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Piotr Kacprzyk (Warszawa)

ALMOST GLOBAL SOLUTIONS OF THE FREE BOUNDARY PROBLEM FOR THE EQUATIONS OF A MAGNETOHYDRODYNAMIC INCOMPRESSIBLE FLUID

Abstract. Almost global in time existence of solutions for equations describing the motion of a magnetohydrodynamic incompressible fluid in a domain bounded by a free surfaced is proved. In the exterior domain we have an electromagnetic field which is generated by some currents which are located on a fixed boundary. We prove that a solution exists for $t \in (0,T)$, where $T > 0$ is large if the data are small.

1. Introduction. In this paper we prove the existence of almost global in time solutions for small data to equations describing the motion of a magnetohydrodynamic incompressible fluid in a domain $\Omega_t \subset \mathbb{R}^3$ bounded by a free surface S_t. In the domain $D_t \subset \mathbb{R}^3$ which is exterior to Ω_t we have a gas under constant pressure p_0. Moreover in the domain D_t we have an electromagnetic field which is generated by some currents which are located on a fixed boundary B on D_t.

In the domain Ω_t the motion is described by the following problem:

$$
\begin{aligned}
&v_t + v\cdot\nabla v - \operatorname{div}\mathbb{T}(v,p) - \mu_1 \overset{1}{H}\cdot\nabla\overset{1}{H} + \frac{1}{2}\mu_1\nabla\overset{1}{H}{}^2 = f && \text{in } \widetilde{\Omega}^T,\\
&\operatorname{div} v = 0 && \text{in } \widetilde{\Omega}^T,\\
&\mu_1\overset{1}{H}_t = -\operatorname{rot}\overset{1}{E} && \text{in } \widetilde{\Omega}^T,\\
&\operatorname{rot}\overset{1}{H} = \sigma_1(\overset{1}{E} + \mu_1 v\times\overset{1}{H}) && \text{in } \widetilde{\Omega}^T,\\
&\operatorname{div}(\mu_1\overset{1}{H}) = 0 && \text{in } \widetilde{\Omega}^T.
\end{aligned}
\tag{1.1}
$$

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where $\widetilde{\Omega}^T = \bigcup_{0\le t\le T} \Omega_t \times \{t\}$, $v = v(x,t)$ is the velocity of the fluid, $p = p(x,t)$ is the pressure, $\overset{1}{H} = \overset{1}{H}(x,t)$ is the magnetic field, $f = f(x,t)$ is the external force field per unit mass, μ_1 is the constant magnetic permeability, σ_1 is the constant electric conductivity, $\overset{1}{E} = \overset{1}{E}(x,t)$ is the electric field, and

$$\mathbb{T}(v,p) = \{\nu(\partial_{x_i} v_j + \partial_{x_j} v_i) - p\delta_{ij}\} \tag{1.2}$$

is the stress tensor, where ν is the viscosity of the fluid. Moreover, by

$$\mathbb{D}(v) = \{\nu(\partial_{x_i} v_j + \partial_{x_j} v_i)\} \tag{1.3}$$

we denote the dilatation tensor.

In the domain D_t which is a dielectric (gas) we assume that there is no fluid motion inside ($v = 0$). Therefore we have only the electromagnetic field described by the following system:

$$\begin{aligned} &\mu_2 \overset{1}{H}_t = -\operatorname{rot} \overset{2}{E} && \text{in } \widetilde{D}^T, \\ &\operatorname{rot} \overset{2}{H} = \sigma_2 \overset{2}{E} && \text{in } \widetilde{D}^T, \\ &\operatorname{div}(\mu_2 \overset{2}{H}) = 0 && \text{in } \widetilde{D}^T, \end{aligned} \tag{1.4}$$

where $\widetilde{D}^T = \bigcup_{0\le t\le T} D_t \times \{t\}$.

On $S_t = \partial\Omega_t \cap \partial D_t$ we assume the following transmission and boundary conditions:

$$\begin{aligned} &n\mathbb{T}(v,p) = -p_0 n && \text{on } \widetilde{S}^T, \\ &\frac{1}{\sigma_1}\overset{1}{H} = \frac{1}{\sigma_2}\overset{2}{H} && \text{on } \widetilde{S}^T, \\ &\overset{1}{E}\cdot\tau_\alpha = \overset{2}{E}\cdot\tau_\alpha, \quad \alpha = 1,2, && \text{on } \widetilde{S}^T, \\ &vn = -\frac{\phi_t}{|\nabla\phi_t|} && \text{on } \widetilde{S}^T, \end{aligned} \tag{1.5}$$

where $\widetilde{S}^T = \bigcup_{0\le t\le T} S_t \times \{t\}$, n is the unit outward vector to Ω_t and normal to S_t, τ_α, $\alpha = 1,2$, is any tangent vector to S_t, $\phi(x,t) = 0$ describes S_t at least locally.

