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ASYMPTOTIC COVARIANCES FOR THE GENERALIZED GAMMA DISTRIBUTION

Abstract. The five-parameter generalized gamma distribution is one of the most flexible distributions in statistics. In this note, for the first time, we provide asymptotic covariances for the parameters using both the method of maximum likelihood and the method of moments.

1. Introduction and summary. The *generalized gamma distribution* (GGD) also known as *power-gamma distribution* due to Stacy (1962) and Stacy and Mihram (1965) is that of the five-parameter family

$$X = \lambda_1 + \{\beta(G + \lambda_2)\}^{1/c}, \tag{1.1}$$

where $G \sim \text{gamma}(\gamma)$, that is, G has density $p_G(g) = g^{\gamma-1}\exp(-g)/\Gamma(\gamma)$ on $(0,\infty)$. Assume that $\beta c > 0$ so that X is an increasing function of G. Then X has density $p_X(x) = p_G(g)\partial g/\partial x$ on (x_0,∞), where $x_0 = \lambda_1 + (\beta\lambda_2)^{1/2}$ and $g = (x-\lambda_1)^c/\beta-\lambda_2$. So, $\partial g/\partial x = (x-\lambda_1)^{c-1}c/\beta$. Section 8.4 of Johnson and Kotz (1970) assumes $\lambda_2 = 0$ and reparametrizes (1.1) as $X = \lambda_1 + \beta' G^{1/c}$, where $\beta' = \beta^{1/c}$.

Many authors have developed estimation procedures for the GGD. Stacy (1962) discusses several distributional properties of the GGD but no estimation procedures are given. Stacy and Mihram (1965) consider maximum likelihood and moment estimators with formulas for variances given for some very special cases. Harter (1967), Lawless (1980), Wingo (1987) and Wong (1993) consider maximum likelihood estimation but give no expressions for the asymptotic covariance matrix. Hager and Bain (1970) and Parr and Webster (1965) consider maximum likelihood estimation and give formulas

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for asymptotic covariances for the particular case $\lambda_1 = 0$. Stacy (1973) proposes new quasimaximum likelihood estimators for the usual two-parameter gamma distribution as alternatives for the maximum likelihood and moments estimators. Huang and Hwang (2006) propose an estimation method based on a characterization of the GGD, and show it to be more convenient and more efficient than the maximum likelihood estimator for small samples. Gomes et al. (2008) propose a new algorithm for estimation referred to as iteration transformation estimation validation. Song (2008) presents fast and globally convergent algorithms for estimation based on novel scale-independent shape estimation equations.

None of the papers in the literature have given formulas for asymptotic covariances of the estimates for the GGD. It would be helpful to have explicit expressions to help construct tests and confidence intervals for functions of the parameters. The aim of this note is to provide asymptotic covariances for the maximum likelihood estimates (Section 2) and for the moment estimates for $\log X$ (Section 3), which we call the *log-moments*. We assume that either $\lambda_1 = 0$ or $\lambda_2 = 0$. We use the estimate $\widehat{x}_0 = \min(X_1, \ldots, X_n)$, where $X_1, \ldots, X_n$ are the sample values, assumed i.i.d. from (1.1). That is, we use $\widehat{\lambda}_1 = \widehat{x}_0$ assuming $\lambda_2 = 0$, or $\widehat{\lambda}_2 = \widehat{x}_0 \widehat{c}/\widehat{\beta}$ assuming $\lambda_1 = 0$, where $\widehat{c}$, $\widehat{\beta}$, $\widehat{\gamma}$ are the maximum likelihood estimates or log-moment estimates computed as if λ_2 were known.

Note that $\widehat{x}_0$ is superefficient, that is, $E(x_0 - \widehat{x}_0)^2 = O(n^{-2})$ not just $O(n^{-1})$. Consequently, the asymptotic covariance of $(\widehat{\lambda}, \widehat{c}, \widehat{\beta}, \widehat{\gamma})$, where $\widehat{\lambda} = \widehat{\lambda}_1$ or $\widehat{\lambda}_2$, is to $O(n^{-2})$ just that for the case when $\widehat{x}_0$ is replaced by x_0. So, a consistent confidence region for any smooth function of $(\widehat{c}, \widehat{\beta}, \widehat{\gamma})$ assuming $\lambda_2 = 0$, or of $(\widehat{\lambda}_2, \widehat{c}, \widehat{\beta}\widehat{\gamma})$ assuming $\lambda_1 = 0$, is obtainable by this device, effectively reducing us to the three-parameter problem, where x_0 is known.

