p-\varepsilon_0}{p_{n+\tau}} - \dfrac{x_{n+\tau}}{p_{n+\tau}} + \dfrac{n-1+\tau}{p_{n+\tau}} \displaystyle\sum_{s=n-1+\tau}^{\infty} (q_s x_{s-\sigma} - r_s x_{s-\lambda}) \\ \qquad + \dfrac{1}{p_{n+\tau}} \displaystyle\sum_{s=N}^{n-2+\tau} s(q_s x_{s-\sigma} - r_s x_{s-\lambda}), \qquad n \geq N, \\ (Tx)_N, \qquad N-\mu \leq n \leq N. \end{cases}$$

Clearly, T is continuous. For every $x \in \Omega$ and $n \geq N$, using (3), (8) and (10) we get

$$\begin{aligned} (Tx)_n &\leq 1 - \frac{N_1}{p+\varepsilon_0} + \frac{1}{p-\varepsilon_0} \sum_{s=n-1+\tau}^{\infty} s(q_s N_2 - r_s N_1) \\ &\quad + \frac{1}{p-\varepsilon_0} \sum_{s=N}^{n-2+\tau} s(q_s N_2 - r_s N_1) \\ &= 1 - \frac{N_1}{p+\varepsilon_0} + \frac{1}{p-\varepsilon_0} \sum_{s=N}^{\infty} s(q_s N_2 - r_s N_1) \\ &\leq 1 - \frac{N_1}{p+\varepsilon_0} + \frac{pN_2 - p}{p-\varepsilon_0} = N_2 + \frac{\varepsilon_0[(p+\varepsilon_0)(N_2-1)+N_1] - N_1 p}{(p-\varepsilon_0)(p+\varepsilon_0)} \\ &\leq N_2 + \frac{\varepsilon_0[2p(N_2-1)+N_1] - N_1 p}{(p-\varepsilon_0)(p+\varepsilon_0)} \leq N_2. \end{aligned}$$

Furthermore, in view of (3), (8) and (11) we have

$$\begin{aligned} (Tx)_n &\geq \frac{p-\varepsilon_0}{p+\varepsilon_0} - \frac{N_2}{p-\varepsilon_0} \geq \frac{p-\varepsilon_0}{p+\varepsilon_0} - \frac{\alpha(N_1-1)}{p-\varepsilon_0} \\ &= N_1 + \frac{p(1-N_1)(p-\alpha) - [(1-N_1)\alpha + 2p]\varepsilon_0 + (N_1+1)\varepsilon_0^2}{p^2 - \varepsilon_0^2} \\ &\geq N_1 + \frac{p(1-N_1)(p-\alpha) - [(1-N_1)\alpha + 2p]\varepsilon_0}{p^2 - \varepsilon_0^2} \geq N_1. \end{aligned}$$

Thus, we proved that $T\Omega \subseteq \Omega$.

Now, for $x^1, x^2 \in \Omega$ and $n \geq N$ we have

$$
\begin{aligned}
&|(Tx^1)_n - (Tx^2)_n| \\
&\leq \frac{1}{p_{n+\tau}} |x^1_{n-\tau} - x^2_{n-\tau}| + \frac{1}{p_{n+\tau}} \sum_{s=n-1+\tau}^{\infty} sq_s |x^1_{n-\sigma} - x^2_{n-\sigma}| \\
&\quad + \frac{1}{p_{n+\tau}} \sum_{s=n-1+\tau}^{\infty} sr_s |x^1_{s-\lambda} - x^2_{s-\lambda}| + \frac{1}{p_{n+\tau}} \sum_{s=N}^{n-2+\tau} sq_s |x^1_{n-\sigma} - x^2_{n-\sigma}| \\
&\quad + \frac{1}{p_{n+\tau}} \sum_{s=N}^{n-2+\tau} sr_s |x^1_{n-\lambda} - x^2_{n-\lambda}| \\
&\leq \frac{1}{p-\varepsilon_0} \|x^1 - x^2\| + \|x^1 - x^2\| \frac{1}{p-\varepsilon_0} \Big[\sum_{s=n-1+\tau}^{\infty} s(q_s + r_s) + \sum_{s=N}^{n-2+\tau} s(q_s + r_s) \Big] \\
&\quad = \|x^1 - x^2\| \Big\{ \frac{1}{p-\varepsilon_0} \Big[1 + \sum_{s=N}^{\infty} s(q_s + r_s) \Big] \Big\}.
\end{aligned}
$$

This immediately implies that

$$\|Tx^1 - Tx^2\| \leq q_1 \|x^1 - x^2\|,$$

where

$$q_1 = \frac{1}{p-\varepsilon_0} \Big[1 + \sum_{s=N}^{\infty} s(q_s + r_s) \Big] < \frac{1 + p - \varepsilon_0 - 1}{p - \varepsilon_0} = 1.$$

Hence, T is a contraction mapping. The contraction principle shows that T has a unique fixed point x, which is obviously a positive solution of (E). This completes the proof in Case 2.

The proof of the theorem is complete.

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References

[1] R. P. Agarwal and J. W. Wong, *Advanced Topics in Difference Equations*, Kluwer, Dordrecht, 1997.

[2] I.-G. E. Kordonis, C. G. Philos and I. K. Purnaras, *On the oscillation of some linear differential equations with periodic coefficients*, J. Comput. Appl. Math. 84 (1997), 219–241.

[3] V. Lakshmikantham and D. Trigante, *Theory of Difference Equations*: *Numerical Methods and Applications*, Academic Press, New York, 1988.

[4] B. S. Lalli and B. G. Zhang, *On existence of positive solutions and bounded oscillations for neutral difference equations*, J. Math. Anal. Appl. 166 (1992), 227–287.

[5] H. J. Li and C. C. Yeh, *Nonoscillation in nonlinear difference equations*, Comput. Math. Appl. 28 (1994), 203–208.

[6] G. Zhang and S. S. Cheng, *Positive solutions of a nonlinear neutral difference equation*, Nonlinear Anal. 38 (1997), 729–738.

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