Next we assume the boundary conditions on B:

$$\begin{aligned} &\overset{2}{H} = H_* && \text{on } B, \\ &\overset{2}{E} = E_* && \text{on } B. \end{aligned} \tag{1.6}$$

Finally, we assume the initial conditions

$$\begin{aligned} &\Omega_{t|t=0} = \Omega, \quad S_{t|t=0} = S, \quad D_{t|t=0} = D, \\ &v_{|t=0} = v_0, \quad \overset{1}{H}_{|t=0} = \overset{1}{H}_0 \quad \text{in } \Omega, \\ &\overset{2}{H}_{|t=0} = \overset{2}{H}_0, \qquad\qquad \text{in } D. \end{aligned} \tag{1.7}$$

To prove existence of solutions to the above problem we introduce the Lagrangian coordinates $\xi \in \Omega$. They are the initial data for the Cauchy problem

$$\frac{dx}{dt} = v(x,t), \quad x_{|t=0} = \xi \in \Omega. \tag{1.8}$$

Therefore $x_v(\xi,t) = \xi + \int_0^t \overline{v}(\xi,\tau)\,d\tau$, where

$$\overline{v}(\xi,t) = v(x_v(\xi,t),t).$$

To introduce the Lagrangian coordinates in D_t we extend v onto D_t. Let us denote the extended function by v'. Then we define $\xi \in D$ to be the Cauchy data for the problem

$$\frac{dx}{dt} = v'(x,t), \quad x_{|t=0} = \xi \in D. \tag{1.9}$$

Therefore $x_{v'}(\xi,t) = \xi + \int_0^t \overline{v}'(\xi,\tau)\,d\tau$, where $\overline{v}'(\xi,t) = v'(x_{v'}(\xi,t),t)$.

Then by $(1.1)_5$,

$$\begin{aligned} \Omega_t &= \{x \in \mathbb{R}^3 : x = x_v(\xi,t),\ \xi \in \Omega\}, \\ S_t &= \{x \in \mathbb{R}^3 : x = x_v(\xi,t),\ \xi \in S\}. \end{aligned}$$

Since S_t is determined at least locally by the equation $\phi(x,t) = 0$, S is described by $\phi(x_v(\xi,t),t)_{|t=0} = 0$. Moreover we have

$$\overline{n}_v = n(x_v(\xi,t),t) = \frac{\nabla_x \phi(x,t)}{|\nabla_x \phi(x,t)|}\bigg|_{x=x_v(\xi,t)}.$$

To simplify considerations we introduce the following notation:

$$\begin{aligned} &\|u\|_{l,Q} = \|u\|_{H^l(Q)}, \quad Q \in \{\Omega, S, D, \Pi, B\},\ 0 \le l \in \mathbb{Z}, \\ &\|u\|_{k,p,q,Q^T} = \|u\|_{L_q(0,T,W_p^k(Q))}, \quad Q \in \{\Omega, S, D, \Pi, B\}, \\ &p,q \in [1,\infty], \quad 0 \le k \in \mathbb{Z}, \end{aligned}$$

where $Q^t = Q \times (0,t)$,

$$|u|_{p,Q} = \|u\|_{L_p(Q)}, \quad Q \in \{\Omega, S, D, \Pi, B\},\ p \in [1,\infty].$$