2. The maximum likelihood estimate. Here, we consider the case $\lambda_2 = 0$, as assumed by Johnson and Kotz (1970). That is $X = \lambda_1 + (\beta G)^{1/c}$, where $G \sim \text{gamma}(\gamma)$. We take $\widehat{\lambda}_1 = \min(X_1, \ldots, X_n)$. As noted in Section 1, the asymptotic covariance of $(\widehat{c}, \widehat{\beta}, \widehat{\gamma})$ is the same to $O(n^{-2})$ as for the case $\widehat{\lambda}_1 = \lambda_1$ known. So, for a test or confidence region for (c, β, γ) without loss of generality we assume $\lambda_1 = 0$, that is, $X = (\beta G)^{1/c}$. So,

$$\log p_X(x) = \log\{c\beta^{-\gamma}/\Gamma(\gamma)\} + (c\gamma - 1)\log x - q_c(x)/\beta,$$

where $q_c(x) = x^c$, and the mean log likelihood of a random sample $X_1, \ldots, X_n$ from $p_X(x)$ is

$$\overline{L} = \log\{c\beta^{-\gamma}/\Gamma(\gamma)\} + (c\gamma - 1)\bar{l} - \overline{q}_c/\beta,$$

where

$$\bar{l} = n^{-1} \sum_{i=1}^{n} \log X_i = \int \log x \, dF_n(x), \qquad \overline{q}_c = n^{-1} \sum_{i=1}^{n} q_c(X_i) = \int q_c(x) \, dF_n(x),$$

where $F_n(x)$ is the empirical distribution. The maximum likelihood estimate $\widehat{\boldsymbol{\theta}} = (\widehat{c}, \widehat{\gamma}, \widehat{\beta})'$ satisfies

$$\begin{aligned} 0 &= \partial \overline{L} / \partial \beta^{-1} = \gamma / \beta^{-1} - \overline{q}_c, \\ 0 &= \partial \overline{L} / \partial c = c^{-1} + \gamma \bar{l} - \overline{q}_{c1} / \beta, \\ 0 &= \partial \overline{L} / \partial \gamma = -\log \beta - \psi(\gamma) + c \bar{l}, \end{aligned}$$

where $\psi(\gamma) = (d/d\gamma) \log \Gamma(\gamma)$ and

$$\overline{q}_{ci} = (\partial / \partial c)^i \overline{q}_c = \int q_{ci}(x) \, dF_n(x),$$

where the ith derivative with respect to c of $q_c(x) = x^c$ is $q_{ci}(x) = x^c (\log x)^i$. So, $\boldsymbol{\theta} = \widehat{\boldsymbol{\theta}}$ satisfies

$$\beta = \overline{q}_c / \gamma, \qquad \gamma = c^{-1} (\overline{q}_{c1} \overline{q}_c^{-1} - \bar{l})^{-1}, \tag{2.1}$$

$$0 = c \bar{l} - \log c - \log(\overline{q}_{c1} - \bar{l} \overline{q}_c) - \psi(\gamma). \tag{2.2}$$

Equations (2.1) and (2.2) can be viewed as one equation in $\nu = \widehat{c} = c$ or as two equations in $\boldsymbol{\nu} = (\widehat{\gamma}, \widehat{c})$ or as three equations in $\boldsymbol{\nu} = \widehat{\boldsymbol{\theta}}$. In each case, one can solve them by Newton's method for solving $\mathbf{h}(\boldsymbol{\nu}) = \mathbf{0}$, starting from an initial value $\boldsymbol{\nu}_0$, and iterating using

$$\boldsymbol{\nu}_{i+1} = \boldsymbol{\nu} - \epsilon \dot{\mathbf{h}}(\boldsymbol{\nu})^{-1} \mathbf{h}(\boldsymbol{\nu})$$

at $\boldsymbol{\nu} = \boldsymbol{\nu}_i$, where $\dot{\mathbf{h}}(\boldsymbol{\nu}) = \partial \mathbf{h}(\boldsymbol{\nu}) / \partial \boldsymbol{\nu}'$ is $p \times p$, and $\mathbf{h}$, $\boldsymbol{\nu}$ have the same dimension $p = 1$, 2 or 3. Here, ϵ is an arbitrary damping parameter in $(0, 1]$, say 0.5, to lessen overshoot. Now

