Jianhui Huang (Edmonton)
Na Li (Hong Kong)

VALUATION AND OPTIMAL DESIGN
TO DEFAULTABLE SECURITY

Abstract. Herein, we develop a backward stochastic differential equation (BSDE) valuation of securities with default risk. Consequently, the optimal recovery problem with quasi-linear utility functions is discussed with the help of the stochastic maximum principle. Finally, two important examples: the exponential and power utility cases are studied and their business implications are considered.

1. Introduction. Credit risk (or default risk) has become the central interest of the financial community, with the development of new financial instruments exposed to default risk in financial engineering. Most of the extant credit risk literature concerns the determination of credit spread dynamics whereas the optimal design of defaultable securities has been traditionally neglected although it should deserve more attention. Our paper aims to fill this gap. We discuss the optimal design of corporate bond, precisely speaking, the optimal recovery problem which consists of seeking and characterizing the optimal recovery policy in bond issuance under the recovery of market value (RMV) scheme of Duffie and Singleton [1999].

One contribution of our work is in the category of default risk valuation which acts as the preliminary part to the credit risk analysis. Our modeling setup belongs to the reduced form framework which goes back to Artzner and Delbaen [1995], Jarrow and Turnbull [1995] and Duffie and Singleton [1999]. Under the RMV scheme, we develop a valuation representation of defaultable securities in terms of backward stochastic differen-

tial equations (BSDE) and our result can be extended to the more general
forward-backward stochastic differential equation (FBSDE) case.

Another contribution of our work lies in the optimal security design, an
indispensable building block to the overall credit risk management. Williams
[2003] investigates the principle-agent problems in continuous time with
stochastic control techniques and gives some sufficient condition for the op-
timal contract design. But to the best of our knowledge, this paper is the
first research in optimization of debt financing within the dynamic recovery
environment. We focus on a risk-neutral enterprise which aims to maximize
its objective functional based on some time additive utilities. We transform
this optimization problem into the framework of stochastic control by view-
ing the recovery rate in RMV as a controlled variable. Afterwards, we apply
the stochastic maximum principle to derive the optimal control.

The rest of this paper is organized as follows. Section 2 introduces the
basic economy setting in which the valuation and optimal design are dis-
cussed. Section 3 is devoted to the BSDE valuation of defaultable securities
where the recovery rate is given. The optimal recovery problem as well as its
optimal solution are spelled out in Section 4. Some further implications to
negative exponential and power utilities are presented in Section 5. Section
6 concludes with some possible extensions.

2. Reduced form model. We consider a continuous time financial
model defined on some filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ which sat-
isfies the "usual conditions" (see Protter [1990] for technical details). The
filtration $\mathcal{F}_t$ represents all the information that arrived to the market up
to time t. A short-rate process r_t is also fixed and assumed to be bounded
and progressively measurable. The probability measure P is assumed to be
the equivalent martingale measure introduced by Harrison and Kreps [1979]
and its existence is essentially equivalent to the arbitrage-free assumption
on the market. Any defaultable security in this market, with maturity T,
can be represented by a triple (ξ, Ψ, τ) where:

- ξ is an $\mathcal{F}_T$-measurable random variable satisfying $E\xi^2 < \infty$ and it
  represents the payoff received by the security holder at T provided no
  default has occurred before. The default time τ is assumed to be an
  $\mathcal{F}_t$-stopping time.
- The process Ψ_t stands for the amount of money received by the se-
  curity holder if default occurs at some time $t < T$. Following Duffie
  and Singleton [1999], throughout this paper, we will assume the RMV
  scheme for Ψ, that is ([1]), $\Psi_t = (1 - L_t)Y_t$.

([1]) Similar to Duffie and Singleton [1999], here we assume the price process admits
no jumps, that is, $\Delta Y_t = 0$.

- The fraction loss rate L_t is a predictable (^{2}) process and because in no situation, can the security holder lose more than the security's total market value Y_t, we have $L_t \leq 1$. However, L_t can take negative values as explained later.

REMARK 2.1. In principle, we can consider the "general recovery rule" suggested by Bielecki and Rutkowski [2002], where the payoff process Ψ satisfies

$$\Psi_t = p(t, Y_t).$$

Here, $p(t, x)$ is Lipschitz continuous with respect to t and $p(t, 0) = 0$.

Given a defaultable security (ξ, Ψ, τ), its *default process* is defined as $N_t = 1_{\{\tau \leq t\}}$, which is a stochastic process taking value 1 if default has occurred and 0 otherwise. Unlike the traditional structural models stemming from the pioneering work of Black and Scholes [1973] and Merton [1974], the reduced-form approach models the default process with a presumed default intensity process given below. As a standard result, the Doob–Meyer decomposition theorem enables us to decompose N_t as

$$N_t = A_t + M_t,$$

where the compensator A_t should be some predictable increasing process and M_t a martingale. Furthermore, suppose there exists some progressively measurable and nonnegative process λ_t such that

$$A_t = \int\limits_0^{t \wedge \tau} \lambda_u \, du = \int\limits_0^t 1_{\{t \leq \tau\}} \lambda_t \, du.$$

Then λ_t is called the *default intensity process* (or *hazard rate process*) as it describes the conditional default rate, or intensity in the sense that conditional on no default prior to t, the likelihood of default during the time interval $(t, t + dt)$ is just $\lambda_t dt$. As the key feature distinguishing them from the structure model, in the reduced-form models, the default intensity λ_t is assumed to be exogenous, in other words, the default is an unpredictable event depending on the whole economy situation instead of the issuer's market value itself.