2. Weak solutions. Weak solutions to problem (1.1)–(1.7) are defined in Lagrangian coordinates.

Definition 2.1. By *weak solutions* to problem (1.1)–(1.7) we mean functions $\overline{v}, \overline{H}$ which satisfy the integral identities

$$(2.1)\qquad \int_0^T\int_\Omega (-\overline{v}\overline{\varphi}_t + \mathbb{D}_v(\overline{v})\mathbb{D}_v(\overline{\varphi}))\,d\xi\,dt - \int_0^T\int_\Omega \left(\mu_1 \overset{1}{\overline{H}}\nabla_v \overset{1}{\overline{H}}\,\overline{\varphi} - \frac{1}{2}\mu_1\nabla_v \overset{1}{\overline{H}}{}^2\overline{\varphi}\right) d\xi\,dt = \int_0^T\int_\Omega \overline{f}\,\overline{\varphi}\,d\xi\,dt - \int_0^T\int_S p_0\overline{n}_v\varphi\,d\xi_S\,dt - \int_\Omega \overline{v}_0\overline{\varphi}(0)\,d\xi,$$

$$(2.2)\qquad \int_0^T\int_\Pi \left(-\mu\overline{H}\,\overline{\psi}_t - \mu\overline{v}\nabla_v\overline{H}\,\overline{\psi} + \frac{1}{\sigma}\operatorname{rot}_v \overline{H}\operatorname{rot}_v\overline{\psi}\right) d\xi\,dt - \int_0^T\int_\Omega \mu_1(\overline{v}\times \overset{1}{\overline{H}})\operatorname{rot}_v\overline{\psi}\,d\xi\,dt = \frac{1}{\sigma_2}\int_0^T\int_B (\overline{n}_v\times\overline{E}_*)\overline{\psi}\,d\xi_B\,dt - \mu\int_\Pi \overline{H}_0\overline{\psi}(0)\,d\xi,$$

where φ,ψ are sufficiently regular with $\varphi(x,T)=\psi(x,T)=0$, and $\overline{n}_v$ is the unit outward vector normal to S or B.

In (2.1), (2.2) we use the notation $\overline{A}\xi,t) = A(x_v(\xi,t),t)$,

$$\overline{H}_{|\Omega} = \overset{1}{H},\quad \overline{H}_{|D} = \overset{2}{\overline{H}},\quad \sigma_{|\Omega}=\sigma_1,\quad \sigma_{|D}=\sigma_2,\quad \Pi=\Omega\cup D,$$
$$\mu_{|\Omega}=\mu_1,\quad \mu_{|D}=\mu_2,$$

in (2.2) v is extended over Π,

$$\mathbb{D}_v(\overline{v}) = \{\mu(\partial_{x_i}\xi_k\nabla_{\xi_k}\overline{v}_j + \partial_{x_j}\xi_k\nabla_{\xi_k}\overline{v}_i)\},\qquad \operatorname{rot}_v\overline{v} = \nabla_v\times\overline{v},$$
$$\nabla_v = \partial_x\xi_i\nabla_{\xi_i},\quad \operatorname{div}_v\overline{v} = \nabla_v\cdot\overline{v} = \partial_{x_i}\xi_k\nabla_{\xi_k}\overline{v}_i,\quad \partial_{\xi_i} = \nabla_{\xi_i}.$$

Let A be the Jacobi matrix of the transformation $x = x_v(\xi,t)$. Then $\det A = \exp(\int_0^t \operatorname{div}_v\overline{v}d\tau) = 1$. Moreover

$$x^i_{\xi j} = \delta_{ij} + \int_0^t \partial_{\xi j}\overline{v}^i(\xi,\tau)\,d\tau,\qquad \xi_x = x_\xi^{-1}.$$

Hence we get

$$\sup_{\xi\in\Omega}|x_\xi| \le 1 + \sup_{\xi\in\Omega}\int_0^t |\overline{v}(\xi,\tau)|\,d\tau \le 1 + c\int_0^t \|\overline{v}\|_{3,\Omega}\,d\tau \le 1 + c\sqrt{t}\Big(\int_0^t \|\overline{v}\|^2_{3,\Omega}\,d\tau\Big)^{1/2} \le 1 + c\sqrt{t}\,\|\overline{v}\|_{3,2,2,\Omega^t},$$

and $\sup_{x\in\Omega_t}|\xi_x| \le \varphi(a)$, where $a = \sqrt{t}\,\|\overline{v}\|_{3,2,2,\Omega^t}$ and φ is an increasing positive function.