$$\text{covar}\, \widehat{\boldsymbol{\theta}} = \mathbf{V} n^{-1} + O(n^{-2}),$$

where $\mathbf{V} = \mathbf{I}(\boldsymbol{\theta}, F)^{-1}$ and

$$\mathbf{I}(\boldsymbol{\theta}, F) = -E \partial^2 \log p_X(X) / \partial \boldsymbol{\theta} \partial \boldsymbol{\theta}' = -\int \partial^2 \log p_X(x) / \partial \boldsymbol{\theta} \partial \boldsymbol{\theta}' \, dF(x),$$

Fisher's information matrix, and $F(x) = F_{\boldsymbol{\theta}}(x)$ say is the cumulative distribution of $X \sim p_X(x)$. Note that $\mathbf{V}$ may be estimated by either $\widehat{\mathbf{V}} = \mathbf{I}(\widehat{\boldsymbol{\theta}}, F_{\widehat{\boldsymbol{\theta}}})^{-1}$ or $\widehat{\mathbf{V}} = \mathbf{I}(\widehat{\boldsymbol{\theta}}, F_n)^{-1}$.

A confidence region of level $\alpha + O(n^{-1})$ for any smooth function say $\mathbf{t} : \mathbb{R}^3 \to \mathbb{R}^2$, where $1 \leq q \leq 3$, is then given by

$$n \{ \mathbf{t}(\boldsymbol{\theta}) - \mathbf{t}(\widehat{\boldsymbol{\theta}}) \}^T \mathbf{C}_{\mathbf{t}}(\widehat{\boldsymbol{\theta}}, \widehat{\mathbf{V}})^{-1} \{ \mathbf{t}(\boldsymbol{\theta}) - \mathbf{t}(\widehat{\boldsymbol{\theta}}) \} \leq \chi^2_{q,\alpha}, \tag{2.3}$$

where $\chi^2_{q,\alpha}$ is the αth quantile of χ^2_q, and

$$\mathbf{C}_{\mathbf{t}}(\boldsymbol{\theta}, \mathbf{V}) = \lim_{n \to \infty} n \, \text{covar}\, \mathbf{t}(\widehat{\boldsymbol{\theta}}) = \dot{\mathbf{t}}(\boldsymbol{\theta}) \mathbf{V} \dot{\mathbf{t}}(\boldsymbol{\theta})'$$

for $\dot{\mathbf{t}}(\boldsymbol{\theta}) = \partial\mathbf{t}(\boldsymbol{\theta})/\partial\boldsymbol{\theta}'$. If we work with $\boldsymbol{\theta} = (c, \gamma, \beta^{-1})$ rather than (c, γ, β), the elements of $\mathbf{I} = \mathbf{I}(\boldsymbol{\theta}, F_n) = -\partial^2\overline{L}/\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}'$ are

$$I_{11} = c^{-2} + \beta^{-1}\overline{q}_{c2}, \quad I_{12} = -\overline{l}, \quad I_{13} = \overline{q}_{c1},$$
$$I_{22} = \dot{\psi}(\gamma), \quad I_{23} = -(\beta^{-1})^{-1} = -\beta, \quad I_{33} = \gamma(\beta^{-1})^{-2} = \gamma\beta^2,$$

and $\mathbf{I}$ has inverse $\mathbf{J}/\det\mathbf{I}$, where

$$J_{11} = I_{22}I_{33} - I_{23}^2 = (\dot{\psi}(\gamma)\gamma - 1)\beta^2,$$
$$J_{12} = I_{13}I_{23} - I_{12}I_{33}, \quad J_{13} = I_{12}I_{23} - I_{13}I_{22},$$
$$J_{22} = I_{11}I_{33} - I_{13}^2, \quad J_{23} = I_{12}I_{13} - I_{11}I_{23}, \quad J_{33} = I_{11}I_{22} - I_{12}^2,$$

and $\det\mathbf{I} = I_{11}J_{11} + I_{12}J_{12} + I_{13}J_{13}$.

