APPLICATIONES MATHEMATICAE
39,3 (2012), pp. 257–272

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ASYMPTOTIC DISTRIBUTION OF THE ESTIMATED PARAMETERS OF AN ARMA(p,q) PROCESS IN THE PRESENCE OF EXPLOSIVE ROOTS

Abstract. We consider an autoregressive moving average process of order (p,q) (ARMA(p,q)) with stationary, white noise error variables having uniformly bounded fourth order moments. The characteristic polynomials of both the autoregressive and moving average components involve stable and explosive roots. The autoregressive parameters are estimated by using the instrumental variable technique while the moving average parameters are estimated through a derived autoregressive process using the same sample. The asymptotic distribution of the estimators is then derived.

1. Introduction. Consider the ARMA(p,q) model,

$$\mathrm{X}_t - \alpha_1 \mathrm{X}_{t-1} - \alpha_2 \mathrm{X}_{t-2} - \cdots - \alpha_p \mathrm{X}_{t-p} = e_t - \beta_1 e_{t-1} - \cdots - \beta_q e_{t-q}, \tag{1.1}$$

where X_t is the observation at time t, $t = 1, \ldots, \mathrm{N}$, and e_t is a sequence of identically and independently distributed $(0, \sigma^2)$ random variables with

$$\mathrm{E}(e_t^{4+\delta}) < \infty \quad \text{for some } \delta > 0. \tag{1.2}$$

The initial conditions are assumed to be zero, that is, $e_t = 0$ for $t \le 0$.

The autoregressive (AR) component is said to be *stable* or *explosive* according as the roots of the characteristic polynomial $\Phi(z) = 1 - \alpha_1 z - \alpha_2 z^2 - \cdots - \alpha_p z^p$ are greater than or less than unity in absolute value. Similarly the moving average (MA) component is *stable* or *explosive* according as the roots of the characteristic polynomial $\Theta(z) = 1 - \beta_1 z - \beta_2 z^2 - \cdots - \beta_q z^q$ are greater than or less than unity in absolute value.

2010 *Mathematics Subject Classification*: Primary 62M10, 62E20.
Key words and phrases: ARMA process, explosive roots, asymptotic distribution.

DOI: 10.4064/am39-3-1

By using a backward shift operator B, model (1.1) can be rewritten as

$$\Phi(\mathrm{B})\mathrm{X}_t = \Theta(\mathrm{B})e_t. \tag{1.3}$$

For an autoregressive process with identically and independently distributed errors, the limiting distribution of the least squares estimators under stable and explosive roots has been studied by several authors, like Mann and Wald (1943), White (1958), Anderson (1959) and Jeganathan (1988). Chan and Wei (1988) derived the asymptotic distribution of the least square estimators in the presence of stable and explosive roots. Basu and Sen Roy (1993) considered all forms of roots and derived the asymptotic distribution of the estimator assuming ϕ-mixing error variables. However, such studies have not been extended to ARMA processes.

In the present work our aim is to find the limiting distribution of the estimated parameters when the characteristic polynomials of the AR and MA components have both stable and explosive roots. Since the ordinary least squares estimator of the AR parameters is inconsistent, we use the instrumental variable technique to estimate these parameters. The MA parameters are estimated through a derived AR process as proposed by Tsay (1993).

In studying the limiting distribution, a componentwise break-up according to stable and explosive roots is made using techniques similar to those of Chan and Wei (1988). Then using suitably chosen norming matrices, the limiting distribution of each component is found separately. The results are then put together in the final theorem.

In Section 2 a componentwise break-up of the process is made. Section 3 considers the asymptotic distributions of the estimators componentwise, while Section 4 contains the main theorem. Some concluding remarks are given in Section 5.

In the following, I_n and $\mathbf{0}_n$ respectively denote the identity matrix of order n and the n-dimensional vector of zero elements. $\mathrm{diag}(\cdot)$ denotes a block diagonal matrix. The norm of a vector refers to euclidean norm, while for a matrix A,

$$\|\mathrm{A}\| = \sup_{\|x\|=1} \|\mathrm{A}x\|.$$

Finally, c_i, $i = 0, 1, \ldots,$ denote constants.

2. A componentwise break-up of the process. For $r + s = p$ and $|\rho_i| > 1$, $i = 1, \ldots, r$, and $|\gamma_j| < 1$, $j = 1, \ldots, s$, $\Phi(z)$ can be rewritten as

$$\Phi(z) = \prod_{i=1}^{r}(1 - \rho_i^{-1}z) \prod_{j=1}^{s}(1 - \gamma_j^{-1}z) \tag{2.1}$$

where ρ_i are the r stable roots and γ_j are the s explosive roots of $\Phi(z) = 0$.

Similarly $\Theta(z)$ can be written as

$$\Theta(z) = \prod_{i=1}^{c}(1 - \pi_i^{-1} z) \prod_{j=1}^{d}(1 - \eta_j^{-1} z) \tag{2.2}$$

where π_i are the stable roots and η_j are the explosive roots of $\Theta(z)$, with $|\pi_i| > 1$, $i = 1, \ldots, c$, $|\eta_j| < 1$, $j = 1, \ldots, d$, and $c + d = q$. All roots are assumed to be distinct.

Model (1.3) can be rewritten as

$$\Phi(\mathrm{B})\mathrm{X}_t = u_t, \tag{2.3}$$

where

$$u_t = \Theta(\mathrm{B}) e_t \tag{2.4}$$

is an MA(q) process.

Defining

$$\mathbf{x}_t = (\mathrm{X}_t, \ldots, \mathrm{X}_{t-p+1})', \qquad \mathbf{u}_t = (u_t, \mathbf{0}'_{p-1})', \qquad \mathbf{A} = \begin{pmatrix} \alpha_1 \; \ldots \; \alpha_{p-1} & \alpha_p \\ \mathrm{I}_{p-1} & \mathbf{0}_{p-1} \end{pmatrix},$$

(2.3) can be rewritten as

$$\mathbf{x}_t = \mathbf{A}\mathbf{x}_{t-1} + \mathbf{u}_t, \qquad t = 1, 2, \ldots. \tag{2.5}$$

Since $\mathbf{x}_{t-1}$ is correlated with u_t through $e_{t-1}, \ldots, e_{t-q}$, the least squares estimator of the AR parameter $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_p)'$ will be inconsistent. Taking $n = \mathrm{N} - q - 1$ and following Basu, Sen Roy and Bhattacharya (2005), the instrumental variable estimator of $\boldsymbol{\alpha}$ is

$$\hat{\boldsymbol{\alpha}}_n = \Big(\sum_{t=1}^{n} \mathbf{x}_t \mathbf{x}'_{t+q}\Big)^{-1} \Big(\sum_{t=1}^{n} \mathbf{x}_t \mathrm{X}_{t+q+1}\Big). \tag{2.6}$$

To estimate the parameters of the MA component, let $\mathrm{Y}_{t-i} = de_t/d\beta_i$ be the partial derivative of e_t with respect to β_i. Then following Tsay (1993) we obtain the derived AR(q) process

$$\Theta(\mathrm{B})\mathrm{Y}_t = e_t, \qquad t = 1, 2, \ldots. \tag{2.7}$$

Defining

$$\mathbf{y}_t = (\mathrm{Y}_t, \ldots, \mathrm{Y}_{t-q+1})', \qquad \mathbf{v}_t = (e_t, \mathbf{0}'_{q-1})', \qquad \mathbf{C} = \begin{pmatrix} \beta_1 \; \ldots \; \beta_{q-1} & \beta_q \\ \mathrm{I}_{q-1} & \mathbf{0} \end{pmatrix},$$