To prove the existence of a solution to the above problem we introduce Lagrangian coordinates connected with a given divergence-free function u. Moreover we linearize the terms with v in (1.1) by writing them in the form $u\nabla v$ and $u \times \overset{1}{H}$. Then from (2.1), (2.2) we get

$$
\begin{aligned}
(2.3)\qquad & \int_0^T\int_\Omega (-\overline{v}\,\overline{\varphi}_t + \mathbb{D}_u(\overline{v})\mathbb{D}_u(\overline{\varphi}))\,d\xi\,dt - \int_0^T\int_\Omega (\mu_1 \overset{1}{\overline{H}}{}' \nabla_u \overset{1}{\overline{H}}{}'\overline{\varphi} \\
& -\mu_1\nabla_u \overset{1}{\overline{H}}{}'^2\overline{\varphi})\,d\xi\,dt = \int_0^T\int_\Omega \bar{f}\,\overline{\varphi}\,d\xi\,dt - \int_0^T\int_S p_0\overline{n}_u\varphi\,d\xi_S\,dt - \int_\Omega \overline{v}_0\overline{\varphi}(0)\,d\xi,
\end{aligned}
$$

$$
\begin{aligned}
(2.4)\qquad & \int_0^T\int_\Pi \Big(-\mu\overline{H}\,\overline{\psi}_t - \mu\overline{u}\nabla_u\overline{H}\,\overline{\psi} + \frac{1}{\sigma}\operatorname{rot}_u\overline{H}\operatorname{rot}_u\overline{\psi}\Big)\,d\xi\,dt \\
& \qquad - \int_0^T\int_\Omega \mu_1(\overline{u}\times \overset{1}{H})\operatorname{rot}_u\overline{\psi}\,d\xi\,dt \\
& \qquad = \frac{1}{\sigma_2}\int_0^T\int_B (\overline{n}_u \times \overline{E}_*)\overline{\psi}\,d\xi_B\,dt - \mu\int_\Pi \overline{H}_0\overline{\psi}(0)\,d\xi,
\end{aligned}
$$

where $\overset{1}{\overline{H}}{}'$ is a given function.

We write $(1.5)_1$ in the form $\overline{n}\mathbb{T}(\overline{v},\overline{p}') = 0$, where $\overline{p} = \overline{p}' + p_0$. Then from Lemmas 3.1–3.9 of [2] we get

$$
\begin{aligned}
(2.5)\qquad & \|\overline{v}_t\|_{0,\Omega}^2 + \|\overline{v}\|_{1,\Omega}^2 + \|\overline{v}\|_{3,2,2,\Omega^t}^2 + \|\overline{v}_t\|_{2,2,2,\Omega^t}^2 + \|\overline{v}_{tt}\|_{1,2,2,\Omega^t}^2 \\
& + \|\overline{v}_{tt}\|_{0,\Omega}^2 + \|\overline{p}'\|_{2,2,2,\Omega^t}^2 + \|\overline{p}_t'\|_{1,2,2,\Omega^t}^2 \\
& \le \alpha(a,t)[(t\|\overline{u}_t\|_{2,2,2,\Omega^t}^2 + \|\overline{u}(0)\|_{0,\Omega}^2)(X_1 + \overline{F}) + (\varepsilon(X_1+\overline{F}) \\
& + c(\varepsilon)t(\overline{F} + \|\overline{v}(0)\|_{0,\Omega}^2))\|\overline{u}_t\|_{1,2,\infty,\Omega^t}^2 \\
& + (\varepsilon(\|\overset{1}{\overline{H}}{}'_t\|_{2,2,2,\Omega^t}^2 + \|\overset{1}{\overline{H}}{}'\|_{3,2,2,\Omega^t}^2) + c(\varepsilon)t(\|\overset{1}{\overline{H}}{}'\|_{0,2,2,\Omega^t}^2 + \|\overset{1}{\overline{H}}{}'_{tt}\|_{0,2,2,\Omega^t}^2 \\
& + \|\overset{1}{\overline{H}}{}'(0)\|_{0,\Omega}^2))(\|\overset{1}{\overline{H}}{}'\|_{2,2,\infty,\Omega^t}^2(1+\|\overline{u}\|_{2,2,\infty,\Omega^t}^2) + \|\overset{1}{\overline{H}}{}'_t\|_{2,2,2,\Omega^t}^2) \\
& + \|\overset{1}{\overline{H}}{}'\|_{2,2,\infty,\Omega^t}^2(\varepsilon\|\overline{u}_t\|_{2,2,2,\Omega^t}^2 + c(\varepsilon)t(\|\overline{u}\|_{0,2,2,\Omega^t}^2 + \|\overline{u}_t(0)\|_{0,\Omega}^2)) \\
& + \|\bar{f}_{tt}\|_{0,2,2,\Omega^t}^2 + \|v_{tt}(0)\|_{0,\Omega}^2 + \|\bar{f}_t\|_{0,2,2,\Omega^t}^2 + \overline{X}_1 + \overline{F}],
\end{aligned}
$$