3. Log-moment estimates. Again we assume that $\lambda_2 = 0$ and take $\widehat{\lambda}_1 = \min(X_1, \ldots, X_n)$. So, for a test or confidence region for any smooth function $\mathbf{t}(\boldsymbol{\theta})$, where $\boldsymbol{\theta} = (c, \gamma, \beta)$, we may without loss of generality assume $\lambda_1 = 0$. A confidence region for $\mathbf{t}(\boldsymbol{\theta})$ of level $\alpha + O(n^{-1})$ is given by (2.3), where now $\widehat{\boldsymbol{\theta}}$ is the log-moment estimate.

Set $\mu = E\log X$ and $\mu_r = E(\log X - \mu)^r$. Then from Johnson and Kotz (1970),

$$\mu_3\mu_2^{-3/2} = \ddot{\psi}(\gamma)\dot{\psi}(\gamma)^{-3/2} = a(\gamma) \quad \text{say,} \tag{3.1}$$
$$c = \mu_2\mu_3^{-1}b(\gamma), \tag{3.2}$$
$$\beta = \exp\{\mu c - \psi(\gamma)\}, \tag{3.3}$$

where $b(\gamma) = \ddot{\psi}(\gamma)\dot{\psi}(\gamma)^{-1}$. The log-moment estimates $\widehat{\boldsymbol{\theta}}$ for $\boldsymbol{\theta}$ are obtained by replacing μ, μ_2, μ_3 by their sample values. Also

$$\operatorname{covar}\widehat{\boldsymbol{\theta}} = \mathbf{V}n^{-1} + O(n^{-2}),$$

where $\mathbf{V} = \mathbf{V}(\boldsymbol{\theta}, F)$ is given in Appendix A. It may be estimated by $\widehat{\mathbf{V}} = \mathbf{V}(\widehat{\boldsymbol{\theta}}, F_{\widehat{\boldsymbol{\theta}}})$ or $\mathbf{V}(\widehat{\boldsymbol{\theta}}, F_n)$. As for the maximum likelihood estimates, refinements to the confidence region (2.3) may be made using the method of Withers (1989) to bring the error in the nominal level α down from $O(n^{-1})$ to $O(n^{-k})$ for any fixed k.

4. Fractional log-moment estimates. In Section 3, we gave estimates for (c, γ, β) based on the first three moments of $\log X$. These estimates downweight the upper tail so will be more robust to outliers than the estimates based on the first three moments of X.

Even more robust would be estimates based on the first three moments of say $(\log X)^{1/2}$, a quantity that may be imaginary. Since $X = (\beta G)^{1/c}$,

$$\log X = c^{-1}(\log\beta + \log G). \tag{4.1}$$

Since

$$EX^r = \beta^{r/c}\Gamma(r/c+\gamma)/\Gamma(\gamma) \tag{4.2}$$

for $\operatorname{Re} r/c + \gamma > 0$, one has

$$\begin{aligned} E(\log X)^j &= (\partial/\partial t)^j EX^t|_{t=0} \\ &= c^{-j}(\partial/\partial t)^j \beta^t \Gamma(t+\gamma)/\Gamma(\gamma)|_{t=0} \\ &= c^{-j}\sum_{k=0}^{j}\binom{j}{k}\Gamma^{(k)}(\gamma)\Gamma(\gamma)^{-1}(\log\beta)^{j-k} \end{aligned}$$

by Leibniz's rule. Similarly, from (4.2), the rth cumulant of $\log X$ is

$$\kappa_r(\log X) = (\partial/\partial t)^r\{\log EX^t\}|_{t=0}/r! = \delta_{r1}c^{-1}\log\beta + c^{-r}\psi^{(r-1)}(\gamma),$$

which proves (3.1). This is not enough to give us the moments of $(\log X)^{1/2}$. From (4.1),

$$(\log X)^t = c^{-t}\sum_{j=0}^{\infty}\binom{t}{j}(\log\beta)^{t-j}(\log G)^j$$

if $|\log G| < |\log\beta|$, so

$$E(\log X)^t = c^{-t}\sum_{j=0}^{\infty}\binom{t}{j}(\log\beta)^{t-j}g_j(\gamma) \tag{4.3}$$

if convergent, where

$$g_j(\gamma) = E(\log G)^j = (\partial/\partial t)^j EG^t|_{t=0} = \Gamma^{(j)}(\gamma)/\Gamma(\gamma).$$

