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WEIGHTING, LIKELIHOOD RATIO ORDER AND LIFE DISTRIBUTIONS

Abstract. We use weighted distributions with a weight function being a ratio of two densities to obtain some results of interest concerning life and residual life distributions. Our theorems are corollaries from results of Jain *et al.* (1989) and Bartoszewicz and Skolimowska (2006).

1. Preliminaries. Let X and Y be two random variables, F and G their respective probability distribution functions, and f and g their density functions, if they exist. Denote by $\overline{F} = 1 - F$ the tail (or survival function) of F, by $F^{-1}(u) = \inf\{x : F(x) \geq u\}$, $u \in (0,1)$, the quantile (or reversed) function and by $F^{-1}(0)$ and $F^{-1}(1)$ the lower and upper bounds of the support of F respectively, and analogously for G. We identify the distribution functions F and G with the respective probability distributions and denote their supports by S_F, S_G respectively. We use *increasing* in place of *nondecreasing* and *decreasing* in place *nonincreasing*.

1.1. *Classes of life distributions.* A distribution F is said to be IFR (resp. DFR) [*increasing* (resp. *decreasing*) *failure rate*] if $\log \overline{F}$ is concave (resp. convex) on S_F which is an interval. A distribution F with $S_F = [a, b]$, $-\infty \leq a < b < \infty$, is said to be IRFR (*increasing reversed failure rate*) if $\log F$ is convex on S_F. A distribution F is said to be DRFR (*decreasing reversed failure rate*) if $\log F$ is concave on S_F. It is well known that each DFR distribution is DRFR and each IRFR distribution is IFR.

A distribution F with $F(0) = 0$ and S_F being an interval is said to

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be IFRA (resp. DFRA) [*increasing* (resp. *decreasing*) *failure rate average*] if $-\log \overline{F}(x)/x$ is increasing (or decreasing) on S_F, or equivalently: $\overline{F}^{\alpha}(x) \leq (\geq)\ \overline{F}(\alpha x)$ for every $\alpha \in (0,1)$ and $x \in S_F$.

A distribution F with $S_F = [0,\infty)$ is said to be

- NBU (resp. NWU) [*new better* (resp. *worse*) *than used*] if $\overline{F}(x+y) \leq (\geq)\ \overline{F}(x)\overline{F}(y)$ for all $x, y, x+y \in S_F$;
- DMRL (resp. IMRL) [*decreasing* (resp. *increasing*) *mean residual life*] if $E[X-t \mid X>t]$ is decreasing (resp. increasing) in $t>0$;
- NBUE (resp. NWUE) [*new better* (resp. *worse*) *than used in expectation*] if $\int_t^{\infty} \overline{F}(x)\,dx \leq (\geq)\ E(X)\overline{F}(t)$, $t \geq 0$, provided that $E(X)$ exists.

It is well known that

$$\mathrm{IFR} \subset \mathrm{IFRA} \subset \mathrm{NBU} \quad \text{and} \quad \mathrm{DFR} \subset \mathrm{DFRA} \subset \mathrm{NWU}$$

and

$$\mathrm{IFR} \subset \mathrm{DMRL} \subset \mathrm{NBUE} \quad \text{and} \quad \mathrm{DFR} \subset \mathrm{IMRL} \subset \mathrm{NWUE}.$$

1.2. *Stochastic orders.* We will deal with some stochastic orders. We recall their definitions and some properties for completeness. Similarly to Shaked and Shanthikumar (1994) we use notation involving random variables. However, stochastic orders are relations between probability distributions.

We say that:

- X *is smaller than* Y *in the likelihood ratio order* ($X \leq_{\mathrm{lr}} Y$) if $g(x)/f(x)$ is increasing;
- X *is smaller than* Y *in the hazard rate order* ($X \leq_{\mathrm{hr}} Y$) if $\overline{G}(x)/\overline{F}(x)$ is increasing or $r_F(x) \geq r_G(x)$ for every x if F and G are absolutely continuous, where $r_F(x) = f(x)/\overline{F}(x)$ is the *hazard rate function* of F (and analogously for r_G);
- X *is smaller than* Y *in the reversed hazard rate order* ($X \leq_{\mathrm{rh}} Y$) if $G(x)/F(x)$ is increasing or $\breve{r}_F(x) \leq \breve{r}_G(x)$ for every x if F and G are absolutely continuous, where $\breve{r}_F(x) = f(x)/F(x)$ is the *reversed hazard rate function* of F (and analogously for $\breve{r}_G$);
- X is *stochastically smaller than* Y ($X \leq_{\mathrm{st}} Y$) if $F(x) \geq G(x)$ for every x, or equivalently, if $\overline{F}(x) \leq \overline{G}(x)$ for every x.