where

$$(2.6)\quad \overline{F}=\alpha_1(a,t)[\varepsilon\|\overset{1}{\overline{H}}{}'\|^2_{3,2,2,\Omega^t}+c(\varepsilon)t(\|\overset{1}{\overline{H}}{}'(0)\|^2_{0,\Omega}+\|\overset{1}{\overline{H}}{}'_t\|^2_{0,2,2,\Omega^t}))$$
$$\cdot(\|\overset{1}{\overline{H}}{}'\|^2_{0,2,\infty,\Omega^t}+\|\overset{1}{\overline{H}}{}'\|^2_{2,2,\infty,\Omega^t})$$
$$+(\varepsilon\|\overline{u}\|^2_{3,2,2,\Omega^t}+c(\varepsilon)t(\|\overline{u}_t\|^2_{0,2,2,\Omega^t}+\|\overline{u}(0)\|^2_{0,\Omega}))\|\overset{1}{\overline{H}}{}'\|^4_{1,2,\infty,\Omega^t}$$
$$+c(\varepsilon)(\|\overline{f}\|^2_{0,2,2,\Omega^t}+\|\overline{f}_t\|^2_{0,2,2,\Omega^t})+\|\overline{v}(0)\|^2_{1,\Omega}+\|\overline{v}_t(0)\|^2_{0,\Omega}],$$

$$(2.7)\quad X_1=\alpha_2(a,t)[(\|\overset{1}{\overline{H}}{}'\|^2_{3,2,2,\Omega^t}+\|\overset{1}{\overline{H}}{}'\|^2_{2,2,\infty,\Omega^t})(\varepsilon\|\overset{1}{\overline{H}}{}'\|^2_{3,2,2,\Omega^t}$$
$$+c(\varepsilon)(\|\overset{1}{\overline{H}}{}'(0)\|^2_{0,\Omega}+\|\overset{1}{\overline{H}}{}'\|^2_{0,2,2,\Omega^t}))+\|\overline{f}\|^2_{1,2,2,\Omega^t}].$$

From Lemmas 4.1–4.8 of [2] we get

$$(2.8)\quad \|\overline{H}_t\|^2_{0,\Pi}+\|\overline{H}\|^2_{1,\Pi}+\|\overline{H}\|^2_{3,2,2,\Pi^t}+\|\overline{H}_t\|^2_{2,2,2,\Pi^t}$$
$$+\|\overline{H}_{tt}\|^2_{1,2,2,\Pi^t}+\|\overline{H}_{tt}\|^2_{0,\Pi}$$
$$\le\alpha(a,t)[(\varepsilon(\overline{Y}_1+\|\overline{H}_*\|^2_{3,2,2,B^t}+\|\overline{u}_t\|^2_{2,2,2,\Pi^t})$$
$$+c(\varepsilon)t(\overline{G}+\overline{Y}_2+\|\overline{H}_{*t}\|^2_{0,2,2,B^t}+\|\overline{H}_{*tt}\|^2_{0,2,2,B^t}+\|\overline{u}_t\|^2_{0,2,2,\Pi^t}$$
$$+\|\overline{u}_{tt}\|^2_{0,2,2,\Pi^t}++\|\overline{u}(0)\|^2_{0,\Pi}+\|\overline{u}_t(0)\|^2_{0,\Pi}+\|\overline{H}(0)\|^2_{0,\Pi}+\|\overline{H}_t(0)\|^2_{0,\Pi}))$$
$$\cdot(\|\overline{u}_t\|^2_{2,2,2,\Pi^t}+\|\overline{u}_t\|^2_{1,2,\infty,\Pi^t}+\|\overline{u}(0)\|^2_{2,\Pi}+\|\overline{H}(0)\|^2_{2,\Pi})+\varepsilon\|\overline{u}_t\|^4_{2,2,2\Pi^t}$$
$$+\|\overline{H}_{*t}\|^2_{1,2,\infty,B^t}+\|\overline{H}_{*t}\|^4_{1,2,2,B^t}+\|H_{*t}\|^2_{2,2,2,B^t}$$
$$+(t\|u_t\|^2_{2,2,2,\Pi^t}+\|\overline{u}(0)\|^2_{2,\Pi})\overline{Y}_1+\overline{G}^2+\overline{G}],$$