(2.7) can be rewritten as

$$\mathbf{y}_t = \mathbf{C}\mathbf{y}_{t-1} + \mathbf{v}_t, \qquad t = 1, 2, \ldots. \tag{2.8}$$

Then the least squares estimator of $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_q)'$, based on n observations, is

$$\hat{\boldsymbol{\beta}}_n = \Big(\sum_{t=1}^{n} \mathbf{y}_{t+q}\mathbf{y}'_{t+q}\Big)^{-1}\Big(\sum_{t=1}^{n} \mathbf{y}_{t+q}\mathrm{Y}_{t+q+1}\Big). \tag{2.9}$$

Let $\boldsymbol{\theta} = (\boldsymbol{\alpha}', \boldsymbol{\beta}')'$, $\hat{\boldsymbol{\theta}}_n = (\hat{\boldsymbol{\alpha}}'_n, \hat{\boldsymbol{\beta}}'_n)'$, $\mathbf{z}_t = (\mathbf{x}'_t u_{t+q+1}, \mathbf{y}'_{t+q} e_{t+q+1})'$, and

$$\mathbf{D}_n = \begin{pmatrix} \sum_{t=1}^{n} \mathbf{x}_t\mathbf{x}'_{t+q} & \mathbf{0} \\ \mathbf{0} & \sum_{t=1}^{n} \mathbf{y}_{t+q}\mathbf{y}'_{t+q} \end{pmatrix}.$$

Then

$$\hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta} = \mathbf{D}_n^{-1}\Big(\sum_{t=1}^{n} \mathbf{z}_t\Big). \tag{2.10}$$

Denote by B the backshift operator. Then the different components are segregated as

$$\mathrm{R}_t = \Phi(\mathrm{B})\prod_{i=1}^{r}(1-\rho_i^{-1}\mathrm{B})^{-1}\mathrm{X}_t, \tag{2.11}$$

$$\mathrm{S}_t = \Phi(\mathrm{B})\prod_{i=1}^{s}(1-\gamma_i^{-1}\mathrm{B})^{-1}\mathrm{X}_t, \tag{2.12}$$

$$\mathrm{Q}_t = \Theta(\mathrm{B})\prod_{i=1}^{c}(1-\pi_i^{-1}\mathrm{B})^{-1}\mathrm{Y}_t, \tag{2.13}$$

$$\mathrm{P}_t = \Theta(\mathrm{B})\prod_{i=1}^{d}(1-\eta_i^{-1}\mathrm{B})^{-1}\mathrm{Y}_t. \tag{2.14}$$

Let $\mathbf{r}_t = (\mathrm{R}_t, \ldots, \mathrm{R}_{t-r+1})$, $\mathbf{s}_t = (\mathrm{S}_t, \ldots, \mathrm{S}_{t-s+1})$, $\mathbf{q}_t = (\mathrm{Q}_t, \ldots, \mathrm{Q}_{t-c+1})$ and $\mathbf{p}_t = (\mathrm{P}_t, \ldots, \mathrm{P}_{t-d+1})$. By (2.1) and (2.11), R_t can be written as

$$\mathrm{R}_t = \prod_{i=1}^{s}(1-\gamma_i^{-1}\mathrm{B})\mathrm{X}_t = \mathrm{X}_t - \gamma_1^*\mathrm{X}_{t-1} - \cdots - \gamma_s^*\mathrm{X}_{t-s} \tag{2.15}$$

so that for the $r \times p$ matrix

$$\mathrm{T}_1 = \begin{pmatrix} 1 & -\gamma_1^* & \ldots & -\gamma_s^* & 0 & 0 & 0 & 0 \\ 0 & 1 & -\gamma_1^* & \ldots & -\gamma_s^* & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & 0 & 1 & -\gamma_1^* & \ldots & -\gamma_s^* \end{pmatrix}$$

we have $\mathrm{T}_1\mathbf{x}_t = \mathbf{r}_t$. Again by (2.1) and (2.12) we may find an $s \times p$ matrix T_2 such that $\mathrm{T}_2\mathbf{x}_t = \mathbf{s}_t$. Hence there exists a $p \times p$ matrix $\mathrm{T}^{(1)} = (\mathrm{T}'_1, \mathrm{T}'_2)'$ such that $\mathrm{T}^{(1)}\mathbf{x}_t = (\mathbf{r}'_t, \mathbf{s}'_t)'$. Similarly, using (2.2), (2.13) and (2.14) we may define a $q \times q$ matrix $\mathrm{T}^{(2)}$ such that $\mathrm{T}^{(2)}\mathbf{y}_t = (\mathbf{q}_t', \mathbf{p}_t')'$. Finally let $\mathbf{T} = \mathrm{diag}(\mathrm{T}^{(1)}, \mathrm{T}^{(2)})$. We next derive the componentwise limiting distributions.

3. Componentwise asymptotic distributions

3.1. The AR stable component. We first consider the stable component of the autoregressive part, $\prod_{i=1}^{r}(1-\rho_i^{-1}\mathrm{B})\mathrm{R}_t = u_t$. Following (2.11), this can be reconstructed as

$$\mathrm{R}_t = \rho_1^*\mathrm{R}_{t-1} + \cdots + \rho_r^*\mathrm{R}_{t-r} + u_t, \tag{3.1}$$

where $\boldsymbol{\rho}^* = (\rho_1^*, \ldots, \rho_r^*)$ are the parameters of the process with roots ρ_j, $j = 1, \ldots, r$. Define

$$\mathbf{L}_1 = \begin{pmatrix} \rho_1^* & \cdots & \rho_{r-1}^* & \rho_r^* \\ & \mathrm{I}_{r-1} & & \mathbf{0}_{r-1} \end{pmatrix} \quad \text{and} \quad \mathbf{u}_{1t} = (u_t, \mathbf{0}_{r-1}').$$

Then (3.1) can be rewritten as

$$\mathbf{r}_t = \mathbf{L}_1\mathbf{r}_{t-1} + \mathbf{u}_{1t}, \qquad t = 1, 2, \ldots. \tag{3.2}$$

Let $\check{\rho}_1 = \max_{1\le j\le r}|\rho_j^{-1}| < 1$. Then

$$\|\mathbf{L}_1^n\| \sim c_0\check{\rho}_1^n \qquad \text{as } n \to \infty. \tag{3.3}$$

Let $\mathbf{J}_n = n^{-1/2}\mathrm{I}_n$ and $\Sigma_1 = \mathrm{E}(\mathbf{r}_n\mathbf{r}_{n+q}')$; then Σ_1 is positive definite. Define $\mathbf{w}_t = \mathbf{r}_t'u_{t+q+1}$ and $\mathbf{R}_n = n^{-1}\sum_{t=1}^{n}\mathbf{r}_t\mathbf{r}_{t+q}'$.

Lemma 3.1. *Under* (3.3) *and bounded fourth order moments of the innovations,*

$$n^{-1/2}\sum_{t=1}^{n}\mathbf{w}_t \xrightarrow{d} \mathrm{N}(0, \Sigma_1^*), \tag{3.4}$$

where

$$\Sigma_1^* = \mathrm{E}(\mathbf{w}_1\mathbf{w}_1') + \sum_{k=1}^{\infty}\mathrm{E}(\mathbf{w}_1\mathbf{w}_{k+1}') + \sum_{k=1}^{\infty}\mathrm{E}(\mathbf{w}_{k+1}\mathbf{w}_1'), \tag{3.5}$$

and the two series in Σ_1^* are convergent.