3. BSDE valuation with given recovery rate. The main result in this section is that, under some mild conditions, the price process of (ξ, Ψ, τ) can be represented as the state solution of some BSDE provided the recovery rate is given. The BSDE, first introduced by Bismut [1973] for the linear

---

(^{2}) See Protter [1990] for the definition.

case and Pardoux and Peng [1990] for the general case, is

(3.1)
$$\begin{cases} -dY_t = f(t, Y_t, Z_t)dt - Z_t dW_t, \\ Y_T = \xi, \end{cases}$$

where W is the standard Brownian motion with respect to $\mathcal{F}_t$ and the functional f is always called the *driver* of the BSDE. Unlike its counterpart in the "forward" stochastic differential equation (SDE), the solution to the BSDE has two components: the state solution Y and volatility solution Z, where the additional component Z is just the risk adjustment factor in hedging theory.

THEOREM 3.1. *Under the general recovery rule, suppose $p(t, x)$ is Lipschitz continuous with respect to t, and r_t, λ_t are bounded and progressively measurable processes. Then the price process Y_t of (ξ, Ψ, τ) satisfies the following BSDE*:

(3.2)
$$\begin{cases} dY_t = [(r_t + \lambda_t)Y_t - p(t, Y_t)\lambda_t]dt + Z_t dW_t, \\ Y_T = \xi. \end{cases}$$

Proof. The defaultable security (ξ, Ψ, τ) can be written uniformly as D_t where

$$D_t = \begin{cases} \Psi_\tau 1_{\{\tau \leq t\}}, & t < T, \\ \Psi_\tau 1_{\{\tau \leq T\}} + \xi 1_{\tau > T}, & t \geq T. \end{cases}$$

Then recall the decomposition of the default process N_t:

$$N_t = 1_{\{\tau \leq t\}} = A_t + M_t = \int_0^{t \wedge \tau} \lambda_u \, du + M_t = \int_0^t 1_{\{u \leq \tau\}} \lambda_u \, du + M_t.$$

For $t > 0$, we have

$$D_t = \int_0^{t \wedge T} \Psi_s 1_{\{s \leq \tau\}} \lambda_s \, ds + \int_0^{t \wedge T} \Psi_s \, dM_s + \xi 1_{\{\tau > T, t \geq T\}}.$$

From Harrison and Kreps [1979], the price process Y_t of (ξ, Ψ, τ) should be

$$Y_t = E\Big[\int_t^T \exp\Big(-\int_t^u r_s \, ds\Big)\Psi_u 1_{\{u < \tau\}} \lambda_u \, du + \exp\Big(-\int_t^T r_s \, ds\Big)\xi 1_{\{\tau > T\}} \,\Big|\, \mathcal{F}_t\Big].$$

Because $\Psi_t = p(t, Y_t)$, we have

$$Y_t = E\Big[\int_t^T \exp\Big(-\int_t^u r_s \, ds\Big)p(u, Y_u) 1_{\{u \leq \tau\}} \lambda_u \, du + \exp\Big(-\int_t^T r_s \, ds\Big)\xi 1_{\{\tau > T\}} \,\Big|\, \mathcal{F}_t\Big].$$

From the nonlinear Feynman–Kac formula,

$$Y_t = E\Big[\int_t^T (p(u, Y_u)\lambda_u - r_u Y_u) \, du + \xi 1_{\{\tau > T\}} \,\Big|\, \mathcal{F}_t\Big].$$

As discussed in Duffie, Schroder and Skiadas [1996], we can eliminate the indicator function $1_{\{\tau > T\}}$ and get the following recursive valuation formula:

$$Y_t = E\Big[\int_t^T [(p(u, Y_u) - Y_u)\lambda_u - r_u Y_u]\, du + \xi \,\Big|\, \mathcal{F}_t\Big].$$

Now consider the following BSDE:

(3.3)
$$\begin{cases} dY_t = [(r_t + \lambda_t)Y_t - p(t, Y_t)\lambda_t]dt + Z_t dW_t, \\ Y_T = \xi. \end{cases}$$

As we assumed $p(t, x)$ is Lipschitz continuous and r_t, λ_t are both bounded and predictable, from Pardoux and Peng [1990], we know the above BSDE has a unique adapted solution (Y, Z). If we write the above BSDE in integral form, we have

$$Y_t = \xi - \int_t^T [(r_t + \lambda_t)Y_t - p(t, Y_t)\lambda_t]\, dt - \int_t^T Z_t\, dW_t.$$

After taking the conditional expectation, the last term vanishes due to the martingale property of Itô integration,

$$Y_t = E\Big[\int_t^T [(p(u, Y_u) - Y_u)\lambda_u - r_u Y_u]\, du + \xi \,\Big|\, \mathcal{F}_t\Big].$$

This completes the proof. ∎

As in the special case when RMV is assumed, $\Psi_t = p(t, Y_t) = (1 - L_t)Y_t$, we have the more explicit result than that obtained in Duffie and Singleton [1999].