It is also well known that

$$\begin{array}{ccc} X \leq_{\mathrm{lr}} Y & \Longrightarrow & X \leq_{\mathrm{hr}} Y \\ \Downarrow & & \Downarrow \\ X \leq_{\mathrm{rh}} Y & \Longrightarrow & X \leq_{\mathrm{st}} Y \end{array}$$

Let X and Y be positive random variables with finite expectations. Denote by

$$L_X(p) = \frac{1}{E(X)} \int_0^p F^{-1}(u)\,du, \qquad 0 \le p \le 1,$$

the *Lorenz curve* of X and set $\overline{L}_X = 1 - L_X$ (and analogously for Y). It is well known that L_X is a distribution function and is convex (and hence also star-shaped) on $(0,1)$.

For properties of classes of life distributions and stochastic orders we refer to Barlow and Proschan (1975), Shaked and Shanthikumar (1994) and Müller and Stoyan (2002).

1.3. *The equilibrium renewal distribution.* Let X be a nonnegative random variable and $0 < E(X) < \infty$. A distribution function

$$F_e(x) = \int_0^x \frac{\overline{F}(t)}{E(X)}\,dt, \qquad t \ge 0,$$

is called the *equilibrium renewal distribution associated with* F (see Cox, 1962). It is evident that its density is of the form $f_e(x) = \overline{F}(x)/E(X)$, $x \ge 0$. Denote by X_e a random variable with distribution F_e. Similarly we define G_e, g_e and Y_e.

The following lemma is well known; see e.g. Shaked and Shanthikumar (1994) and Müller and Stoyan (2002).

Lemma 1. *Let X be a nonnegative random variable and* $0 < E(X) < \infty$. *Then*:

(a) F *is IFR* (*resp. DFR*) $\Leftrightarrow X_e \le_{\mathrm{lr}} X$ (*resp.* $X \le_{\mathrm{lr}} X_e$);

(b) F *is DMRL* (*resp. IMRL*) $\Leftrightarrow X_e \le_{\mathrm{hr}} X$ (*resp.* $X \le_{\mathrm{hr}} X_e$);

(c) F *is NBUE* (*resp. NWUE*) $\Leftrightarrow X_e \le_{\mathrm{st}} X$ (*resp.* $X \le_{\mathrm{st}} X_e$).

1.4. *Weighted distributions.* Let $w : \mathbb{R} \to \mathbb{R}^+$ be a function for which $0 < E[w(X)] < \infty$. Then

$$\widehat{F}_w(x) = \frac{1}{E[w(X)]} \int_{-\infty}^x w(u)\,dF(u) = \frac{1}{E[w(X)]} \int_0^{F(x)} wF^{-1}(z)\,dz$$

is a distribution function, called a *weighted distribution* associated with F. If a density f of F exists, then

$$\widehat{f}_w(x) = \frac{w(x)f(x)}{E[w(X)]}$$

is a density of $\widehat{F}_w$.

We refer to Patil and Rao (1977, 1978) and to Rao (1985) for a survey of statistical applications of weighting, especially to the analysis of data relating to human populations and ecology. Mahfoud and Patil (1982) developed a theory of weighted distributions. Jain *et al.* (1989) and Nanda and Jain (1999) studied relations of weighted distributions with classes of life distributions. Navarro *et al.* (2001) developed characterizations through reliability measures from weighted distributions. Bartoszewicz and Skolimowska (2006) obtained some results about preservation of classes of life distributions and stochastic orders under weighting.

Jain *et al.* (1989) have proved the following theorem.

THEOREM 1. *If the weight function w is increasing and concave* (*resp. decreasing and convex*) *and F is IFR* (*resp. DFR*), *then $\widehat{F}_w$ is also IFR* (*resp. DFR*).

Using a representation of weighted distributions by the Lorenz curve Bartoszewicz and Skolimowska (2006) have proved the following two theorems.

