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SEMILINEAR ELLIPTIC PROBLEMS IN UNBOUNDED DOMAINS

Abstract. We investigate the existence of positive solutions and their continuous dependence on functional parameters for a semilinear Dirichlet problem. We discuss the case when the domain is unbounded and the nonlinearity is smooth and convex on a certain interval only.

1. Introduction. In this paper we are dealing with the following boundary value problem for second order PDE of elliptic type:

$$\begin{cases} -\Delta x(y) = F_x(y, x(y)) & \text{for a.e. } y \in \Omega, \\ x \in W_0^{1,2}(\Omega, \mathbb{R}), \end{cases} \tag{1.1}$$

for Ω being an unbounded domain in $\mathbb{R}^n$ with boundary $\partial\Omega$ and F_x denoting the derivative of F with respect to x. We are looking for a nonnegative and nontrivial weak solution $x \in W_0^{1,2}(\Omega, \mathbb{R})$ of this problem such that $\Delta x(\cdot)$ belongs to $L^2(\Omega, \mathbb{R})$.

There are numerous papers concerning similar equations for a bounded domain Ω (see, among others, [1]–[5]). In the vast existing literature we can also find results on radial solutions for our problem in an exterior domain (see [9], [10], [17]–[19]). More precisely, [17] was devoted to both radial and nonradial cases for an exterior domain with sublinear nonlinearities. In the first part of [17], the authors presented the results for the radial case. Then they obtained sub- and supersolutions of (1.1) as radial solutions of a problem associated to (1.1). Finally, they derived the existence of positive nonradial solutions for (1.1) using the sub- and supersolution methods based on the theory due to Noussair ([11]) for Ω being the exterior of a ball.

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Here we do not impose any symmetry condition on Ω, and we cover both sub- and superlinear cases. Similar boundary value problems on unbounded domains have been discussed e.g. in [11]–[14]. In [12]–[14] (for systems of equations) the authors investigated a semilinear elliptic problem of the form

$$\begin{cases} \mathbf{L}u = \lambda \mathbf{f}(y,u) & \text{for } y \in \Omega, \\ u(y) = 0 & \text{for } y \in \partial\Omega, \end{cases} \tag{1.2}$$

where $\mathbf{L}$ is a uniformly elliptic operator in Ω, $n > 2$, $\lambda > 0$ and Ω is a smooth unbounded domain in $\mathbb{R}^n$. They obtained the existence and nonexistence results for (1.2) provided that, among other things, f is locally Lipschitz continuous on $(\Omega \cup \partial\Omega) \times [0,\infty)$ and $f(x,t) < 0$ for all $x \in \Omega$ and sufficiently large t. Here we consider the case when the nonlinearity is increasing and smooth with respect to the second variable on a certain interval $\widetilde{I}$ only. So there is no information concerning its behavior and smoothness outside $\widetilde{I}$.

2. The existence results. We propose an approach based on the following assumptions:

- **(Ω)** Ω is an unbounded domain in $\mathbb{R}^n$ with a locally Lipschitz boundary $\partial\Omega$.
- **(G1)** There exist $M, M_0 \in W^{1,2}(\Omega,\mathbb{R}) \cap L^\infty(\Omega,\mathbb{R})$ such that $0 < M_0(y) < M(y)$ for a.e. $y \in \Omega$, $M_0|_{\partial\Omega}, M|_{\partial\Omega} \geq 0$, $\Delta M_0(\cdot) \in L^2(\Omega,\mathbb{R}) \cap L^\infty(\Omega,\mathbb{R})$ and for each bounded set $\Omega' \subset \Omega$,