THEOREM 3.2. *Under the RMV scheme, the price Y_t of a defaultable claim (ξ, Ψ, τ) satisfies the following BSDE*:

(3.4)
$$\begin{cases} dY_t = (r_t + \lambda_t L_t)Y_t dt + Z_t dW_t, \\ Y_T = \xi. \end{cases}$$

The advantage of our BSDE valuation is that it gives a dynamical rather than static representation of the price behavior; and based on this, we can use the stochastic control techniques to solve the optimization problem recursively.

4. Optimal security design

4.1. *Optimal recovery policy.* Assume that a firm with a project prefers to choose debt as its financing strategy. Suppose it plans to issue a zero-coupon bond and commits to the RMV scheme where it can choose its recovery policy dynamically. To simplify the analysis, here we first provide a heuristic introduction of the objective functional in a discrete-time setting

and then reformulate it in a continuous-time setting. Consider a discrete-time market with N time instants on which the firm's bond is introduced with payment ξ at the expiry time N. Moreover, for $n = 1, \ldots, N$:

- If no default occurs before time n, then let λ_n be the conditional probability of default between times n and $n + 1$.
- β_n and Y_n are respectively the discount-free short rate and the bond's market value at time n.
- L_n is the recovery rate in case the default occurs at time n.

According to RMV, if the default really occurs at time n, the firm needs only pay $(1 - L_n)Y_n$ to its bond holder. Because the bond's market value at time n is Y_n, this recovery policy helps the firm save as much as $Y_n - (1 - L_n)Y_n = L_nY_n$ compared to its default-free situation. Therefore, L_nY_n is the benefit to the firm due to the recovery policy $\{L_n\}_{n=1}^{N}$ and we call it the *intertemporal gain* process. Following Hojgaard and Taksar [2001], we can treat it as a controlled variable and consider the following functional in discrete time:

$$J^N(L) = E\Big[\sum_{n=1}^{N} e^{-\sum_{j=1}^{n}\beta_j}U_1(L_nY_n)\lambda_n + U_2(Y_0)\Big],$$

where U_1, U_2 are two utility functions. Now we discuss its business implications:

- The terms $U_1(L_nY_n)$, $n = 1, \ldots, N$, denote the firm's preference to *instantaneous gain* at time n. Although the default may occur at any time from 1 to N, it cannot occur at two different times simultaneously, as for a given bond, there is at most one default. Therefore, we should take account of these terms in a somewhat "average" sense by alternatively considering $U_1(L_nY_n) \cdot \lambda_n$ for each $n = 1, \ldots, N$, where the "weight" λ_n is included here to represent their respective likelihoods. Two of the most typical examples will be respectively: (1) $\lambda_n \equiv \lambda$ where λ is some constant; (2) λ_n is some deterministic function on n. In general, our default process is "double stochastic", that is, λ_n and L_nY_n are both stochastic processes (see Cox [1955] for the details). Thus, our objective functional still needs to involve the expectation under P to capture such "double stochastic" feature. Therefore, on the whole, the first term in $J^N(L)$ measures the "cumulative" satisfaction to the *instantaneous gain* process L_nY_n.
- $U_2(Y_0)$ denotes the firm's preference to the initial bond price Y_0 which is exactly the amount of money initially raised per share and it changes conversely to the bond's yield to maturity rate (YTM).

In the continuous-time case, given the recovery policy $1 - L_t$ and utility functions U_1, U_2, $\lambda_t dt$ is just the likelihood of default during the infinitesimal

time interval $(t, t + dt)$ if no default occurs prior to t. Therefore, similarly to the reasoning in the discrete-time case, we can define the objective functional

$$J(L) = E\left[\int_0^T e^{-\int_0^t \beta_s \, ds} U_1(L_t Y_t) \lambda_t \, dt + U_2(Y_0)\right],$$

which represents the firm's expected utility over its *intertemporal gain* process and the initial financing amount Y_0. Here, β is the discount rate over time. The firm aims to maximize this time-additive objective functional $J(L)$ over the admissible recovery policy set $\mathcal{U}$ which we will specify soon.

For the sake of implementation and tractability, here we consider the objective functional defined on the claim's whole life horizon $[0, T]$. A more realistic objective functional, defined on a random time interval, is briefly discussed at the end of this subsection.

Intuitively, the lower L_t, the higher Y_0 (equivalently the lower YTM the investor required), hence the heavier the recovery obligation $(1 - L_t)Y_t$ to the firm in case of default, and this will offset the benefits from the lower YTM somewhat. Conversely, the higher L_t, the lower Y_0, but on the other hand, the higher L_t will release the firm's recovery burden on default. Therefore, on the whole, $J(L)$ trades off the firm's gratification between the instantaneous gain process and the initially raised capital, and the firm's objective can be formulated as the following *optimal recovery problem* in the setting of stochastic control. In the following, we will replace L by u to be consistent with the conventions of control theory.