THEOREM 2. *Let w be a monotone left continuous function.*

(a) *If $w(x)$ is increasing and $w(x)r_F(x)$ is decreasing, then $\widehat{F}_w$ is DFR.*
(b) *If $w(x)$ is decreasing and $w(x)r_F(x)$ is increasing, then $\widehat{F}_w$ is IFR.*
(c) *If $w(x)$ is increasing and $w(x)\breve{r}_F(x)$ is decreasing, then $\widehat{F}_w$ is DRFR.*
(d) *If $w(x)$ is decreasing and $w(x)\breve{r}_F(x)$ is increasing, then $\widehat{F}_w$ is IRFR.*
(e) *If $w(x)r_F(x)$ is decreasing, then $\widehat{F}_w$ is DRFR.*
(f) *If $w(x)\breve{r}_F(x)$ is increasing, then $\widehat{F}_w$ is IFR.*

THEOREM 3. *Let F be absolutely continuous with $F(0) = 0$ and S_F be an interval.*

(a) *If F is IFRA* (*resp. NBU*) *and $w(x)\overline{F}(x)$ is increasing, then $\widehat{F}_w$ is IFRA* (*resp. NBU*).
(b) *Let w be decreasing left continuous. If F is DFRA* (*resp. NWU*) *and $w(x)/L_{w(X)}(\overline{F}(x))$ is decreasing, then $\widehat{F}_w$ is DFRA* (*resp. NWU*).

In this note some results following from these three theorems are discussed.

2. Results

2.1. *Weighting by monotone likelihood ratio.* Let X and Y be random variables with absolutely continuous distributions F and G with densities f and g respectively and such that $S_G \subseteq S_F$. Let F be fixed. Then it is obvious that the distribution G may be represented as the weighted distribution induced by F with the weight function $w(x) = g(x)/f(x)$. We have obviously

$$E[w(X)] = \int_{-\infty}^{\infty} \frac{g(x)}{f(x)} f(x)\,dx = \int_{-\infty}^{\infty} g(x)\,dx = 1.$$

It is easy to notice that the Lorenz curves of $U = w(X)$ are of the form

$$L_U(p) = G(F^{-1}(p)), \qquad p \in [0,1],$$

if w is left continuous increasing, and

$$L_U(p) = 1 - G(F^{-1}(1-p)), \qquad p \in [0,1], \tag{1}$$

if w is left continuous decreasing.

Immediately from Theorem 1 and the definition of the likelihood ratio order we have the following result.

Theorem 4.

(a) *If F is IFR, $X \le_{\mathrm{lr}} Y$ and $g(x)/f(x)$ is concave on S_F then G is also IFR.*
(b) *If F is DFR, $Y \le_{\mathrm{lr}} X$ and $g(x)/f(x)$ is convex on S_F then G is also DFR.*

Theorem 4 may be applied as a useful criterion for verifying the IFR/DFR property. Consider the following example.

Example 1. It is well known that the gamma distribution with density

$$f(x;p) = \frac{x^{p-1}e^{-x}}{\Gamma(p)}, \qquad x > 0,\ p > 0,$$

is DFR for $0 < p \le 1$ and IFR for $p \ge 1$. Barlow and Proschan (1975) proved this fact directly, writing $1/r_F$ as an integral. It is well known that the family of gamma distributions, indexed by the shape parameter $p \in (0,\infty)$, is ordered with respect to the likelihood ratio order. Then immediately from Theorem 4, by comparison with the exponential distribution, the DFR property follows for $p \in (0,1]$, and IFR for $p \in [1,2)$ and then by induction for $p \in [kp,(k+1)p)$, $k = 2,\ldots.$

Similarly, putting $w(x) = g(x)/f(x)$ in Theorem 2(a)–(d) we obtain the following useful results.

Theorem 5. *Let $g(x)/f(x)$ be a left continuous function.*

(a) *If $X \le_{\mathrm{lr}} Y$ (resp. $Y \le_{\mathrm{lr}} X$) and $g(x)/\overline{F}(x)$ (resp. $f(x)/\overline{G}(x)$) is decreasing, then F and G are both DFR.*
(b) *If $X \le_{\mathrm{lr}} Y$ (resp. $Y \le_{\mathrm{lr}} X$) and $f(x)/\overline{G}(x)$ (resp. $g(x)/\overline{F}(x)$) is increasing, then F and G are both IFR.*
(c) *If $X \le_{\mathrm{lr}} Y$ (resp. $Y \le_{\mathrm{lr}} X$) and $g(x)/F(x)$ (resp. $f(x)/G(x)$) is decreasing, then F and G are both DRFR.*

(d) *If* $X \leq_{\mathrm{lr}} Y$ (*resp.* $Y \leq_{\mathrm{lr}} X$) *and* $f(x)/G(x)$ (*resp.* $g(x)/F(x)$) *is increasing*, *then* F *and* G *are both IRFR.*

In this particular statement of Theorem 2, we omit parts (e) and (f), since their new versions are weaker than parts (a) and (d) of Theorem 5, respectively.