Proof. Observe that by (1.2) and (3.3),

$$\begin{aligned} \|\mathrm{E}(\mathbf{r}_t\mathbf{r}_t'u_{t+q+1}^2)\| &\le c_1\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\|\mathrm{L}_1^j\|\cdot\|\mathrm{L}_1^k\|\cdot\mathrm{E}|u_{t-j}u_{t-k}| \\ &\le c_2(1-\|\mathrm{L}_1\|)^{-2} < \infty. \end{aligned} \tag{3.6}$$

Defining $\vartheta(\mathbf{w}_t)$ as the dispersion matrix for the sequence $\mathbf{w}_t$ we have

$$\|\vartheta(\mathbf{w}_t)\| < \infty. \tag{3.7}$$

Truncating $\mathbf{w}_t = \sum_{j=0}^{\infty}(\mathrm{L}_1^j)'\mathbf{u}'_{1,t-j}u_{t+q+1}$ to a finite number of $l+1$ terms, define $\mathbf{w}_t^* = \sum_{j=0}^{l}(\mathrm{L}_1^j)'\mathbf{u}'_{1,t-j}u_{t+q+1}$ so that

$$\sum_{l=1}^{\infty}[\mathrm{E}\|\mathbf{w}_t - \mathbf{w}_t^*\|^2]^{1/2} = \sum_{l=1}^{\infty}\Big[\mathrm{E}\Big\|\sum_{j=l+1}^{\infty}\mathrm{L}_1^j\mathbf{u}_{1,t-j}u_{t+q+1}\Big\|^2\Big]^{1/2}. \tag{3.8}$$

By (1.2),

$$\mathrm{E}\Big\|\sum_{j=l+1}^{\infty}\mathrm{L}_1^j\mathbf{u}_{1,t-j}u_{t+q+1}\Big\|^2 \le c_3\Big(\sum_{j=l+1}^{\infty}\|\mathrm{L}_1^l\|\Big)^2, \tag{3.9}$$

so that using (3.3),

$$\begin{aligned}\sum_{l=1}^{\infty}[\mathrm{E}\|\mathbf{w}_t - \mathbf{w}_t^*\|^2]^{1/2} &\le \sum_{l=1}^{\infty}\Big[c_4\Big\{\sum_{j=l+1}^{\infty}\|\mathrm{L}_1^j\|\Big\}^2\Big]^{1/2} \\ &\le c_5\sum_{l=1}^{\infty}\sum_{j=l+1}^{\infty}\|\mathrm{L}_1^j\| < \infty.\end{aligned} \tag{3.10}$$

Thus from (3.7) and (3.10) and using the multivariate version of Theorem 21.1 of Billingsley (1968) on the sequence $\mathbf{w}_t$, the lemma follows. ∎

LEMMA 3.2. *Under bounded fourth order moments of the innovations, for some constant c_6 and for all $\epsilon > 0$,*

$$\mathrm{P}[\|\mathbf{R}_n - \Sigma_1\| > \epsilon] < c_6 n^{-1}\epsilon^{-1}. \tag{3.11}$$

Proof. Writing

$$\begin{aligned}\mathbf{r}_{t+1}\mathbf{r}'_{t+q+1} - \mathbf{r}_t\mathbf{r}'_{t+q} = \mathrm{L}_1\mathbf{r}_t\mathbf{r}'_{t+q}\mathrm{L}'_1 + \mathbf{u}_{1,t+1}\mathbf{r}'_{t+q}\mathrm{L}'_1 + \mathrm{L}_1\mathbf{r}_t\mathbf{u}'_{1,t+q+1} \\ + \mathbf{u}_{1,t+1}\mathbf{u}'_{1,t+q+1} - \mathbf{r}_t\mathbf{r}'_{t+q}\end{aligned}$$

we have

$$\begin{aligned}&(\mathbf{I} - \mathrm{L}_1\otimes\mathrm{L}_1)\Big[\mathrm{Vec}\Big(n^{-1}\sum_{t=1}^{n}\mathbf{r}_t\mathbf{r}'_{t+q}\Big) - \mathrm{Vec}\,\Sigma_1\Big] \\ &= n^{-1}\mathrm{Vec}[\mathbf{r}_1\mathbf{r}'_{1+q} - \mathrm{E}(\mathbf{r}_1\mathbf{r}'_{1+q})] - n^{-1}\mathrm{Vec}[\mathbf{r}_{n+1}\mathbf{r}'_{n+q+1} - \mathrm{E}(\mathbf{r}_{n+1}\mathbf{r}'_{n+q+1})] \\ &+ n^{-1}\mathrm{Vec}\Big(\sum_{t=1}^{n}\mathrm{L}_1\mathbf{r}_t\mathbf{u}'_{1,t+q+1}\Big) + n^{-1}\mathrm{Vec}\Big[\sum_{t=1}^{n}\{\mathbf{u}_{1,t+1}\mathbf{r}'_{t+q}\mathrm{L}'_1 - \mathrm{E}(\mathbf{u}_{1,t+1}\mathbf{r}'_{t+q}\mathrm{L}'_1)\}\Big] \\ &+ n^{-1}\mathrm{Vec}\Big[\sum_{t=1}^{n}\{\mathbf{u}_{1,t+1}\mathbf{u}'_{1,t+q+1} - \mathrm{E}(\mathbf{u}_{1,t+1}\mathbf{u}'_{1,t+q+1})\}\Big]\end{aligned} \tag{3.12}$$

where $\mathbf{I} - \mathrm{L}_1\otimes\mathrm{L}_1$ is nonsingular. Now by (1.2) and (3.3),

$$\mathrm{E}\|\mathbf{R}_n\mathbf{R}'_{n+q}\| \le \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\|\mathrm{L}_1^j\|\cdot\|\mathrm{L}_1^k\|\cdot\mathrm{E}|u_{n-j}u_{n+q-k}| < \infty, \tag{3.13}$$

so that for some $c_7 > 0$ the first term of (3.12) satisfies

$$\text{(3.14)} \quad \begin{aligned} \mathrm{P}[\|\{\mathbf{r}_1\mathbf{r}'_{1+q} - \mathrm{E}(\mathbf{r}_1\mathbf{r}'_{1+q})\}\| > n\epsilon/5] &\leq 10n^{-1}\epsilon^{-1}[\mathrm{E}\|\mathbf{r}_1\mathbf{r}'_{1+q}\|] \\ &\leq c_7 n^{-1}\epsilon^{-1}. \end{aligned}$$

Similarly the second term in (3.12) can be shown to be of the same order as (3.14).