$$-F_x(y, M(y)) \geq \Delta M_0(y) \quad \text{a.e. in } \Omega'. \tag{2.1}$$

- **(G2)** $F(y,\cdot) \in C^1(\widetilde{I})$ and is convex in $\tilde{I}$ for a.e. $y \in \Omega$, $F(\cdot,x)$ is measurable in Ω for all $x \in \tilde{I}$, where $\tilde{I}$ is a certain neighborhood of $I := [0,a]$, with $a := \operatorname{ess\,sup}_{y\in\Omega} M(y)$.
- **(G3)** $F_x(y,\cdot)$ is nonnegative in I for a.e. $y \in \Omega$, $F_x(\cdot,a) \in L^2(\Omega,\mathbb{R}) \cap L^\infty(\Omega,\mathbb{R})$;
- **(G4)** $\int_\Omega F_x(y,0)\,dy \neq 0$, $\left|\int_\Omega F(y,0)\,dy\right| < \infty$.

Let us define

$$X := \{x \in W_0^{1,2}(\Omega,\mathbb{R}) : 0 \leq x(y) \leq M(y) \text{ a.e. on } \Omega \text{ and } \Delta x(\cdot) \in L^2(\Omega,\mathbb{R})\}.$$

We will prove the existence of solutions to (1.1) in X and their properties in two steps. First we shall construct a sequence of solutions of the corresponding problems in bounded domains. Then a solution of (1.1) will be obtained as the limit of this sequence (precisely, of a subsequence). Let us consider the sequence of bounded sets

$$\Omega_m := \{y = (y_1, \ldots, y_n) \in \Omega : |y_i| < m \text{ for each } i = 1, \ldots, n\}, \qquad m \in \mathbb{N}.$$

There exists an $m_0 \in \mathbb{N}$ such that $\Omega_m \neq \emptyset$ for all $m \in N_0 := \{m \in \mathbb{N} : m \geq m_0\}$. For each $m \in N_0$, we will use the Schauder fixed point theorem to prove the existence of a solution $x_m \in X_m$ of the problem

$$\begin{cases} -\Delta x(y) = F_x(y, x(y)) & \text{for a.e. } y \in \Omega_m, \\ x \in W_0^{1,2}(\Omega_m, \mathbb{R}), \end{cases} \tag{2.2}$$

with

$$X_m = \{x \in W_0^{1,2}(\Omega_m, \mathbb{R}) : 0 \leq x(y) \leq M(y) \text{ a.e. on } \Omega_m \\ \text{and } \Delta x(y) \in L^2(\Omega_m, \mathbb{R})\}.$$

Thus, we fix $m \in N_0$ and consider a map T_m defined in X_m as follows:

$$T_m x(y) = \int_{\Omega_m} \mathbf{G}_m(y, z) \widetilde{F}_x(z, x(z))\, dz \qquad \text{for } x \in X_m,$$

where $\mathbf{G}_m$ is the Green's function corresponding to the linear homogeneous problem associated with (2.2), and

$$\widetilde{F}_x(z, x) := \begin{cases} F_x(z, 0) & \text{for } x < 0 \text{ and } z \in \Omega_m, \\ F_x(z, x) & \text{for } 0 \leq x \leq a \text{ and } z \in \Omega_m, \\ F_x(z, a) & \text{for } x > a \text{ and } z \in \Omega_m, \end{cases}$$

where a was given in **(G2)**. By the above assumptions T_m is well defined on $L^2(\Omega_m, \mathbb{R})$ and is continuous and compact.

It is clear that our problem is equivalent to the existence of a fixed point of T_m in X_m. So we have to show that T_m maps X_m into X_m. To this end we prove the following lemma:

LEMMA 2.1. *For each $m \in N_0$ and each $x_0 \in X_m$ there exists $\overline{x} \in X_m$ such that*

$$\begin{cases} -\Delta \overline{x}(y) = F_x(y, x_0(y)) & \textit{for a.e. } y \in \Omega_m, \\ x \in W_0^{1,2}(\Omega_m, \mathbb{R}). \end{cases}$$

Proof. Since $M_0|_{\Omega_m} \in X_m$ we get $X_m \neq \emptyset$. Let us fix $x_0 \in X_m$ and investigate the existence of solution for the linear problem

$$\begin{cases} -\Delta x(y) = F_x(y, x_0(y)) & \text{for a.e. } y \in \Omega_m, \\ x \in W_0^{1,2}(\Omega_m, \mathbb{R}). \end{cases} \tag{2.3}$$