DEFINITION 4.1. The *optimal recovery problem* is to find

$$\sup_{u \in \mathcal{U}} J(u) = \sup_{u \in \mathcal{U}} E\left[\int_0^T e^{-\int_0^t \beta_s \, ds} U_1(u_t Y_t) \lambda_t \, dt + U_2(Y_0)\right],$$

equivalently ([3])

$$\inf_{u \in \mathcal{U}} -J(u) = \inf_{u \in \mathcal{U}} E\left[\int_0^T -e^{-\int_0^t \beta_s \, ds} U_1(u_t Y_t) \lambda_t \, dt - U_2(Y_0)\right],$$

where

(4.1)
$$\begin{cases} dY_t = (r_t + \lambda_t u_t) Y_t dt + Z_t dW_t, \\ Y_T = \xi, \end{cases}$$

and $\lambda_t > 0, r_t$ are bounded and progressively measurable processes. The

___

([3]) Note that $\sup_{u \in \mathcal{U}} J(u) = -\inf_{u \in \mathcal{U}} [-J(u)]$; we state the optimal recovery problem in this form to relate it directly to control theory which typically considers the minimization of the objective functional.

admissible control set $\mathcal{U}$ is

$$\mathcal{U} = \left\{ u : E \int\limits_0^T u^2(s)\, ds < \infty,\ u(\cdot) \in \mathcal{C} = (-\infty, 1) \text{ a.e.} \right\}.$$

REMARK 4.2. We set the admissible control domain $\mathcal{C}$ to be $(-\infty, 1)$, so the loss rate u can take a negative value. This is reasonable as, in some cases, the firm itself would like to compensate more than the bond market value to its bond holder to keep its business reputation.

The *optimal recovery policy* is the admissible control $u^* \in \mathcal{U}$ which maximizes the objective functional. If Y^* is the price process corresponding to u^*, then (u^*, Y^*) will be called the *optimal control pair*, a terminology borrowed from control theory.

In this paper, we focus on the situations where the utility functions U_1, U_2 are of the quasi-linear form of Hansen [1985] and Rogerson [1988]. The quasi-linear utility is one of the benchmark macroeconomic models and it is used to describe the preference to two different incomes x, y through

$$U(x, y) = U_1(x) + Ky, \quad K > 0.$$

That is, U is additive in x and y, and linear in y, where U_1 satisfies the Inada condition. In the optimal recovery problem, we assume the firm's preference to the initial price Y_0 is linear as in the quasi-linear case ([4])

$$U_2(Y_0) = KY_0, \quad K > 0.$$

A nontrivial example is when $U_1(x) = -e^{-x}$ is the exponential utility,

$$J(u) = E\left[\int\limits_0^T -e^{-\int_0^t \beta_s\, ds} e^{-u_t Y_t} \lambda_t\, dt + KY_0 \right].$$

Several features distinguish our optimal control problem from other stochastic control problems applied in finance and business, and these features make our problem more complicated to solve:

- Unlike the *investment-consumption* proposed by Merton [1974], the state variable Y in our problem follows a BSDE rather than the usual SDE.
- Our objective functional $J(u)$ relies on the initial value Y_0 instead of the terminal value Y_T.
- The control variable u enters nonlinearly into both the state equation and the objective functional.

([4]) Here, the preference is a minor modification to the standard quasi-linear one in that the utility of $u_t Y_t$ is in its integration form over $[0, T]$.

As mentioned before, to be more realistic, sometimes we can also investigate the following objective functional:

$$J(L) = E\left[\int_0^\tau e^{-\int_0^t \beta_s \, ds} U_1(L_t Y_t)\lambda_t \, dt + U_2(Y_0)\right],$$

where τ is the default time to characterize the firm's preference. Such an objective functional involves random time duration which makes its optimization problem more complicated. Here, we just sketch the key point towards its solution and postpone the full details to our future work: analogous to the proof of Theorem 3.1, we can transform the integration of the above functional into some integration with fixed time interval $[0, T]$; as a tradeoff, we need to introduce some extra terms with respect to λ_t into our integrand.

4.2. *Stochastic maximum principle.* To ease the notation, hereafter we denote $-U_1$ by l, $-U_2$ by γ, and rewrite the optimal recovery problem as

$$\inf_{u\in\mathcal{U}} -J(u) = \inf_{u\in\mathcal{U}} E\left[\int_0^T e^{-\int_0^t \beta_s \, ds} l(u_t Y_t)\lambda_t \, dt + \gamma(Y_0)\right].$$

As a preliminary step to solve our problem, we introduce the Hamiltonian function $H : [0, T] \times \mathbb{R} \times \mathcal{C} \times \mathbb{R}^+ \to \mathbb{R}$ defined as

$$H(t, q, u, y) = q_t(r_t + \lambda_t u_t)y_t + e^{-\int_0^t \beta_s \, ds} l(u_t y_t)\lambda_t.$$

According to the stochastic maximum principle, the optimal recovery policy u^* maximizes the Hamiltonian function as well. A key object in the stochastic maximum principle is the adjoint variable q, which is dual to the state variable Y and its dynamics is defined via the dual equation

$$(4.2) \qquad \begin{cases} -dq_t = H_y(t, q_t, u_t, Y_t) \, dt, \\ q_0 = -\gamma_y(Y_0). \end{cases}$$