For the third term in (3.12) let $\mathbf{r}_t^* = \mathrm{L}_1\mathbf{r}_t u_{t+q+1}$. Since u_t is a $(q+1)$-dependent process and hence ϕ-mixing with $\phi_n = 0$ for $n > q+1$, $\mathbf{r}_t^*$ is also ϕ-mixing with mean zero and

$$\text{(3.15)} \qquad \|\vartheta(\mathbf{r}_t^*)\| < \infty.$$

Again defining $\tilde{\mathbf{r}}_t = \sum_{j=0}^{l} \mathrm{L}_1^{j+1}\mathbf{u}_{1,t-j}u_{t+q+1}$, from (1.2) and (3.3) it follows that

$$\text{(3.16)} \qquad \sum_{l=1}^{\infty}[\mathrm{E}\|\mathbf{r}_t^* - \tilde{\mathbf{r}}_t\|^2]^{1/2} < \infty.$$

Using (3.15), (3.16) and the multivariate version of Theorem 21.1 of Billingsley (1968), we obtain

$$\text{(3.17)} \qquad n^{-1/2}\sum_{t=1}^{n}\mathrm{L}_1\mathbf{r}_t u_{t+q+1} \xrightarrow{d} \mathrm{N}(0, \widetilde{\Sigma}_1),$$

where

$$\text{(3.18)} \quad \begin{aligned} \widetilde{\Sigma}_1 = \mathrm{E}(\mathrm{L}_1\mathbf{r}_1\mathbf{r}'_1\mathrm{L}'_1u^2_{q+2}) &+ \sum_{k=1}^{\infty}\mathrm{E}(\mathrm{L}_1\mathbf{r}_1\mathbf{r}'_{k+1}\mathrm{L}'_1u_{q+2}u_{k+q+2}) \\ &+ \sum_{k=1}^{\infty}\mathrm{E}(\mathrm{L}_1\mathbf{r}_{k+1}\mathbf{r}'_1\mathrm{L}'_1u_{q+2}u_{k+q+2}). \end{aligned}$$

Hence for large n and some $c_8 > 0$,

$$\text{(3.19)} \qquad \mathrm{P}\Big[\Big\|n^{-1}\mathrm{Vec}\Big(\sum_{t=1}^{n}\mathrm{L}_1\mathbf{r}_t\mathbf{u}'_{1,t+q+1}\Big)\Big\| > \epsilon/5\Big] \leq c_8 e^{-\sqrt{n}\epsilon/5}.$$

Similarly since $\mathrm{L}_1\mathbf{r}_{t+q}u_{t+1} - \mathrm{E}(\mathrm{L}_1\mathbf{r}_{t+q}u_{t+1})$ and $u_{t+1}u_{t+q+1} - \mathrm{E}(u_{t+1}u_{t+q+1})$ are zero mean ϕ-mixing processes, the last two terms of (3.12) are also of order $e^{-\sqrt{n}\epsilon/5}$.

Combining the above results, (3.11) follows. ■

We next state the main theorem of this section.

THEOREM 3.1. *Under conditions* (1.2) *and* (3.3),

$$\mathbf{J}_n \sum_{t=1}^{n} \mathbf{r}_t \mathbf{r}'_{t+q} \mathbf{J}'_n \xrightarrow{p} \Sigma_1, \tag{3.20}$$

$$(\mathbf{J}'_n)^{-1} \Big(\sum_{t=1}^{n} \mathbf{r}_t \mathbf{r}'_{t+q} \Big)^{-1} \Big(\sum_{t=1}^{n} \mathbf{r}_t u_{t+q+1} \Big) \xrightarrow{d} \mathrm{N}_r(0, \Sigma_1^{-1} \Sigma_1^* \Sigma_1^{-1}). \tag{3.21}$$

Proof. This follows from Lemmas 3.1 and 3.2. ∎

3.2. The AR explosive component. Next consider the explosive component of the autoregressive part, $\prod_{i=1}^{s}(1-\gamma_i^{-1}\mathrm{B})\mathrm{S}_t = u_t$, which from (2.12) can be rewritten as

$$\mathrm{S}_t = \gamma_1^* \mathrm{S}_{t-1} + \cdots + \gamma_s^* \mathrm{S}_{t-s} + u_t \quad \text{for } t = 1, 2, \ldots, \tag{3.22}$$

where $\boldsymbol{\gamma}^* = (\gamma_1^*, \ldots, \gamma_s^*)$ are the parameters of the process with roots γ_j for $j = 1, \ldots, s$. Defining

$$\mathbf{F} = \begin{pmatrix} \gamma_1^* & \cdots & \gamma_{s-1}^* & \gamma_s^* \\ & \mathrm{I}_{s-1} & & \mathbf{0} \end{pmatrix} \quad \text{and} \quad \mathbf{u}_{2t} = (u_t, \mathbf{0}'_{s-1})',$$

the model (3.22) can be rewritten as

$$\mathbf{s}_t = \mathbf{F}\mathbf{s}_{t-1} + \mathbf{u}_{2t}, \quad t = 1, 2, \ldots. \tag{3.23}$$

Let $\check{\gamma}_1 = \min_{1\le j\le s} |\gamma_j^{-1}| > 1$ and $\check{\gamma}_2 = \max_{1\le j\le s} |\gamma_j^{-1}| > 1$. Then $\|\mathbf{F}^n\| \sim c_9 \check{\gamma}_2^n$ and

$$\|\mathbf{F}^{-n}\| \sim c_{10} \check{\gamma}_1^{-n} \quad \text{as } n \to \infty. \tag{3.24}$$

Writing the first column of $\mathbf{F}^{-(t-1)}$ as $\mathbf{f}_t$, let

$$\mathbf{s}_n^* = \mathbf{F}^{-(n-1)} \mathbf{s}_n = \sum_{t=1}^{n} \mathbf{F}^{-(t-1)} \mathbf{u}_{2t} = \sum_{t=1}^{n} \mathbf{f}_t u_t. \tag{3.25}$$

By the results of Longnecker and Serfling (1978), and because of (1.2) and $\sum_{t=1}^{\infty} \|\mathbf{F}^{-t}\| < \infty$, it follows that $\mathbf{s}_n^*$ converges a.s. Let

$$\lim_{n\to\infty} \mathbf{s}_n^* = \mathbf{s}^* = \sum_{t=1}^{\infty} \mathbf{F}^{-(t-1)} \mathbf{u}_{2t}. \tag{3.26}$$

LEMMA 3.3. $\mathbf{s}_n^* \xrightarrow{L_2} \mathbf{s}^*$ *and hence* $\mathbf{s}_n^* \xrightarrow{p} \mathbf{s}^*$.

Proof. Under the condition of bounded second order moments and (3.24),

$$\mathrm{E}\|(\mathbf{s}_n^* - \mathbf{s}^*)(\mathbf{s}_n^* - \mathbf{s}^*)'\| \le c_{11} \Big(\sum_{t=n}^{\infty} \mathbf{F}^{-t} \Big)^2 \to 0 \quad \text{as } n \to \infty. \ ∎$$

Let

$$\mathbf{d}_n = \mathbf{F}^{-(n-1)} \sum_{t=1}^{n} \mathbf{s}_t u_{t+q+1} \quad \text{and} \quad \mathbf{h}_n = \sum_{t=1}^{n} \mathbf{F}^{-(t-1)} \mathbf{s}_n^* u_{n+q+2-t}.$$

LEMMA 3.4. $\mathbf{d}_n - \mathbf{h}_n \xrightarrow{p} 0$.