From assumptions **(G1)**–**(G3)** we can derive that

$$0 \leq F_x(y, x_0(y)) \leq F_x(y, M(y)) \leq -\Delta M_0(y) \tag{2.4}$$

a.e. in Ω_m and $F_x(\cdot, x_0(\cdot)) \in L^2(\Omega_m, \mathbb{R})$. It is well known that problem (2.3) has a unique solution $\overline{x} \in W_0^{1,2}(\Omega_m, \mathbb{R}) \cap W_{\mathrm{loc}}^{2,2}(\Omega_m, \mathbb{R})$ (see e.g. [5, Th. 8.9]).

Our task is now to show that $\overline{x} \in X_m$. To this end we can observe that, by **(G3)**, $\Delta\overline{x} \leq 0$ a.e. in Ω_m. Applying the weak maximum principle (see e.g. [5, Th. 8.1]) we infer that $\overline{x} \geq 0$ a.e. in Ω_m. On the other hand, taking into account (2.4), we obtain

$$-\Delta\overline{x}(y) = F_x(y, x_0(y)) \leq -\Delta M_0(y)$$

a.e. in Ω_m, so that

$$\Delta(\overline{x}(y) - M_0(y)) \geq 0.$$

Moreover we know that $\overline{x} - M_0 \leq 0$ in $\partial\Omega_m$. Finally, using again the weak maximum principle, we find that $\overline{x} \leq M_0$ a.e. in Ω_m and further $0 \leq \overline{x} \leq M$ a.e. in Ω_m. Thus $\overline{x} \in X_m$. ∎

By the above lemma, for each $m \in N_0$, the continuous and compact operator T_m maps the convex set $X_m \subset L^2(\Omega_m, \mathbb{R})$ into itself. Now Schauder's fixed point theorem gives the existence of a fixed point $x_m \in X_m$ of T_m. Thus we have proved the following result.

THEOREM 2.2. *If hypotheses* **(Ω)** *and* **(G1)**–**(G4)** *are satisfied then for each $m \in N_0$, there exists a solution $x_m \in X_m$ for* (2.2).

Now we define the sequence $\{\overline{x}_m\}_{m \in N_0}$ as follows: for each $m \in N_0$,

$$\overline{x}_m(y) = \begin{cases} x_m(y) & \text{for } y \in \Omega_m, \\ 0 & \text{for } y \in \Omega \setminus \Omega_m, \end{cases}$$

where x_m is a solution for (2.2). Its existence follows from Theorem 2.2. Our task is to prove that the weak limit of a certain subsequence of $\{\overline{x}_m\}_{m \in N_0}$ is a solution for (1.1). A similar approach was also used e.g. by Noussair, and Noussair and Swanson (see [11]–[13]). However, we shall consider a quite different class of elliptic problems.

Now we formulate our main result:

THEOREM 2.3. *Assume hypotheses* **(Ω)** *and* **(G1)**–**(G4)**. *Then there exists a solution $x_0 \in X$ of the problem*

$$\begin{cases} -\Delta x(y) = F_x(y, x(y)) & \text{for a.e. } y \in \Omega, \\ x \in W_0^{1,2}(\Omega, \mathbb{R}). \end{cases} \tag{2.5}$$

Proof. For each $m \in N_0$, Theorem 2.2 yields the existence of $x_m \in X_m$ such that

$$\begin{cases} -\Delta x_m(y) = F_x(y, x_m(y)) & \text{for a.e. } y \in \Omega_m, \\ x_m \in W_0^{1,2}(\Omega_m, \mathbb{R}). \end{cases} \tag{2.6}$$