where

$$(2.9)\quad \overline{G}=\alpha_1(a,t)[(\varepsilon\|\overline{u}\|^2_{3,2,2,\Pi^t}+c(\varepsilon)t(\|\overline{u}_t\|^2_{0,2,2,\Pi^t}+\|u(0)\|^2_{0,\Pi}))$$
$$\cdot\|\overline{E}_*\|^2_{1,2,\infty,B^t}+\|\overline{H}(0)\|^2_{1,\Pi}+\|\overline{H}_t(0)\|^2_{0,\Pi}+\|\overline{E}_{*t}\|^2_{0,2,2,B^t}],$$

$$(2.10)\quad \overline{Y}_1=\alpha_2(a,t)[((1+\|\overline{u}\|^2_{2,2,\infty,\Pi^t}+\|\overline{u}\|^2_{3,2,2,\Pi^t}$$
$$+\|\overline{u}\|^4_{2,2,\infty,\Pi^t}+\|\overline{u}_t\|^2_{2,2,2,\Pi^t})$$
$$\cdot(\varepsilon\|\overline{u}\|^2_{3,2,2,\Pi^t}+c(\varepsilon)t(\|\overline{u}_t\|^2_{0,2,2,\Pi^t}$$
$$+\|\overline{u}(0)\|^2_{0,\Pi}))(\|\overline{u}\|^2_{2,2,\infty,\Pi^t}+1)$$
$$+\|\overline{H}(0)\|^2_{0,\Pi}+\|\overline{H}_*\|^2_{3,2,2,B^t}+\|\overline{H}_*(0)\|^2_{0,B})(t\|\overline{u}_t\|^2_{2,2,2,\Pi^t}$$
$$+\|\overline{u}(0)\|^2_{2,\Pi}+1)^2+t\|\overline{u}_t\|^2_{2,2,2,\Pi^t}+\|\overline{u}(0)\|^2_{2,\Pi}+\varepsilon\|\overline{u}_t\|^2_{2,2,2,\Pi^t}$$
$$+c(\varepsilon)t(\|\overline{u}_{tt}\|^2_{0,2,2,\Pi^t}+\|\overline{u}_t(0)\|^2_{0,\Pi})+\|\overline{H}_{*t}\|^2_{2,2,2,B^t}$$
$$+\|\overline{H}_{*tt}\|^2_{0,2,2,B^t}+\|\overline{H}(0)\|_{1,\Pi}],$$

$$
\begin{aligned}
(2.11)\qquad \overline{Y}_2 = \alpha_3(a,t)[&\varepsilon(\|\overline{u}_{tt}\|^4_{1,2,2,\Pi^t} + \|\overline{u}_t\|^4_{1,2,\infty,\Pi^t}\|\overline{u}\|^4_{2,2,\infty,\Pi^t}\\
&+ \|\overline{u}_t\|^4_{1,2,\infty,\Pi^t} + \|\overline{u}\|^4_{2,2,\infty,\Pi^t}\|\overline{u}_{tt}\|^4_{1,2,2,\Pi^t})(t^2\overline{G}^2 + \|\overline{H}(0)\|^4_{1,\Pi})\\
&+ (\varepsilon\|\overline{u}\|^2_{3,2,2,\Pi^t} + c(\varepsilon)t(\|\overline{u}\|^2_{0,2,2,\Pi^t}\\
&+ \|\overline{u}_t(0)\|^2_{0,\Pi}))(\|\overline{u}_t\|^2_{1,2,\infty,\Pi^t} + 1)\\
&\cdot \|\overline{H}_{tt}(0)\|^2_{0,\Pi} + \|\overline{u}_t\|^2_{1,2,\infty,\Pi^t}(\varepsilon\overline{Y}_1 + c(\varepsilon)t(\overline{G} + \|\overline{H}(0)\|^2_{0,\Pi}))\\
&+ \|\overline{E}_*\|^2_{0,2,2,B^t} + \|\overline{E}_{*tt}\|^2_{0,2,2,B^t}],
\end{aligned}
$$

where α_i, $i = 1, 2, 3$, are increasing positive functions.