Proof. We have

$$\mathbf{d}_n - \mathbf{h}_n = \mathbf{F}^{-(n-1)} \sum_{t=1}^{n} \mathbf{F}^{(n-t)} (\mathbf{s}_{n-t+1}^* - \mathbf{s}_n^*) u_{n+q+2-t}.$$

Then from (3.26) it follows that

$$\mathrm{E}\|\mathbf{d}_n - \mathbf{h}_n\| \le \|\mathbf{F}^{-(n-1)}\| \sum_{t=1}^{n} \mathrm{E}\|\mathbf{F}^{(n-t)} (\mathbf{s}_{n-t+1}^* - \mathbf{s}_n^*) u_{n+q+2-t}\|$$

$$\le c_{12} \|\mathbf{F}^{-(n-1)}\| \sum_{t=1}^{n} \Big[\Big\{ \sum_{j=0}^{t-2} \|\mathbf{F}^{-(j+1)}\| \Big\}^2 \Big]^{1/2} = c_{13} n \|\mathbf{F}^{-n}\| \to 0 \quad \text{as } n \to \infty. \blacksquare$$

Let $\mathbf{K}$ be a nonsingular matrix such that $\mathbf{KFK}^{-1} = \mathrm{diag}(\gamma_1^{-1}, \ldots, \gamma_s^{-1})$. Writing $\mathbf{G} = \mathrm{diag}(\gamma_1, \ldots, \gamma_s)$, we have $\mathbf{F}^{-n} = \mathbf{K}^{-1}\mathbf{G}^n\mathbf{K}$ where

$$\|\mathbf{G}^n\| \sim c_{14} \check{\gamma}_1^{-n} \quad \text{as } n \to \infty. \tag{3.27}$$

Also let $\mathbf{S}_n$ and $\mathbf{S}$ be $s \times s$ diagonal matrices with ith diagonal element equal to the ith element of $\mathbf{Ks}_n^*$ and $\mathbf{Ks}^*$ respectively, and let $\boldsymbol{\vartheta}_n = (v_1, \ldots, v_s)'$, where $v_j = \sum_{i=1}^{n} \gamma_j^{(i-1)} u_{n+q+2-i}$ for $j = 1, \ldots, s$. Then $\mathbf{h}_n$ can be written in the form

$$\mathbf{h}_n = \mathbf{K}^{-1} \sum_{t=1}^{n} \mathbf{G}^{t-1} \mathbf{Ks}_n^* u_{n+q+2-t} = \mathbf{K}^{-1} \mathbf{S}_n \boldsymbol{\vartheta}_n. \tag{3.28}$$

Define the $s \times s$ diagonal matrix $\mathbf{S}_n^*$ with ith diagonal element equal to the ith element of $\mathbf{K} \sum_{t=1}^{[n/2]} \mathbf{f}_t u_t$, and $\boldsymbol{\vartheta}_n^* = (v_1^*, \ldots, v_s^*)$ with

$$v_j^* = \sum_{i=1}^{[n/2]} \gamma_j^{(i-1)} u_{n+q+2-i}, \qquad j = 1, \ldots, s.$$

Here $\mathbf{S}_n^*$ and $\boldsymbol{\vartheta}_n^*$ are partial sums consisting of only $[n/2]$ of the u_i's. However $\mathbf{S}_n^*$ depends on the first $[n/2]$ observations of u_t, while $\boldsymbol{\vartheta}_n^*$ depends on the last $[n/2]$ observations. Since $\mathbf{S}_n^*$ and $\boldsymbol{\vartheta}_n^*$ are separated by $q + 1$ intervening u_i's, they are independently distributed.

LEMMA 3.5. *$\mathbf{S}_n$ and $\boldsymbol{\vartheta}_n$ are asymptotically independent.*

Proof. From (1.2) and (3.24) we have

$$\mathrm{E}\|(\mathbf{S}_n - \mathbf{S}_n^*)(\mathbf{S}_n - \mathbf{S}_n^*)'\| \le c_{15} \|\mathbf{F}^{-2[n/2]}\| \to 0 \quad \text{as } n \to \infty.$$

Hence $\mathbf{S}_n - \mathbf{S}_n^* \xrightarrow{L_2} 0$, which implies $\mathbf{S}_n - \mathbf{S}_n^* \xrightarrow{p} 0$.

Again by (1.2) and (3.27),

$$\mathrm{E}\|(\boldsymbol{\vartheta}_n - \boldsymbol{\vartheta}_n^*)(\boldsymbol{\vartheta}_n - \boldsymbol{\vartheta}_n^*)'\| \le c_{16}\|\mathbf{G}^{2[n/2]}\| \to 0 \quad \text{as } n \to \infty.$$

Hence, $\boldsymbol{\vartheta}_n - \boldsymbol{\vartheta}_n^* \xrightarrow{L_2} 0$, which implies $\boldsymbol{\vartheta}_n - \boldsymbol{\vartheta}_n^* \xrightarrow{p} 0$.

Since $\mathbf{S}_n^*$ and $\boldsymbol{\vartheta}_n^*$ are mutually independent random variables, $\mathbf{S}_n$ and $\boldsymbol{\vartheta}_n$ are asymptotically independent. ∎

LEMMA 3.6. *$\mathbf{S}_n^* \xrightarrow{L_2} \mathbf{S}$ and $\boldsymbol{\vartheta}_n^* \xrightarrow{L_2} \boldsymbol{\vartheta}$, where $\boldsymbol{\vartheta} = (\bar{v}_1, \ldots, \bar{v}_s)$ with $\bar{v}_j = \sum_{i=1}^{\infty} \gamma_j^{(i-1)} u_{n+q+2-i}$ for $j = 1, \ldots, s$.*

Proof. The proof is similar to that of Lemma 3.5. ∎

Next, define

$$\Gamma = \begin{pmatrix} (1-\gamma_1^2)^{-1} & (1-\gamma_1\gamma_2)^{-1} & \ldots & (1-\gamma_1\gamma_s)^{-1} \\ \ldots & \ldots & \ldots & \ldots \\ (1-\gamma_1\gamma_s)^{-1} & (1-\gamma_2\gamma_s)^{-1} & \ldots & (1-\gamma_s^2)^{-1} \end{pmatrix},$$

$$\mathbf{F}^* = \sum_{i=1}^{\infty} \mathbf{F}^{-(i-1)} \mathbf{s}^* (\mathbf{s}^*)' (\mathbf{F}^{-(i-1)})'.$$

Then with $\gamma^{(i-1)} = (\gamma_1^{i-1}, \ldots, \gamma_s^{i-1})'$ we observe that

$$\mathbf{F}^* = \mathbf{K}^{-1} \sum_{i=1}^{\infty} \mathbf{S}\gamma^{(i-1)} (\gamma^{(i-1)})' \mathbf{S}' (\mathbf{K}^{-1})' = \mathbf{K}^{-1} \mathbf{S} \Gamma \mathbf{S}' (\mathbf{K}^{-1})'. \tag{3.29}$$

Taking $\mathbf{K}_n = \mathbf{F}^{-(n+q-1)}$ we have the following theorem.

THEOREM 3.2. *Under* (1.2) *and* (3.24),

$$\mathbf{K}_{n-q+1} \sum_{t=1}^{n} \mathbf{s}_t \mathbf{s}_{t+q}' \mathbf{K}_n' \xrightarrow{p} \mathbf{F}^*. \tag{3.30}$$

If in addition e_t's are Gaussian, then $\mathbf{F}^*$ is positive definite a.s. and

$$(\mathbf{K}_n')^{-1} \Big(\sum_{t=1}^{n} \mathbf{s}_t \mathbf{s}_{t+q}' \Big)^{-1} \Big(\sum_{t=1}^{n} \mathbf{s}_t u_{t+q+1} \Big) \xrightarrow{d} \mathrm{N}_1^*, \tag{3.31}$$

where $\mathrm{N}_1^* = \mathbf{K}'\mathbf{S}^{-1}\Gamma^{-1}\boldsymbol{\vartheta}$, $\boldsymbol{\vartheta}$ being an s-variate Gaussian variable with mean zero and dispersion matrix $\mathbf{V} = ((v_{ij}))$ with

$$v_{ij} = \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} \gamma_i^{(k-1)} \gamma_j^{(l-1)} \mathrm{E}(u_{n+q+2-k} u_{n+q+2-l}).$$

Also $\boldsymbol{\vartheta}$ is independent of $\mathbf{K}'\mathbf{S}^{-1}\Gamma^{-1}$.