By the definitions of X_m and $\overline{x}_m$ we have

$$0 \leq \overline{x}_m(y) \leq M(y) \quad \text{a.e. in } \Omega. \tag{2.7}$$

Moreover using (2.6), the monotonicity of $\widetilde{I} \ni x \mapsto F_x(y,x)$ and the fact that $F_x(\cdot, M(\cdot)) \in L^2(\Omega, \mathbb{R})$, we can derive that for each $m \in N_0$,

$$\tag{2.8} \int_{\Omega} |\nabla \overline{x}_m(y)|^2 \, dy = \int_{\Omega_m} \langle \nabla \overline{x}_m(y), \nabla \overline{x}_m(y) \rangle \, dy$$
$$= \int_{\Omega_m} F_x(y, \overline{x}_m(y)) \overline{x}_m(y) \, dy \le \Big[\int_{\Omega} (F_x(y, M(y))^2 \, dy \Big]^{1/2} \Big[\int_{\Omega} (M(y))^2 \, dy \Big]^{1/2}.$$

Taking into account (2.8) we derive that the sequence $\{\nabla \overline{x}_m\}_{m \in N_0}$ is bounded in $L^2(\Omega, \mathbb{R}^n)$, so (up to a subsequence) $\{\nabla \overline{x}_m\}_{m \in N_0}$ tends weakly in $L^2(\Omega, \mathbb{R}^n)$ to a certain $v \in L^2(\Omega, \mathbb{R}^n)$. Thus we obtain the existence of $\overline{x}_1 \in W_0^{1,2}(\Omega, \mathbb{R})$ such that $v = \nabla \overline{x}_1$ in $L^2(\Omega, \mathbb{R}^n)$ and further (up to a subsequence again) $\{\overline{x}_m(y)\}_{m \in N_0}$ tends to $\overline{x}_1(y)$ a.e. in Ω, so $\overline{x}_1(y) \le M(y)$ a.e. in Ω.

Now we claim that

$$\Delta \overline{x}_m \rightharpoonup p_1 \quad \text{(weakly) in } L^2(\Omega, \mathbb{R}).$$

Indeed, from **(G2)** and the definition of $\overline{x}_m$ one obtains the estimate

$$|\Delta \overline{x}_m(y)| \le F_x(y, \overline{x}_m(y)) \le F_x(y, M(y)) \quad \text{a.e. on } \Omega,$$

for each $m \in N_0$. Therefore $\{\Delta \overline{x}_m\}_{m \in N_0}$ is bounded in $L^2(\Omega, \mathbb{R})$, and consequently, passing to a subsequence if necessary, it tends weakly to a certain element p_1 in $L^2(\Omega, \mathbb{R})$. So for any $h \in C_{\mathrm{c}}^{\infty}(\Omega, \mathbb{R})$,

$$\int_{\Omega} \langle \nabla \overline{x}_1(y), \nabla h(y) \rangle \, dy = \lim_{m \to \infty} \int_{\Omega} \langle \nabla \overline{x}_m(y), \nabla h(y) \rangle \, dy$$
$$= -\lim_{m \to \infty} \int_{\Omega} \Delta \overline{x}_m(y) h(y) \, dy = -\int_{\Omega} p_1(y) h(y) \, dy,$$

which means that $\Delta \overline{x}_1(y) = p_1(y)$ for a.e. $y \in \Omega$. On the other hand, by (2.6), we obtain, for $h \in C_{\mathrm{c}}^{\infty}(\mathbb{R}^n, \mathbb{R})$,

$$\tag{2.9} \int_{\Omega} -\Delta \overline{x}_1(y) h(y) \, dy = \lim_{m \to \infty} \int_{\Omega} -\Delta \overline{x}_m(y) h(y) \, dy$$
$$= \lim_{m \to \infty} \int_{\Omega_m} -\Delta \overline{x}_m(y) h(y) \, dy = \lim_{m \to \infty} \int_{\Omega_m} F_x(y, \overline{x}_m(y)) h(y) \, dy$$
$$= \lim_{m \to \infty} \Big[\int_{\Omega} F_x(y, \overline{x}_m(y)) h(y) \, dy - \int_{\Omega \setminus \Omega_m} F_x(y, \overline{x}_m(y)) h(y) \, dy \Big]$$
$$= \lim_{m \to \infty} \Big[\int_{\Omega} F_x(y, \overline{x}_m(y)) h(y) \, dy - \int_{\Omega \setminus \Omega_m} F_x(y, 0) h(y) \, dy \Big].$$