3. Existence of solutions to (1.1)–(1.7)

LEMMA 3.1. *Let $\varepsilon = \sqrt{t}$ and $\overline{u} = \overline{v}_m$, $\overline{H}' = \overline{H}_m$. Then from* (2.5)–(2.11) *we get*

$$
\begin{aligned}
(3.1)\qquad \beta_{m+1}(t)&\\
&\le \sum_{i=2}^{6}\alpha_i(t^\gamma B, t^\gamma\varphi(0), B, \varphi(0))\beta^i_m(t) + ct^\gamma(B + \varphi(0))\beta_m(t)\\
&\quad + \alpha_1(t^\gamma B, t^\gamma\varphi(0), B, \varphi(0)),
\end{aligned}
$$

where

$$
\begin{aligned}
(3.2)\quad B &= \|\overline{E}_*\|_{0,2,2,B^t} + \|\overline{E}_{*t}\|^2_{0,2,2,B^t} + \|\overline{H}_*\|^2_{3,2,2,B^t} + \|\overline{H}_{*t}\|^2_{2,2,2,B^t}\\
&\quad + \|\overline{E}_{*tt}\|^2_{0,2,2,B^t} + \|\overline{E}_*\|^2_{1,2,\infty,B^t} + \|\overline{H}_{*tt}\|^2_{0,2,2,B^t} + \|\bar{f}_t\|^2_{0,2,2,\Omega^t}\\
&\quad + \|\bar{f}\|^2_{1,2,2,\Omega^t} + \|\bar{f}_{tt}\|^2_{0,2,2,\Omega^t},\\
\varphi(0) &= \sum_{i+k\le 2}(\|\partial^i_t\overline{v}(0)\|^2_{k,\Omega} + \|\partial^i_t\overline{H}(0)\|^2_{k,\Pi}),\\
\beta_m(t) &= \|\overline{v}_{mt}\|^2_{0,\Omega} + \|\overline{v}_m\|^2_{1,\Omega} + \|\overline{v}_m\|^2_{3,2,2,\Omega^t} + \|\overline{v}_{mt}\|^2_{2,2,2,\Omega^t}\\
&\quad + \|\overline{v}_{mtt}\|^2_{1,2,2,\Omega^t} + \|\overline{v}_{mtt}\|^2_{0,\Omega} + \|\overline{p}'_m\|^2_{2,2,2,\Omega^t} + \|\overline{p}'_{tm}\|^2_{1,2,2,\Omega^t}\\
&\quad + \|\overline{H}_{mt}\|^2_{0,\Pi} + \|\overline{H}_m\|^2_{1,\Pi} + \|\overline{H}_m\|^2_{3,2,2,\Pi^t}\\
&\quad + \|\overline{H}_{mt}\|^2_{2,2,2,\Pi^t} + \|\overline{H}_{mtt}\|^2_{1,2,2,\Pi^t}
\end{aligned}
$$

and α_i, $i = 1, \dots, 6$, are polynomial functions ($\alpha_1(0,0,0,0) = 0$), *and $\gamma > 0$, $c > 0$ are some constants.*

REMARK. In (2.5)–(2.11) we use the inequalities

$$
\begin{aligned}
\|\overline{u}\|^2_{2,2,\infty,\Omega^t} &\le ct(\|\overline{u}_t\|^2_{2,2,2,\Omega^t} + \|\overline{u}(0)\|^2_{2,\Omega}),\\
\|\overline{H}'\|^2_{2,2,\infty,\Pi^t} &\le ct(\|\overline{H}'_t\|^2_{2,2,2,\Pi^t} + \|\overline{H}'(0)\|^2_{2,\Pi}).
\end{aligned}
$$

Lemma 3.2. *Let $A > 0$ be sufficiently small and let $B, \varphi(0)$ be such that*

$$\begin{aligned} &\alpha_1(t^\gamma B, t^\gamma \varphi(0), B, \varphi(0)) \le \frac{1}{3}A, \\ &ct^\gamma (B + \varphi(0)) \le \frac{1}{3}, \\ &\sum_{i=2}^{6} \alpha_i(t^\gamma B, t^\gamma \varphi(0), B, \varphi(0)) A^{i-1} \le \frac{1}{3}. \end{aligned} \tag{3.3}$$

Then $\beta_m(t) \le A$, $m = 1, 2, \ldots$

Proof. Assume that $\beta_m(t) \le A$. Then (3.1), (3.3) imply that $\beta_{m+1}(t) \le A$.

Remark. From (3.3) we obtain the qualitative formula

$$t^\gamma \sim \frac{A}{B + \varphi(0)}. \tag{3.4}$$

Then for given A and small B, $\varphi(0)$ we can choose t large.