Proof. Since $\mathbf{s}^*$ is convergent a.s, under (3.24) and using Lemma 3.6,

$$\begin{aligned}
&\left\|\mathbf{F}^{-(n-1)}\sum_{t=1}^{n}\mathbf{s}_t\mathbf{s}'_{t+q}(\mathbf{F}^{-(n+q-1)})' - \mathbf{F}^*\right\| \\
&\qquad \le \left\|\sum_{t=n}^{\infty}\mathbf{F}^{-t}\mathbf{s}^*(\mathbf{s}^*)'(\mathbf{F}^{-t})'\right\| + \left\|\sum_{t=0}^{n}\mathbf{F}^{-t}(\mathbf{s}^*_{n-t}(\mathbf{s}^*_{n+q-t})' - \mathbf{s}^*(\mathbf{s}^*)')(\mathbf{F}^{-t})'\right\| \\
&\qquad \le \|\mathbf{s}^*(\mathbf{s}^*)'\|\sum_{t=n}^{\infty}\|\mathbf{F}^{-t}\|^2 + \sum_{t=0}^{\infty}\|\mathbf{F}^{-t}\|^2\|\mathbf{s}^*_{n-t}(\mathbf{s}^*_{n+q-t})' - \mathbf{s}^*(\mathbf{s}^*)'\| \xrightarrow{p} 0
\end{aligned}$$

as $n \to \infty$. Since e_t's and hence u_t's are Gaussian, $\mathbf{s}^*_n$ being a linear transform of Gaussian variables is also Gaussian. As $\mathbf{s}^*_n \xrightarrow{L_2} \mathbf{s}^*$, $\mathbf{s}^*$ is also Gaussian. Hence, $\mathrm{P}(\mathbf{t}'\mathbf{s}^* = 0) = 0$ for all $\mathbf{t} \in \mathbb{R}^s - \{\mathbf{0}\}$. Thus by Lai and Wei (1983), $\mathbf{F}^*$ is positive definite a.s.

Define $\widetilde{\mathbf{u}}_n = (u_n, \dots, u_{q+2})'$ and

$$\widetilde{\Sigma}_n = \begin{pmatrix} \mathrm{E}(u_n^2) & \dots & \mathrm{E}(u_n u_{q+2}) \\ \dots & \dots & \dots \\ \mathrm{E}(u_n u_{q+2}) & \dots & \mathrm{E}(u_{q+2}^2) \end{pmatrix}, \quad \widetilde{\mathbf{G}}_n = \begin{pmatrix} 1 & \gamma_1 & \dots & \gamma_1^{n-1} \\ \dots & \dots & \dots & \dots \\ 1 & \gamma_s & \dots & \gamma_s^{n-1} \end{pmatrix}.$$

Thus $\boldsymbol{\vartheta}_n = \widetilde{\mathbf{G}}_n\widetilde{\mathbf{u}}_n$. As the roots of the explosive component are distinct, the rows of $\widetilde{\mathbf{G}}_n$ are linearly independent. Again since $\widetilde{\mathbf{u}}_n$ is Gaussian with mean zero and dispersion matrix $\widetilde{\Sigma}_n$, $\boldsymbol{\vartheta}_n$ is Gaussian with mean zero and dispersion matrix $\widetilde{\mathbf{G}}_n\widetilde{\Sigma}_n\widetilde{\mathbf{G}}'_n$. Hence as $\boldsymbol{\vartheta}_n \xrightarrow{L_2} \boldsymbol{\vartheta}$, $\boldsymbol{\vartheta}$ is also Gaussian with mean zero and dispersion matrix $\mathbf{V}$.

By Lemmas 3.4–3.6, $\mathbf{d}_n \xrightarrow{d} \mathbf{K}^{-1}\bar{\mathbf{S}}\boldsymbol{\vartheta}$. Using this and (3.30) in

$$\begin{aligned}
&((\mathbf{F}^{-(n+q-1)})')^{-1}\Big(\sum_{t=1}^{n}\mathbf{s}_t\mathbf{s}'_{t+q}\Big)^{-1}\sum_{t=1}^{n}\mathbf{s}_t u_{t+q+1} \\
&\qquad\qquad = \Big(\mathbf{F}^{-(n-1)}\sum_{t=1}^{n}\mathbf{s}_t\mathbf{s}'_{t+q}(\mathbf{F}^{-(n+q-1)})'\Big)^{-1}\mathbf{d}_n,
\end{aligned}$$

the theorem follows. ∎

3.3. The MA stable component. From (2.13), writing

$$\mathbf{L}_2 = \begin{pmatrix} \pi_1^* & \dots & \pi_{c-1}^* & \pi_c^* \\ & \mathrm{I}_{c-1} & & \mathbf{0}_{c-1} \end{pmatrix} \quad \text{and} \quad \mathbf{v}_{1t} = (e_t, \mathbf{0}'_{c-1})',$$

the stable part of the moving average component can be written as

(3.32) $$\mathbf{q}_t = \mathbf{L}_2\mathbf{q}_{t-1} + \mathbf{v}_{1t}, \quad t = 1, 2, \dots.$$

Let $\breve{\pi}_1 = \max_{1\le j\le c} |\pi_j^{-1}| < 1$. Then

$$\|\mathbf{L}_2^n\| \sim c_{17}\breve{\pi}_1^n \quad \text{as } n \to \infty. \tag{3.33}$$

Let $\mathbf{M}_n = n^{-1/2}\mathrm{I}_n$ and $\Sigma_2 = \mathrm{E}(\mathbf{q}_n\mathbf{q}_n')$.

THEOREM 3.3. *Under the conditions* (1.2) *and* (3.33),

$$\mathbf{M}_n \sum_{i=1}^{n} \mathbf{q}_{t+q}\mathbf{q}_{t+q}'\mathbf{M}_n' \xrightarrow{p} \Sigma_2, \tag{3.34}$$

$$(\mathbf{M}_n')^{-1}\Big(\sum_{i=1}^{n} \mathbf{q}_{t+q}\mathbf{q}_{t+q}'\Big)^{-1}\Big(\sum_{i=1}^{n} \mathbf{q}_{t+q}e_{t+q+1}\Big) \xrightarrow{d} \mathrm{N}_c(0, \Sigma_2^{-1}\Sigma_2^*\Sigma_2^{-1}), \tag{3.35}$$

where $\Sigma_2^* = \mathrm{E}(\mathbf{q}_{q+1}\mathbf{q}_{q+1}'e_{q+2}^2)$.