Taking into account **(G2)–(G3)**, the Lebesgue dominated convergence theorem leads to

$$\lim_{m\to\infty}\int_{\Omega} F_x(y,\overline{x}_m(y))h(y)\,dy = \int_{\Omega} F_x(y,\overline{x}_1(y))h(y)\,dy. \tag{2.10}$$

Moreover, by the continuity of the integral as a function of a set, and the fact that $\bigcup_{n=n_0}^{\infty}\Omega_m = \Omega$ and $\Omega_m \subset \Omega_{m+1} \subset \Omega$ for all $m \in N_0$, we have

$$\lim_{m\to\infty}\int_{\Omega\setminus\Omega_m} F_x(y,0)h(y)\,dy = 0. \tag{2.11}$$

Combining (2.9) with (2.10) and (2.11) we obtain

$$\int_{\Omega} -\Delta\overline{x}_1(y)h(y)\,dy = \int_{\Omega} F_x(y,\overline{x}_1(y))h(y)\,dy.$$

Since $h \in C_c^{\infty}(\mathbb{R}^n,\mathbb{R})$ was arbitrary we infer that $\overline{x}_1 \in X$ satisfies (2.5). ∎

3. Applications

EXAMPLE 1. Let us consider (1.1) with $\Omega = \{y = (y_1,y_2) \in \mathbb{R}^2 : 1/10 < y_1 < 1/2 \text{ and } y_2 < 6\}$, and

$$F(y,x) = \frac{25}{11}\ln|x+5| - \frac{36}{11}\ln|6-x| - x + \left(\frac{1}{4}x^4 + x\right)\frac{1}{y^4}$$

for $y \in \Omega$ and all $x \in \mathbb{R}\setminus\{-5,6\}$. Then the problem

$$\begin{cases} -\Delta x(y) = \dfrac{(x(y))^2}{(6-x(y))(x(y)+5)} + \dfrac{(x(y))^3+1}{(y_2)^4} & \text{for a.e. } y \in \Omega,\\ x \in W_0^{1,2}(\Omega,\mathbb{R}), \end{cases} \tag{3.1}$$

has at least one positive solution x_0 such that $x_0(y) \le M$ a.e. on Ω.

Proof. Our task is to find $0 < M_0 \le M$ a.e. on Ω such that (2.1) holds. Let us consider

$$M_0(y_1,y_2) = \frac{1}{2}\left[\frac{y_1}{(y_1)^4 + 1/20} + \frac{1}{(y_2)^4}\right]$$

and $M(y_1,y_2) = 1.1M_0(y_1,y_2)$. It is easy to check that $M_0 \in W^{1,2}(\Omega,\mathbb{R}) \cap L^{\infty}(\Omega,\mathbb{R})$, $\Delta M_0(\cdot) \in L^2(\Omega,\mathbb{R}) \cap L^{\infty}(\Omega,\mathbb{R})$ and

$$-F_x(y,M(y)) \ge \Delta M_0(y) \quad \text{a.e. in } \Omega,$$

where

$$F_x(y,x) = \frac{x^2}{(6-x)(x+5)} + \frac{x^3+1}{(y_2)^4}.$$

Since $0 \le M(y_1,y_2) \le 3.5$ on Ω and $F(y,\cdot)$ is smooth and convex, e.g. in $(-1,4)$, assumptions **(G2)–(G4)** are satisfied. Thus, by Theorem 2.3 there exists a nonnegative, nontrivial and bounded solution of (3.1). ∎

Of course our results can also be applied to sublinear problems.