PROPOSITION 4.3 (Stochastic Maximum Principle, Peng [1993]). *Suppose the admissible control domain $\mathcal{C}$ of the optimal recovery problem is convex, (u^*, Y^*) is the optimal control pair, and q^* the corresponding adjoint variable. Then the following necessary condition holds true almost surely:*

$$H_u(t, q_t^*, u_t^*, Y_t^*) \cdot (v - u_t^*) \geq 0 \quad \forall v \in \mathcal{C}.$$

When we consider the situation in which the admissible control domain $\mathcal{C}$ is an open set, the admissible control will be an interior control and the above stochastic maximum principle will take a simpler form so we have the following result.

THEOREM 4.4. *If the admissible control domain $\mathcal{C}$ is open, then the optimal control pair (u^*, Y^*) for the optimal recovery problem satisfies*

$$l'(u_t^* Y_t^*) = \gamma_y(Y_0^*)e^{\int_0^t (\beta_s - r_s)\, ds}.$$

Proof. The Hamiltonian function H for our problem is

$$H(t, q_t^*, u_t^*, Y_t^*) = q_t^*(r_t + \lambda_t u_t^*)Y_t^* + e^{-\int_0^t \beta_s\, ds} l(u_t^* Y_t^*)\lambda_t.$$

Due to the stochastic maximum principle, the optimal control u^* should satisfy

$$H_u(t, q_t^*, u_t^*, Y_t^*) \cdot (v - u_t^*) \geq 0 \quad \forall v \in \mathcal{C},$$

where q^* is given by the dual equation ([5])

(4.3)
$$\begin{cases} -dq_t^* = H_y(t, q_t^*, u_t^*, Y_t^*)dt, \\ q_0^* = -\gamma_y(Y_0^*). \end{cases}$$

Note that

(4.4)
$$\begin{cases} H_u(t, q, u, y) = q_t \lambda_t y_t + e^{-\int_0^t \beta_s\, ds} l'(u_t y_t)\lambda_t y_t, \\ H_y(t, q, u, y) = q_t(r_t + \lambda_t u_t) + e^{-\int_0^t \beta_s\, ds} l'(u_t y_t)\lambda_t u_t. \end{cases}$$

Therefore

(4.5)
$$\begin{cases} -dq_t^* = q_t^*(r_t + \lambda_t u_t^*)dt + e^{-\int_0^t \beta_s\, ds} l'(u_t^* Y_t^*)\lambda_t u_t^* dt, \\ q_t^* = -\gamma_y(Y_0^*), \end{cases}$$

(4.6) $\quad (q_t^* \lambda_t Y_t^* + e^{-\int_0^t \beta_s\, ds} l'(u_t^* Y_t^*)\lambda_t Y_t^*)(v - u_t^*) \geq 0 \quad \forall v \in \mathcal{C}.$

Because the control domain $\mathcal{C} = (-\infty, 1)$ is an open set, any point in $\mathcal{C}$ will be an interior point, so necessarily

(4.7) $\qquad q_t^* \lambda_t Y_t^* + e^{-\int_0^t \beta_s\, ds} l'(u_t^* Y_t^*)\lambda_t Y_t^* = 0.$

In view of $\lambda_t Y_t^* > 0$,

(4.8) $\qquad q_t^* + e^{-\int_0^t \beta_s\, ds} l'(u_t^* Y_t^*) = 0.$

Recalling the dual equation, we have

(4.9)
$$\begin{cases} -dq_t^* = q_t^* r_t dt, \\ q_0^* = -\gamma_y(Y_0^*). \end{cases}$$

Its solution is

$$q_t^* = -\gamma_y(Y_0^*)e^{-\int_0^t r_s\, ds}.$$

As U_1 is strictly concave, so $h = (-l')^{-1} = (U_1')^{-1}$ exists, we have

(4.10) $\qquad u_t^* Y_t^* = h(e^{\int_0^t \beta_s\, ds} q_t^*) = h(-\gamma_y(Y_0^*)e^{\int_0^t (\beta_s - r_s)\, ds}).$

---

[5] The initial condition of the dual equation in Peng [1993] should be $-\gamma(Y_0^*)$.

It follows that

$$(4.11) \qquad \begin{cases} dY_t^* = (r_t Y_t^* + \lambda_t h(e^{\int_0^t \beta_s \, ds} q_t^*)) dt + Z_t^* dW_t, \\ Y_T^* = \xi. \end{cases}$$

From Pardoux and Peng [1990], the above BSDE has a unique adapted solution (Y_t^*, Z_t^*) and the optimal recovery policy u^* will be

$$(4.12) \qquad u_t^* = \frac{h(e^{\int_0^t \beta_s \, ds} q_t^*)}{Y_t^*}. \quad \blacksquare$$