There is no longer a simple expression for the cumulant generating function or cumulants of $(\log X)^{1/2}$, so one is forced to solve (4.3) for $t = 1/2$, 1, 3/2 by numerical means. Set

$$h(t) = \sum_{j=0}^{\infty}\binom{t}{j}(\log\beta)^{t-j}g_j(\gamma), \qquad m(t) = E(\log X)^t.$$

So,

$$m(0.5) = c^{-1/2}h(0.5), \qquad m(1) = c^{-1}h(1), \qquad m(1.5) = c^{-3/2}h(1.5),$$

where $h(1) = \log\beta + \psi(\gamma)$. So,

$$m(1)m(0.5)^{-2} = h(1)h(1.5)^{-2} = H_1(\beta,\delta) \tag{4.4}$$

say, and

$$m(1.5)m(1)^{-3/2} = h(1.5)h(1)^{-3/2} = H_2(\beta,\delta) \tag{4.5}$$

say. One can now apply Newton's method to solve (4.4) and (4.5) for β, γ and hence find the estimates in terms of the sample versions of $E(\log)^{j/2}$, $j = 1, 2, 3$.

The above method assumes that $\beta > 1$ so that $(\log\beta)^t$ is real for t real. Now suppose that $\beta < 1$. Then the right hand side of (4.4) is complex. So, the real and imaginary parts of (4.4) provide two equations and (4.5) is not needed, provided the right hand side of (4.4) is not purely imaginary. For the sample versions, which method is used is determined by whether the sample version of $m(1)m(0.5)^{-2}$ is real or complex but not purely imaginary. If $m(1)m(0.5)^{-2}$ is purely imaginary, then one needs to include (4.5).

Appendix A. Here, we give the asymptotic covariance of the log-moment estimates of Section 3. Writing

$$\mu(F) = \mu = \int x\, dF(x), \qquad \mu_r(F) = \mu_r = \int (x - \mu(F))^r\, dF(x),$$

(3.1)–(3.3) give equations for $\boldsymbol{\theta}(F) = \boldsymbol{\theta}$. So, $\widehat{\boldsymbol{\theta}} = \boldsymbol{\theta}(F_n)$ has covariance $\mathbf{V}n^{-1} + O(n^{-2})$, where

$$\mathbf{V} = \int \boldsymbol{\theta}_x\, \boldsymbol{\theta}'_x, \tag{A.1}$$

where

$$\int g(x) = \int g(x)\, dF(x),$$

and $\boldsymbol{\theta}_x = \boldsymbol{\theta}_F(x)$ is the influence function, that is, the first von Mises derivative of $\boldsymbol{\theta}(F)$. This is obtained by differentiating both sides of (3.1)–(3.3):

$$\begin{aligned} a_1\gamma_x &= -3\mu_{2x}\mu_3\mu_2^{-5/2}/2 + \mu_{3x}\mu_2^{-3/2}, \\ c_x &= (\mu_{2x}\mu_3^{-1} - \mu_{3x}\mu_2\mu_3^{-2})b_0 + \gamma_x\mu_3^{-1}\mu_2 b_1, \\ \beta_x &= (\mu_x c + c_x\mu - \gamma_x\psi_1)\beta, \end{aligned}$$

where $a_1 = \dot{a}(\gamma)$, $b_0 = b(\gamma)$, $b_1 = \dot{b}(\gamma)$, $\psi_1 = \dot{\psi}(\gamma)$, $\mu_x = \mu_F(x) = x - \mu$, the influence function of μ, and $\mu_{rx} = \mu_{rF}(x) = (x-\mu)^r - \mu_r - r\mu_x\mu_{r-1}$, the influence function of μ_r: see, for example, Withers (1983). Set

$$\begin{aligned} &\sigma = \mu_2^{1/2}, \qquad \lambda_r = \mu_r\sigma^{-r}, \\ &u_r = \int \mu_x\mu_{rx} = \mu_{r+1} - r\mu_2\mu_{r-1}, \\ &\nu_{rs} = \int \mu_{rx}\mu_{sx} = \mu_{r+s} - \mu_r\mu_s - r\mu_{r-1}\mu_{s+1} - s\mu_{s-1}\mu_{r+1} + rs\mu_2\mu_{r-1}\mu_{s-1}. \end{aligned}$$