Proof. The proof is similar to that of Theorem 3.1 but somewhat simpler because, unlike the u_t's in (3.1), the e_t's in (3.32) are uncorrelated. ∎

3.4. The MA explosive component. From (2.14), writing

$$\widetilde{\mathbf{F}} = \begin{pmatrix} \eta_1^* & \ldots & \eta_{d-1}^* & \eta_d^* \\ & \mathrm{I}_{d-1} & & \mathbf{0}_{d-1} \end{pmatrix} \quad \text{and} \quad \mathbf{v}_{2t} = (e_t, \mathbf{0}_{d-1}')',$$

the explosive component of the moving average part can be written as

$$\mathbf{p}_t = \widetilde{\mathbf{F}}\mathbf{p}_{t-1} + \mathbf{v}_{2t}, \qquad t = 1, 2, \ldots. \tag{3.36}$$

Let $\breve{\eta}_1 = \min_{1\le j\le d} |\eta_j^{-1}| > 1$ and $\breve{\eta}_2 = \max_{1\le j\le d} |\eta_j^{-1}| > 1$. Then $\|\widetilde{\mathbf{F}}^n\| \sim c_{18}\breve{\eta}_2^n$ and

$$\|\widetilde{\mathbf{F}}^{-n}\| \sim c_{19}\breve{\eta}_1^{-n} \quad \text{as } n \to \infty. \tag{3.37}$$

Let $\widetilde{\mathbf{K}}$ be a nonsingular matrix such that $\widetilde{\mathbf{K}}\widetilde{\mathbf{F}}\widetilde{\mathbf{K}}^{-1} = \mathrm{diag}(\eta_1^{-1}, \ldots, \eta_d^{-1})$ and $\widetilde{\mathbf{s}} = \sum_{t=1}^{\infty} \widetilde{\mathbf{F}}^{-(t-1)}\mathbf{v}_{2t}$. Define the $d \times d$ diagonal matrix $\widetilde{\mathbf{S}}$ whose ith diagonal element is the ith element of $\widetilde{\mathbf{K}}\widetilde{\mathbf{s}}$. Let

$$\mathbf{N}_n = \widetilde{\mathbf{F}}^{-(n+q-1)}, \qquad \Lambda = \begin{pmatrix} (1-\eta_1^2)^{-1} & \ldots & (1-\eta_1\eta_d)^{-1} \\ \ldots & \ldots & \ldots \\ (1-\eta_1\eta_d)^{-1} & \ldots & (1-\eta_d^2)^{-1} \end{pmatrix},$$

$$\widetilde{\mathbf{F}}^* = \sum_{i=1}^{\infty} \widetilde{\mathbf{F}}^{-(i-1)}\widetilde{\mathbf{s}}\widetilde{\mathbf{s}}'(\widetilde{\mathbf{F}}^{-(i-1)})' = \widetilde{\mathbf{K}}^{-1}\widetilde{\mathbf{S}}\Lambda\widetilde{\mathbf{S}}'(\widetilde{\mathbf{K}}^{-1})'.$$

THEOREM 3.4. *Under* (1.2) *and* (3.37),

$$\mathbf{N}_n \sum_{t=1}^{n} \mathbf{p}_{t+q}\mathbf{p}_{t+q}'\mathbf{N}'_n \xrightarrow{p} \widetilde{\mathbf{F}}^*. \tag{3.38}$$

In addition if e_t's are Gaussian, then $\widetilde{\mathbf{F}}^*$ *is positive definite a.s. Also*

$$(\mathbf{N}_n')^{-1}\Big(\sum_{t=1}^n \mathbf{p}_{t+q}\mathbf{p}_{t+q}'\Big)^{-1}\sum_{t=1}^n \mathbf{p}_{t+q}e_{t+q+1} \xrightarrow{d} \mathrm{N}_2^*, \tag{3.39}$$

where $\mathrm{N}_2^* = \widetilde{\mathbf{K}}'\widetilde{\mathbf{S}}^{-1}\Lambda^{-1}\widetilde{\boldsymbol{\vartheta}}$, $\widetilde{\boldsymbol{\vartheta}}$ *being a d-variate Gaussian variable with mean zero and dispersion matrix* $\tilde{\mathbf{V}} = ((\tilde{v}_{ij}))$ *with* $\tilde{v}_{ij} = \sigma^2\sum_{k=1}^\infty\sum_{l=1}^\infty \eta_i^{(k-1)}\eta_j^{(l-1)}$. *Also* $\widetilde{\boldsymbol{\vartheta}}$ *is independent of* $\widetilde{\mathbf{K}}'\widetilde{\mathbf{S}}^{-1}\Lambda^{-1}$.

Proof. Similar to that of Theorem 3.2. ∎

4. The Main Theorem

THEOREM 4.1. *Under conditions* (1.2), (3.3), (3.24), (3.34) *and* (3.39), *as* $n \to \infty$,

$$\mathbf{G}_n\mathbf{T}\mathbf{D}_n\mathbf{T}'\mathbf{G}_n' \overset{p}{\sim} \operatorname{diag}(\Sigma_1, \mathbf{F}^*, \Sigma_2, \widetilde{\mathbf{F}}^*), \tag{4.1}$$

$$(\mathbf{T}'\mathbf{G}_n')^{-1}(\hat{\boldsymbol{\theta}}_n - \boldsymbol{\theta}) \overset{d}{\sim} (\mathrm{N}_r, \mathrm{N}_1^*, \mathrm{N}_c, \mathrm{N}_2^*)', \tag{4.2}$$

where the stable and explosive components are asymptotically independent of each other, but the two stable components and the two explosive components of the AR and MA parts are not.

To prove Theorem 4.1 we will require the following lemmas.

LEMMA 4.1. *Under conditions* (1.2), (3.3), (3.24), (3.34) *and* (3.39),

(i) $\mathbf{J}_n\sum_{t=1}^n \mathbf{r}_t\mathbf{s}_{t+q}'\mathbf{K}_n' \xrightarrow{p} 0$,

(ii) $\mathbf{M}_n\sum_{t=1}^n \mathbf{q}_{t+q}\mathbf{p}_{t+q}'\mathbf{N}_n' \xrightarrow{p} 0$.

Proof. Under condition (1.2),

$$\mathrm{E}\|n^{-1/2}\mathbf{r}_t\|^2 \le n^{-1}c_{20}\Big(\sum_{j=0}^{t-1}\|\mathbf{L}_1^j\|\Big)^2 \to 0 \quad \text{as } n \to \infty \text{ uniformly in } t,$$

so that

$$\sup_t n^{-1/2}\mathbf{r}_t \xrightarrow{p} 0. \tag{4.3}$$

Again since $\mathbf{F}^{-(n+q-1)}\mathbf{S}_{t+q}$ is bounded in probability, there exists an increasing sequence m_n' ($m_n'/n \to 1$ as $n \to \infty$) such that

$$n^{-1/2}\sum_{t=m_n'+1}^n \mathbf{r}_t\mathbf{s}_{t+q}'(\mathbf{F}^{-(n+q-1)})' \xrightarrow{p} 0. \tag{4.4}$$

Also under (3.24), as $n \to \infty$,

$$\sup_t \mathrm{E}\Big\| n^{-1/2} \sum_{t=1}^{m_n'} \mathbf{r}_t \mathbf{s}_{t+q}' (\mathbf{F}^{-(n+q-1)})' \Big\| \\ \leq \sup_t n^{-1/2} \sum_{t=1}^{m_n'} \mathrm{E}\|\mathbf{r}_t\|^2 \mathrm{E}\|\mathbf{s}_{t+q}^*\|^2 \|\mathbf{F}^{-n+t}\| \to 0$$

so that

$$n^{-1/2} \sum_{t=1}^{m_n'} \mathbf{r}_t \mathbf{s}_{t+q}' (\mathbf{F}^{-(n+q-1)})' \xrightarrow{p} 0. \tag{4.5}$$

Combining (4.4) and (4.5), (i) follows. (ii) follows similarly. ∎

Lemma 4.2. *$\mathbf{J}_n \sum_{t=1}^n \mathbf{r}_t u_{t+q+1}$ and $\mathbf{K}_n \sum_{t=1}^n \mathbf{s}_t u_{t+q+1}$ are asymptotically independent.*

Proof. Using Lemma 3.4 and the expression in (3.28), it is enough to show that $\mathbf{h}_n$ is asymptotically independent of $n^{-1/2} \sum_{t=1}^n \mathbf{r}_t u_{t+q+1}$.