To find the optimal recovery policy u^*, we need first to solve the BSDE for the optimal pair (u^*, Y^*); for the technical background on the numerical solution of BSDE, the reader may refer to Ma *et al.* [2002]. Roughly, from the structure of the BSDE, we can see the optimal recovery policy is changing in the opposite direction to the bond price: the higher the bond price, the lower the recovery rate, and vice versa. Moreover, to make our optimal recovery policy u^* implementable or admissible, we need $u_t^* < 1$, that is,

$$u_t^* = \frac{h(e^{\int_0^t \beta_s \, ds} q_t^*)}{Y_t^*} < 1$$

or equivalently

$$Y_t^* > h(e^{\int_0^t \beta_s \, ds} q_t^*).$$

Define

$$K_t^* = Y_t^* - h(e^{\int_0^t \beta_s \, ds} q_t^*) > 0.$$

Using the Itô formula, K_t^* should satisfy the following BSDE:

$$(4.13) \quad \begin{cases} dK_t^* = [r_t K_t^* + (r_t + \lambda_t) h(S_t^*) - (\beta_t - r_t) h'(S_t^*) S_t^*] dt + Z_t^* dW_t, \\ K_T^* = \xi - h(e^{\int_0^T \beta_s \, ds} q_T^*), \end{cases}$$

where

$$S_t^* = e^{\int_0^t \beta_s \, ds} q_t^* = -\gamma_y(Y_0^*) e^{\int_0^t (\beta_s - r_s) \, ds}.$$

From the comparison theorem for BSDE (see Peng [1990] for details), we get the following result which leads to $u_t^* < 1$ immediately.

PROPOSITION 4.5. *The optimal solution u^* satisfies $u^* < 1$ whenever*

$$(4.14) \qquad (r_t + \lambda_t) h(S_t^*) - (\beta_t - r_t) h'(S_t^*) S_t^* < 0,$$

and

$$(4.15) \qquad K_T^* = Y_T^* - h(-\gamma_y(Y_0^*) e^{\int_0^T (\beta_t - r_t) \, dt}) > 0.$$

In particular, when $\beta_t = r_t$, because $r_t + \lambda_t > 0$, conditions (4.15) and (4.16) take a simpler form

$$h(-\gamma_y(Y_0^*)) < 0.$$

5. Examples. In this section, we discuss some interesting and insightful examples and show how to explore the optimal recovery policy using the general results we derived in the previous section.

5.1. *Negative exponential utility.* First consider the case when U_1 is a negative exponential utility and U_2 remains linear ([6]),

$$U_1(x) = -e^{-x}, \quad U_2(y) = Ky, \quad K > 0.$$

Then the objective functional takes the form

$$J(u) = E\Big[\int_0^T -e^{-\int_0^t \beta_s \, ds} e^{-u_t Y_t} \lambda_t \, dt + KY_0\Big],$$

and the corresponding optimal recovery problem is

$$\inf_{u \in \mathcal{U}} -J(u) = \inf_{u \in \mathcal{U}} E\Big[\int_0^T e^{-\int_0^t \beta_s \, ds} e^{-u_t Y_t} \lambda_t \, dt - KY_0\Big],$$

where

- $l = e^{-\int_0^t \beta_s \, ds} e^{-uy} \lambda, \quad \gamma = -Ky,$
- $H = q[(r + \lambda u)y] + e^{-\int_0^t \beta_s \, ds} e^{-uy} \lambda.$

The admissible control domain $\mathcal{C} = (-\infty, 1)$ is an open set, so the stochastic maximum principle becomes

$$e^{\int_0^t \beta_s \, ds} q_t^* = -\gamma_y(Y_0^*) e^{\int_0^t (\beta_s - r_s) \, ds} = e^{u_t^* Y_t^*}.$$

Therefore

$$u_t^* Y_t^* = -\ln(-\gamma_y(Y_0^*) e^{\int_0^t (\beta_s - r_s) \, ds}) = h(e^{\int_0^t \beta_s \, ds} q_t^*).$$

From the comparison theorem for BSDE, or the properties of linear BSDE, we know that a sufficient condition to have $Y_t^* \leq 1$ is

$$(r_t + \lambda_t)h(S_t^*) - (\beta_t - r_t)h'(S_t^*)S_t^* < 0.$$

Its market implications are as follows:

- If $\beta_t = r_t$, then the optimal recovery policy u^* exists whenever

$$\ln(-\gamma_y(Y_0^*)) > 0,$$

  that is,

$$K > 1.$$

  This tells us that if the linear utility coefficient is $K > 1$, the optimal recovery policy can be designed in the financial market. Such market characteristics become apparent when we consider the asymptotic

---

([6]) We can generalize this to the case

$$U_1(x) = -\kappa e^{\alpha x}, \quad \alpha, \kappa < 0, \quad U_2(y) = Ky, \quad K > 0.$$

behavior, that is, let $K \to \infty$. In this limit case, the objective functional $J(u)$ is simplified by only considering the initial capital Y_0, and eliminating the instantaneous gain process $u_t Y_t$. It is obvious that the optimal control in this situation should be $u^* = 0$, that is, the firm should compensate all the loss to the bond holder once the default occurs. This means the bond is default-free from the perspective of the bond holder. Keep in mind that the higher the initial price Y_0, the lower the YTM. Therefore, our result can be understood that if the issuer cares more (when $K > 1$) about the initial price, then the optimal recovery policy can be dynamically designed in this market model; however, if the firm cares less about the initial price Y_0 (when $K < 1$), and more about the instantaneous gain process, their preference style will prohibit them from finding the optimal recovery policy in such a market.