Putting now $w(x) = g(x)/f(x)$ in Theorem 3 and using (1) we easily obtain results for IFRA, DFRA, NBU and NWU distributions.

THEOREM 6. *Let* F *and* G *be absolutely continuous with supports being intervals*, $F(0) = G(0) = 0$ *and* $g(x)/f(x)$ *left continuous.*

(a) *If* F *is IFRA* (*resp. NBU*) *and* $g(x)/r_F(x)$ *is increasing*, *then* $X \leq_{\mathrm{lr}} Y$ *and* G *is also IFRA* (*resp. NBU*).
(b) *If* F *is DFRA* (*resp. NWU*) *and* $f(x)/r_G(x)$ *is increasing*, *then* $Y \leq_{\mathrm{lr}} X$ *and* G *is also DFRA* (*resp. NWU*).

REMARK 1. It is easy to notice that Theorem 5 may be proved directly, without using weighting and Theorem 2, and without the continuity assumption on $g(x)/f(x)$. For example we prove Theorem 5(a).

Let $X \leq_{\mathrm{lr}} Y$. We have

$$r_F(x) = \frac{f(x)}{\overline{F}(x)} = \frac{f(x)}{g(x)}\,\frac{g(x)}{\overline{F}(x)}$$

and so r_F is decreasing as the product of two positive decreasing functions. Since the likelihood ratio order implies the hazard rate order, the ratio $\overline{G}(x)/\overline{F}(x)$ is increasing and hence

$$r_G(x) = \frac{g(x)}{\overline{F}(x)}\,\frac{\overline{F}(x)}{\overline{G}(x)}$$

is decreasing. Other cases may be proved in a similar way.

REMARK 2. Notice that the assumptions of Theorem 5 imply that the distributions F and G are of the same type. For example, if $w(x) = g(x)/f(x)$ is increasing and $w(x)r_F(x) = g(x)\,/\,\overline{F}(x) = [g(x)\overline{G}(x)]\,/\,[\overline{G}(x)\overline{F}(x)] = r_G(x)[\overline{G}(x)/\overline{F}(x)]$ is decreasing, then both F and G must evidently be DFR. This is a consequence of the fact that the likelihood ratio order implies the hazard rate and reversed hazard rate orders.

2.2. *Results obtained without weighting assumptions.* The possibility of proving Theorem 5 without the use of weighting implies a simple observation that, the assumptions of this theorem may be weakened by replacing the likelihood ratio order by the hazard rate order in (a) and (b), and the reversed hazard rate order in (c) and (d). However, in these cases only one distribution has the required property in the assertions. We may formulate the following theorem.

THEOREM 7.

(a) *If* $X \leq_{\mathrm{hr}} Y$ (*resp.* $Y \leq_{\mathrm{hr}} X$) *and* $g(x)/\overline{F}(x)$ *is decreasing* (*resp. increasing*), *then* G *is DFR* (*resp. IFR*).
(b) *If* $X \leq_{\mathrm{hr}} Y$ (*resp.* $Y \leq_{\mathrm{hr}} X$) *and* $f(x)/\overline{G}(x)$ *is increasing* (*resp. decreasing*), *then* F *is IFR* (*resp. DFR*).
(c) *If* $X \leq_{\mathrm{rh}} Y$ (*resp.* $Y \leq_{\mathrm{rh}} X$) *and* $g(x)/F(x)$ *is decreasing* (*resp. increasing*), *then* G *is DRFR* (*resp. IRFR*).
(d) *If* $X \leq_{\mathrm{rh}} Y$ (*resp.* $Y \leq_{\mathrm{rh}} X$) *and* $f(x)/G(x)$ *is increasing* (*resp. decreasing*), *then* F *is IRFR* (*resp. DRFR*).

There is also another way to prove Theorem 5(a), (b) as well as to extend Theorem 7, by using relations between the equilibrium renewal distribution and classes of life distributions. Moreover, we can obtain results for NBUE and NWUE classes. In these cases the continuity of $g(x)/f(x)$ is not required, but finiteness of the expectations of X and Y is needed. We will not formulate these new versions of Theorem 5(a), (b), which would be weaker and the proof would be similar to that of the next theorem.

Combining Theorem 7 and Lemma 1(b), we obtain new versions of Theorems 7(a) and 7(b).