Consider any sequence $k_n \uparrow \infty$ such that $k_n/n \to 0$ as $n \to \infty$. By Lemma 3.5, for each n, $\mathbf{S}_n$ depends on e_i for $1 \leq i \leq k_n$ and $\boldsymbol{\vartheta}_n$ depends on e_i for $n - k_n + 1 \leq i \leq n$. Hence the lemma will follow if it can be shown that for large n, $n^{-1/2} \sum_{t=1}^n \mathbf{r}_t u_{t+q+1}$ depends only on e_i, $k_n + 1 \leq i \leq n - k_n$.

We first show that

$$n^{-1/2} \sum_{t=1}^{n} \mathbf{r}_t u_{t+q+1} - n^{-1/2} \sum_{t=k_n+q+1}^{n-k_n-q-1} \mathbf{r}_t u_{t+q+1} \xrightarrow{p} 0. \tag{4.6}$$

Since $\|\mathbf{r}_t u_{t+q+1}\|$ is a mixing process, by Theorem 20.1 of Billingsley (1968),

$$\mathrm{E}\Big\| n^{-1} \Big(\sum_{t=1}^{k_n+q} \mathbf{r}_t u_{t+q+1} \Big) \Big(\sum_{t=1}^{k_n+q} \mathbf{r}_t u_{t+q+1} \Big)' \Big\| \\ \leq \frac{k_n + q}{n} \mathrm{E}\Big(\frac{1}{\sqrt{k_n + q}} \sum_{t=1}^{k_n+q} \|\mathbf{r}_t u_{t+q+1}\| \Big)^2 \to 0 \quad \text{as } n \to \infty, \tag{4.7}$$

the term within bracket being bounded. Similarly,

$$\mathrm{E}\Big\| n^{-1} \Big(\sum_{t=n-k_n-q}^{n} \mathbf{r}_t u_{t+q+1} \Big) \Big(\sum_{t=n-k_n-q}^{n} \mathbf{r}_t u_{t+q+1} \Big)' \Big\| \to 0 \quad \text{as } n \to \infty. \tag{4.8}$$

Combining (4.7) and (4.8) yields (4.6).

Next define $\mathbf{r}_j^* = \mathbf{u}_{1j} + \mathbf{L}_1 \mathbf{u}_{1,j-1} + \cdots + \mathbf{L}_1^{j-k_n-1-q} \mathbf{u}_{1,k_n+1+q}$. Then under (1.2),

$$\mathrm{E}\Big\|n^{-1}\Big(\sum_{j=k_n+q+1}^{n}(\mathbf{r}_j-\mathbf{r}_j^*)u_{j+q+1}\Big)\Big(\sum_{j=k_n+q+1}^{n}(\mathbf{r}_j-\mathbf{r}_j^*)u_{j+q+1}\Big)'\Big\|$$
$$\le c_{21}n^{-1}\sum_{j=k_n+q+1}^{\infty}\sum_{k=k_n+q+1}^{\infty}\|\mathbf{L}_1^{j-k_n-q-1}\|\cdot\|\mathbf{L}_1^{k-k_n-q-1}\|\to 0 \quad \text{as } n\to\infty.$$

Hence

$$n^{-1/2}\sum_{t=k_n+q+1}^{n-k_n-q-1}\mathbf{r}_t u_{t+q+1}-n^{-1/2}\sum_{t=k_n+q+1}^{n-k_n-q-1}\mathbf{r}_t^* u_{t+q+1}\xrightarrow{p}0,$$

and the assertion follows. ■

Corollary 4.1. *The following are asymptotically independent:*

(i) $\mathbf{M}_n\sum_{t=1}^{n}\mathbf{q}_{t+q}e_{t+q+1}$ *and* $\mathbf{K}_n\sum_{t=1}^{n}\mathbf{s}_t u_{t+q+1}$,
(ii) $\mathbf{J}_n\sum_{t=1}^{n}\mathbf{r}_t u_{t+q+1}$ *and* $\mathbf{N}_n\sum_{t=1}^{n}\mathbf{p}_{t+q}e_{t+q+1}$,
(iii) $\mathbf{M}_n\sum_{t=1}^{n}\mathbf{q}_{t+q}e_{t+q+1}$ *and* $\mathbf{N}_n\sum_{t=1}^{n}\mathbf{p}_{t+q}e_{t+q+1}$.

Proof. The proofs are similar to that of Lemma 4.2.

Corollary 4.2. *The following terms are dependent even for large n:*

(i) $\mathbf{J}_n\sum_{t=1}^{n}\mathbf{r}_t u_{t+q+1}$ *and* $\mathbf{M}_n\sum_{t=1}^{n}\mathbf{q}_{t+q}e_{t+q+1}$,
(ii) $\mathbf{K}_n\sum_{t=1}^{n}\mathbf{s}_t u_{t+q+1}$ *and* $\mathbf{N}_n\sum_{t=1}^{n}\mathbf{p}_{t+q}e_{t+q+1}$.

Proof. Following the arguments in Lemma 4.2, for large n, both the terms in (i) depend on e_i, $k_n+1\le i\le n-k_n$, and hence are dependent.

Similarly, the terms in (ii) are dependent since for large n, $\mathbf{S}_n$ and $\widetilde{\mathbf{S}}_n$ depend on e_i for $1\le i\le k_n$ and $\boldsymbol{\vartheta}_n$ and $\widetilde{\boldsymbol{\vartheta}}_n$ depend on e_i for $n-k_n+1\le i\le n$. ■

Proof of Theorem 4.1. The theorem follows from Theorems 3.1–3.4, Lemmas 4.1 and 4.2 and Corollaries 4.1 and 4.2.

5. Concluding remarks. In this paper we have derived the asymptotic distribution of the estimated ARMA parameters taking the instrumental variable estimator for the AR component and the derived AR process estimator for the MA component. The latter is unobservable and hence cannot be directly used to estimate β. As suggested by Chan and Tsay (1996), the derived process $Y_t(\beta^0)$ can be constructed from an initial value β^0 of β and then iterated to obtain the final solution $\hat{\beta}$.

The stable components are shown to be asymptotically normal while the explosive components are mixtures of normals. However, although the stable components are independent of the explosive components, between themselves the stable and explosive components are dependent. An implication of this is that even for large samples, inferences regarding each component cannot be done independently of the corresponding stable or explosive component.

The proofs indicate that the results would hold even if the e_t's are a sequence of martingale differences. However, for more complex dependent structures condition (1.2) may not suffice and some additional conditions may need to be imposed.

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Received on 20.4.2011;
revised version on 17.8.2011 (2084)