EXAMPLE 2. The sublinear elliptic BVP

$$\begin{cases} -\Delta x(y) = \dfrac{(x(y))^2}{(4-x(y))(5+x(y))} + \sqrt{x(y)+1}\,\dfrac{y_1}{(y_2)^6} & \text{a.e. in } \Omega, \\ x \in W_0^{1,2}(\Omega,\mathbb{R}), \end{cases} \tag{3.2}$$

with Ω given as in Example 1, has at least one positive solution.

Proof. One can easily check that for M_0 and M from Example 1, assumption **(G1)** is satisfied. Moreover

$$F(y,x) = -x - \frac{16}{9}\ln|4-x| + \frac{25}{9}\ln|x+5| + \frac{2}{3}(x+1)^{3/2}\,\frac{y_1}{(y_2)^6}$$

is continuously differentiable and convex in x, e.g. in $\widetilde{I} = \left(-\frac{1}{2}, 3\frac{1}{2}\right)$. Finally, **(G2)**–**(G4)** hold. Thus Theorem 2.3 gives the existence of a nonnegative, nontrivial and bounded solution of (3.2). ■

4. Continuous dependence on parameters. Continuous dependence of solutions for elliptic problems has been widely discussed by S. Walczak since the 1990's (see e.g. [6]–[8], [20]–[22]). It was also studied in [15] (for bounded Ω) and in [16] (for an exterior domain).

This section is devoted to the following PDE:

$$\begin{cases} -\Delta x(y) = F_x(y,x(y)) + u(y) & \text{for a.e. } y \in \Omega, \\ x \in W_0^{1,2}(\Omega,\mathbb{R}), \end{cases} \tag{4.1}$$

with functional parameters u from a certain subset U of $L^2(\Omega,\mathbb{R}_+)$. We introduce the following assumption:

(G1u) there exists $M_0 \in W^{1,2}(\Omega,\mathbb{R}) \cap L^\infty(\Omega,\mathbb{R})$ such that for each $u \in U$ there exist $M_u, M_{0u} \in W^{1,2}(\Omega,\mathbb{R}) \cap L^\infty(\Omega,\mathbb{R})$ such that

$$0 < M_{0u}(y) < M_u(y) \le M_0(y)$$

for a.e. $y \in \Omega$, and $\Delta M_{0u}(\cdot) \in L^2(\Omega,\mathbb{R}) \cap L^\infty(\Omega,\mathbb{R})$ and for each bounded set $\Omega' \subset \Omega$,

$$-F_x(y, M_u(y)) \ge \Delta M_{0u}(y) \tag{4.2}$$

a.e. in Ω', $M_u|_{\partial\Omega}, M_{0u}|_{\partial\Omega} \ge 0$.

We shall consider the case when **(Ω)**, **(G2)**–**(G4)** hold for $M = M_0$ a.e. in Ω.

THEOREM 4.1. *Assume hypotheses* **(Ω)**, **(G1u)** *and* **(G2)**–**(G4)**. *Suppose that $\{u_m\}_{m\in\mathbb{N}} \subset U$ tends weakly to 0 in $L^2(\Omega,\mathbb{R}_+)$. For each $m \in \mathbb{N}$, denote by $x_m \in X_{u_m}$ a solution of* (4.1) *corresponding to u_m, namely*

$$-\Delta x_m(y)) = F_x(y, x_m(y)) + u_m(y) \tag{4.3}$$

for a.e. $y \in \Omega$, *with*

$$X_{u_m} = \{x \in W_0^{1,2}(\Omega,\mathbb{R}) : 0 \le x(y) \le M_{u_m}(y) \text{ a.e. on } \Omega \text{ and } \Delta x \in L^2(\Omega,\mathbb{R})\}.$$