From Lemma 5.2 of [2] we get

$$\begin{aligned} \|\overline{\mathcal{V}}_{m+1}\|^2_{0,\Pi} + \|\overline{\mathcal{V}}_{m+1}\|^2_{2,2,2,\Pi^t} + \|\overline{\mathcal{V}}_{m+1t}\|^2_{0,2,2,\Pi^t} + \|\overline{\mathcal{V}}_{m+1}\|^2_{1,\Pi} \\ + \|\overline{\mathcal{P}}'_{m+1}\|^2_{1,2,2,\Omega^t} + \|\overline{\mathcal{H}}_{m+1}\|^2_{0,\Pi} + \|\overline{\mathcal{H}}_{m+1}\|^2_{1,2,2,\Pi^t} \\ \le \alpha(A) t(t+1)(\|\overline{\mathcal{V}}_m\|^2_{2,2,2,\Pi^t} + \|\overline{\mathcal{H}}_m\|^2_{1,2,2,\Pi^t}), \end{aligned} \tag{3.5}$$

where $\overline{\mathcal{V}}_{m+1} = \overline{v}_{m+1} - \overline{v}_m$, $\overline{\mathcal{P}}'_{m+1} = \overline{p}'_{m+1} - \overline{p}'_m$, $\overline{\mathcal{H}}_{m+1} = \overline{H}_{m+1} - \overline{H}_m$ and $\alpha(0) = 0$.

Lemma 3.3. *Let the assumptions of Lemma 3.2 be satisfied. Then for $t \le T^*$, where $\alpha(A)T^*(T^*+1) < 1$, we have convergence of the sequence $(\overline{v}_m, \overline{p}'_m, \overline{H}_m)$.*

Theorem 3.1. *Let the assumptions of Lemmas 3.1, 3.2 be satisfied. Then for $T \le T^*$ there exists a solution to problem (1.1)–(1.7) such that*

$$\begin{aligned} &\overline{v} \in L_2(0, T, H^3(\Omega)) \cap L_\infty(0, T, H^1(\Omega)), \\ &\overline{v}_t \in L_\infty(0, T, H^1(\Omega)) \cap L_2(0, T, H^2(\Omega)), \\ &\overline{v}_{tt} \in L_2(0, T, H^1(\Omega)) \cap L_\infty(0, T, L_2(\Omega)), \\ &\overline{p}' \in L_2(0, T, H^2(\Omega)), \\ &\overline{p}'_t \in L_2(0, T, H^1(\Omega)), \\ &\overline{H} \in L_2(0, T, H^3(\Pi)) \cap L_\infty(0, T, H^1(\Pi)), \\ &\overline{H}_t \in L_\infty(0, T, H^1(\Pi)) \cap L_2(0, T, H^2(\Pi)), \\ &\overline{H}_{tt} \in L_2(0, T, H^1(\Pi)) \cap L_\infty(0, T, L_2(\Pi)) \end{aligned}$$

and

$$\|\overline{v}_t\|^2_{1,2,\infty,\Omega^T} + \|\overline{v}\|^2_{1,2,\infty,\Omega^T} + \|\overline{v}\|^2_{3,2,2,\Omega^T} + \|\overline{v}_t\|^2_{2,2,2,\Omega^T} + \|\overline{v}_{tt}\|^2_{1,2,2,\Omega^T} + \|\overline{v}_{tt}\|^2_{0,2,\infty,\Omega^T} + \|\overline{p}'\|^2_{2,2,2,\Omega^T} + \|\overline{p}'_t\|^2_{1,2,2,\Omega^T} + \|\overline{H}_t\|^2_{1,2,\infty,\Pi^T} + \|\overline{H}\|^2_{1,2,\infty,\Pi^T} + \|\overline{H}\|^2_{3,2,2,\Pi^T} + \|\overline{H}_t\|^2_{2,2,2,\Pi^T} + \|\overline{H}_{tt}\|^2_{0,2,\infty,\Pi^T} + \|\overline{H}_{tt}\|^2_{1,2,2,\Pi^T} \le A.$$

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Institute of Mathematics and Cryptology
Cybernetics Faculty
Military University of Technology
S. Kaliskiego 2
00-908 Warszawa, Poland
E-mail: p.kacprzyk@imbo.wat.edu.pl

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