THEOREM 7′. *Let* X *and* Y *be nonnegative random variables with absolutely continuous distributions and* $0 < E(X) < \infty$ *and* $0 < E(Y) < \infty$.

(a) *If* $X \leq_{\mathrm{hr}} Y$ (*resp.* $Y \leq_{\mathrm{hr}} X$) *and* $g(x)/\overline{F}(x)$ *is decreasing* (*resp. increasing*), *then* F *is IMRL* (*resp. DMRL*) *and* G *is DFR* (*resp. IFR*).
(b) *If* $X \leq_{\mathrm{hr}} Y$ (*resp.* $Y \leq_{\mathrm{hr}} X$) *and* $f(x)/\overline{G}(x)$ *is increasing* (*resp. decreasing*), *then* F *is IFR* (*resp. DFR*) *and* G *is DMRL* (*resp. IMRL*).

Proof. (a) Let $g(x)/\overline{F}(x)$ be decreasing. This is equivalent to $Y \leq_{\mathrm{lr}} X_e$, since $\overline{F}(x)/E(X)$ is the density of the equilibrium distribution associated with F. The likelihood order implies the hazard one, so we have $X \leq_{\mathrm{hr}} Y$ and $Y \leq_{\mathrm{hr}} X_e$. Hence $X \leq_{\mathrm{hr}} X_e$ and from Lemma 1(b), F is IMRL. Moreover, G is DFR since its failure rate is decreasing as the product of two positive decreasing functions: $r_G(x) = [g(x)/\overline{F}(x)][\overline{F}(x)/\overline{G}(x)]$.

The case when $Y \leq_{\mathrm{hr}} X$ and $g(x)/\overline{F}(x)$ is increasing and part (b) can be proved in a similar manner.

Lemma 1(c) implies the following theorem.

THEOREM 8. *Let* X *and* Y *be nonnegative random variables with absolutely continuous distribution and* $0 < E(X) < \infty$ *and* $0 < E(Y) < \infty$.

(a) *If* $X \leq_{\mathrm{st}} Y$ (*resp.* $Y \leq_{\mathrm{st}} X$) *and* $g(x)/\overline{F}(x)$ *is decreasing* (*resp. increasing*), *then* F *is NWUE* (*resp. NBUE*).

(b) *If* $X \leq_{\mathrm{st}} Y$ (*resp.* $Y \leq_{\mathrm{st}} X$) *and* $f(x)/\overline{G}(x)$ *is increasing* (*resp. decreasing*), *then* G *is NBUE* (*resp. NWUE*).

2.3. *Residual life distribution.* Let us consider the residual life X_t of a life random variable X, where $t > 0$ is fixed. It is well known that

$$F_t(x) = P(X_t \leq x) = \frac{F(x+t) - F(t)}{\overline{F}(t)}, \qquad x > 0,\, t > 0,$$

the survival function is

$$\overline{F}_t(x) = \frac{\overline{F}(x+t)}{\overline{F}(t)}, \qquad x > 0,\, t > 0,$$

and a density of X_t is of the form

$$f_t(x) = \frac{f(x+t)}{\overline{F}(t)}, \qquad x > 0,\, t > 0,$$

provided that the density f exists. Therefore in the absolutely continuous case we may represent the distribution F_t as the weighted distribution induced by F with density

$$f_t(x) = \frac{1}{\overline{F}(t)} \frac{f(x+t)}{f(x)} f(x)$$

and weight function

$$w_t(x) = \frac{1}{\overline{F}(t)} \frac{f(x+t)}{f(x)}.$$

Notice that monotonicity of w_t means that $\log f$ is convex if w_t is increasing, and concave if w_t is decreasing. It is well known (see Barlow and Proschan, 1975) that if $\log f$ is convex, then F is DFR, and if $\log f$ is concave, then F is IFR, i.e. convexity or concavity of $\log f$ are stronger properties than DFR and IFR respectively. It is known that if F is IFR (resp. DFR), then F_t is also IFR (resp. DFR); see e.g. Müller and Stoyan (2002). From Theorems 2(d) and 2(e) we obtain the following results:

THEOREM 9.

(a) *If* $\log f$ *is concave and* $f(x+t)/F(x)$ *is increasing in* x *for every* $t > 0$, *then* F_t *is IRFR.*
(b) *If* $\log f$ *is concave and* $f(x+t)/\overline{F}(x)$ *is decreasing in* x *for every* $t > 0$, *then* F_t *is DRFR.*

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