Then, using

$$\begin{aligned} &\int \mu_{rx}\gamma_x = a_1^{-1}(\nu_{r3}\mu_2^{-3/2} - 3\nu_{r2}\mu_3\mu_2^{-5/2}/2), \\ &\int \mu_{rx}c_x = b_0(\nu_{r2}\mu_3^{-1} - \nu_{r3}\mu_3^{-2}\mu_2) + b_1\mu_3^{-1}\mu_2 \int \mu_{rx}\gamma_x, \\ &\int \mu_x c_x = b_0 P_1 + b_1 a_1^{-1} P_2, \\ &\int \mu_x\gamma_x = \sigma a_1^{-1} P_3 \end{aligned}$$

for $P_1 = 1-(\lambda_4-3)\lambda_3^{-2}$, $P_2 = \lambda_3^{-1}(\lambda_4-3)-3\lambda_3/2$, and $P_3 = \lambda_4-3-3\lambda_3^2/2$, (A.1) gives

$$\begin{aligned}
V_{11} &= \sigma^{-2}(b_0^2 A_0 + 2b_0 b_1 a_1^{-1} A_1 + b_1^2 \lambda_3^{-2} V_{22}),\\
V_{12} &= \sigma^{-1}(a_1 b_0 A_2 + a_1^{-2} b_1 \lambda_3^{-1} A_3),\\
V_{13} &= \sigma^{-1}\lambda_3^{-1}(b_0 \beta B_1 + b_1 \beta B_2 - \lambda_3^{-1} B_3),\\
V_{22} &= a_1^{-2} A_3,\\
V_{23} &= a_1^{-1}\{c\sigma(\lambda_4 - 3) + \mu\sigma^{-1} B_4\} - \psi_1 a_1^{-2} A_5/2,\\
V_{33} &= \beta^2 B_5
\end{aligned}$$

for

$$\begin{aligned}
A_0 &= \lambda_3^{-4}(\lambda_4\lambda_3^2 - 2\lambda_5\lambda_3 + \lambda_6 + 6\lambda_3^2 - 6\lambda_4 + 9),\\
A_1 &= \lambda_3^{-3}(-\lambda_6 + 6\lambda_4 - 9) + \lambda_3^{-2}(3\lambda_5 + 2\lambda_4 - 3\lambda_4\lambda_3 + 5\lambda_3)/2,\\
A_2 &= \lambda_3^{-2}(-\lambda_6 + 6\lambda_4 - 9) + \lambda_3^{-1}(5\lambda_5 - 3\lambda_4\lambda_3 - 15\lambda_3)/2,\\
A_3 &= \lambda_6 - 6\lambda_4 + 9 - 3\lambda_5\lambda_3 + \lambda_3^2(9\lambda_4 - 35)/4,\\
B_1 &= c\sigma\lambda_3 + \mu\sigma^{-1} b_0\{-\lambda_3^{-2}\lambda_5 + \lambda_3^{-1}(\lambda_4 + 3)\}\\
&\quad + \mu\sigma^{-1} b_1 a_1^{-2}\{\lambda_4(2\lambda_3^{-1} - 3) - 5\}/2,\\
B_2 &= c\sigma a_1^{-1}(\lambda_4 - 3 - 3\lambda_3^2/2) + \mu\sigma^{-1}(a_1 b_0 \lambda_3^{-2} A_2 + a_1^{-2} b_1 \lambda_3^{-1} A_3) - \psi_1 a_1^{-2} A_3,\\
B_3 &= c\sigma(\lambda_4 - 3) + \mu\sigma^{-1} b_0 A_4,\\
A_4 &= \lambda_3^{-2}(-\lambda_6 + 6\lambda_4 - 9) + \lambda_3^{-1}(\lambda_5 - 3\lambda_3),\\
B_4 &= b_0 A_6 + b_1 \lambda_3^{-1} a_1 A_5/2,\\
A_5 &= \lambda_4(2 - 3\lambda_3) - 5\lambda_3,\\
A_6 &= -\lambda_5\lambda_3^{-2} + (\lambda_4 + 3)\lambda_3^{-1},\\
B_5 &= c^2\mu_2 + 2c\mu(b_0 P_1 + a_1^{-1} b_1 P_2) - 2\psi_1 a_1^{-1} c\sigma P_3.
\end{aligned}$$

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