Then $\{x_m\}_{m\in\mathbb{N}}$ *(up to a subsequence) tends weakly to* x_0 *in* $W_0^{1,2}(\Omega,\mathbb{R})$, *where* $x_0 \in X_0$ *is a solution of the equation*

$$-\Delta x(y) = F_x(y, x(y)) \quad \text{for a.e. } y \in \Omega. \tag{4.4}$$

Proof. We start with the observation that **(G1u)**, the properties of F_x and (4.3) yield

$$\begin{aligned}\int_\Omega |\nabla x_m(y)|^2\,dy &= \int_\Omega (-\Delta x_m(y)x_m(y))\,dy \\ &= \int_\Omega F_x(y, x_m(y))x_m(y)\,dy + \int_\Omega u_m(y)x_m(y)\,dy \\ &\le \Big[\int_\Omega (F_x(y, M_0(y)))^2\,dy\Big]^{1/2}\Big[\int_\Omega (M_0(y))^2\,dy\Big]^{1/2} + \int_\Omega u_m(y)M_0(y)\,dy\end{aligned} \tag{4.5}$$

for each $m \in N_0$. Combining (4.5) with the weak convergence of $\{u_m\}_{m\in\mathbb{N}}$ to 0 in $L^2(\Omega,\mathbb{R}_+)$ we infer that $\{\nabla x_m\}_{m\in\mathbb{N}}$ is bounded in $L^2(\Omega,\mathbb{R})$, and consequently, it is (up to a subsequence) weakly convergent in $L^2(\Omega,\mathbb{R})$ to a certain $v \in L^2(\Omega,\mathbb{R})$. This yields the existence of $x_0 \in W_0^{1,2}(\Omega,\mathbb{R})$ such that $v = \nabla x_0$ in $L^2(\Omega,\mathbb{R}^n)$. We can also derive that some subsequence of $\{x_m\}_{m\in\mathbb{N}}$ (still denoted by $\{x_m\}_{m\in\mathbb{N}}$) tends to x_0 a.e. on Ω, which implies that $x_0 \le M_0$ a.e. in Ω.

Our task is to show that x_0 is a solution for (4.4). To see this, we use again (4.3), monotonicity of $F_x(y,\cdot)$ and the fact that $u_m \rightharpoonup 0$ in $L^2(\Omega,\mathbb{R}_+)$, and obtain the boundedness of $\{\Delta x_m\}_{m\in\mathbb{N}}$ in $L^2(\Omega,\mathbb{R})$. So (up to a subsequence) $\{\Delta x_m\}_{m\in\mathbb{N}}$ is weakly convergent to p in $L^2(\Omega,\mathbb{R})$. Analysis similar to that in the proof of Theorem 2.3 shows that $p = \Delta x_0$ a.e. on Ω. Taking into account (4.3) and the weak convergence of $\{u_m(\cdot)\}_{m\in\mathbb{N}}$ to 0 in $L^2(\Omega,\mathbb{R}_+)$, and employing the scheme used in the proof of (2.9), we get, for any $h \in C_c^\infty(\Omega,\mathbb{R})$,

$$\begin{aligned}\int_\Omega -\Delta x_0(y)h(y)\,dy &= \lim_{m\to\infty}\int_\Omega -\Delta x_m(y)h(y)\,dy \\ &= \lim_{m\to\infty}\int_\Omega (F_x(y, x_m(y)) + u_m(y))h(y)\,dy = \int_\Omega F_x(y, x_0(y))h(y)\,dy.\end{aligned} \tag{4.6}$$

Since $h \in C_c^\infty(\Omega,\mathbb{R})$ was arbitrary we conclude that $x_0 \in X$ satisfies (4.4).

Summarizing we have proved that the sequence $\{x_m\}_{m\in\mathbb{N}}$ of solutions corresponding to the sequence $\{u_m\}_{m\in\mathbb{N}}$ of parameters tends weakly in $W_0^{1,2}(\Omega,\mathbb{R})$ (up to a subsequence) to x_0 provided that $u_m(\cdot) \rightharpoonup 0$ in $L^2(\Omega,\mathbb{R}_+)$ as $m \to \infty$. ∎

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