- If $\beta_t = r_t$, notice that

$$u_t^* Y_t^* = h(S_t^*) = h(-\gamma_y(Y_0^*)) = -\ln(K).$$

Of course, the price process satisfies $Y_t^* > 0$, so the above equality tells us that

$$u_t^* = \frac{-\ln(K)}{Y_t^*} < 0.$$

Hence, another market implication is as follows: recall the financial meaning of $u_t^* = L_t^*$ is just the loss rate in RMV scheme, therefore, the firm should choose a negative recovery policy to maximize its utility. The negative recovery policy here means the firm should make more compensation than the loss to the bond holder once the default occurs. Further, the bigger the linear coefficient K, the greater the compensation. The total amount compensated to the bond holder when the default occurs at time t will be $u_t^* Y_t^* = \ln(K)$, which is independent of the default time $\tau = t$.

- On the other hand, note that

$$U_1(x) = -e^{-x}, \quad U_2(y) = Ky, \quad K > 0.$$

Correspondingly,

$$U_1'(x) = e^{-x}, \quad U_2'(y) = K, \quad K > 0.$$

As $x, y \to \infty$, $U_1'(x) \to 0$ while $U_2'(y) \equiv K > 0$. Based on this observation, we know that, compared to the instantaneous gain process $u_t Y_t$ with utility U_1, the bond issuer cares more about the initially raised money Y_0 with U_2 as its utility. Consequently, the firm would like to adopt the negative recovery policy $u_t < 0$ to minimize its discounted rate $r_t + \lambda_t u_t$, or equivalently, maximize Y_0. In fact, if the bond issuer commits himself to making more compensation than the loss to the

bond holder whenever the default occurs, then the bond itself will be more "creditable" or "safer" than the risk-free treasure bond. In this case, the bond holder will only require $r_t + \lambda_t u_t < r_t$ as his return rate which is strictly less than the risk-free rate r_t. As a result, the firm will raise more money. In short, through adopting the negative recovery policy, the bond issuer succeeds in maximizing his utility where U_2 far outweighs U_1. The negative recovery policy is also discussed in Duffie and Singleton [1999].

5.2. *Power utility.* To complete the spectrum of important utility functions, in this subsection we will consider the power utility, which provides an interesting scenario different from that for the negative exponential utility:

$$U_1(x) = \kappa x^\alpha, \quad 0 < \alpha < 1, \kappa > 0, \quad U_2(y) = Ky, \quad K > 0.$$

To simplify the discussion, assume U_1 takes the special form

$$U_1(x) = \frac{1}{p} x^p, \quad x \geq 0, \, p \in (0, 1).$$

REMARK 5.1. For the power utility, $p > 0$ is called the *Arrow–Pratt relative risk-aversion coefficient.* If $p = 1$, we define the log utility $U_1(x) = \log x$ for $x > 0$. If $x < 0$, the utility is set to $-\infty$, thus effectively restricting x from falling below zero.

The objective functional is

$$J(u) = E\left[\int_0^T e^{-\int_0^t \beta_s \, ds} \frac{(u_t Y_t)^p}{p} \lambda_t \, dt + KY_0\right],$$

and the optimal recovery problem becomes

$$\inf_{u \in \mathcal{U}} -J(u) = \inf_{u \in \mathcal{U}} E\left[\int_0^T -e^{-\int_0^t \beta_s \, ds} \frac{(u_t Y_t)^p}{p} \lambda_t \, dt - KY_0\right].$$

Moreover, we know that

- $l = -e^{-\int_0^t \beta_s \, ds} \frac{(yu)^p}{p} \lambda, \quad p \in (-\infty, 1), \, p \neq 0, \, \gamma = -Ky,$

- $H = q[(r + \lambda u)y] - e^{-\int_0^t \beta_s \, ds} \frac{(yu)^p}{p} \lambda.$

So the stochastic maximum principle becomes

$$(q_t^* - e^{-\int_0^t \beta_s \, ds}(u_t^* Y_t^*)^{p-1})(v - u_t^*) > 0 \quad \forall v \in \mathcal{U}.$$

Because the admissible control domain $\mathcal{C} = (-\infty, 1)$ is open, we have

$$e^{\int_0^t \beta_s ds} q_t^* = (u_t^* Y_t^*)^{p-1}.$$

Then

$$h(S_t^*) = u_t^* Y_t^* = [-\gamma_y(Y_0^*)]^{1/(p-1)} \exp \frac{\int_0^t (\beta_s - r_s)\, ds}{p-1} > 0.$$

To make the optimal recovery policy admissible, it is sufficient that

(5.1) $$(r_t + \lambda_t)h(S_t^*) - (\beta_t - r_t)h'(S_t^*)S_t^* < 0.$$

- If $\beta_t = r_t$, then

$$(r_t + \lambda_t)h(S_t^*) < 0.$$

This contradicts $h(S_t^*) > 0$, hence the optimal recovery policy does not exist in this situation.

- Note that $h(x) = x^{1/(p-1)}$, therefore (5.1) becomes

$$\left[r_t + \lambda_t - \frac{\beta_t - r_t}{p-1}\right](S_t^*)^{1/(p-1)} < 0.$$

As $S_t^* > 0$, the optimal recovery exists whenever

$$r_t + \lambda_t - \frac{\beta_t - r_t}{p-1} < 0,$$

that is,

(5.2) $$r_t + \lambda_t < \frac{\beta_t - r_t}{p-1}.$$

If $r_t > 0$, $\lambda_t > 0$, $p-1 < 0$ are fixed, to make condition (5.2) hold true, it is necessary to have $\beta_t < r_t$. Once again, if $\beta_t = r_t$, it is impossible to design the optimal recovery policy due to the positivity of r_t and λ_t.

6. Conclusion and extension. In this paper, we establish some important results concerning the credit risk pricing and the associated optimal recovery problem. Firstly, with the BSDE representation, we can fully explore the impact of the loss rate L_t on the price of a credit-risky bond, as well as the objective functional involving the instantaneous gain process $u_t Y_t$ and initial value Y_0. Secondly, applying the stochastic maximum principle, we obtain the optimality condition for the optimal recovery problem, which possesses a significant econometric interpretation in financial market. Thirdly, we focus on two important cases where the utilities are negative exponential and power respectively. We determine the optimal recovery policy dynamically in these two cases and we propose some useful criteria to guarantee the implementation of the optimal recovery policy. It is remarkable that the procedures we employed here also work for general additive utility functions such as CARA, CRRA and HARA. In this respect, our work proposes a new framework that should be thought of as a basis for further practical applications.

On the other hand, in this paper we concentrate on the optimal recovery problem where the state variable has BSDE evolution dynamics. The results we obtain can be generalized to the FBSDE case where the objective functional involves both the forward and backward variables. One illustrating example will be the case when L takes small values; it will influence the firm's value Y, so some additional adjoint equation should be introduced. This will be the subject of future research. It would also be of interest to apply our technique to the problem of contract design in which the objective functional involves the stochastic differential utility (SDU) of Duffie and Epstein [1992].

References

P. Artzner and F. Delbaen (1995), *Defaultable risk insurance and incomplete market*, Math. Finance 5, 187–195.

T. Bielecki and M. Rutkowski (2002), *Credit Risk*: *Modeling, Valuation and Hedging*, Springer, Berlin.

J. M. Bismut (1973), *Conjugate convex functions in optimal stochastic control*, J. Math. Anal. Appl. 44, 383–404.

F. Black and M. Scholes (1973), *The pricing of options and corporate liabilities*, J. Political Economy 81, 637–654.

D. R. Cox (1955), *Some statistical methods connected with series of events*, J. Roy. Statist. Soc. Ser. B 17, 129–164.

D. Duffie and L. Epstein (1992), *Stochastic differential utility*, Econometrica 60, 353–394.

D. Duffie, M. Schroder and C. Skiadas (1996), *Recursive valuation of defaultable securities and the timing of resolution of uncertainty*, Ann. Appl. Probab. 6, 1075–1090.

D. Duffie and J. Singleton (1999), *Modeling term structure of defaultable bonds*, Rev. Financial Stud. 12, 687–720.

G. Hansen (1985), *Indivisible labor and the business cycle*, J. Monetary Economics 16, 309–328.

M. Harrison and D. Kreps (1979), *Martingales and arbitrage in multiperiod security markets*, J. Economic Theory 20, 381–408.

B. Hojgaard and M. Taksar (2001), *Optimal risk control for a large corporation in the presense of returns on investment*, Finance Stoch. 5, 527–547.

R. Jarrow and S. Turnbull (1995), *Pricing derivatives on financial securities subject to credit risk*, J. Finance 50, 53–84.

J. Ma, P. Protter, J. S. Martin and S. Torres (2002), *Numerical methods for backward stochastic differential equations*, Ann. Appl. Probab. 12, 302–316.

R. C. Merton (1974), *On the pricing of corporate debt and the risk structure of interest rates*, J. Finance 29, 449–470.

E. Pardoux and S. G. Peng (1990), *Adapted solution of a backward stochastic differential equation*, System Control Lett. 14, 55–61.

S. G. Peng (1990), *A General Stochastic Maximum Principle for Optimal Control Problems*, SIAM J. Control Optim. 28, 966–979.

S. G. Peng (1993), *Backward stochastic differential equations and applications to optimal control*, Appl. Math Optim. 27, 125–144.

P. Protter (1990), *Stochastic Integration and Differential Equations*, Springer, New York.

R. Rogerson (1988), *Indivisible labor, lotteries and equilibrium*, J. Monetary Economics 21, 3–16.

N. Williams (2003), *On dynamic principle-agent problems in continuous time*, working paper.

CAB 632
Department of Mathematics
and Statistics
University of Alberta
Edmonton, Alberta
Canada T6G 2G1
E-mail: jhuang@stat.ualberta.ca

Department of Real Estate
and Construction
The University of Hong Kong
Pokfulam, Hong Kong
E-mail: h0499024@hkusua.hku.hk